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Algebraic surfaces and hyperbolic geometry

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We describe the Kawamata–Morrison cone conjecture on the structure of Calabi–Yau varieties and more generally klt Calabi–Yau pairs. The conjecture is true in dimension 2. We also show that the automorphism group of a K3 surface need not be commensurable with an arithmetic group, which answers a question by Mazur.

1. Introduction

Many properties of a projective algebraic variety can be encoded by convex cones, such as the ample cone and the cone of curves. This is especially useful when these cones have only finitely many edges, as happens for Fano varieties. For a broader class of varieties which includes Calabi–Yau varieties and many rationally connected varieties, the Kawamata–Morrison cone conjecture predicts the structure of these cones. I like to think of this conjecture as what comes after the abundance conjecture. Roughly speaking, the cone theorem of Mori, Kawamata, Shokurov, Kollár, and Reid describes the structure of the curves on a projective variety X on which the canonical bundle K_X has negative degree; the abundance conjecture would give strong information about the curves on which K_X has degree zero; and the cone conjecture fully describes the structure of the curves on which K_X has degree zero.

We give a gentle summary of the proof of the cone conjecture for algebraic surfaces, with plenty of examples [Totaro 2010]. For algebraic surfaces, these cones are naturally described using hyperbolic geometry, and the proof can also be formulated in those terms.

Example 7.3 shows that the automorphism group of a K3 surface need not be commensurable with an arithmetic group. This answers a question by Barry Mazur [1993, Section 7].

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2. The main trichotomy

Let X be a smooth complex projective variety. There are three main types of varieties. (Not every variety is of one of these three types, but minimal model theory relates every variety to one of these extreme types.)

Fano. This means that $-K_X$ is ample. (We recall the definition of ampleness in Section 3.)

Calabi–Yau. We define this to mean that K_X is numerically trivial.

ample canonical bundle. This means that K_X is ample; it implies that X is of "general type."

Here, for X of complex dimension n, the *canonical bundle* K_X is the line bundle Ω^n_X of n-forms. We write $-K_X$ for the dual line bundle K^*_X, the determinant of the tangent bundle.

Example 2.1. Let X be a curve, meaning that X has complex dimension 1. Then X is Fano if it has genus zero, or equivalently if X is isomorphic to the complex projective line $\mathbb{P}^1$; as a topological space, this is the 2-sphere. Next, X is Calabi–Yau if X is an elliptic curve, meaning that X has genus 1. Finally, X has ample canonical bundle if it has genus at least 2.

Example 2.2. Let X be a smooth surface in $\mathbb{P}^3$. Then X is Fano if it has degree at most 3. Next, X is Calabi–Yau if it has degree 4; this is one class of *K3 surfaces*. Finally, X has ample canonical bundle if it has degree at least 5.

Belonging to one of these three classes of varieties is equivalent to the existence of a Kähler metric with Ricci curvature of a given sign [Yau 1978]. Precisely, a smooth projective variety is Fano if and only if it has a Kähler metric with positive Ricci curvature; it is Calabi–Yau if and only if it has a Ricci-flat Kähler metric; and it has ample canonical bundle if and only if it has a Kähler metric with negative Ricci curvature.

We think of Fano varieties as the most special class of varieties, with projective space as a basic example. Strong support for this idea is provided by Kollár, Miyaoka, and Mori's theorem that smooth Fano varieties of dimension n form a bounded family [Kollár et al. 1992]. In particular, there are only finitely many diffeomorphism types of smooth Fano varieties of a given dimension.

Example 2.3. Every smooth Fano surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or to a blow-up of $\mathbb{P}^2$ at at most 8 points. The classification of smooth Fano 3-folds is also known, by Iskovskikh, Mori, and Mukai; there are 104 deformation classes [Iskovkikh and Prokhorov 1999].

By contrast, varieties with ample canonical bundle form a vast and uncontrollable class. Even in dimension 1, there are infinitely many topological types of varieties with ample canonical bundle (curves of genus at least 2). Calabi–Yau varieties are on the border in terms of complexity. It is a notorious open question whether there are only finitely many topological types of smooth Calabi–Yau varieties of a given dimension. This is true in dimension at most 2. In particular, a smooth Calabi–Yau surface is either an abelian surface, a K3 surface, or a quotient of one of these surfaces by a free action of a finite group (and only finitely many finite groups occur this way).

3. Ample line bundles and the cone theorem

After a quick review of ample line bundles, this section states the cone theorem and its application to Fano varieties. Lazarsfeld's book [2004] is an excellent reference on ample line bundles.

Definition 3.1. A line bundle L on a projective variety X is *ample* if some positive multiple nL (meaning the line bundle $L^{\otimes n}$) has enough global sections to give a projective embedding

$$X \hookrightarrow \mathbb{P}^N.$$

(Here $N = \dim_{\mathbb{C}} H^0(X, nL) - 1$.)

One reason to investigate which line bundles are ample is in order to classify algebraic varieties. For classification, it is essential to know how to describe a variety with given properties as a subvariety of a certain projective space defined by equations of certain degrees.

Example 3.2. For X a curve, L is ample on X if and only if it has positive degree. We write $L \cdot X = \deg(L|_X) \in \mathbb{Z}$.

An $\mathbb{R}$-*divisor* on a smooth projective variety X is a finite sum

$$D = \sum a_i D_i$$

with $a_i \in \mathbb{R}$ and each D_i an irreducible divisor (codimension-one subvariety) in X. Write $N^1(X)$ for the "Néron–Severi" real vector space of $\mathbb{R}$-divisors modulo *numerical equivalence*: $D_1 \equiv D_2$ if $D_1 \cdot C = D_2 \cdot C$ for all curves C in X. (For me, a *curve* is irreducible.)

We can also define $N^1(X)$ as the subspace of the cohomology $H^2(X, \mathbb{R})$ spanned by divisors. In particular, it is a finite-dimensional real vector space. The dual vector space $N_1(X)$ is the space of 1-cycles $\sum a_i C_i$ modulo numerical equivalence, where C_i are curves on X. We can identify $N_1(X)$ with the subspace of the homology $H_2(X, \mathbb{R})$ spanned by algebraic curves.

Definition 3.3. The *closed cone of curves* $\overline{\mathrm{Curv}}(X)$ is the closed convex cone in $N_1(X)$ spanned by curves on X.

Definition 3.4. An $\mathbb{R}$-divisor D is *nef* if $D \cdot C \geq 0$ for all curves C in X. Likewise, a line bundle L on X is *nef* if the class $[L]$ of L (also called the first Chern class $c_1(L)$) in $N^1(X)$ is nef. That is, L has nonnegative degree on all curves in X.

Thus $\mathrm{Nef}(X) \subset N^1(X)$ is a closed convex cone, the *dual cone* to $\overline{\mathrm{Curv}}(X) \subset N_1(X)$.

Theorem 3.5 (Kleiman). *A line bundle L is ample if and only if $[L]$ is in the interior of the nef cone in $N^1(X)$.*

This is a *numerical* characterization of ampleness. It shows that we know the ample cone $\mathrm{Amp}(X) \subset N^1(X)$ if we know the cone of curves $\overline{\mathrm{Curv}}(X) \subset N_1(X)$. The following theorem gives a good understanding of the "K-negative" half of the cone of curves [Kollár and Mori 1998, Theorem 3.7]. A *rational* curve means a curve that is birational to $\mathbb{P}^1$.

Theorem 3.6. *(Cone theorem: Mori, Shokurov, Kawamata, Reid, Kollár). Let X be a smooth projective variety. Write $K_X^{<0} = \{u \in N_1(X) : K_X \cdot u < 0\}$. Then every extremal ray of $\overline{\mathrm{Curv}}(X) \cap K_X^{<0}$ is isolated, spanned by a rational curve, and can be contracted.*

In particular, every extremal ray of $\overline{\mathrm{Curv}}(X) \cap K_X^{<0}$ is rational (meaning that it is spanned by a $\mathbb{Q}$-linear combination of curves, not just an $\mathbb{R}$-linear combination), since it is spanned by a single curve. A *contraction* of a normal projective variety X means a surjection from X onto a normal projective variety Y with connected fibers. A contraction is determined by a face of the cone of curves $\overline{\mathrm{Curv}}(X)$, the set of elements of $\overline{\mathrm{Curv}}(X)$ whose image under the pushforward map $N_1(X) \to N_1(Y)$ is zero. The last statement in the cone theorem means that every extremal ray in the K-negative half-space corresponds to a contraction of X.

Corollary 3.7. *For a Fano variety X, the cone of curves $\overline{\mathrm{Curv}}(X)$ (and therefore the dual cone $\mathrm{Nef}(X)$) is rational polyhedral.*

A rational polyhedral cone means the closed convex cone spanned by finitely many rational points.

Proof. Since $-K_X$ is ample, K_X is negative on all of $\overline{\mathrm{Curv}}(X) - \{0\}$. So the cone theorem applies to all the extremal rays of $\overline{\mathrm{Curv}}(X)$. Since they are isolated and live in a compact space (the unit sphere), $\overline{\mathrm{Curv}}(X)$ has only finitely many extremal rays. The cone theorem also gives that these rays are rational. □

It follows, in particular, that a Fano variety has only finitely many different contractions. A simple example is the blow-up X of $\mathbb{P}^2$ at one point, which is Fano. In this case, $\overline{\mathrm{Curv}}(X)$ is a closed strongly convex cone in the two-dimensional real vector space $N_1(X)$, and so it has exactly two 1-dimensional faces. We can write down two contractions of X, $X \to \mathbb{P}^2$ (contracting a (-1)-curve) and $X \to \mathbb{P}^1$ (expressing X as a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$). Each of these morphisms must contract one of the two 1-dimensional faces of $\overline{\mathrm{Curv}}(X)$. Because the cone has no other nontrivial faces, these are the only nontrivial contractions of X.

4. Beyond Fano varieties

"Just beyond" Fano varieties, the cone of curves and the nef cone need not be rational polyhedral. Lazarsfeld's book [2004] gives many examples of this type, as do other books on minimal model theory [Debarre 2001; Kollár and Mori 1998].

Example 4.1. Let X be the blow-up of $\mathbb{P}^2$ at n very general points. For $n \leq 8$, X is Fano, and so $\overline{\mathrm{Curv}}(X)$ is rational polyhedral. In more detail, for $2 \leq n \leq 8$, $\overline{\mathrm{Curv}}(X)$ is the convex cone spanned by the finitely many (-1)-curves in X. (A (-1)-curve on a surface X means a curve C isomorphic to $\mathbb{P}^1$ with self-intersection number $C^2 = -1$.) For example, when $n = 6$, X can be identified with a cubic surface, and the (-1)-curves are the famous 27 lines on X.

But for $n \geq 9$, X is not Fano, since $(-K_X)^2 = 9-n$ (whereas a projective variety has positive degree with respect to any ample line bundle). For $p_1, \ldots, p_n$ very general points in $\mathbb{P}^2$, X contains infinitely many (-1)-curves; see [Hartshorne 1977, Exercise V.4.15]. Every curve C with $C^2 < 0$ on a surface spans an isolated extremal ray of $\overline{\mathrm{Curv}}(X)$, and so $\overline{\mathrm{Curv}}(X)$ is not rational polyhedral.

Notice that a (-1)-curve C has $K_X \cdot C = -1$, and so these infinitely many isolated extremal rays are on the "good" (K-negative) side of the cone of curves, in the sense of the cone theorem. The K-positive side is a mystery. It is conjectured (Harbourne–Hirschowitz) that the closed cone of curves of a very general blow-up of $\mathbb{P}^2$ at $n \geq 10$ points is the closed convex cone spanned by the (-1)-curves and the "round" positive cone $\{x \in N_1(X) : x^2 \geq 0 \text{ and } H \cdot x \geq 0\}$, where H is a fixed ample line bundle. This includes the famous Nagata conjecture [Lazarsfeld 2004, Remark 5.1.14] as a special case. By de Fernex, even if the Harbourne–Hirschowitz conjecture is correct, the intersection of $\overline{\mathrm{Curv}}(X)$ with the K-positive half-space, for X a very general blow-up of $\mathbb{P}^2$ at $n \geq 11$ points, is bigger than the intersection of the positive cone with the K-positive half-space, because the (-1)-curves stick out a lot from the positive cone [de Fernex 2010].

Example 4.2. Calabi–Yau varieties (varieties with $K_X \equiv 0$) are also "just beyond" Fano varieties ($-K_X$ ample). Again, the cone of curves of a Calabi–Yau variety need not be rational polyhedral.

For example, let X be an abelian surface, so $X \cong \mathbb{C}^2/\Lambda$ for some lattice $\Lambda \cong \mathbb{Z}^4$ such that X is projective. Then $\overline{\mathrm{Curv}}(X) = \mathrm{Nef}(X)$ is a round cone, the positive cone

$$\{x \in N^1(X) : x^2 \geq 0 \text{ and } H \cdot x \geq 0\},$$

where H is a fixed ample line bundle. (Divisors and 1-cycles are the same thing on a surface, and so the cones $\overline{\mathrm{Curv}}(X)$ and $\mathrm{Nef}(X)$ lie in the same vector space $N^1(X)$.) Thus the nef cone is not rational polyhedral if X has Picard number $\rho(X) := \dim_{\mathbb{R}} N^1(X)$ at least 3 (and sometimes when $\rho = 2$).

For a K3 surface, the closed cone of curves may be round, or may be the closed cone spanned by the (-2)-curves in X. (One of those two properties must hold, by [Kovács 1994].) There may be finitely or infinitely many (-2)-curves. See Section 5.1 for an example.

5. The cone conjecture

But there is a good substitute for the cone theorem for Calabi–Yau varieties, the *Morrison–Kawamata cone conjecture*. In dimension 2, this is a theorem of Sterk, Looijenga, and Namikawa [Sterk 1985; Namikawa 1985; Kawamata 1997]. We call this Sterk's theorem for convenience:

Theorem 5.1. *Let X be a smooth complex projective Calabi–Yau surface* (*meaning that K_X is numerically trivial*). *Then the action of the automorphism group* $\mathrm{Aut}(X)$ *on the nef cone* $\mathrm{Nef}(X) \subset N^1(X)$ *has a rational polyhedral fundamental domain.*

Remark 5.2. For any variety X, if $\mathrm{Nef}(X)$ is rational polyhedral, then the group $\mathrm{Aut}^*(X) := \mathrm{im}(\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)))$ is finite. This is easy: the group $\mathrm{Aut}^*(X)$ must permute the set consisting of the smallest integral point on each extremal ray of $\mathrm{Nef}(X)$. Sterk's theorem implies the remarkable statement that the converse is also true for Calabi–Yau surfaces. That is, if the cone $\mathrm{Nef}(X)$ is not rational polyhedral, then $\mathrm{Aut}^*(X)$ must be infinite. Note that $\mathrm{Aut}^*(X)$ coincides with the discrete part of the automorphism group of X up to finite groups, because $\ker(\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)))$ is an algebraic group and hence has only finitely many connected components.

Sterk's theorem should generalize to Calabi–Yau varieties of any dimension (the Morrison–Kawamata cone conjecture). But in dimension 2, we can visualize it better, using hyperbolic geometry.

Indeed, let X be any smooth projective surface. The intersection form on $N^1(X)$ always has signature $(1, n)$ for some n (the Hodge index theorem). So $\{x \in N^1(X) : x^2 > 0\}$ has two connected components, and the positive cone $\{x \in N^1(X) : x^2 > 0 \text{ and } H \cdot x > 0\}$ is the standard round cone. As a result, we can identify the quotient of the positive cone by $\mathbb{R}^{>0}$ with *hyperbolic n-space*.

One way to see this is that the negative of the Lorentzian metric on $N^1(X) = \mathbb{R}^{1,n}$ restricted to the quadric $\{x^2 = 1\}$ is a Riemannian metric with curvature -1.

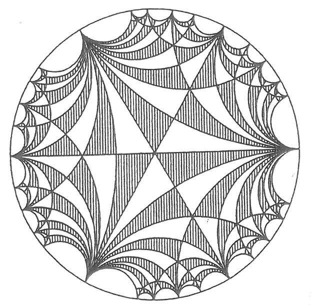
For any projective surface X, $\mathrm{Aut}(X)$ preserves the intersection form on $N^1(X)$. So $\mathrm{Aut}^*(X)$ is always a group of isometries of hyperbolic n-space, where $n = \rho(X) - 1$.

By definition, two groups G_1 and G_2 are *commensurable*, written $G_1 \doteq G_2$, if some finite-index subgroup of G_1 is isomorphic to a finite-index subgroup of G_2. A group *virtually* has a given property if some subgroup of finite index has the property. Since the groups we consider are all virtually torsion-free, we are free to replace a group G by G/N for a finite normal subgroup N (that is, G and G/N are commensurable).

5.1. ***Examples.*** For an abelian surface X with Picard number at least 3, the cone $\mathrm{Nef}(X)$ is round, and so $\mathrm{Aut}^*(X)$ must be infinite by Sterk's theorem. (For abelian surfaces, the possible automorphism groups were known long before; see [Mumford 1970, Section 21].)

For example, let $X = E \times E$ with E an elliptic curve (not having complex multiplication). Then $\rho(X) = 3$, with $N^1(X)$ spanned by the curves $E \times 0$, $0 \times E$, and the diagonal Δ_E in $E \times E$. So $\mathrm{Aut}^*(X)$ must be infinite. In fact,

$$\mathrm{Aut}^*(X) \cong \mathrm{PGL}(2, \mathbb{Z}).$$

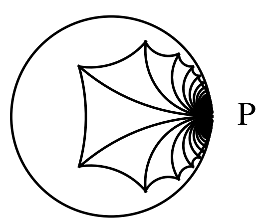


Here $\mathrm{GL}(2, \mathbb{Z})$ acts on $E \times E$ as on the direct sum of any abelian group with itself. This agrees with Sterk's theorem, which says that $\mathrm{Aut}^*(X)$ acts on the hyperbolic plane with a rational polyhedral fundamental domain; a fundamental domain for $\mathrm{PGL}(2, \mathbb{Z})$ acting on the hyperbolic plane (not preserving orientation) is given by any of the triangles in the figure.

For a K3 surface, the cone $\mathrm{Nef}(X)$ may or may not be the whole positive cone. For any projective surface, the nef cone modulo scalars is a convex subset of hyperbolic space. A finite polytope in hyperbolic space (even if some vertices are at infinity) has finite volume. So Sterk's theorem implies that, for a Calabi–Yau surface, $\mathrm{Aut}^*(X)$ acts with *finite covolume* on the convex set $\mathrm{Nef}(X)/\mathbb{R}^{>0}$ in hyperbolic space.

For example, let X be a K3 surface such that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}^3$ with intersection form

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$



Such a surface exists, since Nikulin showed that every even lattice of rank at most 10 with signature $(1, *)$ is the Picard lattice of some complex projective K3 surface [Nikulin 1979, Section 1.12]. Using the ideas of Section 6, one computes that the nef cone of X modulo scalars is the convex subset of the hyperbolic plane shown in the figure. The surface X has a unique elliptic fibration $X \to \mathbb{P}^1$, given by a nef line bundle P with $\langle P, P\rangle = 0$. The line bundle P appears in the figure as the point where $\text{Nef}(X)/\mathbb{R}^{>0}$ meets the circle at infinity. And X contains infinitely many (-2)-curves, whose orthogonal complements are the codimension-1 faces of the nef cone. Sterk's theorem says that $\text{Aut}(X)$ must act on the nef cone with rational polyhedral fundamental domain. In this example, one computes that $\text{Aut}(X)$ is commensurable with the Mordell–Weil group of the elliptic fibration (Pic^0 of the generic fiber of $X \to \mathbb{P}^1$), which is isomorphic to $\mathbb{Z}$. One also finds that all the (-2)-curves in X are sections of the elliptic fibration. The Mordell–Weil group moves one section to any other section, and so it divides the nef cone into rational polyhedral cones as in the figure.

6. Outline of the proof of Sterk's theorem

We discuss the proof of Sterk's theorem for K3 surfaces. The proof for abelian surfaces is the same, but simpler (since an abelian surface contains no (-2)-curves), and these cases imply the case of quotients of K3 surfaces or abelian surfaces by a finite group. For details, see [Kawamata 1997], based on the earlier [Sterk 1985; Namikawa 1985].

The proof of Sterk's theorem for K3 surfaces relies on the Torelli theorem of Piatetski-Shapiro and Shafarevich. That is, any isomorphism of Hodge structures between two K3s is realized by an isomorphism of K3s if it maps the nef cone into the nef cone. In particular, this lets us construct automorphisms of a K3 surface X: up to finite index, every element of the integral orthogonal group $O(\text{Pic}(X))$ that preserves the cone $\text{Nef}(X)$ is realized by an automorphism of X. (Here $\text{Pic}(X) \cong \mathbb{Z}^\rho$, and the intersection form has signature $(1, \rho(X) - 1)$ on $\text{Pic}(X)$.)

Moreover, $\text{Nef}(X)/\mathbb{R}^{>0}$ is a very special convex set in hyperbolic space $H_{\rho-1}$: it is the closure of a Weyl chamber for a discrete reflection group W acting on $H_{\rho-1}$. We can define W as the group generated by all reflections in vectors $x \in \text{Pic}(X)$ with $x^2 = -2$, or (what turns out to be the same) the group generated by reflections in all (-2)-curves in X. By the first description, W is a *normal* subgroup of $O(\text{Pic}(X))$. In fact, up to finite groups, $O(\text{Pic}(X))$ is the semidirect product group

$$O(\text{Pic}(X)) \doteq \text{Aut}(X) \ltimes W.$$

By general results on arithmetic groups going back to Minkowski, $O(\text{Pic}(X))$ acts on the positive cone in $N^1(X)$ with a rational polyhedral fundamental domain D. (This fundamental domain is not at all unique.) And the reflection group W acts on the positive cone with fundamental domain the nef cone of X. Therefore, after we arrange for D to be contained in the nef cone, $\text{Aut}(X)$ must act on the nef cone with the same rational polyhedral fundamental domain D, up to finite index. Sterk's theorem is proved.

7. Nonarithmetic automorphism groups

In this section, we show for the first time that the discrete part of the automorphism group of a smooth projective variety need not be commensurable with an arithmetic group. (Section 5 defines commensurability.) This answers a question raised by Mazur [1993, Section 7]. Corollary 7.2 applies to a large class of K3 surfaces.

An *arithmetic group* is a subgroup of the group of $\mathbb{Q}$-points of some $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}}$ which is commensurable with $H(\mathbb{Z})$ for some integral structure on $H_{\mathbb{Q}}$; this condition is independent of the integral structure [Serre 1979]. We view arithmetic groups as abstract groups, not as subgroups of a fixed Lie group.

Borcherds gave an example of a K3 surface whose automorphism group is not isomorphic to an arithmetic group [Borcherds 1998, Example 5.8]. But, as he says, the automorphism group in his example has a nonabelian free subgroup of finite index, and so it is commensurable with the arithmetic group $\text{SL}(2, \mathbb{Z})$. Examples of K3 surfaces with explicit generators of the automorphism group have been given by Keum, Kondo, Vinberg, and others; see [Dolgachev 2008, Section 5] for a survey.

Although they need not be commensurable with arithmetic groups, the automorphism groups G of K3 surfaces are very well-behaved in terms of geometric group theory. More generally this is true for the discrete part G of the automorphism group of a surface X which can be given the structure of a klt Calabi–Yau pair, as defined in Section 8. Namely, G acts cocompactly on a CAT(0) space (a precise notion of a metric space with nonpositive curvature). Indeed, the nef cone modulo scalars is a closed convex subset of hyperbolic space, and thus a CAT(-1) space [Bridson and Haefliger 1999, Example II.1.15]. Removing a G-invariant set of disjoint open horoballs gives a CAT(0) space on which G acts properly and cocompactly, by the proof of [Bridson and Haefliger 1999, Theorem II.11.27]. This implies all the finiteness properties one could want, even though G need not be arithmetic. In particular: G is finitely presented, a finite-index subgroup of G has a finite CW complex as classifying space, and

G has only finitely many conjugacy classes of finite subgroups [Bridson and Haefliger 1999, Theorem III.Γ.1.1].

For smooth projective varieties in general, very little is known. For example, is the discrete part G of the automorphism group always finitely generated? The question is open even for smooth projective rational surfaces. About the only thing one can say for an arbitrary smooth projective variety X is that G modulo a finite group injects into $\mathrm{GL}(\rho(X), \mathbb{Z})$, by the comments in Section 5.

In Theorem 7.1, a *lattice* means a finitely generated free abelian group with a symmetric bilinear form that is nondegenerate $\otimes\mathbb{Q}$.

Theorem 7.1. *Let M be a lattice of signature $(1, n)$ for $n \geq 3$. Let G be a subgroup of infinite index in $O(M)$. Suppose that G contains $\mathbb{Z}^{n-1}$ as a subgroup of infinite index. Then G is not commensurable with an arithmetic group.*

Corollary 7.2. *Let X be a $K3$ surface over any field, with Picard number at least 4. Suppose that X has an elliptic fibration with no reducible fibers and a second elliptic fibration with Mordell–Weil rank positive. (For example, the latter property holds if the second fibration also has no reducible fibers.) Suppose also that X contains a (-2)-curve. Then the automorphism group of X is a discrete group that is not commensurable with an arithmetic group.*

Example 7.3. Let X be the double cover of $\mathbb{P}^1 \times \mathbb{P}^1 = \{([x, y], [u, v])\}$ ramified along the following curve of degree $(4, 4)$:

$$0 = 16x^4u^4 + xy^3u^4 + y^4u^3v - 40x^4u^2v^2 - x^3yu^2v^2 - x^2y^2uv^3 + 33x^4v^4 - 10x^2y^2v^4 + y^4v^4.$$

Then X is a K3 surface whose automorphism group (over $\mathbb{Q}$, or over $\overline{\mathbb{Q}}$) is not commensurable with an arithmetic group.

Proof of Theorem 7.1. We can view $O(M)$ as a discrete group of isometries of hyperbolic n-space. Every solvable subgroup of $O(M)$ is virtually abelian [Bridson and Haefliger 1999, Corollary II.11.28 and Theorem III.Γ.1.1]. By the classification of isometries of hyperbolic space as elliptic, parabolic, or hyperbolic [Alekseevskij et al. 1993], the centralizer of any subgroup $\mathbb{Z} \subset O(M)$ is either commensurable with $\mathbb{Z}$ (if a generator g of $\mathbb{Z}$ is hyperbolic) or commensurable with $\mathbb{Z}^a$ for some $a \leq n - 1$ (if g is parabolic). These properties pass to the subgroup G of $O(M)$. Also, G is not virtually abelian, because it contains $\mathbb{Z}^{n-1}$ as a subgroup of infinite index, and $\mathbb{Z}^{n-1}$ is the largest abelian subgroup of $O(M)$ up to finite index. Finally, G acts properly and not cocompactly on hyperbolic n-space, and so G has virtual cohomological dimension at most $n - 1$ [Brown 1982, Proposition VIII.8.1].

Suppose that G is commensurable with some arithmetic group Γ. Thus Γ is a subgroup of the group of $\mathbb{Q}$-points of some $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}}$, and Γ is commensurable with $H(\mathbb{Z})$ for some integral structure on $H_{\mathbb{Q}}$. We freely change Γ by finite groups in what follows. So we can assume that $H_{\mathbb{Q}}$ is connected. After replacing $H_{\mathbb{Q}}$ by the kernel of some homomorphism to a product of copies of the multiplicative group G_m over $\mathbb{Q}$, we can assume that Γ is a *lattice* in the real group $H(\mathbb{R})$ (meaning that $\mathrm{vol}(H(\mathbb{R})/\Gamma) < \infty$), by Borel and Harish-Chandra [Borel and Harish-Chandra 1962, Theorem 9.4].

Every connected $\mathbb{Q}$-algebraic group $H_{\mathbb{Q}}$ is a semidirect product $R_{\mathbb{Q}} \ltimes U_{\mathbb{Q}}$ where $R_{\mathbb{Q}}$ is reductive and $U_{\mathbb{Q}}$ is unipotent [Borel and Serre 1964, Theorem 5.1]. By the independence of the choice of integral structure, we can assume that $\Gamma = R(\mathbb{Z}) \ltimes U(\mathbb{Z})$ for some arithmetic subgroups $R(\mathbb{Z})$ of $R_{\mathbb{Q}}$ and $U(\mathbb{Z})$ of $U_{\mathbb{Q}}$. Since every solvable subgroup of G is virtually abelian, $U_{\mathbb{Q}}$ is abelian, and $U(\mathbb{Z}) \cong \mathbb{Z}^a$ for some a. The conjugation action of $R_{\mathbb{Q}}$ on $U_{\mathbb{Q}}$ must be trivial; otherwise Γ would contain a solvable group of the form $\mathbb{Z} \ltimes \mathbb{Z}^a$ which is not virtually abelian. Thus $\Gamma = R(\mathbb{Z}) \times \mathbb{Z}^a$. But the properties of centralizers in G imply that any product group of the form $W \times \mathbb{Z}$ contained in G must be virtually abelian. Therefore, $a = 0$ and $H_{\mathbb{Q}}$ is reductive.

Modulo finite groups, the reductive group $H_{\mathbb{Q}}$ is a product of $\mathbb{Q}$-simple groups and tori, and Γ is a corresponding product modulo finite groups. Since any product group of the form $W \times \mathbb{Z}$ contained in G is virtually abelian, $H_{\mathbb{Q}}$ must be $\mathbb{Q}$-simple. Since the lattice Γ in $H(\mathbb{R})$ is isomorphic to the discrete subgroup G of $O(M) \subset O(n, 1)$ (after passing to finite-index subgroups), Prasad showed that $\dim(H(\mathbb{R})/K_H) \le \dim(O(n, 1)/O(n)) = n$, where K_H is a maximal compact subgroup of $H(\mathbb{R})$. Moreover, since G has infinite index in $O(M)$ and hence infinite covolume in $O(n, 1)$, Prasad showed that either $\dim(H(\mathbb{R})/K_H) \le n-1$ or else $\dim(H(\mathbb{R})/K_H) = n$ and there is a homomorphism from $H(\mathbb{R})$ onto $\mathrm{PSL}(2, \mathbb{R})$ [Prasad 1976, Theorem B].

Suppose that $\dim(H(\mathbb{R})/K_H) \le n - 1$. We know that Γ acts properly on $H(\mathbb{R})/K_H$ and that Γ contains $\mathbb{Z}^{n-1}$. The quotient $\mathbb{Z}^{n-1}\backslash H(\mathbb{R})/K_H$ is a manifold of dimension $n - 1$ with the homotopy type of the $(n - 1)$-torus (in particular, with nonzero cohomology in dimension $n - 1$), and so it must be compact. So $\mathbb{Z}^{n-1}$ has finite index in Γ, contradicting our assumption.

So $\dim(H(\mathbb{R})/K_H) = n$ and $H(\mathbb{R})$ maps onto $\mathrm{PSL}(2, \mathbb{R})$. We can assume that $H_{\mathbb{Q}}$ is simply connected. Since H is $\mathbb{Q}$-simple, H is equal to the restriction of scalars $R_{K/\mathbb{Q}}L$ for some number field K and some absolutely simple and simply connected group L over K [Tits 1966, Section 3.1]. Since $H(\mathbb{R})$ maps onto $\mathrm{PSL}(2, \mathbb{R})$, L must be a form of $\mathrm{SL}(2)$. We showed that $G \cong \Gamma$ has virtual cohomological dimension at most $n-1$, and so Γ must be a noncocompact subgroup of $H(\mathbb{R})$. Equivalently, H has $\mathbb{Q}$-rank greater than zero [Borel and Harish-Chandra 1962, Lemma 11.4, Theorem 11.6], and so $\mathrm{rank}_K(L) = \mathrm{rank}_{\mathbb{Q}}(H)$ is greater than zero. Therefore, L is isomorphic to $\mathrm{SL}(2)$ over K.

It follows that Γ is commensurable with $\mathrm{SL}(2, o_K)$, where o_K is the ring of integers of K. So we can assume that Γ contains the semidirect product

$$o_K^* \ltimes o_K = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \right\} \subset \mathrm{SL}(2, o_K).$$

If the group of units o_K^* has positive rank, then $o_K^* \ltimes o_K$ is a solvable group which is not virtually abelian. So the group of units is finite, which means that K is either $\mathbb{Q}$ or an imaginary quadratic field, by Dirichlet. If K is imaginary quadratic, then $H_{\mathbb{Q}} = R_{K/\mathbb{Q}} \mathrm{SL}(2)$ and $H(\mathbb{R}) = \mathrm{SL}(2, \mathbb{C})$, which does not map onto $\mathrm{PSL}(2, \mathbb{R})$. Therefore $K = \mathbb{Q}$ and $H_{\mathbb{Q}} = \mathrm{SL}(2)$. It follows that Γ is commensurable with $\mathrm{SL}(2, \mathbb{Z})$. So Γ is commensurable with a free group. This contradicts that $G \cong \Gamma$ contains $\mathbb{Z}^{n-1}$ with $n \geq 3$. □

Proof of Corollary 7.2. Let M be the Picard lattice of X, that is, $M = \mathrm{Pic}(X)$ with the intersection form. Then M has signature $(1, n)$ by the Hodge index theorem, where $n \geq 3$ since X has Picard number at least 4.

For an elliptic fibration $X \to \mathbb{P}^1$ with no reducible fibers, the Mordell–Weil group of the fibration has rank $\rho(X) - 2 = n - 1$ by the Shioda–Tate formula [Shioda 1972, Corollary 1.5], which is easy to check in this case. So the first elliptic fibration of X gives an inclusion of $\mathbb{Z}^{n-1}$ into $G = \mathrm{Aut}^*(X)$. The second elliptic fibration gives an inclusion of $\mathbb{Z}^a$ into G for some $a > 0$. In the action of G on hyperbolic n-space, the Mordell–Weil group of each elliptic fibration is a group of parabolic transformations fixing the point at infinity that corresponds to the class $e \in M$ of a fiber (which has $\langle e, e \rangle = 0$). Since a parabolic transformation fixes only one point of the sphere at infinity, the subgroups $\mathbb{Z}^{n-1}$ and $\mathbb{Z}^a$ in G intersect only in the identity. It follows that the subgroup $\mathbb{Z}^{n-1}$ has infinite index in G.

We are given that X contains a (-2)-curve C. I claim that C has infinitely many translates under the Mordell–Weil group $\mathbb{Z}^{n-1}$. Indeed, any curve with finitely many orbits under $\mathbb{Z}^{n-1}$ must be contained in a fiber of $X \to \mathbb{P}^1$. Since all fibers are irreducible, the fibers have self-intersection 0, not -2. Thus X contains infinitely many (-2)-curves. Therefore the group

$$W \subset O(M)$$

generated by reflections in (-2)-vectors is infinite. Here W acts simply transitively on the Weyl chambers of the positive cone (separated by hyperplanes $v^{\perp}$ with v a (-2)-vector), whereas $G = \mathrm{Aut}^*(X)$ preserves one Weyl chamber, the ample cone of X. So G and W intersect only in the identity. Since W is infinite, G has infinite index in $O(M)$. By Theorem 7.1, G is not commensurable with an arithmetic group. □

Proof of Example 7.3. The given curve C in the linear system $|O(4,4)| = |-2K_{\mathbb{P}^1\times\mathbb{P}^1}|$ is smooth. One can check this with Macaulay 2, for example. Therefore the double cover X of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along C is a smooth K3 surface. The two projections from X to $\mathbb{P}^1$ are elliptic fibrations. Typically, such a double cover $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ would have Picard number 2, but the curve C has been chosen to be tangent at 4 points to each of two curves of degree $(1, 1)$, $D_1 = \{xv = yu\}$ and $D_2 = \{xv = -yu\}$. (These points are $[x, y] = [u, v]$ equal to $[1, 1], [1, 2], [1, -1], [1, -2]$ on D_1 and $[x, y] = [u, -v]$ equal to $[1, 1], [1, 2], [1, -1], [1, -2]$ on D_2.) It follows that the double covering is trivial over D_1 and D_2, outside the ramification curve C: the inverse image in X of each curve D_i is a union of two curves, $\pi^{-1}(D_i) = E_i \cup F_i$, meeting transversely at 4 points. The smooth rational curves E_1, F_1, E_2, F_2 on X are (-2)-curves, since X is a K3 surface.

The curves D_1 and D_2 meet transversely at the two points $[x, y] = [u, v]$ equal to $[1, 0]$ or $[0, 1]$. Let us compute that the double covering

$$\pi : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

is trivial over the union of D_1 and D_2 (outside the ramification curve C). Indeed, if we write X as $w^2 = f(x, y, z, w)$ where f is the given polynomial of degree $(4, 4)$, then a section of π over $D_1 \cup D_2$ is given by

$$w = 4x^2u^2 - 5x^2v^2 + y^2v^2.$$

We can name the curves E_i, F_i so that the image of this section is $E_1 \cup E_2$ and the image of the section $w = -(4x^2u^2 - 5x^2v^2 + y^2v^2)$ is $F_1 \cup F_2$. Then E_1 and F_2 are disjoint. So the intersection form among the divisors $\pi^*O(1,0), \pi^*O(0,1), E_1, F_2$ on X is given by

$$\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{pmatrix}$$

Since this matrix has determinant -32, not zero, X has Picard number at least 4.

Finally, we compute that the two projections from $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ are each ramified over 24 distinct points in $\mathbb{P}^1$. It follows that all fibers of our two elliptic fibrations $X \to \mathbb{P}^1$ are irreducible. By Corollary 7.2, the automorphism group of X (over $\mathbb{C}$, or equivalently over $\overline{\mathbb{Q}}$) is not commensurable with an arithmetic group. Our calculations have all worked over $\mathbb{Q}$, and so Corollary 7.2 also gives that $\mathrm{Aut}(X_{\mathbb{Q}})$ is not commensurable with an arithmetic group. □

8. Klt pairs

We will see that the previous results can be generalized from Calabi–Yau varieties to a broader class of varieties using the language of pairs. For the rest of the paper, we work over the complex numbers.

A normal variety X is $\mathbb{Q}$-*factorial* if for every point p and every codimension-one subvariety S through p, there is a regular function on some neighborhood of p that vanishes exactly on S (to some positive order).

Definition 8.1. A pair (X, Δ) is a $\mathbb{Q}$-factorial projective variety X with an effective $\mathbb{R}$-divisor Δ on X.

Notice that Δ is an actual $\mathbb{R}$-divisor $\Delta = \sum a_i \Delta_i$, not a numerical equivalence class of divisors. We think of $K_X + \Delta$ as the canonical bundle of the pair (X, Δ). The following definition picks out an important class of "mildly singular" pairs.

Definition 8.2. A pair (X, Δ) is *klt* (Kawamata log terminal) if the following holds. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities. Suppose that the union of the exceptional set of π (the subset of $\widetilde{X}$ where π is not an isomorphism) with $\pi^{-1}(\Delta)$ is a divisor with simple normal crossings. Define a divisor $\widetilde{\Delta}$ on $\widetilde{X}$ by

$$K_{\widetilde{X}} + \widetilde{\Delta} = \pi^*(K_X + \Delta).$$

We say that (X, Δ) is klt if all coefficients of $\widetilde{\Delta}$ are less than 1. This property is independent of the choice of resolution.

Example 8.3. A surface $X = (X, 0)$ is klt if and only if X has only quotient singularities [Kollár and Mori 1998, Proposition 4.18].

Example 8.4. For a smooth variety X and Δ a divisor with simple normal crossings (and some coefficients), the pair (X, Δ) is klt if and only if Δ has coefficients less than 1.

All the main results of minimal model theory, such as the cone theorem, generalize from smooth varieties to klt pairs. For example, the Fano case of the cone theorem becomes [Kollár and Mori 1998, Theorem 3.7]:

Theorem 8.5. *Let* (X, Δ) *be a klt Fano pair, meaning that* $-(K_X + \Delta)$ *is ample. Then* $\overline{\text{Curv}}(X)$ (*and hence the dual cone* $\text{Nef}(X)$) *is rational polyhedral.*

Notice that the conclusion does not involve the divisor Δ. This shows the power of the language of pairs. A variety X may not be Fano, but if we can find an $\mathbb{R}$-divisor Δ that makes (X, Δ) a klt Fano pair, then we get the same conclusion (that the cone of curves and the nef cone are rational polyhedral) as if X were Fano.

Example 8.6. Let X be the blow-up of $\mathbb{P}^2$ at any number of points on a smooth conic. As an exercise, the reader can write down an $\mathbb{R}$-divisor Δ such that (X, Δ) is a klt Fano pair. This proves that the nef cone of X is rational polyhedral, as proved by other methods in [Galindo and Monserrat 2005, Corollary 3.3; Mukai 2005; Castravet and Tevelev 2006]. These surfaces are definitely not Fano if we blow up 6 or more points. Their Betti numbers are unbounded, in contrast to the smooth Fano surfaces.

More generally, Testa, Várilly-Alvarado, and Velasco proved that every smooth projective rational surface X with $-K_X$ big has finitely generated Cox ring [Testa et al. 2009]. Finite generation of the Cox ring (the ring of all sections of all line bundles) is stronger than the nef cone being rational polyhedral, by the analysis of [Hu and Keel 2000]. Chenyang Xu showed that a rational surface with $-K_X$ big need not have any divisor Δ with (X, Δ) a klt Fano pair [Testa et al. 2009]. I do not know whether the blow-ups of higher-dimensional projective spaces considered by in [Mukai 2005] and [Castravet and Tevelev 2006] have a divisor Δ with (X, Δ) a klt Fano pair.

It is therefore natural to extend the Morrison–Kawamata cone conjecture from Calabi–Yau varieties to *Calabi–Yau pairs* (X, Δ), meaning that $K_X + \Delta \equiv 0$. The conjecture is reasonable, since we can prove it in dimension 2 [Totaro 2010].

Theorem 8.7. *Let* (X, Δ) *be a klt Calabi–Yau pair of dimension* 2*. Then* $\mathrm{Aut}(X, \Delta)$ (*and also* $\mathrm{Aut}(X)$) *acts with a rational polyhedral fundamental domain on the cone* $\mathrm{Nef}(X) \subset N^1(X)$.

Here is a more concrete consequence of Theorem 8.7:

Corollary 8.8 [Totaro 2010]. *Let* (X, Δ) *be a klt Calabi–Yau pair of dimension* 2*. Then there are only finitely many contractions of X up to automorphisms of X. Related to that*: $\mathrm{Aut}(X)$ *has finitely many orbits on the set of curves in X with negative self-intersection.*

This was shown in one class of examples [Dolgachev and Zhang 2001]. These results are false for surfaces in general, even for some smooth rational surfaces:

Example 8.9. Let X be the blow-up of $\mathbb{P}^2$ at 9 very general points. Then $\mathrm{Nef}(X)$ is not rational polyhedral, since X contains infinitely many (-1)-curves. But $\mathrm{Aut}(X) = 1$ [Gizatullin 1980, Proposition 8], and so the conclusion fails for X.

Moreover, let Δ be the unique smooth cubic curve in $\mathbb{P}^2$ through the 9 points, with coefficient 1. Then $-K_X \equiv \Delta$, and so (X, Δ) is a *log-canonical* (and even canonical) Calabi–Yau pair. The theorem therefore fails for such pairs.

We now give a classical example (besides the case $\Delta = 0$ of Calabi–Yau surfaces) where Theorem 8.7 applies.

Example 8.10. Let X be the blow-up of $\mathbb{P}^2$ at 9 points $p_1, \dots, p_9$ which are the intersection of two cubic curves. Then taking linear combinations of the two cubics gives a $\mathbb{P}^1$-family of elliptic curves through the 9 points. These curves become disjoint on the blow-up X, and so we have an elliptic fibration $X \to \mathbb{P}^1$. This morphism is given by the linear system $|-K_X|$. Using that, we see that the (-1)-curves on X are exactly the sections of the elliptic fibration $X \to \mathbb{P}^1$.

In most cases, the Mordell–Weil group of $X \to \mathbb{P}^1$ is $\doteq \mathbb{Z}^8$. So X contains infinitely many (-1)-curves, and so the cone $\mathrm{Nef}(X)$ is not rational polyhedral. But $\mathrm{Aut}(X)$ acts transitively on the set of (-1)-curves, by translations using the group structure on the fibers of $X \to \mathbb{P}^1$. That leads to the proof of Theorem 8.7 in this example. (The theorem applies, in the sense that there is an $\mathbb{R}$-divisor Δ with (X, Δ) klt Calabi–Yau: let Δ be the sum of two smooth fibers of $X \to \mathbb{P}^1$ with coefficients $\frac{1}{2}$, for example.)

9. The cone conjecture in dimension greater than 2

In higher dimensions, the cone conjecture also predicts that a klt Calabi–Yau pair (X, Δ) has only finitely many small $\mathbb{Q}$-factorial modifications $X \dashrightarrow X_1$ *up to pseudo-automorphisms of* X. (See [Kawamata 1997; Totaro 2010] for the full statement of the cone conjecture in higher dimensions.) A pseudo-automorphism means a birational automorphism which is an isomorphism in codimension 1.

More generally, the conjecture implies that X has only finitely many birational contractions $X \dashrightarrow Y$ modulo pseudo-automorphisms of X, where a birational contraction means a dominant rational map that extracts no divisors. There can be infinitely many small modifications if we do not divide out by the group $\mathrm{PsAut}(X)$ of pseudo-automorphisms of X.

Kawamata proved a relative version of the cone conjecture for a 3-fold X with a K3 fibration or elliptic fibration $X \to S$ [Kawamata 1997]. Here X can have infinitely many minimal models (or small modifications) over S, but it has only finitely many modulo $\mathrm{PsAut}(X/S)$.

This is related to other finiteness problems in minimal model theory. We know that a klt pair (X, Δ) has only finitely many minimal models if Δ is big [Birkar et al. 2010, Corollary 1.1.5]. It follows that a variety of general type has a finite set of minimal models. A variety of nonmaximal Kodaira dimension can have infinitely many minimal models [Reid 1983, Section 6.8; 1997]. But it is conjectured that every variety X has only finitely many minimal models *up to isomorphism*, meaning that we ignore the birational identification with X. Kawamata's results on Calabi–Yau fiber spaces imply at least that 3-folds of positive Kodaira dimension have only finitely many minimal models up to isomorphism [Kawamata 1997, Theorem 4.5]. If the abundance conjecture

[Kollár and Mori 1998, Corollary 3.12] holds (as it does in dimension 3), then every nonuniruled variety has an Iitaka fibration where the fibers are Calabi–Yau. The cone conjecture for Calabi–Yau fiber spaces (plus abundance) implies finiteness of minimal models up to isomorphism for arbitrary varieties.

The cone conjecture is wide open for Calabi–Yau 3-folds, despite significant results by Oguiso and Peternell [1998], Szendrői [1999], Uehara [2004], and Wilson [1994]. Hassett and Tschinkel recently [2010] checked the conjecture for a class of holomorphic symplectic 4-folds.

10. Outline of the proof of Theorem 8.7

The proof of Theorem 8.7 gives a good picture of the Calabi–Yau pairs of dimension 2. We summarize the proof from [Totaro 2010]. In most cases, if (X, Δ) is a Calabi–Yau pair, then X turns out to be rational. It is striking that the most interesting case of the theorem is proved by reducing properties of certain rational surfaces to the Torelli theorem for K3 surfaces.

Let (X, Δ) be a klt Calabi–Yau pair of dimension 2. That is, $K_X + \Delta \equiv 0$, or equivalently

$$-K_X \equiv \Delta,$$

where Δ is effective. We can reduce to the case where X is smooth by taking a suitable resolution of (X, Δ).

If $\Delta = 0$, then X is a smooth Calabi–Yau surface, and the result is known by Sterk, using the Torelli theorem for K3 surfaces. So assume that $\Delta \neq 0$. Then X has Kodaira dimension

$$\kappa(X) := \kappa(X, K_X)$$

equal to $-\infty$. With one easy exception, Nikulin showed that our assumptions imply that X is *rational* [Alexeev and Mori 2004, Lemma 1.4]. So assume that X is rational from now on.

We have three main cases for the proof, depending on whether the Iitaka dimension $\kappa(X, -K_X)$ is 0, 1, or 2. (It is nonnegative because $-K_X \sim_{\mathbb{R}} \Delta \geq 0$.) By definition, the Iitaka dimension $\kappa(X, L)$ of a line bundle L is $-\infty$ if $h^0(X, mL) = 0$ for all positive integers m. Otherwise, $\kappa(X, L)$ is the natural number r such that there are positive integers a, b and a positive integer m_0 with $am^r \leq h^0(X, mL) \leq bm^r$ for all positive multiples m of m_0 [Lazarsfeld 2004, Corollary 2.1.38].

10.1. ***Case where $\kappa(X, -K_X) = 2$.*** That is, $-K_X$ is big. In this case, there is an $\mathbb{R}$-divisor Γ such that (X, Γ) is klt Fano. Therefore $\mathrm{Nef}(X)$ is rational polyhedral by the cone theorem, and hence the group $\mathrm{Aut}^*(X)$ is finite. So Theorem 8.7 is true in a simple way. More generally, for (X, Γ) klt Fano of any dimension, the

Cox ring of X is finitely generated, by Birkar, Cascini, Hacon, and M^{c}Kernan [2010].

This proof illustrates an interesting aspect of working with pairs: rather than Fano being a different case from Calabi–Yau, Fano becomes a special case of Calabi–Yau. That is, if (X, Γ) is a klt Fano pair, then there is another effective $\mathbb{R}$-divisor Δ with (X, Δ) a klt Calabi–Yau pair.

10.2. ***Case where $\kappa(X, -K_X) = 1$.*** In this case, some positive multiple of $-K_X$ gives an elliptic fibration $X \to \mathbb{P}^1$, not necessarily minimal. Here $\text{Aut}^*(X)$ equals the Mordell–Weil group of $X \to \mathbb{P}^1$ up to finite index, and so $\text{Aut}^*(X) \doteq \mathbb{Z}^n$ for some n. This generalizes the example of $\mathbb{P}^2$ blown up at the intersection of two cubic curves.

The (-1)-curves in X are multisections of $X \to \mathbb{P}^1$ of a certain fixed degree. One shows that $\text{Aut}(X)$ has only finitely many orbits on the set of (-1)-curves in X. This leads to the statement of Theorem 8.7 in terms of cones.

10.3. ***Case where $\kappa(X, -K_X) = 0$.*** This is the hardest case. Here $\text{Aut}^*(X)$ can be a fairly general group acting on hyperbolic space; in particular, it can be highly nonabelian.

Here $-K_X \equiv \Delta$ where the intersection pairing on the curves in Δ is negative definite. We can contract all the curves in Δ, yielding a singular surface Y with $-K_Y \equiv 0$. Note that Y is klt and hence has quotient singularities, but it must have worse than ADE singularities, because it is a singular Calabi–Yau surface that is rational.

Let I be the "global index" of Y, the least positive integer with IK_Y Cartier and linearly equivalent to zero. Then

$$Y = M/(\mathbb{Z}/I)$$

for some Calabi–Yau surface M with ADE singularities. The minimal resolution of M is a smooth Calabi–Yau surface. Using the Torelli theorem for K3 surfaces, this leads to the proof of the theorem for M and then for Y, by Oguiso and Sakurai [2001, Corollary 1.9].

Finally, we have to go from Y to its resolution of singularities, the smooth rational surface X. Here $\text{Nef}(X)$ is more complex than $\text{Nef}(Y)$: X typically contains infinitely many (-1)-curves, whereas Y has none (because $K_Y \equiv 0$). Nonetheless, since we know "how big" $\text{Aut}(Y)$ is (up to finite index), we can show that the group

$$\text{Aut}(X, \Delta) = \text{Aut}(Y)$$

has finitely many orbits on the set of (-1)-curves. This leads to the proof of Theorem 8.7 for (X, Δ). QED

11. Example

Here is an example of a smooth rational surface with a highly nonabelian (discrete) automorphism group, considered by Zhang [1991, Theorem 4.1, p. 438], Blache [1995, Theorem C(b)(2)], and [2010, Section 2]. This is an example of the last case in the proof of Theorem 8.7, where $\kappa(X, -K_X) = 0$. We will also see a singular rational surface whose nef cone is round, of dimension 4.

Let X be the blow-up of $\mathbb{P}^2$ at the 12 points: $[1, \zeta^i, \zeta^j]$ for $i, j \in \mathbb{Z}/3$, $[1, 0, 0]$, $[0, 1, 0]$, $[0, 0, 1]$. Here ζ is a cube root of 1. (This is the dual of the "Hesse configuration" [Dolgachev 2004, Section 4.6]. There are 9 lines $L_1, \dots, L_9$ through quadruples of the 12 points in $\mathbb{P}^2$.)

On $\mathbb{P}^2$, we have

$$-K_{\mathbb{P}^2} \equiv 3H \equiv \sum_{i=1}^{9} \tfrac{1}{3} L_i.$$

On the blow-up X, we have

$$-K_X \equiv \sum_{i=1}^{9} \tfrac{1}{3} L_i,$$

where $L_1, \dots, L_9$ are the proper transforms of the 9 lines, which are now disjoint and have self-intersection number -3. Thus $\big(X, \sum_{i=1}^{9} \frac{1}{3} L_i\big)$ is a klt Calabi–Yau pair.

Section 10.3 shows how to analyze X: contract the 9 (-3)-curves L_i on X. This gives a rational surface Y with 9 singular points (of type $\frac{1}{3}(1, 1)$) and $\rho(Y) = 4$. We have $-K_Y \equiv 0$, so Y is a klt Calabi–Yau surface which is rational. We have $3K_Y \sim 0$, and so $Y \cong M/(\mathbb{Z}/3)$ with M a Calabi–Yau surface with ADE singularities. It turns out that M is smooth, $M \cong E \times E$ where E is the Fermat cubic curve

$$E \cong \mathbb{C}/\mathbb{Z}[\zeta] \cong \{[x, y, z] \in \mathbb{P}^2 : x^3 + y^3 = z^3\},$$

and $\mathbb{Z}/3$ acts on $E \times E$ as multiplication by (ζ, ζ) [Totaro 2010, Section 2].

Since E has endomorphism ring $\mathbb{Z}[\zeta]$, the group $\mathrm{GL}(2, \mathbb{Z}[\zeta])$ acts on the abelian surface $M = E \times E$. This passes to an action on the quotient variety $Y = M/(\mathbb{Z}/3)$ and hence on its minimal resolution X (which is the blow-up of $\mathbb{P}^2$ at 12 points we started with). Thus the infinite, highly nonabelian discrete group $\mathrm{GL}(2, \mathbb{Z}[\zeta])$ acts on the smooth rational surface X. This is the whole automorphism group of X up to finite groups (*loc. cit.*).

Here $\mathrm{Nef}(Y) = \mathrm{Nef}(M)$ is a round cone in $\mathbb{R}^4$, and so Theorem 8.7 says that $\mathrm{PGL}(2, \mathbb{Z}[\zeta])$ acts with finite covolume on hyperbolic 3-space. In fact, the quotient of hyperbolic 3-space by an index-24 subgroup of $\mathrm{PGL}(2, \mathbb{Z}[\zeta])$ is familiar to topologists as the complement of the figure-eight knot [Maclachlan and Reid 2003, 1.4.3, 4.7.1].

References

[Alekseevskij et al. 1993] D. V. Alekseevskij, È. B. Vinberg, and A. S. Solodovnikov, "Geometry of spaces of constant curvature", pp. 1–138 in *Geometry*, vol. II, Encyclopaedia Math. Sci. **29**, Springer, Berlin, 1993.

[Alexeev and Mori 2004] V. Alexeev and S. Mori, "Bounding singular surfaces of general type", pp. 143–174 in *Algebra, arithmetic and geometry with applications* (West Lafayette, IN, 2000), Springer, Berlin, 2004.

[Birkar et al. 2010] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, "Existence of minimal models for varieties of log general type", *J. Amer. Math. Soc.* **23**:2 (2010), 405–468.

[Blache 1995] R. Blache, "The structure of l.c. surfaces of Kodaira dimension zero. I", *J. Algebraic Geom.* **4**:1 (1995), 137–179.

[Borcherds 1998] R. E. Borcherds, "Coxeter groups, Lorentzian lattices, and $K3$ surfaces", *Internat. Math. Res. Notices* **1998**:19 (1998), 1011–1031.

[Borel and Harish-Chandra 1962] A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups", *Ann. of Math.* (2) **75** (1962), 485–535.

[Borel and Serre 1964] A. Borel and J.-P. Serre, "Théorèmes de finitude en cohomologie galoisienne", *Comment. Math. Helv.* **39** (1964), 111–164.

[Bridson and Haefliger 1999] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften **319**, Springer, Berlin, 1999.

[Brown 1982] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, New York, 1982.

[Castravet and Tevelev 2006] A.-M. Castravet and J. Tevelev, "Hilbert's 14th problem and Cox rings", *Compos. Math.* **142**:6 (2006), 1479–1498.

[Debarre 2001] O. Debarre, *Higher-dimensional algebraic geometry*, Springer, New York, 2001.

[Dolgachev 2004] I. V. Dolgachev, "Abstract configurations in algebraic geometry", pp. 423–462 in *The Fano Conference*, Univ. Torino, Turin, 2004.

[Dolgachev 2008] I. V. Dolgachev, "Reflection groups in algebraic geometry", *Bull. Amer. Math. Soc.* (*N.S.*) **45**:1 (2008), 1–60.

[Dolgachev and Zhang 2001] I. V. Dolgachev and D.-Q. Zhang, "Coble rational surfaces", *Amer. J. Math.* **123**:1 (2001), 79–114.

[de Fernex 2010] T. de Fernex, "The Mori cone of blow-ups of the plane", preprint, 2010. arXiv 1001.5243

[Galindo and Monserrat 2005] C. Galindo and F. Monserrat, "The total coordinate ring of a smooth projective surface", *J. Algebra* **284**:1 (2005), 91–101.

[Gizatullin 1980] M. H. Gizatullin, "Rational G-surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **44**:1 (1980), 110–144. In Russian; translated in *Math. USSR Izv.* **16** (1981), 103–134.

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[Hassett and Tschinkel 2010] B. Hassett and Y. Tschinkel, "Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces", *J. Inst. Math. Jussieu* **9**:1 (2010), 125–153.

[Hu and Keel 2000] Y. Hu and S. Keel, "Mori dream spaces and GIT", *Michigan Math. J.* **48** (2000), 331–348.

[Iskovkikh and Prokhorov 1999] V. Iskovkikh and Y. Prokhorov, "Fano varieties", pp. 1–247 in *Algebraic geometry*, vol. V, Encyclopaedia of Mathematical Sciences **47**, Springer, Berlin, 1999.

[Kawamata 1997] Y. Kawamata, "On the cone of divisors of Calabi–Yau fiber spaces", *Internat. J. Math.* **8**:5 (1997), 665–687.

[Kollár and Mori 1998] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge University Press, Cambridge, 1998.

[Kollár et al. 1992] J. Kollár, Y. Miyaoka, and S. Mori, "Rational connectedness and boundedness of Fano manifolds", *J. Differential Geom.* **36**:3 (1992), 765–779.

[Kovács 1994] S. J. Kovács, "The cone of curves of a $K3$ surface", *Math. Ann.* **300**:4 (1994), 681–691.

[Lazarsfeld 2004] R. Lazarsfeld, *Positivity in algebraic geometry, I: classical setting: line bundles and linear series*, Ergebnisse der Math. (3) **48**, Springer, Berlin, 2004.

[Maclachlan and Reid 2003] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics **219**, Springer, New York, 2003.

[Mazur 1993] B. Mazur, "On the passage from local to global in number theory", *Bull. Amer. Math. Soc.* (*N.S.*) **29**:1 (1993), 14–50.

[Mukai 2005] S. Mukai, "Finite generation of the Nagata invariant rings in A-D-E cases", preprint 1502, RIMS, 2005.

[Mumford 1970] D. Mumford, *Abelian varieties*, Tata Studies in Mathematics **5**, Tata Institute of Fundamental Research, Bombay, 1970.

[Namikawa 1985] Y. Namikawa, "Periods of Enriques surfaces", *Math. Ann.* **270**:2 (1985), 201–222.

[Nikulin 1979] V. V. Nikulin, "Integer symmetric bilinear forms and some of their geometric applications", *Izv. Akad. Nauk SSSR Ser. Mat.* **43**:1 (1979), 111–177. In Russian; translated in *Math. USSR Izv.* **14** (1979), 103–167.

[Oguiso and Peternell 1998] K. Oguiso and T. Peternell, "Calabi–Yau threefolds with positive second Chern class", *Comm. Anal. Geom.* **6**:1 (1998), 153–172.

[Oguiso and Sakurai 2001] K. Oguiso and J. Sakurai, "Calabi–Yau threefolds of quotient type", *Asian J. Math.* **5**:1 (2001), 43–77.

[Prasad 1976] G. Prasad, "Discrete subgroups isomorphic to lattices in Lie groups", *Amer. J. Math.* **98**:4 (1976), 853–863.

[Reid 1983] M. Reid, "Minimal models of canonical 3-folds", pp. 131–180 in *Algebraic varieties and analytic varieties* (Tokyo, 1981), Adv. Stud. Pure Math. **1**, North-Holland, Amsterdam, 1983.

[Serre 1979] J.-P. Serre, "Arithmetic groups", pp. 105–136 in *Homological group theory* (Durham, 1977), London Math. Soc. Lecture Note Ser. **36**, Cambridge Univ. Press, Cambridge, 1979. Reprinted in his *Œuvres*, v. 3, Springer (1986), 503–534.

[Shioda 1972] T. Shioda, "On elliptic modular surfaces", *J. Math. Soc. Japan* **24** (1972), 20–59.

[Sterk 1985] H. Sterk, "Finiteness results for algebraic $K3$ surfaces", *Math. Z.* **189**:4 (1985), 507–513.

[Szendrői 1999] B. Szendrői, "Some finiteness results for Calabi–Yau threefolds", *J. London Math. Soc.* (2) **60**:3 (1999), 689–699.

[Testa et al. 2009] D. Testa, A. Várilly-Alvarado, and M. Velasco, "Big rational surfaces", preprint, 2009. To appear in *Math. Ann.* arXiv 0901.1094

[Tits 1966] J. Tits, “Classification of algebraic semisimple groups”, pp. 33–62 in *Algebraic groups and discontinuous subgroups* (Boulder, CO, 1965), Amer. Math. Soc., Providence, RI, 1966, 1966.

[Totaro 2010] B. Totaro, “The cone conjecture for Calabi–Yau pairs in dimension 2”, *Duke Math. J.* **154**:2 (2010), 241–263.

[Uehara 2004] H. Uehara, “Calabi–Yau threefolds with infinitely many divisorial contractions”, *J. Math. Kyoto Univ.* **44**:1 (2004), 99–118.

[Wilson 1994] P. M. H. Wilson, “Minimal models of Calabi–Yau threefolds”, pp. 403–410 in *Classification of algebraic varieties* (L’Aquila, 1992), Contemp. Math. **162**, Amer. Math. Soc., Providence, RI, 1994.

[Yau 1978] S. T. Yau, “On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I”, *Comm. Pure Appl. Math.* **31**:3 (1978), 339–411.

[Zhang 1991] D.-Q. Zhang, “Logarithmic Enriques surfaces”, *J. Math. Kyoto Univ.* **31**:2 (1991), 419–466.

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