Current Developments in Algebraic Geometry
MSRI Publications
Volume **59**, 2011

Quotients by finite equivalence relations

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APPENDIX BY CLAUDIU RAICU

We study the existence of geometric quotients by finite set-theoretic equivalence relations. We show that such geometric quotients always exist in positive characteristic but not in characteristic 0. The appendix gives some examples of unexpected behavior for scheme-theoretic equivalence relations.

Let $f : X \to Y$ be a finite morphism of schemes. Given Y, one can easily describe X by the coherent sheaf of algebras $f_*\mathbb{O}_X$. Here our main interest is the converse. Given X, what kind of data do we need to construct Y? For this question, the surjectivity of f is indispensable.

The fiber product $X \times_Y X \subset X \times X$ defines an equivalence relation on X, and one might hope to reconstruct Y as the quotient of X by this equivalence relation. Our main interest is in the cases when f is not flat. A typical example we have in mind is when Y is not normal and X is its normalization. In these cases, the fiber product $X \times_Y X$ can be rather complicated. Even if Y and X are pure-dimensional and CM, $X \times_Y X$ can have irreducible components of different dimension and its connected components need not be pure-dimensional. None of these difficulties appear if f is flat [Raynaud 1967; SGA 3 1970] or if Y is normal (Lemma 21).

Finite equivalence relations appear in moduli problems in two ways. First, it is frequently easier to construct or to understand the normalization $\bar{M}$ of a moduli space M. Then one needs to construct M as a quotient of $\bar{M}$ by a finite equivalence relation. This method was used in [Kollár 1997] and finite equivalence relations led to some unsolved problems in [Viehweg 1995, Section 9.5]; see also [Kollár 2011].

Second, in order to compactify moduli spaces of varieties, one usually needs nonnormal objects. The methods of the minimal model program seem to apply naturally to their normalizations. It is quite subtle to descend information from the normalization to the nonnormal variety, see [Kollár 2012, Chapter 5].

In Sections 1, 2, 3 and 6 of this article we give many examples, review (and correct) known results and pose some questions. New results concerning finite equivalence relations are in Sections 4 and 5 and in the Appendix.

1. Definition of equivalence relations

Definition 1 (equivalence relations). Let X be an S-scheme and $\sigma : R \to X \times_S X$ a morphism (or $\sigma_1, \sigma_2 : R \rightrightarrows X$ a pair of morphisms). We say that R is an *equivalence relation* on X if, for every scheme $T \to S$, we get a (set-theoretic) equivalence relation

$$\sigma(T) : \mathrm{Mor}_S(T, R) \hookrightarrow \mathrm{Mor}_S(T, X) \times \mathrm{Mor}_S(T, X).$$

Equivalently, the following conditions hold:

(1) σ is a monomorphism (Definition 31).

(2) (reflexive) R contains the diagonal Δ_X.

(3) (symmetric) There is an involution τ_R on R such that $\tau_{X\times X} \circ \sigma \circ \tau_R = \sigma$, where $\tau_{X\times X}$ denotes the involution which interchanges the two factors of $X \times_S X$.

(4) (transitive) For $1 \le i < j \le 3$ set $X_i := X$ and let $R_{ij} := R$ when it maps to $X_i \times_S X_j$. Then the coordinate projection of $R_{12} \times_{X_2} R_{23}$ to $X_1 \times_S X_3$ factors through R_{13}:

$$R_{12} \times_{X_2} R_{23} \to R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3.$$

We say that $\sigma_1, \sigma_2 : R \rightrightarrows X$ is a *finite* equivalence relation if the maps σ_1, σ_2 are finite. In this case, $\sigma : R \to X \times_S X$ is also finite, hence a closed embedding (Definition 31).

Definition 2 (set-theoretic equivalence relations). Let X and R be reduced S-schemes. We say that a morphism $\sigma : R \to X \times_S X$ is a *set-theoretic equivalence relation* on X if, for every geometric point $\mathrm{Spec}\, K \to S$, we get an equivalence relation on K-points

$$\sigma(K) : \mathrm{Mor}_S(\mathrm{Spec}\, K, R) \hookrightarrow \mathrm{Mor}_S(\mathrm{Spec}\, K, X) \times \mathrm{Mor}_S(\mathrm{Spec}\, K, X).$$

Equivalently:

(1) σ is geometrically injective.

(2) (reflexive) R contains the diagonal Δ_X.

(3) (symmetric) There is an involution τ_R on R such that $\tau_{X\times X} \circ \sigma \circ \tau_R = \sigma$, where $\tau_{X\times X}$ denotes the involution which interchanges the two factors of $X \times_S X$.

(4) (transitive) For $1 \le i < j \le 3$ set $X_i := X$ and let $R_{ij} := R$ when it maps to $X_i \times_S X_j$. Then the coordinate projection of $\mathrm{red}(R_{12} \times_{X_2} R_{23})$ to $X_1 \times_S X_3$ factors through R_{13}:

$$\mathrm{red}(R_{12} \times_{X_2} R_{23}) \to R_{13} \xrightarrow{\pi_{13}} X_1 \times_S X_3.$$

Note that the fiber product need not be reduced, and taking the reduced structure above is essential, as shown by Example 3.

It is sometimes convenient to consider finite morphisms $p : R \to X \times_S X$ such that the injection $i : p(R) \hookrightarrow X \times_S X$ is a set-theoretic equivalence relation. Such a $p : R \to X \times_S X$ is called a *set-theoretic pre-equivalence relation.*

Example 3. On $X := \mathbb{C}^2$ consider the $\mathbb{Z}/2$-action $(x, y) \mapsto (-x, -y)$. This can be given by a set-theoretic equivalence relation $R \subset X_{x_1,y_1} \times X_{x_2,y_2}$ defined by the ideal

$$(x_1-x_2,\, y_1-y_2)\cap(x_1+x_2,\, y_1+y_2) = (x_1^2-x_2^2,\, y_1^2-y_2^2,\, x_1y_1-x_2y_2,\, x_1y_2-x_2y_1)$$

in $\mathbb{C}[x_1, y_1, x_2, y_2]$. We claim that this is *not* an equivalence relation. The problem is transitivity. The defining ideal of $R_{12} \times_{X_2} R_{23}$ in $\mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3]$ is

$$\begin{aligned}\bigl(&x_1^2 - x_2^2,\, y_1^2 - y_2^2,\, x_1y_1 - x_2y_2,\, x_1y_2 - x_2y_1,\\ &x_2^2 - x_3^2,\, y_2^2 - y_3^2,\, x_2y_2 - x_3y_3,\, x_2y_3 - x_3y_2\bigr).\end{aligned}$$

This contains $(x_1^2 - x_3^2, y_1^2 - y_3^2, x_1y_1 - x_3y_3)$ but it does not contain $x_1y_3 - x_3y_1$. Thus there is no map $R_{12} \times_{X_2} R_{23} \to R_{13}$. Note, however, that the problem is easy to remedy. Let $R^* \subset X \times X$ be defined by the ideal

$$(x_1^2 - x_2^2,\, y_1^2 - y_2^2,\, x_1y_1 - x_2y_2) \subset \mathbb{C}[x_1, y_1, x_2, y_2].$$

We see that R^* defines an equivalence relation. The difference between R and R^* is one embedded point at the origin.

Definition 4 (categorical and geometric quotients). Given two morphisms

$$\sigma_1, \sigma_2 : R \rightrightarrows X,$$

there is at most one scheme $q : X \to (X/R)^{cat}$ such that $q \circ \sigma_1 = q \circ \sigma_2$ and q is universal with this property. We call $(X/R)^{cat}$ the *categorical quotient* (or *coequalizer*) of $\sigma_1, \sigma_2 : R \rightrightarrows X$.

The categorical quotient is easy to construct in the affine case. Given $\sigma_1, \sigma_2 : R \rightrightarrows X$, the categorical quotient $(X/R)^{cat}$ is the spectrum of the S-algebra

$$\ker\Bigl[\mathbb{O}_X \xrightarrow{\sigma_1^*-\sigma_2^*} \mathbb{O}_R\Bigr].$$

Let $\sigma_1, \sigma_2 : R \rightrightarrows X$ be a finite equivalence relation. We say that $q : X \to Y$ is a *geometric quotient* of X by R if

(1) $q : X \to Y$ is the categorical quotient $q : X \to (X/R)^{cat}$,

(2) $q : X \to Y$ is finite, and

(3) for every geometric point $\operatorname{Spec} K \to S$, the fibers of $q_K : X_K(K) \to Y_K(K)$ are the $\sigma\bigl(R_K(K)\bigr)$-equivalence classes of $X_K(K)$.

The geometric quotient is denoted by X/R.

The main example to keep in mind is the following, which easily follows from Lemma 17 and the construction of $(X/R)^{cat}$ for affine schemes.

Example 5. Let $f : X \to Y$ be a finite and surjective morphism. Set $R := \mathrm{red}(X \times_Y X) \subset X \times X$ and let $\sigma_i : R \to X$ denote the coordinate projections. Then the geometric quotient X/R exists and $X/R \to Y$ is a finite and universal homeomorphism (Definition 32). Therefore, if X is the normalization of Y, then X/R is the weak normalization of Y. (See [Kollár 1996, Section 7.2] for basic results on seminormal and weakly normal schemes.)

By taking the reduced structure of $X \times_Y X$ above, we chose to focus on the set-theoretic properties of Y. However, as Example 16 shows, even if X, Y and $X \times_Y X$ are all reduced, $X/R \to Y$ need not be an isomorphism. Thus X and $X \times_Y X$ do not determine Y uniquely.

In Section 2 we give examples of finite, set-theoretic equivalence relations $R \rightrightarrows X$ such that the categorical quotient $(X/R)^{cat}$ is non-Noetherian and there is no geometric quotient. This can happen even when X is very nice, for instance a smooth variety over $\mathbb{C}$. Some elementary results about the existence of geometric quotients are discussed in Section 3.

An inductive plan to construct geometric quotients is outlined in Section 4. As an application, we prove in Section 5 the following:

Theorem 6. *Let S be a Noetherian $\mathbb{F}_p$-scheme and X an algebraic space which is essentially of finite type over S. Let $R \rightrightarrows X$ be a finite, set-theoretic equivalence relation. Then the geometric quotient X/R exists.*

Remark 7. There are many algebraic spaces which are not of finite type and such that the Frobenius map $F^q : X \to X^{(q)}$ is finite. By a result of Kunz (see [Matsumura 1980, p. 302]) such algebraic spaces are excellent. As the proof shows, Theorem 6 remains valid for algebraic spaces satisfying this property.

In the Appendix, C. Raicu constructs finite scheme-theoretic equivalence relations R on $X = \mathbb{A}^2$ (in any characteristic) such that the geometric quotient X/R exists yet R is strictly smaller than the fiber product $X \times_{X/R} X$. Closely related examples are in [Venken 1971; Philippe 1973].

In characteristic zero, this leaves open the following:

Question 8. Let $R \subset X \times X$ be a scheme-theoretic equivalence relation such that the coordinate projections $R \rightrightarrows X$ are finite.

Is there a geometric quotient X/R?

A special case of the quotient problem, called gluing or pinching, is discussed in Section 6. This follows [Artin 1970], [Ferrand 2003] (which is based on an unpublished manuscript from 1970) and [Raoult 1974].

2. First examples

The next examples show that in many cases, the categorical quotient of a very nice scheme X can be non-Noetherian. We start with a nonreduced example and then we build it up to smooth ones.

Example 9. Let k be a field and consider $k[x, \epsilon]$, where $\epsilon^2 = 0$. Set

$$g_1\big(a(x)+\epsilon b(x)\big)=a(x)+\epsilon b(x) \quad\text{and}\quad g_2\big(a(x)+\epsilon b(x)\big)=a(x)+\epsilon\big(b(x)+a'(x)\big).$$

If $\operatorname{char} k = 0$, the coequalizer is the spectrum of

$$\ker\Big[k[x, \epsilon] \stackrel{g_1-g_2}{\longrightarrow} k[x, \epsilon]\Big] = k + \epsilon k[x].$$

Note that $k + \epsilon k[x]$ is not Noetherian and its only prime ideal is $\epsilon k[x]$.

If $\operatorname{char} k = p$ then the coequalizer is the spectrum of the finitely generated k-algebra

$$\ker\Big[k[x, \epsilon] \stackrel{g_1-g_2}{\longrightarrow} k[x, \epsilon]\Big] = k[x^p] + \epsilon k[x].$$

It is not surprising that set-theoretic equivalence relations behave badly on nonreduced schemes. However, the above example is easy to realize on reduced and even on smooth schemes.

Example 10. (Compare [Holmann 1963, p. 342].) Let $p_i : Z \to Y_i$ be finite morphisms for $i = 1, 2$. We can construct out of them an equivalence relation on $Y_1 \amalg Y_2$, where R is the union of the diagonal with two copies of Z, one of which maps as

$$(p_1, p_2) : Z \to Y_1 \times Y_2 \subset (Y_1 \amalg Y_2) \times (Y_1 \amalg Y_2),$$

the other its symmetric pair. The categorical quotient $\big((Y_1 \amalg Y_2)/R\big)^{cat}$ is also the universal push-out of $Y_1 \stackrel{p_1}{\leftarrow} Z \stackrel{p_2}{\rightarrow} Y_2$. If Z and the Y_i are affine over S, then it is the spectrum of the S-algebra

$$\ker\Big[\mathbb{O}_{Y_1} + \mathbb{O}_{Y_2} \stackrel{p_1^*-p_2^*}{\longrightarrow} \mathbb{O}_Z\Big].$$

For the first example let $Y_1 \cong Y_2 := \operatorname{Spec} k[x, y^2, y^3]$ and $Z := \operatorname{Spec} k[u, v]$ with p_i given by

$$p_1^* : (x, y^2, y^3) \mapsto (u, v^2, v^3) \quad\text{and}\quad p_2^* : (x, y^2, y^3) \mapsto (u + v, v^2, v^3).$$

Since the p_i^* are injective, the categorical quotient is the spectrum of the k-algebra $k[u, v^2, v^3] \cap k[u + v, v^2, v^3]$. Note that

$$\begin{aligned} k[u, v^2, v^3] &= \big\{f_0(u) + \textstyle\sum_{i\geq 2} v^i f_i(u) : f_i \in k[u]\big\},\\ k[u + v, v^2, v^3] &= \big\{f_0(u) + v f_0'(u) + \textstyle\sum_{i\geq 2} v^i f_i(u) : f_i \in k[u]\big\}.\end{aligned}$$

As in Example 9, if $\operatorname{char} k = 0$ then the categorical quotient is the spectrum of the non-Noetherian algebra $k + \sum_{n\geq 2} v^n k[u]$. If $\operatorname{char} k = p$ then the geometric quotient is given by the finitely generated k-algebra

$$k[u^p] + \sum_{n\geq 2} v^n k[u].$$

This example can be embedded into a set-theoretic equivalence relation on a smooth variety.

Example 11. Let $Y_1 \cong Y_2 := \mathbb{A}^3_{xyz}$, $Z := \mathbb{A}^2_{uv}$ and

$$p_1^* : (x_1, y_1, z_1) \mapsto (u, v^2, v^3) \quad \text{and} \quad p_2^* : (x_2, y_2, z_2) \mapsto (u + v, v^2, v^3).$$

By the previous computations, in characteristic zero the categorical quotient is given by

$$k + (y_1, z_1) + (y_2, z_2) \subset k[x_1, y_1, z_1] + k[x_2, y_2, z_2],$$

where (y_i, z_i) denotes the ideal $(y_i, z_i) \subset k[x_i, y_i, z_i]$. A minimal generating set is given by

$$y_1 x_1^m, z_1 x_1^m, y_2 x_2^m, z_2 x_2^m : m = 0, 1, 2, \ldots$$

In positive characteristic the categorical quotient is given by

$$k[x_1^p, x_2^p] + (y_1, z_1) + (y_2, z_2) \subset k[x_1, y_1, z_1] + k[x_2, y_2, z_2].$$

A minimal generating set is given by

$$x_1^p, x_2^p, y_1 x_1^m, z_1 x_1^m, y_2 x_2^m, z_2 x_2^m : m = 0, 1, \ldots, p - 1.$$

Example 12. The following example, based on [Nagata 1969], shows that even for rings of invariants of finite group actions some finiteness assumption on X is necessary in order to obtain geometric quotients.

Let k be a field of characteristic $p > 0$ and $K := k(x_1, x_2, \ldots)$, where the x_i are algebraically independent over k. Let

$$D := \sum_i x_{i+1} \frac{\partial}{\partial x_i} \quad \text{be a derivation of } K.$$

Let $F := \{f \in K \mid D(f) = 0\}$ be the subfield of "constants". Set

$$R = K + \epsilon K \quad \text{where } \epsilon^2 = 0 \quad \text{and} \quad \sigma : f + \epsilon g \mapsto f + \epsilon(g + D(f)).$$

R is a local Artin ring. It is easy to check that σ is an automorphism of R of order p. The fixed ring is $R^\sigma = F + \epsilon K$. Its maximal ideal is $m := (\epsilon K)$ and generating sets of m correspond to F-vector space bases of K. Next we show the x_i are linearly independent over F, which implies that R^σ is not Noetherian.

Assume that we have a relation

$$\sum_{i\le n} f_i x_i = 0.$$

We may assume that $f_n = 1$ and $f_i \in F \cap k(x_1, \dots, x_r)$ for some r. Apply D to get that

$$0 = \sum_{i\le n} f_i D(x_i) = \sum_{i\le n} f_i x_{i+1}.$$

Repeating s times gives that $\sum_{i\le n} f_i x_{i+s} = 0$ or, equivalently,

$$x_{n+s} = -\sum_{i\le n-1} f_i x_{i+s}.$$

This is impossible if $n + s > r$; a contradiction.

It is easy to see that R is not a submodule of any finitely generated R^σ-module.

Example 13. This example of [Nagarajan 1968] gives a 2-dimensional regular local ring R and an automorphism of order 2 such that the ring of invariants is not Noetherian.

Let k be a field of characteristic 2 and $K := k(x_1, y_1, x_2, y_2, \dots)$, where the x_i, y_i are algebraically independent over k. Let $R := K[[u, v]]$ be the power series ring in 2 variables. Note that R is a 2-dimensional regular local ring, but it is not essentially of finite type over k. Define a derivation of K to R by

$$D_K := \sum_i v(x_{i+1}u + y_{i+1}v)\frac{\partial}{\partial x_i} + u(x_{i+1}u + y_{i+1}v)\frac{\partial}{\partial y_i}.$$

This extends to a derivation of R to R by setting

$$D_R|_K = D_K \quad \text{and} \quad D_R(u) = D_R(v) = 0.$$

Note that $D_R \circ D_R = 0$, thus $\sigma : r \mapsto r + D_R(r)$ is an order 2 automorphism of R. We claim that the ring of invariants R^σ is not Noetherian.

To see this, note first that $x_i u + y_i v \in R^\sigma$ for every i.

Claim. *For every n, $x_{n+1}u + y_{n+1}v \notin \big(x_1 u + y_1 v, \dots, x_n u + y_n v\big)R^\sigma$.*

Proof. Assume the contrary and write

$$x_{n+1}u + y_{n+1}v = \sum_{i\le n} r_i(x_i u + y_i v), \quad \text{where } r_i \in R^\sigma.$$

Working modulo $(u, v)^2$ and gathering the terms involving u, we get an equality

$$x_{n+1} \equiv \sum_{i\le n} r_i x_i \mod R^\sigma \cap (u, v)R.$$

Applying D_R and again gathering the terms involving u we obtain

$$x_{n+2} \equiv \sum_{i\le n} r_i x_{i+1} \quad \text{modulo } R^\sigma \cap (u, v)R.$$

Repeating this s times gives

$$x_{n+s+1} = \sum_{i\le n} \bar{r}_i x_{i+s}, \quad \text{where } \bar{r}_i \in K.$$

Since the $\bar{r}_i$ involve only finitely many variables, we get a contradiction for large s. Thus

$$\begin{aligned}(x_1u + y_1v) &\subset (x_1u + y_1v, x_2u + y_2v)\\ &\subset (x_1u + y_1v, x_2u + y_2v, x_3u + y_3v) \subset \cdots\end{aligned}$$

is an infinite increasing sequence of ideals in R^σ. □

The next examples show that, if S is a smooth projective surface, then a geometric quotient S/R can be nonprojective (but proper) and if X is a smooth proper 3-fold, X/R can be an algebraic space which is not a scheme.

Example 14. 1. Let C, D be smooth projective curves and S the blow up of $C \times D$ at a point (c, d). Let $C_1 \subset S$ be the birational transform of $C \times \{d\}$, $C_2 := C \times \{d'\}$ for some $d' \ne d$ and $\mathbb{P}^1 \cong E \subset S$ the exceptional curve.

Fix an isomorphism $\sigma : C_1 \cong C_2$. This generates an equivalence relation R which is the identity on $S \setminus (C_1 \cup C_2)$. As we will see in Proposition 33, S/R is a surface of finite type. Note however that the image of E in S/R is numerically equivalent to 0, thus S/R is not quasiprojective. Indeed, let M be any line bundle on S/R. Then π^*M is a line bundle on S such that $(C_1 \cdot \pi^*M) = (C_2 \cdot \pi^*M)$. Since C_2 is numerically equivalent to $C_1 + E$, this implies that $(E \cdot \pi^*M) = 0$.

2. Take $S \cong \mathbb{P}^2$ and $Z := (x(y^2 - xz) = 0)$. Fix an isomorphism of the line $(x = 0)$ and the conic $(y^2 - xz = 0)$ which is the identity on their intersection. As before, this generates an equivalence relation R which is the identity on their complement. By Proposition 33, $\mathbb{P}^2/R$ exists as a scheme but it is not projective.

   Indeed, if M is a line bundle on $\mathbb{P}^2/R$ then π^*M is a line bundle on $\mathbb{P}^2$ whose degree on a line is the same as its degree on a conic. Thus $\pi^*M \cong \mathcal{O}_{\mathbb{P}^2}$ and so M is not ample.

3. Let $S = S_1 \amalg S_2 \cong \mathbb{P}^2 \times \{1, 2\}$ be 2 copies of $\mathbb{P}^2$. Let $E \subset \mathbb{P}^2$ be a smooth cubic. For a point $p \in E$, let $\sigma_p : E \times \{1\} \to E \times \{2\}$ be the identity composed with translation by $p \in E$. As before, this generates an equivalence relation R which is the identity on their complement.

Let M be a line bundle on S/R. Then $\pi^*M|_{S_i} \cong \mathbb{O}_{\mathbb{P}^2}(m_i)$ for some $m_i > 0$, and we conclude that

$$\mathbb{O}_{\mathbb{P}^2}(m_1)|E \cong \tau_p^*\big(\mathbb{O}_{\mathbb{P}^2}(m_2)|E\big).$$

This holds if and only if $m_1 = m_2$ and $p \in E$ is a $3m_1$-torsion point. Thus the projectivity of S/R depends very subtly on the gluing map σ_p.

Example 15. Hironaka's example in [Hartshorne 1977, B.3.4.1] gives a smooth, proper threefold X and two curves $\mathbb{P}^1 \cong C_1 \cong C_2 \subset X$ such that $C_1 + C_2$ is homologous to 0. Let $g : C_1 \cong C_2$ be an isomorphism and R the corresponding equivalence relation.

We claim that there is no quasiprojective open subset $U \subset X$ which intersects both C_1 and C_2. Assume to the contrary that U is such. Then there is an ample divisor $H_U \subset U$ which intersects both curves but does not contain either. Its closure $H \subset X$ is a Cartier divisor which intersects both curves but does not contain either. Thus $H \cdot (C_1 + C_2) > 0$, a contradiction.

This shows that if $p \in X/R$ is on the image of C_i then p does not have any affine open neighborhood since the preimage of an affine set by a finite morphism is again affine. Thus X/R is not a scheme.

Example 16. [Lipman 1975] Fix a field k and let $a_1, \dots, a_n \in k$ be different elements. Set

$$A := k[x, y]/\prod_i (x - a_i y).$$

Then $Y := \operatorname{Spec} A$ is n lines through the origin. Let $f : X \to Y$ its normalization. Thus $X = \amalg_i \operatorname{Spec} k[x, y]/(x - a_i y)$. Note that

$$k[x, y]/(x - a_i y) \otimes_A k[x, y]/(x - a_j y) = \begin{cases} k[x, y]/(x - a_i y) & \text{if } a_i = a_j, \\ k & \text{if } a_i \neq a_j. \end{cases}$$

Thus $X \times_Y X$ is reduced. It is the union of the diagonal Δ_X and of $f^{-1}(0, 0) \times f^{-1}(0, 0)$. Thus $X/(X \times_Y X)$ is a seminormal scheme which is isomorphic to the n coordinate axes in $\mathbb{A}^n$. For $n \geq 3$, it is not isomorphic to Y.

One can also get similar examples where Y is integral. Indeed, let $Y \subset \mathbb{A}^2$ be any plane curve whose only singularities are ordinary multiple points and let $f : X \to Y$ be its normalization. By the above computations, $X \times_Y X$ is reduced and $X/(X \times_Y X)$ is the seminormalization of Y.

If Y is a reduced scheme with normalization $\bar{Y} \to Y$, then, as we see in Lemma 17, the geometric quotient $\bar{Y}/(\bar{Y} \times_Y \bar{Y})$ exists. It coincides with the strict closure considered in [Lipman 1971]. The curve case was introduced in [Arf 1948].

The related Lipschitz closure is studied in [Pham 1971] and [Lipman 1975].

3. Basic results

In this section we prove some basic existence results for geometric quotients.

Lemma 17. *Let S be a Noetherian scheme. Assume that X is finite over S and let $p_1, p_2 : R \rightrightarrows X$ be a finite, set-theoretic equivalence relation over S. Then the geometric quotient X/R exists.*

Proof. Since $X \to S$ is affine, the categorical quotient is the spectrum of the $\mathcal{O}_S$-algebra

$$\ker\Bigl[\mathcal{O}_X \stackrel{p_1^*-p_2^*}{\longrightarrow} \mathcal{O}_R\Bigr].$$

This kernel is a submodule of the finite $\mathcal{O}_S$-algebra $\mathcal{O}_X$, hence itself a finite $\mathcal{O}_S$-algebra. The only question is about the geometric fibers of $X \to (X/R)^{cat}$. Pick any $s \in S$. Taking the kernel commutes with flat base scheme extensions. Thus we may assume that S is complete, local with closed point s and algebraically closed residue field $k(s)$. We need to show that the reduced fiber of $(X/R)^{cat} \to S$ over s is naturally isomorphic to $\operatorname{red} X_s / \operatorname{red} R_s$.

If $U \to S$ is any finite map then $\mathcal{O}_{\operatorname{red} U_s}$ is a sum of $m(U)$ copies of $k(s)$ for some $m(U) < \infty$. U has $m(U)$ connected components $\{U_i : i = 1, \dots, m(U)\}$ and each $U_i \to S$ is finite. Thus $U \to S$ uniquely factors as

$$U \stackrel{g}{\to} \amalg_{m(U)} S \to S \quad \text{such that} \quad g_s : \operatorname{red} U_s \stackrel{\cong}{\longrightarrow} \amalg_{m(U)} \operatorname{Spec} k(s)$$

is an isomorphism, where $\amalg_m S$ denotes the disjoint union of m copies of S.

Applying this to $X \to S$ and $R \to S$, we obtain a commutative diagram

$$\begin{array}{ccc} R & \stackrel{p_1, p_2}{\rightrightarrows} & X \\ \downarrow & & \downarrow \\ \amalg_{m(R)} S & \stackrel{p_1(s), p_2(s)}{\rightrightarrows} & \amalg_{m(X)} S. \end{array}$$

Passing to global sections we get

$$\begin{array}{ccc} \mathcal{O}_X & \stackrel{p_1^*-p_2^*}{\longrightarrow} & \mathcal{O}_R \\ \uparrow & & \uparrow \\ \mathcal{O}_S^{m(X)} & \stackrel{p_1^*(s)-p_2^*(s)}{\longrightarrow} & \mathcal{O}_S^{m(R)}. \end{array}$$

The kernel of $p_1^*(s) - p_2^*(s)$ is $m := |X_s/R_s|$ copies of $\mathcal{O}_S$, hence we obtain a factorization

$$(X/R)^{cat} \to \amalg_m S \to S \quad \text{such that} \quad \operatorname{red}(X/R)^{cat}_s \to \amalg_m \operatorname{Spec} k(s)$$

is an isomorphism. □

For later reference, we record the following straightforward consequence.

Corollary 18. *Let $R \rightrightarrows X$ be a finite, set-theoretic equivalence relation such that X/R exists. Let $Z \subset X$ be a closed R-invariant subscheme. Then $Z/R|_Z$ exists and $Z/R|_Z \to X/R$ is a finite and universal homeomorphism (Definition 32) onto its image.* □

Example 19. Even in nice situations, $Z/R|_Z \to X/R$ need not be a closed embedding, as shown by the following examples.

(19.1) Set $X := \mathbb{A}^2_{xy} \amalg \mathbb{A}^2_{uv}$ and let R be the equivalence relation that identifies the x-axis with the u-axis.

Let $Z = (y = x^2) \amalg (v = u^2)$. In $Z/R|_Z$ the two components intersect at a node, but the image of Z in X/R has a tacnode.

In this example the problem is clearly caused by ignoring the scheme structure of $R|_Z$. As the next example shows, similar phenomena happen even if $R|_Z$ is reduced.

(19.2) Set $Y := (xyz = 0) \subset \mathbb{A}^3$. Let X be the normalization of Y and $R := X \times_Y X$. Set $W := (x + y + z = 0) \subset Y$ and let $Z \subset X$ be the preimage of W. As computed in Example 16, R and $R|_Z$ are both reduced, $Z/R|_Z$ is the seminormalization of W and $Z/R|_Z \to W$ is not an isomorphism.

Remark 20. The following putative counterexample to Lemma 17 is proposed in [Białynicki-Birula 2004, 6.2]. Consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} k[x, y] & \xrightarrow{p_1} & \operatorname{Spec} k[x, y^2, y^3] \\ p_2 \downarrow & & \downarrow q_2 \\ \operatorname{Spec} k[x+y, x+x^2, y^2, y^3] & \xrightarrow{q_1} & \operatorname{Spec} k[x+x^2, xy^2, xy^3, y^2, y^3] \end{array} \tag{20.1}$$

It is easy to see that the p_i are homeomorphisms but $q_2 p_1 = q_1 p_2$ maps $(0, 0)$ and $(-1, 0)$ to the same point. If (20.1) were a universal push-out, one would get a counterexample to Lemma 17. However, it is not a universal push-out. Indeed,

$$\begin{aligned} \tfrac{1}{3}(x+y)^3 + \tfrac{1}{2}(x+y)^2 &= \left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right) + (x^2+x)y + xy^2 + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 \\ &= -\left(\tfrac{2}{3}x^3 + \tfrac{1}{2}x^2\right) + (x^2+x)(x+y) + xy^2 + \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 \end{aligned}$$

shows that $\frac{2}{3}x^3 + \frac{1}{2}x^2$ is also in the intersection

$$k[x, y^2, y^3] \cap k[x+y, x+x^2, y^2, y^3].$$

Another case where X/R is easy to obtain is the following.

Lemma 21. *Let $p_1, p_2 : R \rightrightarrows X$ be a finite, set-theoretic equivalence relation where X is normal, Noetherian and X, R are both pure-dimensional. Assume*

(1) *X is defined over a field of characteristic* 0, *or*

(2) *X is essentially of finite type over S, or*

(3) *X is defined over a field of characteristic $p > 0$ and the Frobenius map $F^p : X \to X^{(p)}$ of §34 is finite.*

Then the geometric quotient X/R exists as an algebraic space. X/R is normal, Noetherian and essentially of finite type over S in case (2).

Proof. Thus U_x/G_x exists and it is easy to see that the U_x/G_x give étale charts for X/G.

In the general case, it is enough to construct the quotient when X is irreducible. Let m be the separable degree of the projections $\sigma_i : R \to X$.

Consider the m-fold product $X \times \cdots \times X$ with coordinate projections π_i. Let R_{ij} (resp. Δ_{ij}) denote the preimage of R (resp. of the diagonal) under (π_i, π_j). A geometric point of $\bigcap_{ij} R_{ij}$ is a sequence of geometric points $(x_1, \dots, x_m)$ such that any 2 are R-equivalent and a geometric point of $\bigcap_{ij} R_{ij} \setminus \cup_{ij} \Delta_{ij}$ is a sequence $(x_1, \dots, x_m)$ that constitutes a whole R-equivalence class. Let X' be the normalization of the closure of $\bigcap_{ij} R_{ij} \setminus \cup_{ij} \Delta_{ij}$. Note that every $\pi_\ell : \bigcap_{ij} R_{ij} \to X$ is finite, hence the projections $\pi'_\ell : X' \to X$ are finite.

The symmetric group S_m acts on $X \times \cdots \times X$ by permuting the factors and this lifts to an S_m-action on X'. Over a dense open subset of X, the S_m-orbits on the geometric points of X' are exactly the R-equivalence classes.

Let $X^* \subset X'/S_m \times X$ be the image of X' under the diagonal map.

By construction, $X^* \to X$ is finite and one-to-one on geometric points over an open set. Since X is normal, $X^* \cong X$ in characteristic 0 and $X^* \to X$ is purely inseparable in positive characteristic.

In characteristic 0, we thus have a morphism $X \to X'/S_m$ whose geometric fibers are exactly the R-equivalence classes. Thus $X'/S_m = X/R$.

Essentially the same works in positive characteristic, see Section 5 for details. □

Lemma 22. *Let $p_1, p_2 : R \rightrightarrows X$ be a finite, set-theoretic equivalence relation such that $(X/R)^{cat}$ exists.*

(1) *If X is normal and X, R are pure-dimensional then $(X/R)^{cat}$ is also normal.*

(2) *If X is seminormal then $(X/R)^{cat}$ is also seminormal.*

Proof. In the first case, let $Z \to (X/R)^{cat}$ be a finite morphism which is an isomorphism at all generic points of $(X/R)^{cat}$. Since X is normal, $\pi : X \to (X/R)^{cat}$ lifts to $\pi_Z : X \to Z$. By assumption, $\pi_Z \circ p_1$ equals $\pi_Z \circ p_2$ at all generic points of R and R is reduced. Thus $\pi_Z \circ p_1 = \pi_Z \circ p_2$. The universal property of categorical quotients gives $(X/R)^{cat} \to Z$, hence $Z = (X/R)^{cat}$ and $(X/R)^{cat}$ is normal.

In the second case, let $Z \to (X/R)^{cat}$ be a finite morphism which is a universal homeomorphism; see 32. As before, we get liftings $\pi_Z \circ p_1, \pi_Z \circ p_2 : R \rightrightarrows$

$X \to Z$ which agree on closed points. Since R is reduced, we conclude that $\pi_Z \circ p_1 = \pi_Z \circ p_2$, thus $(X/R)^{cat}$ is seminormal. □

The following result goes back at least to E. Noether.

Proposition 23. *Let A be a Noetherian ring, R a Noetherian A-algebra and G a finite group of A-automorphisms of R. Let $R^G \subset R$ denote the subalgebra of G-invariant elements. Assume that*

(1) *$|G|$ is invertible in A, or*

(2) *R is essentially of finite type over A, or*

(3) *R is finite over $A[R^p]$ for every prime p that divides $|G|$.*

Then R^G is Noetherian and R is finite over R^G.

Proof. Assume first that R is a localization of a finitely generated A algebra $A[r_1, \dots, r_m] \subset R$. We may assume that G permutes the r_j. Let σ_{ij} denote the jth elementary symmetric polynomial of the $\{g(r_i) : g \in G\}$. Then

$$A[\sigma_{ij}] \subset A[r_1, \dots, r_m]^G \subset R^G$$

and, with $n := |G|$, each r_i satisfies the equation

$$r_i^n - \sigma_{i1} r_i^{n-1} + \sigma_{i2} r_i^{n-2} - + \cdots = 0.$$

Thus $A[r_1, \dots, r_m]$ is integral over $A[\sigma_{ij}]$, and therefore also over the larger ring $A[r_1, \dots, r_m]^G$.

By assumption $R = U^{-1} A[r_1, \dots, r_m]$, where U is a subgroup of units in $A[r_1, \dots, r_m]$. We may assume that U is G-invariant. If $r/u \in R$, where $r \in A[r_1, \dots, r_m]$ and u a unit in $A[r_1, \dots, r_m]$, then

$$\frac{r}{u} = \frac{r \prod_{g \neq 1} g(u)}{u \prod_{g \neq 1} g(u)},$$

where the product is over the nonidentity elements of G. Thus $r/u = r'/u'$, where $r' \in A[r_1, \dots, r_m]$ and u' is a G-invariant unit in $A[r_1, \dots, r_m]$. Therefore,

$$R = (U^G)^{-1} A[r_1, \dots, r_m] \quad \text{is finite over} \quad (U^G)^{-1} A[\sigma_{ij}].$$

Since R^G is an $(U^G)^{-1} A[\sigma_{ij}]$-submodule of R, it is also finite over $(U^G)^{-1} A[\sigma_{ij}]$, hence the localization of a finitely generated algebra.

Assume next that $|G|$ is invertible in A. We claim that $JR \cap R^G = J$ for any ideal $J \subset R^G$. Indeed, if $a_i \in R^G$, $r_i \in R$ and $\sum r_i a_i \in R^G$ then

$$|G| \cdot \sum_i r_i a_i = \sum_{g \in G} \sum_i g(r_i) g(a_i) = \sum_i a_i \sum_{g \in G} g(r_i) \in \sum_i a_i R^G.$$

If $|G|$ is invertible, this gives

$$R^G \cap \sum a_i R = \sum a_i R^G.$$

Thus the map $J \mapsto JR$ from the ideals of R^G to the ideals of R is an injection which preserves inclusions. Therefore R^G is Noetherian if R is.

If R is an integral domain, then R is finite over R^G by Lemma 24. The general case, which we do not use, is left to the reader.

The arguments in case (3) are quite involved; see [Fogarty 1980]. □

Lemma 24. *Let R be an integral domain and G a finite group of automorphisms of R. Then R is contained in a finite R^G-module. Thus, if R^G is Noetherian, then R is finite over R^G.*

Proof. Let $K \supset R$ and $K^G \supset R^G$ denote the quotient fields. K/K^G is a Galois extension with group G. Pick $r_1, \dots, r_n \in R$ that form a K^G-basis of K. Then any $r \in R$ can be written as

$$r = \sum_i a_i r_i, \quad \text{where } a_i \in K^G.$$

Applying any $g \in G$ to it, we get a system of equations

$$\sum_i g(r_i) a_i = g(r) \quad \text{for } g \in G.$$

We can view these as linear equations with unknowns a_i. The system determinant is $D := \det_{i,g}\bigl(g(r_i)\bigr)$, which is nonzero since its square is the discriminant of K/K^G. The value of D is G-invariant up to sign; thus D^2 is G-invariant hence in R^G. By Kramer's rule, $a_i \in D^{-2} R^G$, hence $R \subset D^{-2} \sum_i r_i R^G$. □

In the opposite case, when the equivalence relation is nontrivial only on a proper subscheme, we have the following general result.

Proposition 25. *Let X be a reduced scheme, $Z \subset X$ a closed, reduced subscheme and $R \rightrightarrows X$ a finite, set-theoretic equivalence relation. Assume that R is the identity on $R \setminus Z$ and that the geometric quotient $Z/R|_Z$ exists. Then X/R exists and is given by the universal push-out diagram*

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ \downarrow & & \downarrow \\ Z/R|_Z & \hookrightarrow & X/R. \end{array}$$

Proof. Let Y denote the universal push-out (Theorem 38). Then $X \to Y$ is finite and so X/R exists and we have a natural map $X/R \to Y$ by Lemma 17. On the other hand, there is a natural map $Z/R|_Z \to X/R$ by Corollary 18, hence the universal property of the push-out gives the inverse $Y \to X/R$. □

4. Inductive plan for constructing quotients

Definition 26. Let $R \rightrightarrows X$ be a finite, set-theoretic equivalence relation and $g : Y \to X$ a finite morphism. Then

$$g^*R := R \times_{(X\times X)} (Y \times Y) \rightrightarrows Y$$

defines a finite, set-theoretic equivalence relation on Y. It is called the *pull-back* of $R \rightrightarrows X$. (Strictly speaking, it should be denoted by $(g \times g)^*R$.)

Note that the g^*R-equivalence classes on the geometric points of Y map injectively to the R-equivalence classes on the geometric points of X.

If X/R exists then, by Lemma 17, Y/g^*R also exists and the natural morphism $Y/g^*R \to X/R$ is injective on geometric points. If, in addition, g is surjective then $Y/g^*R \to X/R$ is a finite and universal homeomorphism; see Definition 32. Thus, if X is seminormal and the characteristic is 0, then $Y/g^*R \cong X/R$.

Let $h : X \to Z$ be a finite morphism. If the geometric fibers of h are subsets of R-equivalence classes, then the composite $R \rightrightarrows X \to Z$ defines a finite, set-theoretic pre-equivalence relation

$$h_*R := (h \times h)(R) \subset Z \times Z,$$

called the *push forward* of $R \rightrightarrows X$. If Z/R exists, then, by Lemma 17, X/R also exists and the natural morphism $X/R \to Z/R$ is a finite and universal homeomorphism.

Lemma 27. *Let X be weakly normal, excellent and $R \rightrightarrows X$ a finite, set-theoretic equivalence relation. Let $\pi : X^n \to X$ be the normalization and $R^n \rightrightarrows X^n$ the pull back of R to X^n. If X^n/R^n exists then X/R also exists and $X/R = X^n/R^n$.*

Proof. Let $X^* \subset (X^n/R^n) \times_S X$ be the image of X^n under the diagonal morphism. Since $X^n \to X$ is a finite surjection, X^* is a closed subscheme of $(X^n/R^n) \times_S X$ and $X^* \to X$ is a finite surjection. Moreover, for any geometric point $\bar{x} \to X$, its preimages $\bar{x}_i \to X^n$ are R^n-equivalent, hence they map to the same point in $(X^n/R^n) \times_S X$. Thus $X^* \to X$ is finite and one-to-one on geometric points, so it is a finite and universal homeomorphism; see Definition 32. $X^n \to X$ is a local isomorphism at the generic point of every irreducible component of X, hence $X^* \to X$ is also a local isomorphism at the generic point of every irreducible component of X. Since X is weakly normal, $X^* \cong X$ and we have a morphism $X \to X^n/R^n$ and thus $X/R = X^n/R^n$. □

Lemma 28. *Let X be normal and of pure dimension d. Let $\sigma : R \rightrightarrows X$ be a finite, set-theoretic equivalence relation and $R^d \subset R$ its d-dimensional part. Then $\sigma^d : R^d \rightrightarrows X$ is also an equivalence relation.*

Proof. The only question is transitivity. Since X is normal, the maps σ^d : $R^d \rightrightarrows X$ are both universally open by Chevalley's criterion; see [EGA IV-3 1966, IV.14.4.4]. Thus the fiber product $R^d \times_X R^d \to X$ is also universally open and hence its irreducible components have pure dimension d. □

Example 29. Let C be a curve with an involution τ. Pick $p, q \in C$ with q different from p and $\tau(p)$. Let C' be the nodal curve obtained from C by identifying p and q. The equivalence relation generated by τ on C' consists of the diagonal, the graph of τ plus the pairs $\big(\tau(p), \tau(q)\big)$ and $\big(\tau(q), \tau(p)\big)$. The 1-dimensional parts of the equivalence relation do not form an equivalence relation.

§30 (Inductive plan). Let X be an excellent scheme that satisfies one of the conditions of Lemma 21. and $R \rightrightarrows X$ a finite, set-theoretic equivalence relation. We aim to construct the geometric quotient X/R in two steps. First we construct a space that, roughly speaking, should be the normalization of X/R and then we try to go from the normalization to the geometric quotient itself.

Step 1. Let $X^n \to X$ be the normalization of X and $R^n \rightrightarrows X^n$ the pull back of R to X^n. Set $d = \dim X$ and let $X^{nd} \subset X^n$ (resp. $R^{nd} \subset R^n$) denote the union of the d-dimensional irreducible components. By Lemma 28, $R^{nd} \rightrightarrows X^{nd}$ is a pure-dimensional, finite, set-theoretic equivalence relation and the geometric quotient X^{nd}/R^{nd} exists by Lemma 21.

There is a closed, reduced subscheme $Z \subset X^n$ of dimension $< d$ such that Z is closed under R^n and the two equivalence relations

$$R^n|_{X^n \setminus Z} \quad \text{and} \quad R^{nd}|_{X^n \setminus Z} \quad \text{coincide.}$$

Let $Z_1 \subset X^{nd}/R^{nd}$ denote the image of Z. $R^n|_Z \rightrightarrows Z$ gives a finite set-theoretic equivalence relation on Z. Since the geometric fibers of $Z \to Z_1$ are subsets of R^n-equivalence classes, by Definition 26, the composite maps $R^n|_Z \rightrightarrows Z \to Z_1$ define a finite set-theoretic pre-equivalence relation on Z_1.

Step 2. In order to go from X^{nd}/R^{nd} to X/R, we make the following

Inductive assumption (30.2.1). The geometric quotient $Z_1/\big(R^n|_Z\big)$ exists.

Then, by Proposition 25, X^n/R^n exists and is given as the universal push-out of the following diagram:

$$\begin{array}{ccc} Z_1 & \hookrightarrow & X^n/R^{nd} \\ \downarrow & & \downarrow \\ Z_1/\big(R^n|_Z\big) & \hookrightarrow & X^n/R^n. \end{array}$$

As in Lemma 27, let $X^* \subset (X^n/R^n) \times_S X$ be the image of X^n under the diagonal morphism. We have established that $X^* \to X$ is a finite and universal homeomorphism (Definition 32) sitting in the following diagram:

$$\begin{array}{ccccccc} Z_1 & \hookrightarrow & X^n/R^{nd} & \leftarrow & X^n & & \\ \downarrow & & \downarrow & \swarrow & \downarrow & \searrow & \\ Z_1/(R^n|_Z) & \to & X^n/R^n & \leftarrow & X^* & \to & X \end{array} \tag{30.2.2}$$

There are now two ways to proceed.

Positive characteristic (30.2.3). Most finite and universal homeomorphisms can be inverted, up to a power of the Frobenius (Proposition 35), and so we obtain a morphism

$$X \to (X^*)^{(q)} \to (X^n/R^n)^{(q)}$$

for some $q = p^m$. X/R is then obtained using Lemma 17. This is discussed in Section 5.

In this case the inductive assumption (30.2.1) poses no extra problems.

Zero characteristic (30.2.4). As the examples of Section 2 show, finite and universal homeomorphisms cause a substantial problem. The easiest way to overcome these difficulties is to assume to start with that X is seminormal. In this case, by Lemma 27, we obtain $X/R = X^n/R^n$.

Unfortunately, the inductive assumption (30.2.1) becomes quite restrictive. By construction Z_1 is reduced, but it need not be seminormal in general. Thus we get the induction going only if we can guarantee that Z_1 is seminormal. Note that, because of the inductive set-up, seminormality needs to hold not only for X and Z_1, but on further schemes that one obtains in applying the inductive proof to $R^n|_Z \rightrightarrows Z_1$, and so on.

It turns out, however, that the above inductive plan works when gluing semi-log-canonical schemes. See [Kollár 2012, Chapters 5 and 8].

Definition 31. A morphism of schemes $f : X \to Y$ is a *monomorphism* if for every scheme Z the induced map of sets $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, Y)$ is an injection.

By [EGA IV-4 1967, IV.17.2.6] this is equivalent to assuming that f is universally injective and unramified.

A proper monomorphism $f : Y \to X$ is a closed embedding. Indeed, a proper monomorphism is injective on geometric points, hence finite. Thus it is a closed embedding if and only if $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is onto. By the Nakayama lemma this is equivalent to $f_x : f^{-1}(x) \to x$ being an isomorphism for every $x \in f(Y)$. By passing to geometric points, we are down to the case when $X = \operatorname{Spec} k$, k is algebraically closed and $Y = \operatorname{Spec} A$, where A is an Artin k-algebra.

If $A \neq k$, there are at least 2 different k maps $A \to k[\epsilon]$; thus Spec $A \to$ Spec k is not a monomorphism.

Definition 32. We say that a morphism of schemes $g : U \to V$ is a *universal homeomorphism* if it is a homeomorphism and for every $W \to V$ the induced morphism $U \times_V W \to W$ is again a homeomorphism. The definition extends to morphisms of algebraic spaces the usual way [Knutson 1971, II.3].

A simple example of a homeomorphism which is not a universal homeomorphism is Spec $K \to$ Spec L, where L/K is a finite field extension and $L \neq K$. A more interesting example is given by the normalization of the nodal curve $\bigl(y^2 = x^2(x+1)\bigr)$ with one of the preimages of the node removed:

$$\mathbb{A}^1 \setminus \{-1\} \to \bigl(y^2 = x^2(x+1)\bigr) \quad \text{given by} \quad t \mapsto \bigl(t^2 - 1, t(t^2-1)\bigr).$$

When g is finite, the notion is pretty much set-theoretic since a continuous proper map of topological spaces which is injective and surjective is a homeomorphism. Thus we see that for a finite and surjective morphism of algebraic spaces $g : U \to V$ the following are equivalent (see [Grothendieck 1971, I.3.7–8]):

(1) g is a universal homeomorphism.

(2) g is surjective and universally injective.

(3) For every $v \in V$ the fiber $g^{-1}(v)$ has a single point v' and $k(v')$ is a purely inseparable field extension of $k(v)$.

(4) g is surjective and injective on geometric points.

One of the most important properties of these morphisms is that taking the fiber product induces an equivalence between the categories

$$(\text{étale morphisms: } * \to V) \stackrel{* \mapsto * \times_V U}{\longrightarrow} (\text{étale morphisms: } * \to U).$$

See [SGA 1 1971, IX.4.10] for a proof. We do not use this in the sequel.

In low dimensions one can start the method of §30 and it gives the following. These results are sufficient to deal with the moduli problem for surfaces.

Proposition 33. *Let S be a Noetherian scheme over a field of characteristic* 0 *and X an algebraic space of finite type over S. Let $R \rightrightarrows X$ be a finite, set-theoretic equivalence relation. Assume that*

(1) *X is* 1*-dimensional and reduced*, *or*

(2) *X is* 2*-dimensional and seminormal*, *or*

(3) *X is* 3*-dimensional*, *normal and there is a closed*, *seminormal $Z \subset X$ such that R is the identity on $X \setminus Z$.*

Then the geometric quotient X/R exists.

Proof. Consider first the case when $\dim X = 1$. Let $\pi : X^n \to X$ be the normalization. We construct X^n/R^{nd} as in §30. Note that since Z is zero-dimensional, it is finite over S. Let $V \subset S$ be its image. Next we make a different choice for Z_1. Instead, we take a subscheme $Z_2 \subset X^n/R^{nd}$ whose support is Z_1 such that the pull back of its ideal sheaf $I(Z_2)$ to X^n is a subsheaf of the inverse image sheaf $\pi^{-1}\mathbb{O}_X \subset \mathbb{O}_{X^n}$.

Then we consider the push-out diagram

$$V \leftarrow Z_2 \hookrightarrow X^n/R^{nd}$$

with universal push-out Y. Then $X \to Y$ is a finite morphism and X/R exists by Lemma 17.

The case when $\dim X = 2$ and X is seminormal is a direct consequence of (30.2.4) since the inductive assumption (30.2.1) is guaranteed by item (1) of Proposition 33.

If $\dim X = 3$, then X is already normal and Z is seminormal by assumption. Thus $Z/\bigl(R|_Z\bigr)$ exists by Proposition 33(2). Therefore X/R is given by the push-out of $Z/\bigl(R|_Z\bigr) \leftarrow Z \hookrightarrow X$. □

5. Quotients in positive characteristic

The main result of this section is the proof of Theorem 6.

§34 (Geometric Frobenius morphism [SGA 5 1977, XIV]). Let S be an $\mathbb{F}_p$-scheme. Fix $q = p^r$ for some natural number r. Then $a \mapsto a^q$ defines an $\mathbb{F}_p$-morphism $F^q : S \to S$. This can be extended to polynomials by the formula

$$f = \sum a_I x^I \mapsto f^{(q)} := \sum a_I^q x^I.$$

Let $U = \operatorname{Spec} R$ be an affine scheme over S. Write

$$R = \mathbb{O}_S[x_1, \dots, x_m]/(f_1, \dots, f_n)$$

and set

$$R^{(q)} := \mathbb{O}_S[x_1^{(q)}, \dots, x_m^{(q)}]/(f_1^{(q)}, \dots, f_n^{(q)}) \quad \text{and} \quad U^{(q)} := \operatorname{Spec} R^{(q)},$$

where the $x_i^{(q)}$ are new variables. Thus we have a surjection $R^{(q)} \twoheadrightarrow R^q \subset R$, where R^q denotes the S-algebra generated by the q-th powers of all elements. $R^{(q)} \twoheadrightarrow R^q$ is an isomorphism if and only if R has no nilpotents.

There are natural morphisms

$$F^q : U \to U^{(q)} \quad \text{and} \quad (F^q)^* : R^{(q)} \to R \quad \text{given by} \quad (F^q)^*(x_i^{(q)}) = x_i^q.$$

It is easy to see that these are independent of the choices made. Thus F^q gives a natural transformation from algebraic spaces over S to algebraic spaces over S. One can define $X^{(q)}$ intrinsically as

$$X^{(q)} = X \times_{S,F^q} S.$$

If X is an algebraic space which is essentially of finite type over $\mathbb{F}_p$ then $F^q : X \to X^{(q)}$ is a finite and universal homeomorphism.

For us the most important feature of the Frobenius morphism is the following universal property:

Proposition 35. *Let S be a scheme essentially of finite type over $\mathbb{F}_p$ and X, Y algebraic spaces which are essentially of finite type over S. Let $g : X \to Y$ be a finite and universal homeomorphism. Then for $q = p^r \gg 1$ the map F^q can be factored as*

$$F^q : X \stackrel{g}{\to} Y \stackrel{\bar{g}}{\to} X^{(q)}.$$

Moreover, for large enough q (depending on $g : X \to Y$), there is a functorial choice of the factorization in the sense that if

$$\begin{array}{ccc} X_1 & \stackrel{g_1}{\to} & Y_1 \\ \downarrow & & \downarrow \\ X_2 & \stackrel{g_2}{\to} & Y_2 \end{array}$$

is a commutative diagram where the g_i are finite and universal homeomorphisms, then, for $q \gg 1$ (depending on the $g_i : X_i \to Y_i$) the factorization gives a commutative diagram

$$\begin{array}{ccccc} X_1 & \stackrel{g_1}{\to} & Y_1 & \stackrel{\bar{g}_1}{\to} & X_1^{(q)} \\ \downarrow & & \downarrow & & \downarrow \\ X_2 & \stackrel{g_2}{\to} & Y_2 & \stackrel{\bar{g}_2}{\to} & X_2^{(q)}. \end{array}$$

Proof. It is sufficient to construct the functorial choice of the factorization in case X and Y are affine schemes over an affine scheme $\operatorname{Spec} C$. Thus we have a ring homomorphism $g^* : A \to B$, where A and B are finitely generated C-algebras. We can decompose g^* into $A \twoheadrightarrow B_1$ and $B_1 \hookrightarrow B$. We deal with them separately.

First consider $B_1 \subset B$. In this case there is no choice involved and we need to show that there is a q such that $B^q \subset B_1$, where B^q denotes the C-algebra generated by the q-th powers of all elements. The proof is by Noetherian induction.

First consider the case when B is Artinian. The residue field of B is finite and purely inseparable over the residue field of B_1. For large enough q, taking qth powers kills all the nilpotents, thus B^q is contained in a field of representatives of B_1.

In the general, we can use the Artinian case over the generic points to obtain that $B_1 \subset B_1 B^q$ is an isomorphism at all generic points for $q \gg 1$. Let $I \subset B_1$

denote the conductor of this extension. That is, $IB_1B^q = I$. By induction we know that there is a q' such that $(B_1B^q/I)^{q'} \subset B_1/I$. Thus we get that

$$B^{(qq')} \to B^{qq'} \subset (B_1B^q)^{q'} \subset B_1 + IB_1B^q = B_1.$$

Next consider $A \twoheadrightarrow B_1$. Here we have to make a good choice. The kernel is a nilpotent ideal $I \subset A$, say $I^m = 0$. Choose q' such that $q' \geq m$. For $b_1 \in B_1$ let $b_1' \in A$ be any preimage. Then $(b_1')^{q'}$ depends only on b_1. The map

$$b_1 \mapsto (b_1')^{q'} \quad \text{defines a factorization} \quad B_1^{(q')} \to A \to B_1.$$

Combining the map $B^{(q)} \to B_1$ with $B_1^{(q')} \to A$ we obtain $B^{(qq')} \to A$. □

§36 (Proof of Theorem 6). The question is local on S, hence we may assume that S is affine. X and R are defined over a finitely generated subring of $\mathbb{O}_S$, hence we may assume that S is of finite type over $\mathbb{F}_p$.

The proof is by induction on $\dim X$. We follow the inductive plan in §30 and use its notation.

If $\dim X = 0$ then X is finite over S and the assertion follows from Lemma 17.

In going from dimension $d-1$ to d, the assumption (30.2.1) holds by induction. Thus (30.2.3) shows that X^n/R^n exists.

Let $X^* \subset (X^n/R^n) \times_S X$ be the image of X^n under the diagonal morphism. As we noted in §30, $X^* \to X$ is a finite and universal homeomorphism. Thus, by Proposition 35, there is a factorization

$$X^* \to X \to X^{*(q)} \to \big(X^n/R^n\big)^{(q)}.$$

Here $X \to \big(X^n/R^n\big)^{(q)}$ is finite and R is an equivalence relation on X over the base scheme $\big(X^n/R^n\big)^{(q)}$. Hence, by Lemma 17, the geometric quotient X/R exists. □

Remark 37. Some of the scheme-theoretic aspects of the purely inseparable case are treated in [Ekedahl 1987] and [SGA 3 1970, Exposé V].

6. Gluing or pinching

The aim of this section is to give an elementary proof of the following.

Theorem 38 [Artin 1970, Theorem 3.1]. *Let X be a Noetherian algebraic space over a Noetherian base scheme A. Let $Z \subset X$ be a closed subspace. Let $g : Z \to V$ be a finite surjection. Then there is a universal push-out diagram of algebraic spaces*

$$\begin{array}{ccc} Z & \hookrightarrow & X \\ g\downarrow & & \downarrow\pi \\ V & \hookrightarrow & Y := X/(Z \to V) \end{array}$$

Furthermore:

(1) *Y is a Noetherian algebraic space over A.*

(2) *$V \to Y$ is a closed embedding and $Z = \pi^{-1}(V)$.*

(3) *The natural map* $\ker[\mathcal{O}_Y \to \mathcal{O}_V] \to \pi_* \ker[\mathcal{O}_X \to \mathcal{O}_Z]$ *is an isomorphism.*

(4) *if X is of finite type over A then so is Y.*

Remark 39. If X is of finite type over A and A itself is of finite type over a field or an excellent Dedekind ring, then this is an easy consequence of the contraction results [Artin 1970, Theorem 3.1]. The more general case above follows using the later approximation results [Popescu 1986]. The main point of [Artin 1970] is to understand the case when $Z \to V$ is proper but not finite. This is much harder than the finite case we are dealing with. An elementary approach following [Ferrand 2003] and [Raoult 1974] is discussed below.

§40. The affine case of Theorem 38 is simple algebra. Indeed, let $q : \mathcal{O}_X \to \mathcal{O}_Z$ be the restriction. By Theorem 41, $q^{-1}(\mathcal{O}_V)$ is Noetherian; set $Y := \operatorname{Spec} q^{-1}(\mathcal{O}_V)$.

If $\bar{r}_i \in \mathcal{O}_X/I(Z)$ generate $\mathcal{O}_X/I(Z)$ as an $\mathcal{O}_V$-module then $r_i \in \mathcal{O}_X$ and $I(Z)$ generate $\mathcal{O}_X$ as a $q^{-1}(\mathcal{O}_V)$-module. Since $I(Z) \subset q^{-1}(\mathcal{O}_V)$, we obtain that $r_i \in \mathcal{O}_X$ and $1 \in \mathcal{O}_X$ generate $\mathcal{O}_X$ as a $q^{-1}(\mathcal{O}_V)$-module. Applying Theorem 41 to $R_1 = \mathcal{O}_X$ and $R_2 = q^{-1}(\mathcal{O}_V)$ gives the rest. □

For the proof of the following result, see [Matsumura 1986, Theorem 3.7] and the proof of Proposition 23.

Theorem 41 (Eakin and Nagata). *Let $R_1 \supset R_2$ be A-algebras with A Noetherian. Assume that R_1 is finite over R_2.*

(1) *If R_1 is Noetherian then so is R_2.*

(2) *If R_1 is a finitely generated A-algebra then so is R_2.* □

Gluing for algebraic spaces, following [Raoult 1974], is easier than the quasi-projective case.

§42 (Proof of Theorem 38). For every $p \in V$ we construct a commutative diagram

$$\begin{array}{ccccc} V_p & \stackrel{g_p}{\leftarrow} & Z_p & \to & X_p \\ \tau_V \downarrow & & \downarrow \tau_Z & & \downarrow \tau_X \\ V & \stackrel{g}{\leftarrow} & Z & \to & X \end{array}$$

where

(1) V_p, Z_p, X_p are affine,

(2) g_p is finite and $Z_p \to X_p$ is a closed embedding,

(3) V_p (resp. Z_p, X_p) is an étale neighborhood of p (resp. $g^{-1}(p)$) and

(4) both squares are fiber products.

Affine gluing (§40) then gives $Y_p := X_p/(Z_p \to V_p)$ and Lemma 44 shows that the Y_p are étale charts on $Y = X/(Z \to V)$.

Start with affine, étale neighborhoods $V_1 \to V$ of p and $X_1 \to X$ of $g^{-1}(p)$. Set $Z_1 := Z \times_X X_1 \subset X_1$. By §43 we may assume that there is a connected component $(Z \times_V V_1)^\circ \subset Z \times_V V_1$ and a (necessarily étale) morphism $(Z \times_V V_1)^\circ \to Z_1$. In general there is no étale neighborhood $X' \to X_1$ extending $(Z \times_V V_1)^\circ \to Z_1$, but there is an affine, étale neighborhood $X_2 \to X_1$ extending $(Z \times_V V_1)^\circ \to Z_1$ over a Zariski neighborhood of $g^{-1}(p)$ (§43).

Thus we have affine, étale neighborhoods $V_2 \to V$ of p, $X_2 \to X$ of $g^{-1}(p)$ and an open embedding $Z \times_X X_2 \hookrightarrow Z_2 := Z \times_V V_2$. Our only remaining problem is that $Z_2 \neq Z \times_X X_2$, hence Z_2 is not a subscheme of X_2. We achieve this by further shrinking V_2 and X_2.

The complement $B_2 := Z_2 \setminus Z \times_X X_2$ is closed, thus $g(B_2) \subset V_2$ is a closed subset not containing p. Pick $\phi \in \Gamma(\mathbb{O}_{V_2})$ that vanishes on $g(B_2)$ such that $\phi(p) \neq 0$. Then $\phi \circ g$ is a function on Z_2 that vanishes on B_2 but is nowhere zero on $g^{-1}(p)$. We can thus extend $\phi \circ g$ to a function Φ on X_2. Thus $V_P := V_2 \setminus (\phi = 0)$, $Z_P := Z_2 \setminus (\phi \circ g = 0)$ and $X_P := X_2 \setminus (\Phi = 0)$ have the required properties. □

§43. During the proof we have used two basic properties of étale neighborhoods.

First, if $\pi : X \to Y$ is finite then for every étale neighborhood $(u \in U) \to (x \in X)$ there is an étale neighborhood $(v \in V) \to (\pi(x) \in Y)$ and a connected component $(v' \in V') \subset X \times_Y V$ such that there is a lifting $(v' \in V') \to (u \in U)$.

Second, if $\pi : X \to Y$ is a closed embedding, $U \to X$ is étale and $P \subset U$ is a finite set of points then we can find an étale $V \to Y$ such that $P \subset V$ and there is an open embedding $(P \subset X \times_Y V) \to (P \subset U)$.

For proofs see [Milne 1980, 3.14 and 4.2–3].

The next result shows that gluing commutes with flat morphisms.

Lemma 44. *For $i = 1, 2$, let X_i be Noetherian affine A-schemes, $Z_i \subset X_i$ closed subschemes and $g_i : Z_i \to V_i$ finite surjections with universal push-outs Y_i. Assume that in the diagram below both squares are fiber products.*

$$\begin{array}{ccccc} V_1 & \stackrel{g_1}{\leftarrow} & Z_1 & \to & X_1 \\ \downarrow & & \downarrow & & \downarrow \\ V_2 & \stackrel{g_2}{\leftarrow} & Z_2 & \to & X_2 \end{array}$$

(1) *If the vertical maps are flat then $Y_1 \to Y_2$ is also flat.*

(2) *If the vertical maps are smooth then $Y_1 \to Y_2$ is also smooth.*

Proof. We may assume that all occurring schemes are affine. Set $R_i := \mathcal{O}_{X_i}$, $I_i := I(Z_i)$ and $S_i := \mathcal{O}_{V_i}$. Thus we have $I_i \subset R_i$ and $S_i \subset R_i/I_i$. Furthermore, R_1 is flat over R_2, $I_1 = I_2 R_1$ and S_1 is flat over S_2. We may also assume that R_2 is local. The key point is the isomorphism

$$(R_1/I_1) \cong (R_2/I_2) \otimes_{R_2} R_1 \cong (R_2/I_2) \otimes_{S_2} S_1. \tag{44.3}$$

This isomorphism is not naturally given; see Remark 45.

We check the local criterion of flatness in [Matsumura 1986, Theorem 22.3]. The first condition we need is that $q_1^{-1}(S_1)/I_1 \cong S_1$ be flat over $q_2^{-1}(S_2)/I_2 \cong S_2$. This holds by assumption. Second, we need that the maps

$$\left(I_2^n/I_2^{n+1}\right) \otimes_{S_2} S_1 \to I_2^n R_1/I_2^{n+1} R_1$$

be isomorphisms. Since R_1 is flat over R_2, the right hand side is isomorphic to

$$\left(I_2^n/I_2^{n+1}\right) \otimes_{R_2/I_2} (R_1/I_1).$$

Using (44.3), we get that

$$\left(I_2^n/I_2^{n+1}\right) \otimes_{R_2/I_2} (R_1/I_1) \cong \left(I_2^n/I_2^{n+1}\right) \otimes_{R_2/I_2} (R_2/I_2) \otimes_{S_2} S_1 \cong \left(I_2^n/I_2^{n+1}\right) \otimes_{S_2} S_1.$$

This settles flatness. In order to prove the smooth case, we just need to check that the fibers of $Y_1 \to Y_2$ are smooth. Outside $V_1 \to V_2$ we have the same fibers as before and $V_1 \to V_2$ is smooth by assumption. □

Remark 45. There is some subtlety in Lemma 44. Consider the simple case when X_2 is a smooth curve over a field k, $Z_2 = \{p, q\}$ two k-points and $V_2 = \operatorname{Spec} k$. Then Y_2 is a nodal curve where p and q are identified.

Let now $X_1 = X_2 \times \{0, 1\}$ as 2 disjoint copies. Then Z_1 consists of 4 points p_0, q_0, p_1, q_1 and V_1 is 2 copies of $\operatorname{Spec} k$. There are two distinct way to arrange g_1. Namely,

- either $g_1'(p_0) = g_1'(q_0)$ and $g_1'(p_1) = g_1'(q_1)$ and then Y_1' consists of 2 disjoint nodal curves,
- or $g_1''(p_0) = g_1''(q_1)$ and $g_1''(p_1) = g_1''(q_0)$ and then Y_1'' consists of a connected curve with 2 nodes and 2 irreducible components.

Both of these are étale double covers of Y_2.

As in §42, the next lemma will be used to reduce quasiprojective gluing to the affine case.

Lemma 46. *Let X be an A-scheme, $Z \subset X$ a closed subscheme and $g : Z \to V$ a finite surjection.*

Let $P \subset V$ be a finite subset and assume that there are open affine subsets $P \subset V_1 \subset V$ and $g^{-1}(P) \subset X_1 \subset X$.

Then there are open affine subsets $P \subset V_P \subset V_1$ and $g^{-1}(P) \subset X_P \subset X_1$ such that g restricts to a finite morphism $g : Z \cap X_P \to V_P$.

Proof. There is an affine subset $g^{-1}(P) \subset X_2 \subset X_1$ such that $g^{-1}(V \setminus V_1)$ is disjoint from X_2. Thus g maps $Z \cap X_2$ to V_1. The problem is that $(Z \cap X_2) \to V_1$ is only quasi finite in general. The set $Z \setminus X_2$ is closed in X and so $g(Z \setminus X_2)$ is closed in V. Since V_1 is affine, there is a function f_P on V_1 which vanishes on $g(Z \setminus X_2) \cap V_1$ but does not vanish on P. Then $f_P \circ g$ is a function on $g^{-1}(V_1)$ which vanishes on $(Z \setminus X_2) \cap g^{-1}(V_1)$ but does not vanish at any point of $g^{-1}(P)$. Since $Z \cap X_1$ is affine, $f_P \circ g$ can be extended to a regular function F_P on X_2.

Set $V_P := V_1 \setminus (f_P = 0)$ and $X_P := X_2 \setminus (F_P = 0)$. The restriction $(Z \cap X_P) \to V_P$ is finite since, by construction, $X_P \cap Z$ is the preimage of V_P. □

Definition 47. We say that an algebraic space X has the *Chevalley–Kleiman property* if X is separated and every finite subscheme is contained in an open affine subscheme. In particular, X is necessarily a scheme.

These methods give the following interesting corollary.

Corollary 48. *Let $\pi : X \to Y$ be a finite and surjective morphism of separated, excellent algebraic spaces. Then X has the Chevalley–Kleiman property if and only if Y has.*

Proof. Assume that Y has the Chevalley–Kleiman property and let $P \subset X$ be a finite subset. Since $\pi(P) \subset Y$ is finite, there is an open affine subset $Y_P \subset Y$ containing $\pi(P)$. Then $g^{-1}(Y_P) \subset X$ is an open affine subset containing P.

Conversely, assume that X has the Chevalley–Kleiman property. By the already established direction, we may assume that X is normal. Next let Y^n be the normalization of Y. Then $X \to Y^n$ is finite and dominant. Fix irreducible components $X_1 \subset X$ and $Y_1 \subset Y^n$ such that the induced map $X_1 \to Y_1$ is finite and dominant. Let $\pi_1' : X_1' \to X_1 \to Y_1$ be the Galois closure of X_1/Y_1 with Galois group G. We already know that X_1' has the Chevalley–Kleiman property, hence there is an open affine subset $X_P' \subset X_1'$ containing $(\pi_1')^{-1}(P)$. Then $U_P' := \bigcap_{g \in G} g(X_P') \subset X_1'$ is affine, Galois invariant and $(\pi_1')^{-1}\bigl(\pi_1'(U_P')\bigr) = U_P'$.

Thus $U_P' \to \pi_1'(U_P')$ is finite and, by Chevalley's theorem [Hartshorne 1977, Exercise III.4.2], $\pi_1'(U_P') \subset Y_1$ is an open affine subset containing P. Thus Y^n has the Chevalley–Kleiman property.

Next consider the normalization map $g : Y^n \to \operatorname{red} Y$. There are lower-dimensional closed subschemes $P \subset V \subset \operatorname{red} Y$ and $Z := g^{-1}(V) \subset Y^n$ such that $g : Y^n \setminus Z \cong \operatorname{red} Y \setminus V$ is an isomorphism. By induction on the dimension, V has the Chevalley–Kleiman property.

By Lemma 46 there are open affine subsets $P \subset V_P \subset V$ and $g^{-1}(P) \subset Y_P^n \subset Y^n$ such that g restricts to a finite morphism $g : Z \cap Y_P^n \to V_P$. Thus, by §40,

$g(Y_P^n) \subset \operatorname{red} Y$ is open, affine and it contains P. Thus $\operatorname{red} Y$ has the Chevalley–Kleiman property.

Finally, $\operatorname{red} Y \to Y$ is a homeomorphism, thus if $U \subset \operatorname{red} Y$ is an affine open subset and $U' \subset Y$ the "same" open subset of Y then U' is also affine by Chevalley's theorem and so Y has the Chevalley–Kleiman property. □

Example 49. Let E be an elliptic curve and set $S := E \times \mathbb{P}^1$. Pick a general $p \in E$ and $g : E \times \{0, 1\} \to E$ be the identity on $E_0 := E \times \{0\}$ and translation by $-p$ on $E_1 := E \times \{1\}$. Where are the affine charts on the quotient Y?

If $P_i \subset E_i$ are 0-cycles then there is an ample divisor H on S such that $(H \cdot E_i) = P_i$ if and only if $\mathcal{O}_{E_0}(P_0) = \mathcal{O}_{E_1}(P_1)$ under the identity map $E_0 \cong E_1$.

Pick any $a, b \in E_0$ and let $a + p, b + p \in E_1$ be obtained by translation by p. Assume next that $2a + b = a + p + 2(b + p)$, or, equivalently, that $3p = a - b$. Let $H(a, b)$ be an ample divisor on S such that $H(a, b) \cap E_0 = \{a, b\}$ and $H(a, b) \cap E_1 = \{a + p, b + p\}$. Then $U(a, b) := S \setminus H(a, b)$ is affine and g maps $E_i \cap U(a, b)$ isomorphically onto $E \setminus \{a, b\}$ for $i = 0, 1$. As we vary a, b (subject to $3p = a - b$) we get an affine covering of Y.

Note however that the curves $H(a, b)$ do not give Cartier divisors on Y. In fact, for nontorsion $p \in E$, every line bundle on Y pulls back from the nodal curve obtained from the $\mathbb{P}^1$ factor by gluing the points 0 and 1 together.

Appendix by Claudiu Raicu

§50. Let A be a noetherian commutative ring and $X = \mathbb{A}^n_S$ the n-dimensional affine space over $S = \operatorname{Spec} A$. Then $\mathcal{O}_X \simeq A[\boldsymbol{x}]$, where $\boldsymbol{x} = (x_1, \dots, x_n)$. To give a finite equivalence relation $R \subset X \times_S X$ is equivalent to giving an ideal $I(\boldsymbol{x}, \boldsymbol{y}) \subset A[\boldsymbol{x}, \boldsymbol{y}]$ which satisfies the following properties:

(1) (reflexivity) $I(\boldsymbol{x}, \boldsymbol{y}) \subset (x_1 - y_1, \dots, x_n - y_n)$.

(2) (symmetry) $I(\boldsymbol{x}, \boldsymbol{y}) = I(\boldsymbol{y}, \boldsymbol{x})$.

(3) (transitivity) $I(\boldsymbol{x}, \boldsymbol{z}) \subset I(\boldsymbol{x}, \boldsymbol{y}) + I(\boldsymbol{y}, \boldsymbol{z})$ in $A[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$.

(4) (finiteness) $A[\boldsymbol{x}, \boldsymbol{y}]/I(\boldsymbol{x}, \boldsymbol{y})$ is finite over $A[\boldsymbol{x}]$.

Suppose now that we have an ideal $I(\boldsymbol{x}, \boldsymbol{y})$ satisfying these four conditions and let R be the equivalence relation it defines. If the geometric quotient exists, then by Definition 4 it is of the form $\operatorname{Spec} A[f_1, \dots, f_m]$ for some polynomials $f_1, \dots, f_m \in A[\boldsymbol{x}]$. It follows that

$$I(\boldsymbol{x}, \boldsymbol{y}) \supset (f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}) : i = 1, 2, \dots, m)$$

and R is said to be *effective* if and only if equality holds.

We are mainly interested in the case when A is $\mathbb{Z}$ or some field k and I is homogeneous. Consider an ideal $I(\boldsymbol{x}, \boldsymbol{y}) = \big(J(\boldsymbol{x}, \boldsymbol{y}), f(\boldsymbol{x}, \boldsymbol{y})\big)$, where J is an ideal of the form

$$J(\boldsymbol{x},\boldsymbol{y}) = \big(f_i(\boldsymbol{x}) - f_i(\boldsymbol{y}) : i = 1, 2, \dots, m\big),$$

with homogeneous $f_i \in A[\boldsymbol{x}]$ such that $A[\boldsymbol{x}]$ is a finite module over $A[f_1, \dots, f_m]$ and $f \in A[\boldsymbol{x},\boldsymbol{y}]$ a homogeneous polynomial that satisfies the cocycle condition

$$f(\boldsymbol{x},\boldsymbol{y}) + f(\boldsymbol{y},\boldsymbol{z}) - f(\boldsymbol{x},\boldsymbol{z}) \in J(\boldsymbol{x},\boldsymbol{y}) + J(\boldsymbol{y},\boldsymbol{z}) \subset A[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]. \tag{50.1}$$

The reason we call (50.1) a cocycle condition is the following. If we let $B = A[f_1, \dots, f_m]$, $C = A[\boldsymbol{x}]$ and consider the complex (starting in degree zero)

$$C \to C \otimes_B C \to \cdots \to C^{\otimes_B m} \to \cdots \tag{50.2}$$

with differentials given by the formula

$$d_{m-1}(c_1 \otimes c_2 \otimes \cdots \otimes c_m) = \sum_{i=1}^{m+1} (-1)^i c_1 \otimes \cdots \otimes c_{i-1} \otimes 1 \otimes c_i \otimes \cdots \otimes c_m,$$

then $C \otimes_B C \simeq A[\boldsymbol{x},\boldsymbol{y}]/J(\boldsymbol{x},\boldsymbol{y})$, $C \otimes_B C \otimes_B C \simeq A[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}]/(J(\boldsymbol{x},\boldsymbol{y}) + J(\boldsymbol{y},\boldsymbol{z}))$, and if the polynomial $f(\boldsymbol{x},\boldsymbol{y})$ satisfies (50.1), then its class in $C \otimes_B C$ is a 1-cocycle in the complex (50.2).

Any ideal $I(x, y)$ defined as above is the ideal of a finite equivalence relation (though the geometric quotient can be different from B). To show that the equivalence relation it defines is noneffective it suffices to check that $f(x, y)$ is not congruent to a difference modulo $J(x, y)$. This can be done using a computer algebra system by computing the finite A-module U of homogeneous forms of the same degree as f which are congruent to differences modulo J, and checking that f is not contained in U. We used Macaulay 2 to check that the following example gives a noneffective equivalence relation (we took $A = \mathbb{Z}$ and $n = 2$):

$$f_1(x) = x_1^2, \quad f_2(x) = x_1x_2 - x_2^2, \quad f_3(x) = x_2^3,$$
$$f(x, y) = (x_1y_2 - x_2y_1)y_2^3,$$
$$I(x, y) = (x_1^2 - y_1^2, x_1x_2 - x_2^2 - y_1y_2 + y_2^2, x_2^3 - y_2^3, (x_1y_2 - x_2y_1)y_2^3).$$

We also claim that this example remains noneffective after any base change. Indeed, the A-module V generated by the forms of degree $5 (= \deg(f))$ in I and the differences $g(x) - g(y)$ with g homogeneous of degree 5, is a direct summand in U. Elements of V correspond to 0-coboundaries in (50.2). The module W consisting of elements of $k[x, y]_5$ whose classes in $k[x, y]/J$ are 1-cocycles is also a direct summand in U. The quotient W/V is a free $\mathbb{Z}$-module H generated by the class of $f(x, y)$. This shows that $W = V \oplus H$, hence for any field k we have $W_k = V_k \oplus H_k$, where for an abelian group G we let $G_k = G \otimes_{\mathbb{Z}} k$. If we denote by d_i^k the differentials in the complex obtained from (50.2) by base changing from $\mathbb{Z}$ to k, then we get that $\operatorname{im} d_0^k = V_k$ and $\ker d_1^k \supset W_k$. It follows that the nonzero elements of H_k will represent nonzero cohomology classes in (50.2) for any field k, hence our example is indeed universal.

By [Raicu 2010, Lemma 4.3], all homogeneous noneffective equivalence relations are contained in a homogeneous noneffective equivalence relation constructed as above.

In the positive direction, we have the following result in the toric case, where a *toric equivalence relation* (over a field k) is a scheme-theoretic equivalence relation R on a (not necessarily normal) toric variety X/k that is invariant under the diagonal action of the torus.

Theorem 51 [Raicu 2010, Theorem 4.1]. *Let k be a field, X/k an affine toric variety, and R a toric equivalence relation on X. Then there exists an affine toric variety Y/k together with a toric map $X \to Y$ such that $R \simeq X \times_Y X$.*

Notice that we do not require the equivalence relation to be finite.

Acknowledgements

We thank D. Eisenbud, M. Hashimoto, C. Huneke, K. Kurano, M. Lieblich, J. McKernan, M. Mustaţă, P. Roberts, Ch. Rotthaus, D. Rydh and R. Skjelnes for useful comments, corrections and references. Partial financial support was provided by the NSF under grant number DMS-0758275.

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