

Rigidity properties of Fano varieties

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We overview some recent results on Fano varieties giving evidence of their rigid nature under small deformations.

1. Introduction

From the point of view of the Minimal Model Program, Fano varieties constitute the building blocks of uniruled varieties. Important information on the biregular and birational geometry of a Fano variety is encoded, via Mori theory, in certain combinatorial data corresponding to the Néron–Severi space of the variety. It turns out that, even when there is actual variation in moduli, much of this combinatorial data remains unaltered, provided that the singularities are “mild” in an appropriate sense. One should regard any statement of this sort as a rigidity property of Fano varieties.

This paper gives an overview of Fano varieties, recalling some of their most important properties and discussing their rigid nature under small deformations. We will keep a colloquial tone, referring the reader to the appropriate references for many of the proofs. Our main purpose is indeed to give a broad overview of some of the interesting features of this special class of varieties. Throughout the paper, we work over the complex numbers.

2. General properties of Fano varieties

A *Fano manifold* is a projective manifold X whose anticanonical line bundle $-K_X := \bigwedge^n T_X$ is ample (here $n = \dim X$).

The simplest examples of Fano manifolds are given by the projective spaces \mathbb{P}^n . In this case, in fact, even the tangent space is ample. (By [Mori 1979], we know that projective spaces are the only manifolds with this property.)

In dimension two, Fano manifolds are known as del Pezzo surfaces. This class of surfaces has been widely studied in the literature (it suffices to mention

The first author was partially supported by NSF CAREER Grant DMS-0847059. The second author was supported by NSF Grant 0757897.

MSC2000: primary 14J45; secondary 14E22, 14D15.

Keywords: Fano varieties, Mori theory.

that several books have been written just on cubic surfaces), and their geometry is quite well understood. There are ten families of del Pezzo surfaces. The following theorem, obtained as a result of a series of papers [Nadel 1990; 1991; Campana 1991; 1992; Kollár et al. 1992a; 1992b], shows that this is a general phenomenon.

Theorem 2.1. *For every n , there are only finitely many families of Fano manifolds of dimension n .*

This theorem is based on the analysis of rational curves on Fano manifolds. In this direction, we should also mention the following important result:

Theorem 2.2 [Campana 1992; Kollár et al. 1992a]. *Fano manifolds are rationally connected.*

Fano varieties arise naturally in the context of the Minimal Model Program. This however leads us to work with possibly singular varieties. The smallest class of singularities that one has to allow is that of \mathbb{Q} -factorial varieties with terminal singularities. However, one can enlarge the class of singularities further, and work with \mathbb{Q} -Gorenstein varieties with log terminal (or, in some cases, even log canonical) singularities. In either case, one needs to consider $-K_X$ as a Weil divisor. The hypothesis guarantees that some positive multiple $-mK_X$ is Cartier (i.e., $\mathcal{O}_X(-mK_X)$ is a line bundle), so that one can impose the condition of ampleness.

For us, a *Fano variety* will be a variety with \mathbb{Q} -Gorenstein log terminal singularities such that $-K_X$ is ample. We will however be mostly interested in the case where the singularities are \mathbb{Q} -factorial and terminal.

The above results are however more delicate in the singular case. By a recent result of Zhang [2006], it is known that Fano varieties are rationally connected (see [Hacon and Mckernan 2007] for a related statement). Boundedness of Fano varieties is instead an open problem. The example of a cone over a rational curve of degree d shows that even for surfaces, we must make some additional assumptions. In this example one has that the minimal log discrepancies are given by $1/d$. One may hope that if we bound the minimal log discrepancies away from 0 the boundedness still holds. More precisely, the BAB conjecture (due to Alexeev, Borisov, Borisov) states that for every $n > 0$ and any $\epsilon > 0$, there are only finitely many families of Fano varieties of dimension n with ϵ -log terminal singularities (in particular, according to this conjecture, for every n there are only finitely many families of Fano varieties with canonical singularities).

Note that, by Theorem 2.2, it follows that Fano manifolds have the same cohomological invariants as rational varieties (namely $h^i(\mathcal{O}_X) = h^0(\Omega_X^q) = 0$ for all $i, q > 0$). On the other hand, by celebrated results of Iskovskikh and Manin [1971] and of Clemens and Griffiths [1972], it is known that there are

examples of Fano manifolds that are nonrational. The search for these examples was motivated by the Lüroth Problem. Note that it is still an open problem to find examples of Fano manifolds that are not unirational.

Perhaps the most important result known to hold for Fano varieties (for mild singularities and independently of their dimension), concerns the combinatorial structure associated to the cone of effective curves. The first instance of this was discovered by Mori [1982] in the smooth case. It is a particular case of the Cone Theorem (which holds for all varieties with log terminal singularities).

Theorem 2.3 (Cone theorem for Fano varieties). *The Mori cone of a Fano variety is rational polyhedral, generated by classes of rational curves.*

Naturally one may also ask if there are similar results concerning the structure of other cones of curves. From a dual perspective, one would like to understand the structure of the various cones of divisors on a Fano variety. The strongest result along these lines was conjectured by Hu and Keel [2000] and recently proved by Birkar, Cascini, Hacon, and McKernan:

Theorem 2.4 [Birkar et al. 2010]. *Fano varieties are Mori dream spaces in the sense of Hu and Keel.*

The meaning and impact of these results will be discussed in the next section.

3. Mori-theoretic point of view

Let X be a normal projective variety and consider the dual \mathbb{R} -vector spaces

$$N_1(X) := (Z_1(X)/\equiv) \otimes \mathbb{R} \quad \text{and} \quad N^1(X) := (\text{Pic}(X)/\equiv) \otimes \mathbb{R},$$

where \equiv denotes numerical equivalence. The *Mori cone* of X is the closure $\overline{\text{NE}}(X) \subset N_1(X)$ of the cone spanned by classes of effective curves. Its dual cone is the *nef cone* $\text{Nef}(X) \subset N^1(X)$, which by Kleiman's criterion is the closure of the cone spanned by ample classes. The closure of the cone spanned by effective classes in $N^1(X)$ is the *pseudo-effective cone* $\text{PEff}(X)$. Sitting in between the nef cone and the pseudo-effective cone is the *movable cone of divisors* $\text{Mov}(X)$, given by the closure of the cone spanned by classes of divisors moving in a linear system with no fixed components. All of these cones,

$$\text{Nef}(X) \subset \text{Mov}(X) \subset \text{PEff}(X) \subset N^1(X),$$

carry important geometric information about the variety X .

The Cone Theorem says that $\overline{\text{NE}}(X)$ is generated by the set of its K_X positive classes $\overline{\text{NE}}(X)_{K_X \geq 0} = \{\alpha \in \overline{\text{NE}}(X) \mid K_X \cdot \alpha \geq 0\}$ and at most countably many K_X negative rational curves $C_i \subset X$ of bounded anti-canonical degree $0 < -K_X \cdot C_i \leq 2 \dim(X)$. In particular the only accumulation points for the curve classes $[C_i]$

in $\overline{\text{NE}}(X)$ lie along the hyperplane determined by $K_X \cdot \alpha = 0$. Thus, for a Fano variety, the Mori cone $\overline{\text{NE}}(X)$ is a rational polyhedral cone. By duality, it follows that the nef cone $\text{Nef}(X) = (\overline{\text{NE}}(X))^\vee$ is also a rational polyhedral cone.

The geometry of X is reflected to a large extent in the combinatorial properties of $\overline{\text{NE}}(X)$. Every extremal face F of $\overline{\text{NE}}(X)$ corresponds to a surjective morphism $\text{cont}_F: X \rightarrow Y$, which is called a *Mori contraction*. The morphism cont_F contracts precisely those curves on X with class in F . Conversely, any morphism with connected fibers onto a normal variety arises in this way.

Remark 3.1. When X is not a Fano variety, $\overline{\text{NE}}(X)_{K_X < 0}$ may fail to be finitely generated, and even in very explicit examples such as blow-ups of \mathbb{P}^2 , the structure of the K_X positive part of the Mori cone is in general unknown. Consider, for example, the long-standing open conjectures of Nagata and Segre–Harbourne–Gimigliano–Hirschowitz. Also, the fact that the external faces of the cone of curves can always be contracted is only known when X is Fano (or, more generally, “log Fano”).

A similar behavior, that we will now describe, also occurs for the cone of nef curves. By definition the cone of nef curves $\overline{\text{NM}}(X) \subset N_1(X)$ is the closure of the cone generated by curves belonging to a covering family (a family of curves that dominates the variety X). It is clear that if $\alpha \in \overline{\text{NM}}(X)$ and D is an effective Cartier divisor on X , then $\alpha \cdot D \geq 0$. It follows that $\alpha \cdot D \geq 0$ for any pseudo-effective divisor D on X . We have this remarkable result:

Theorem 3.2 [Boucksom et al. 2004]. *The cone of nef curves is dual to the cone of pseudo-effective divisors, i.e., $\overline{\text{NM}}(X) = \text{PEff}(X)^\vee$.*

We now turn our attention to the case of \mathbb{Q} -factorial Fano varieties. In this case, the cone of nef curves $\overline{\text{NM}}(X)$ is also rational polyhedral and every extremal ray corresponds to a *Mori fiber space* $X' \rightarrow Y'$ on a model X' birational to X . More precisely:

Theorem 3.3 [Birkar et al. 2010, 1.3.5]. *R is an extremal ray of $\overline{\text{NM}}(X)$ if and only if there exists a \mathbb{Q} -divisor D such that (X, D) is Kawamata log terminal, and a $(K_X + D)$ Minimal Model Program $X \dashrightarrow X'$ ending with a Mori fiber space $X' \rightarrow Y'$, such that the numerical transform of any curve in the fibers of $X' \rightarrow Y'$ (e.g., the proper transform of a general complete intersection curve on a general fiber of $X' \rightarrow Y'$) has class in R .*

We will refer to the induced rational map $X \dashrightarrow Y'$ as a *birational Mori fiber structure* on X . We stress that we only consider Mori fiber structures that are the output of a Minimal Model Program. This was first studied by Batyrev [1992] in dimension three. The picture in higher dimensions was recently established by Birkar–Cascini–Hacon–McKernan for Fano varieties and, in a more general

context, by Araujo [2010] and Lehman [2008]. As a side note, even if it is known that the fibers of any Mori fibration $X' \rightarrow Y'$ are covered by rational curves, it still remains an open question whether the extremal rays of $\overline{NM}(X)$ are spanned by classes of rational curves. This is related to a delicate question on the rational connectivity of the smooth locus of singular varieties.

The dual point of view (looking at $N^1(X)$ rather than $N_1(X)$), also offers a natural way of refining the above results. As mentioned above, if X is a \mathbb{Q} -factorial Fano variety, then it is a *Mori dream space* [Hu and Keel 2000; Birkar et al. 2010]. The movable cone $\text{Mov}(X)$ of a Mori dream space admits a finite decomposition into rational polyhedral cones, called *Mori chambers*. One of these chambers is the nef cone of X . The other chambers are given by nef cones of \mathbb{Q} -factorial birational models $X' \sim_{\text{bir}} X$ which are isomorphic to X in codimension one. Note indeed that any such map gives a canonical isomorphism between $N^1(X)$ and $N^1(X')$. Wall-crossings between contiguous Mori chambers correspond to flops (or flips, according to the choice of the log pair structure) between the corresponding birational models. We can therefore view the Mori chamber decomposition of $\text{Mov}(X)$ as encoding information not only on the biregular structure of X but on its birational structure as well.

There is a way of recovering all this information from the total coordinate ring, or Cox ring, of a Mori dream space X , via a GIT construction. For simplicity, we assume that the map $\text{Pic}(X) \rightarrow N^1(X)$ is an isomorphism and that the class group of Weil divisors $\text{Cl}(X)$ of X is finitely generated. These properties hold if X is a Fano variety. The property that $\text{Pic}(X) \cong N^1(X)$ simply follows by the vanishing of $H^i(X, \mathbb{O}_X)$ for $i > 0$. The finite generation of $\text{Cl}(X)$ is instead a deeper property; a proof can be found in [Totaro 2009]. Specifically, see Theorem 3.1 there, which implies that the natural map $\text{Cl}(X) \rightarrow H_{2n-2}(X, \mathbb{Z})$ is an isomorphism for any n -dimensional Fano variety X (recall that in our definition of Fano variety we assume that the singularities are log terminal).

A *Cox ring* of X is, as defined in [Hu and Keel 2000], a ring of the type

$$R(L_1, \dots, L_r) := \bigoplus_{m \in \mathbb{Z}^\rho} H^0(X, \mathbb{O}_X(m_1 L_1 + \dots + m_\rho L_\rho)),$$

for any choice of line bundles L_1, \dots, L_ρ inducing a basis of $N^1(X)$. Here $\rho = \rho(X)$ is the Picard number of X , and $m = (m_1, \dots, m_r)$. We will call the *full Cox ring* of X the ring

$$R(X) := \bigoplus_{[D] \in \text{Cl}(X)} H^0(X, \mathbb{O}_X(D)).$$

If X is factorial (that is, if the map $\text{Pic}(X) \rightarrow \text{Cl}(X)$ is an isomorphism) and the line bundles L_i induce a basis of the Picard group, then the two rings coincide.

These rings were first systematically studied by Cox [1995] when X is a toric variety. If X is a toric variety and Δ is the fan of X , then the full Cox ring is the polynomial ring

$$R(X) = \mathbb{C}[x_\lambda \mid \lambda \in \Delta(1)],$$

where each x_λ defines a prime toric invariant divisor of X . When X is smooth, this property characterizes toric varieties. More precisely:

Theorem 3.4 [Hu and Keel 2000]. *Assume that X is a smooth Mori dream space. Then $R(X)$ is isomorphic to a polynomial ring if and only if X is a toric variety.*

More generally, Hu and Keel prove that a \mathbb{Q} -factorial Mori dream space X can be recovered from any of its Cox rings via a GIT construction. Moreover, the Mori chamber decomposition of X descends to X via this construction from a chamber decomposition associated to variations of linearizations in the GIT setting. From this perspective, the Cox ring of a Fano variety is a very rich invariant, encoding all essential information on the biregular and birational geometry of the variety.

The above discussion shows how the main features of the geometry of a Fano variety X , both from a biregular and a birational standpoint, are encoded in combinatorial data embedded in the spaces $N_1(X)$ and $N^1(X)$. Loosely speaking, we will say that geometric properties of X that are captured by such combinatorial data constitute the *Mori structure* of X .

In the remaining part of the paper, we will discuss to what extent the Mori structure of a Fano variety is preserved under flat deformations. Any positive result in this direction should be thought of as a rigidity statement.

The following result is the first strong evidence that Fano varieties should behave in a somewhat rigid way under deformations:

Theorem 3.5 [Wiśniewski 1991; 2009]. *The nef cone is locally constant in smooth families of Fano varieties.*

First notice that if $f: X \rightarrow T$ is a smooth family of Fano varieties, then f is topologically trivial, and thus, if we denote by $X_t := f^{-1}(t)$ the fiber over t , the space $N^1(X_t)$, being naturally isomorphic to $H^2(X_t, \mathbb{R})$, varies in a local system. By the polyhedrality of the nef cone, this local system has finite monodromy. This implies that, after suitable étale base change, one can reduce to a setting where the spaces $N^1(X_t)$ are all naturally isomorphic. The local constancy can therefore be intended in the étale topology.

Wiśniewski's result is the underlying motivation for the results that will be discussed in the following sections.

4. Deformations of the Cox rings

The proof of Theorem 3.5 has three main ingredients: the theory of deformations of embedded rational curves, Ehresmann's Theorem, and the Hard Lefschetz Theorem. All these ingredients use in an essential way the fact that the family is smooth. On the other hand, the very definitions involved in the whole Mori structure of a Fano variety use steps in the Minimal Model Program, which unavoidably generate singularities. With this in mind, we will present a different approach to the general problem of studying the deformation of Mori structures. The main ingredients of this approach will be the use of the Minimal Model Program in families, and an extension theorem for sections of line bundles (and, more generally, of divisorial reflexive sheaves). The first implications of such approach will be on the Cox rings. These applications will be discussed in this section. Further applications will then presented in the following section.

When working with families of singular Fano varieties, one needs to be very cautious. This is evident for instance in the simple example of quadric surfaces degenerating to a quadric cone: in this case allowing even the simplest surface singularity creates critical problems (the Picard number dropping in the central fiber), yielding a setting where the questions themselves cannot be posed.

We will restrict ourselves to the smallest category of singularities which is preserved in the Minimal Model Program, that of \mathbb{Q} -factorial terminal singularities. This is the setting considered in [de Fernex and Hacon 2011]. As explained in [Totaro 2009], many of the results presented below hold in fact under weaker assumptions on the singularities.

We consider a small flat deformation $f: X \rightarrow T$ of a Fano variety X_0 . Here T is a smooth curve with a distinguished point $0 \in T$, and $X_0 = f^{-1}(0)$. We assume that X_0 has terminal \mathbb{Q} -factorial singularities. A proof of the following basic result can be found in [de Fernex and Hacon 2011, Corollary 3.2 and Proposition 3.8], where an analogous but less trivial result is also proven to hold for small flat deformations of weak log Fano varieties with terminal \mathbb{Q} -factorial singularities.

Proposition 4.1. *For every t in a neighborhood of 0 in T , the fiber X_t is a Fano variety with terminal \mathbb{Q} -factorial singularities.*

After shrinking T near 0 , we can therefore assume that $f: X \rightarrow T$ is a flat family of Fano varieties with terminal \mathbb{Q} -factorial singularities. If $t \in T$ is a general point, the monodromy on $N^1(X_t)$ has finite order. This can be seen using the fact that the monodromy action preserves the nef cone of X_t , which is finitely generated and spans the whole space. After a suitable base change, one may always reduce to a setting where the monodromy is trivial.

If f is a smooth family, then it is topologically trivial, and we have already noticed that the spaces $N^1(X_t)$ vary in a local system. We have remarked how in general the dimension of these spaces may jump if f is not smooth. Under our assumptions on singularities the property remains however true. The proof of the following result is given in [de Fernex and Hacon 2011, Proposition 6.5], and builds upon results from [Kollár and Mori 1992, (12.10)].

Theorem 4.2. *The spaces $N^1(X_t)$ and $N_1(X_t)$ form local systems on T with finite monodromy. After suitable base change, for every $t \in T$ there are natural isomorphisms $N^1(X/T) \cong N^1(X_t)$ and $N_1(X_t) \cong N_1(X/T)$ induced, respectively, by pull-back and push-forward.*

A similar property holds for the class group, and is stated next. The proof of this property is given in [de Fernex and Hacon 2011, Lemma 7.2], and uses the previous result in combination with a generalization of the Lefschetz hyperplane theorem of Ravindra and Srinivas [2006] (the statement is only given for toric varieties, but the proof works in general). As shown in [Totaro 2009, Theorem 4.1], the same result holds more generally, only imposing that X is a projective variety with rational singularities and $H^1(X, \mathbb{C}_X) = H^2(X, \mathbb{C}_X) = 0$ (these conditions hold for any Fano variety) and that X_0 is smooth in codimension 2 and \mathbb{Q} -factorial in codimension three.

Theorem 4.3. *With the same assumptions as in Theorem 4.2, the class groups $\text{Cl}(X_t)$ form a local system on T with finite monodromy. After suitable base change, for every $t \in T$ there are natural isomorphisms $\text{Cl}(X/T) \cong \text{Cl}(X_t)$ induced by restricting Weil divisors to the fiber (the restriction is well-defined as the fibers are smooth in codimension one and their regular locus is contained in the regular locus of X).*

For simplicity, we henceforth assume that the monodromy is trivial. It follows by the first theorem that one can fix a common grading for Cox rings of the fibers X_t of the type considered in [Hu and Keel 2000]. The second theorem implies that there is also a common grading, given by $\text{Cl}(X/T)$, for the full Cox rings of the fibers. This is the first step needed to control the Cox rings along the deformation. The second ingredient is the following extension theorem.

Theorem 4.4. *With the above assumptions, let L be any Weil divisor on X that does not contain any fiber of f in its support. Then, after possibly restricting T (and consequently X) to a neighborhood of 0, the restriction map*

$$H^0(X, \mathbb{C}_X(L)) \rightarrow H^0(X_0, \mathbb{C}_{X_0}(L|_{X_0}))$$

is surjective (here $L|_{X_0}$ denotes the restriction of L to X_0 as a Weil divisor).

When L is Cartier, this theorem is a small generalization of Siu's invariance of plurigenera for varieties of general type. The formulation for Weil divisors follows by a more general result of [de Fernex and Hacon 2011], which is recalled below in Theorem 5.4.

As a corollary of the above theorems, we obtain the flatness of the Cox rings.

Corollary 4.5. *The full Cox ring $R(X_0)$ of X_0 , or any Cox ring $R(L_{0,1}, \dots, L_{0,\rho})$ of X_0 (provided the line bundles $L_{0,i}$ on X_0 are sufficiently divisible), deforms flatly in the family.*

The flatness of the deformation of the full Cox ring has a very interesting consequence when applied to deformations of toric Fano varieties.

Corollary 4.6. *Simplicial toric Fano varieties with terminal singularities are rigid.*

The proof of this corollary is based on the simple observation that a polynomial ring has no non-isotrivial flat deformations. This theorem appears in [de Fernex and Hacon 2011]. When X is smooth, the result was already known, and follows by a more general result of Bien and Brion on the vanishing of $H^1(X, T_X)$ for any smooth projective variety admitting a toroidal embedding (these are also known as *regular varieties*). The condition that the toric variety is simplicial is the translation, in toric geometry, of the assumption of \mathbb{Q} -factoriality. The above rigidity result holds in fact more in general, only assuming that the toric Fano variety is smooth in codimension 2 and \mathbb{Q} -factorial in codimension 3. This was proven in [Totaro 2009] using the vanishing theorems of Danilov and Mustașă $H^i(\tilde{\Omega}^j \otimes \mathcal{O}(D)^{**}) = 0$ for $i > 0$, $j > 0$ and D an ample Weil divisor on a projective toric variety.

The above results can also be used to show that also the Picard group is locally constant.

Corollary 4.7. *With the same assumptions as in Theorem 4.2, the Picard groups $\text{Pic}(X_t)$ form a local system on T with finite monodromy. After suitable base change, for every $t \in T$ there are natural isomorphisms $\text{Pic}(X/T) \cong \text{Pic}(X_t)$ induced by restriction.*

Proof. After suitable étale base change, we can assume that there is no monodromy on $\text{Cl}(X_t)$. Then, as $\text{Pic}(X_t)$ is a subgroup of $\text{Cl}(X_t)$, in view of Theorem 4.3 it suffices to show that every line bundle on X_0 extends, up to isomorphism, to a line bundle on X . Or, equivalently, that given any Cartier divisor D_0 on X_0 , there exists a Cartier divisor D on X not containing X_0 in its support and such that $D|_{X_0} \sim D_0$. Since any Cartier divisor can be written as the difference of two very ample divisors, we may assume that D_0 is very ample.

By Theorem 4.3, we can find a Weil divisor D on X not containing X_0 in its support and such that $D|_{X_0} \sim D_0$. We need to show that D is Cartier in a neighborhood of X_0 . We can replace D_0 with $D|_{X_0}$. After possibly shrinking T near 0, every section of $\mathbb{C}_{X_0}(D_0)$ extends to a section of $\mathbb{C}_X(D)$ by Theorem 4.4. Since $\mathbb{C}_{X_0}(D_0)$ is generated by its global sections, it follows that the natural homomorphism $\mathbb{C}_X(D) \rightarrow \mathbb{C}_{X_0}(D_0)$ is surjective. As $\mathbb{C}_{X_0}(D_0)$ is invertible, this implies that so is $\mathbb{C}_X(D)$, and thus that D is Cartier. \square

5. Deformations of the Mori structure

The flatness of Cox rings in flat families of Fano varieties with terminal \mathbb{Q} -factorial singularities is already evidence of a strong rigidity property of such varieties. In this section, we consider a flat family $f: X \rightarrow T$ of Fano varieties with terminal \mathbb{Q} -factorial singularities, parametrized by a smooth curve T .

An immediate corollary of Theorem 4.4 is the following general fact.

Corollary 5.1. *For any flat family $f: X \rightarrow T$ of Fano varieties with terminal \mathbb{Q} -factorial singularities over a smooth curve T , the pseudo-effective cones $\text{PEff}(X_t)$ of the fibers of f are locally constant in the family.*

If one wants to further investigate how the Mori structure varies in the family, it becomes necessary to run the Minimal Model Program. This requires us to step out, for a moment, from the setting of families of Fano varieties.

Suppose for now that $f: X \rightarrow T$ is just a flat projective family of normal varieties with \mathbb{Q} -factorial singularities. Let X_0 be the fiber over a point $0 \in T$. We assume that the restriction map $N^1(X) \rightarrow N^1(X_0)$ is surjective, that X_0 has canonical singularities, and that there is an effective \mathbb{Q} -divisor D on X , not containing X_0 in its support, such that $(X_0, D|_{X_0})$ is a Kawamata log terminal pair. Assume furthermore that $D|_{X_0} - aK_{X_0}$ is ample for some $a > -1$. Note that this last condition always holds for Fano varieties.

The following result is crucial for our investigation.

Theorem 5.2. *With the above notation, every step $X^i \dashrightarrow X^{i+1}$ in the Minimal Model Program of (X, D) over T with scaling of $D - aK_X$ is either trivial on the fiber X_0^i of X^i over 0, or it induces a step of the same type (divisorial contraction, flip, or Mori fibration) $X_0^i \dashrightarrow X_0^{i+1}$ in the Minimal Model Program of $(X_0, D|_{X_0})$ with scaling of $D|_{X_0} - aK_{X_0}$. In particular, at each step X_0^i is the proper transform of X_0 .*

For a proof of this theorem, we refer the reader to [de Fernex and Hacon 2011] (specifically, see Theorem 4.1 and the proof of Theorem 4.5 there). The key observation is that, by running a Minimal Model Program with scaling of $D - aK_X$, we can ensure that the property that $D|_{X_0} - aK_{X_0}$ is ample for some

$a > -1$ is preserved after each step of the program. By the semicontinuity of fiber dimensions, it is easy to see that $X^i \dashrightarrow X^{i+1}$ is a Mori fiber space if and only if so is $X_0^i \dashrightarrow X_0^{i+1}$. If $X^i \dashrightarrow X^{i+1}$ is birational, then the main issue is to show that if $X^i \dashrightarrow X^{i+1}$ is a flip and $X^i \rightarrow Z^i$ is the corresponding flipping contraction, then $X_0^i \rightarrow Z_0^i$ is also a flipping contraction. If this were not the case, then $X_0^i \rightarrow Z_0^i$ would be a divisorial contraction and hence Z_0^i would be \mathbb{Q} -factorial. Since $D^i|_{X_0^i} - aK_{X_0^i}$ is nef over Z_0^i , it follows that $-K_{X_0^i}$ is ample over Z_0^i and hence that Z_0^i is canonical. By [de Fernex and Hacon 2011, Proposition 3.5] it then follows that Z^i is \mathbb{Q} -factorial. This is the required contradiction as the target of a flipping contraction is never \mathbb{Q} -factorial. Therefore it follows that $X^i \rightarrow Z^i$ is a flipping contraction if and only if so is $X_0^i \rightarrow Z_0^i$.

Remark 5.3. The theorem implies that $X_0^i \dashrightarrow X_0^{i+1}$ is a divisorial contraction or a Mori fibration if and only if so is $X_t^i \dashrightarrow X_t^{i+1}$ for general $t \in T$. However, there exist flipping contractions $X^i \dashrightarrow X^{i+1}$ which are the identity on a general fiber X_t^i . This follows from the examples of Totaro that we will briefly sketch at the end of the section.

One of the main applications of this result is the following extension theorem, which in particular implies the statement of Theorem 4.4 in the case of families of Fano varieties.

Theorem 5.4 [de Fernex and Hacon 2011, Theorem 4.5]. *With the same notation as in Theorem 5.2, assume that $(X_0, D|_{X_0})$ is canonical and, moreover, that either $D|_{X_0}$ or $K_{X_0} + D|_{X_0}$ is big. Let L be any Weil divisor whose support does not contain X_0 and such that $L|_{X_0} \equiv k(K_X + D)|_{X_0}$ for some rational number $k > 1$. Then the restriction map*

$$H^0(X, \mathbb{O}_X(L)) \rightarrow H^0(X_0, \mathbb{O}_{X_0}(L|_{X_0}))$$

is surjective.

There are versions of the above results where the condition on the positivity of $D|_{X_0} - aK_{X_0}$ is replaced by the condition that the stable base locus of $K_X + D$ does not contain any irreducible component of $D|_{X_0}$ [de Fernex and Hacon 2011, Theorem 4.5]. The advantage of the condition considered here is that it only requires us to know something about the special fiber X_0 . This is a significant point, as after all we are trying to lift geometric properties from the special fiber to the whole space and nearby fibers of an arbitrary flat deformation.

We now come back to the original setting, and hence assume that $f : X \rightarrow T$ is a flat family of Fano varieties with \mathbb{Q} -factorial terminal singularities. After étale base change, we can assume that $N^1(X_t) \cong N^1(X/T)$ for every t .

Corollary 5.1 implies, by duality, that the cones of nef curves $\overline{NM}(X_t)$ are constant in the family. Combining this with Theorem 5.2, we obtain the following

rigidity property of birational Mori fiber structures. Recall that we only consider Mori fiber structures that are the output of some Minimal Model Program.

Theorem 5.5. *The birational Mori fiber structures $X_t \dashrightarrow X'_t \rightarrow Y'_t$ are locally constant in the family $X \rightarrow T$.*

This result is implicit in [de Fernex and Hacon 2011]. As it was not explicitly stated there, we provide a proof.

Proof. Let R_0 be the extremal ray of $\overline{\text{NM}}(X_0)$ corresponding to a given birational Mori fiber structure on X_0 . Note that by [Birkar et al. 2010, 1.3.5] and its proof, there exists an ample \mathbb{R} -divisor A_0 such that the K_{X_0} Minimal Model Program with scaling of A_0 say $X_0 \dashrightarrow X'_0$ ends with Mori fiber space $X'_0 \rightarrow Y'_0$ which is $(K_{X_0} + A_0)$ -trivial. Notice also that if we make a general choice of A_0 in $N^1(X_0)$, then each step of this Minimal Model Program with scaling is uniquely determined since at each step there is a unique $K_{X_0} + t_i A_0$ trivial extremal ray.

We may now assume that there is an ample \mathbb{R} -divisor A on X such that $A_0 = A|_{X_0}$. Consider running the K_X Minimal Model Program over T with scaling of A say $X \dashrightarrow X'$. Since X is uniruled, this ends with a Mori fiber space $X' \rightarrow Y'$. By Theorem 5.2, this induces the Minimal Model Program with scaling on the fiber X_0 considered in the previous paragraph. Moreover, the Minimal Model Program on X terminates with the Mori fiber space $X' \rightarrow Y'$ at the same step when the induced Minimal Model Program on X_0 terminates with the Mori fiber space $X'_0 \rightarrow Y'_0$. This implies that the birational Mori fiber structure $X_0 \dashrightarrow Y'_0$ extends to the birational Mori fiber structure $X \dashrightarrow Y'$, and thus deforms to a birational Mori fiber structure on the nearby fibers. \square

A similar application of Theorems 5.2 and 5.4 leads to the following rigidity result for the cone of moving divisors.

Theorem 5.6 [de Fernex and Hacon 2011]. *The moving cone $\text{Mov}(X_t)$ of divisors is locally constant in the family.*

Proof. The proof is similar to the proof of Theorem 5.5 once we observe that the faces of $\text{Mov}(X)$ are determined by divisorial contractions and that given an extremal contraction $X \rightarrow Z$ over T , this is divisorial if and only if the contraction on the central fiber $X_0 \rightarrow Z_0$ is divisorial. \square

Regarding the behavior of the nef cone of divisors and, more generally, of the Mori chamber decomposition of the moving cone, the question becomes however much harder. In fact, once we allow even the mildest singularities, the rigidity of the whole Mori structure only holds in small dimensions.

Theorem 5.7 [de Fernex and Hacon 2011, Theorem 6.9]. *With the notation above, assume that X_0 is either at most 3-dimensional, or is 4-dimensional*

and Gorenstein. Then the Mori chamber decomposition of $\text{Mov}(X_t)$ is locally constant for t in a neighborhood of $0 \in T$.

Totaro [2009] provides families of examples that show that this result is optimal. In particular, for every $a > b > 1$, he shows that there is a family of terminal \mathbb{Q} -factorial Gorenstein Fano varieties $X \rightarrow T$ such that $X_t \cong \mathbb{P}^a \times \mathbb{P}^b$ for $t \neq 0$ and $\text{Nef}(X_0)$ is strictly contained in $\text{Nef}(X_t)$. The reason for this is that there is a flipping contraction $X \rightarrow Z$ over T which is an isomorphism on the general fiber X_t but contracts a copy of \mathbb{P}^a contained in X_0 . Let $X^+ \rightarrow Z$ be the corresponding flip and fix H^+ a divisor on X^+ which is ample over T . If H is its strict transform on X , then H is negative on flipping curves and hence $H|_{X_0}$ is not ample, however $H|_{X_t} \cong H^+|_{X^+}$ is ample for $t \neq 0$. Therefore, the nef cone of X_0 is strictly smaller than the nef cone of X_t so that the Mori chamber decomposition of $\text{Mov}(X_0)$ is finer than that of $\text{Mov}(X_t)$.

The construction of this example starts from the flip from the total space of $\mathbb{C}_{\mathbb{P}^a}(-1)^{\oplus(b+1)}$ to the total space of $\mathbb{C}_{\mathbb{P}^b}(-1)^{\oplus(a+1)}$. The key idea is to interpret this local setting in terms of linear algebra, by viewing the two spaces as small resolutions of the space of linear maps of rank at most one from \mathbb{C}^{b+1} to \mathbb{C}^{a+1} , and to use such a description to compactify the setting into a family of Fano varieties. Totaro also gives an example in dimension 4, where the generic element of the family is isomorphic to the blow-up of \mathbb{P}^4 along a line, and the central fiber is a Fano variety with \mathbb{Q} -factorial terminal singularities that is not Gorenstein.

Remark 5.8. The fact that the Mori chamber decomposition is not in general locally constant in families of Fano varieties with \mathbb{Q} -factorial terminal singularities is not in contradiction with the flatness of Cox rings. The point is that the flatness of such rings is to be understood only as modules, but it gives no information on the ring structure. The changes in the Mori chamber decomposition are related to jumps of the kernels of the multiplication maps.

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