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Periods and moduli

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This text is an introduction, without proofs and by means of many examples, to some elementary aspects of the theory of period maps, period domains, and their relationship with moduli spaces. We start with the definitions of Jacobians of curves, Prym varieties, and intermediate Jacobians, then move on to Griffiths' construction of period domains and period maps. We review some instances of the Torelli problem and discuss some recent results of Allcock, Carlson, Laza, Looijenga, Swierstra, and Toledo, expressing some moduli spaces as ball quotients.

It has been known since the nineteenth century that there is a group structure on the points of the smooth cubic complex plane curve (called an *elliptic curve*) and that it is isomorphic to the quotient of $\mathbf{C}$ by a lattice. Conversely, any such quotient is an elliptic curve.

The higher-dimensional analogs are *complex tori* V/Γ, where Γ is a lattice in a (finite-dimensional) complex vector space V. The group structure and the analytic structure are obvious, but not all tori are algebraic. For that, we need an additional condition, which was formulated by Riemann: the existence of a positive definite Hermitian form on V whose (skew-symmetric) imaginary part is integral on Γ. An algebraic complex torus is called an *abelian variety.* When this skew-symmetric form is in addition unimodular on Γ, we say that the abelian variety is *principally polarized.* It contains a hypersurface uniquely determined up to translation ("the" *theta divisor*).

The combination of the algebraic and group structures makes the geometry of abelian varieties very rich. This is one of the reasons why it is useful to associate, whenever possible, an abelian variety (if possible principally polarized) to a given geometric situation. This can be done only in a few specific cases, and the theory of periods, mainly developed by Griffiths, constitutes a far-reaching extension.

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Our aim is to present an elementary introduction to this theory. We show many examples to illustrate its diversity, with no pretense at exhaustivity, and no proofs. For those interested in pursuing this very rich subject, we refer to [Carlson et al. 2003] and its bibliography.

Here is a short description of the contents of this text. In section 1, we review some of the classical cases where one can attach an abelian variety to a geometric situation; they are very special, because they correspond to varieties with a Hodge decomposition of level one. In section 2, we show, following Griffiths, how to extend drastically this construction by defining period maps and period domains. In section 3, we review some instances of the Torelli problem: when does the abelian variety associated to a given situation (as in section 1) characterize it? In the framework of section 2, this translates into the question of the injectivity of the period map. There is no general principle here, and we give examples where the answer is yes, and other examples where the period map is not injective, and even has positive-dimensional fibers. In the last section, we briefly discuss various questions relative to moduli spaces; the period map can sometimes be used to relate them to more concrete geometrical objects. There is also the important matter of the construction of compactifications for these moduli spaces and of the extension of the period map to these compactifications.

We work over the field of complex numbers.

1. Attaching an abelian variety to an algebraic object

1.1. ***Curves.*** Given a smooth projective curve C of genus g, we have the Hodge decomposition

$$H^1(C, \mathbf{Z}) \subset H^1(C, \mathbf{C}) = H^{0,1}(C) \oplus H^{1,0}(C),$$

where the right side is a $2g$-dimensional complex vector space and $H^{1,0}(C) = \overline{H^{0,1}(C)}$. The g-dimensional complex torus

$$J(C) = H^{0,1}(C)/H^1(C, \mathbf{Z})$$

is a principally polarized abelian variety (the polarization corresponds to the unimodular intersection form on $H^1(C, \mathbf{Z})$). We therefore have an additional geometric object: the theta divisor $\Theta \subset J(C)$, uniquely defined up to translation.

The geometry of the theta divisor of the Jacobian of a curve has been intensively studied since Riemann. One may say that it is well-known; see [Arbarello et al. 1985].

1.2. ***Prym varieties.*** Given a double étale cover $\pi : \tilde{C} \to C$ between smooth projective curves, one can endow the abelian variety $P(\tilde{C}/C) = J(\tilde{C})/\pi^* J(C)$

with a natural principal polarization. The dimension of $P(\tilde{C}/C)$ is

$$g(\tilde{C}) - g(C) = g(C) - 1;$$

it is called the *Prym variety* attached to π. By results from [Mumford 1974; Tjurin 1975a; Beauville 1977a], we have a rather good understanding of the geometry of the theta divisor of $P(\tilde{C}/C)$.

1.3. *Threefolds.* If X is a smooth projective threefold, the Hodge decomposition is

$$H^3(X, \mathbf{Z}) \subset H^3(X, \mathbf{C}) = \big(H^{0,3}(X) \oplus H^{1,2}(X)\big) \oplus (\text{complex conjugate})$$

and we may again define the *intermediate Jacobian* of X as the complex torus

$$J(X) = \big(H^{0,3}(X) \oplus H^{1,2}(X)\big)/H^3(X, \mathbf{Z}).$$

It is in general not algebraic. In case $H^{0,3}(X)$ vanishes, however, we have again a principally polarized abelian variety.

In some situations, the intermediate Jacobian is a Prym. For example if X has a conic bundle structure $X \to \mathbf{P}^2$ (i.e., a morphism with fibers isomorphic to conics), define the discriminant curve $C \subset \mathbf{P}^2$ as the locus of points whose fibers are reducible conics, i.e., unions of two lines. The choice of one of these lines defines a double cover $\tilde{C} \to C$. Although C may have singular points, we can still define a Prym variety $P(\tilde{C}/C)$. We have $H^{0,3}(X) = 0$, and

$$J(X) \simeq P(\tilde{C}/C)$$

as principally polarized abelian varieties. This isomorphism is a powerful tool for proving nonrationality of some threefolds: the intermediate Jacobian of a rational threefold must have a very singular theta divisor and the theory of Prym varieties can sometimes tell that this does not happen.

Example 1.1 (Cubic threefolds). If $X \subset \mathbf{P}^4$ is a smooth cubic hypersurface, we have $h^{0,3}(X) = 0$ and $h^{1,2}(X) = 5$, so that $J(X)$ is a 5-dimensional principally polarized abelian variety.

Any such X contains a line ℓ. Projecting from this line induces a conic bundle structure $\tilde{X} \to \mathbf{P}^2$ on the blow-up $\tilde{X}$ of ℓ in X. The discriminant curve $C \subset \mathbf{P}^2$ is a quintic and $J(X) \simeq P(\tilde{C}/C)$ (this agrees with the fact that $J(X)$ has dimension $g(C) - 1 = \binom{4}{2} - 1 = 5$). This isomorphism can be used to prove that the theta divisor $\Theta \subset J(X)$ has a unique singular point, which has multiplicity 3 [Beauville 1982]. In particular, it is not "singular enough," and X is not rational [Clemens and Griffiths 1972].

Example 1.2 (Quartic double solids). If $p : X \to \mathbf{P}^3$ is a double cover branched along a quartic surface $B \subset \mathbf{P}^3$ (a *quartic double solid),* we have $h^{0,3}(X) = 0$

and $h^{1,2}(X) = 10$, so that $J(X)$ has dimension 10. There is in general no conic bundle structure on X. However, when B acquires an ordinary double point s, the variety X becomes singular, and there is a (rational) conic bundle structure $X \dashrightarrow \mathbf{P}^2$ obtained by composing p with the projection $\mathbf{P}^3 \dashrightarrow \mathbf{P}^2$ from s. The discriminant curve is a sextic. This degeneration can be used to prove that for X general, the singular locus of the theta divisor $\Theta \subset J(X)$ has dimension 5 [Voisin 1988] and has a unique component of that dimension [Debarre 1990]. Again, this implies that X is not rational.

Example 1.3 (Fano threefolds of degree 10). If $X \subset \mathbf{P}^9$ is the smooth complete intersection of the Grassmannian $G(2, 5)$ in its Plücker embedding, two hyperplanes, and a smooth quadric, we have $h^{0,3}(X) = 0$ and $h^{1,2}(X) = 10$, so that $J(X)$ has dimension 10.

1.4. ***Odd-dimensional varieties.*** Let X be a smooth projective variety of dimension $2n + 1$ whose Hodge decomposition is of the form

$$H^{2n+1}(X, \mathbf{C}) = H^{n,n+1}(X) \oplus H^{n+1,n}(X).$$

We may define the *intermediate Jacobian* of X as

$$J(X) = H^{n,n+1}(X)/H^{2n+1}(X, \mathbf{Z}).$$

This is again a principally polarized abelian variety.

Example 1.4 (Intersections of two quadrics). If $X \subset \mathbf{P}^{2n+3}$ is the smooth base-locus of a pencil Λ of quadrics, its Hodge decomposition satisfies the conditions above, so we can form the principally polarized abelian variety $J(X)$.

The choice of one of the two components of the family of $\mathbf{P}^{n+1}$ contained in a member of Λ defines a double cover $C \to \Lambda$ ramified exactly over the $2n + 4$ points corresponding to the singular members of the pencil. The curve C is smooth, hyperelliptic, of genus $n + 1$, and its Jacobian is isomorphic to $J(X)$ [Reid 1972; Donagi 1980].

Example 1.5 (Intersections of three quadrics). If $X \subset \mathbf{P}^{2n+4}$ is the smooth base-locus of a net of quadrics $\Pi = \langle Q_1, Q_2, Q_3\rangle$, its Hodge decomposition satisfies the conditions above, so we can form the principally polarized abelian variety $J(X)$.

When $n \geq 1$, the variety X contains a line ℓ. The map $X \dashrightarrow \Pi$ defined by sending a point $x \in X$ to the unique quadric in Π that contains the 2-plane $\langle \ell, x\rangle$, is a (rational) *quadric* bundle structure on X. The discriminant curve $C \subset \Pi$ parametrizing singular quadrics has equation

$$\det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3) = 0,$$

hence degree $2n+5$. The choice of a component of the set of $\mathbf{P}^{n+1}$ contained in a singular quadric of the net Π defines a double étale cover $\tilde{C} \to C$, and $J(X) \simeq P(\tilde{C}/C)$ [Beauville 1977b; Tjurin 1975b].

2. Periods and period maps

Assume now that we have a family $\mathscr{X} \to S$ of smooth projective threefolds, whose fibers X_s all satisfy $H^{0,3}(X_s) = 0$. We can construct for each $s \in S$ the intermediate Jacobian $J(X_s)$. Let us look at this from a slightly different point of view. Assume that the base S is simply connected and fix a point 0 in S, with fiber X_0; we can then identify each $H^3(X_s, \mathbf{Z})$ with the fixed rank-$2g$ lattice $H_{\mathbf{Z}} = H^3(X_0, \mathbf{Z})$ and define an algebraic *period map* with values in a Grassmannian:

$$\wp : S \longrightarrow G(g, H_{\mathbf{C}}), \quad s \longmapsto H^{2,1}(X_s),$$

where $H_{\mathbf{C}} = H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{C}$. Letting Q be the skew-symmetric intersection form on $H_{\mathbf{C}}$, the following properties hold:

- $H^{2,1}(X_s)$ is totally isotropic for Q,
- the Hermitian form $i\,Q(\,\cdot\,, \bar{\cdot}\,)$ is positive definite on $H^{2,1}(X_s)$,

so that $\wp$ takes its values into a dense open subset of an isotropic Grassmannian which is isomorphic to the Siegel upper half-space $\mathscr{H}_g = \mathrm{Sp}(2g, \mathbf{R})/\,\mathrm{U}(g)$. If $\wp(s)$ correspond to $\tau(s) \in \mathscr{H}_g$, we have

$$J(X_s) = H_{\mathbf{C}}/(H_{\mathbf{Z}} \oplus \tau(s) H_{\mathbf{Z}}).$$

Back to the case where the base S is general, with universal cover $\tilde{S} \to S$, we obtain a diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\wp}"] \arrow[d] & \mathscr{H}_g \arrow[d] \\
S \arrow[r, "\wp"] & \mathscr{H}_g/\,\mathrm{Sp}(2g, \mathbf{Z}) = \mathscr{A}_g
\end{tikzcd}$$

where $\tilde{\wp}$ is *holomorphic,* $\wp$ is algebraic if S is algebraic, and

$$\mathscr{A}_g = \{\text{ppavs of dimension } g\}/\text{isomorphism}$$

is the *moduli space* of principally polarized abelian varieties of dimension g. It has a natural structure of a quasiprojective variety of dimension $g(g+1)/2$.

We want to generalize this construction to any smooth projective variety X of dimension n. Even if the Hodge decomposition

$$H^k(X, \mathbf{C}) = \bigoplus_{p=0}^{k} H^{p,k-p}(X)$$

does not have level one (i.e., only two pieces), we can still use it to define a period map as follows [Griffiths 1969; 1970]. Choose an ample class $h \in H^2(X, \mathbf{Z}) \cap H^{1,1}(X)$ and define the *primitive cohomology* by

$$H^k(X, \mathbf{C})_{\mathrm{prim}} = \mathrm{Ker}\left(H^k(X, \mathbf{C}) \xrightarrow{\smile h^{n-k+1}} H^{2n-k+2}(X, \mathbf{C})\right).$$

Set $H^{p,q}(X)_{\mathrm{prim}} = H^{p,q}(X) \cap H^k(X, \mathbf{C})_{\mathrm{prim}}$ and

$$F^r = \bigoplus_{p \geq r} H^{p,k-p}(X)_{\mathrm{prim}}.$$

Define a bilinear form on $H^k(X, \mathbf{C})_{\mathrm{prim}}$ by

$$Q(\alpha, \beta) = \alpha \smile \beta \smile h^{n-k}.$$

The associated *period domain* $\mathfrak{D}$ is then the set of flags

$$0 = F^{k+1} \subset F^k \subset \cdots \subset F^1 \subset F^0 = H^k(X, \mathbf{C})$$

satisfying the following conditions:

- $F^r \oplus \overline{F^{k-r+1}} = F^0$,
- $F^r = (F^{k-r+1})^{\perp_Q}$,
- for each p and k, the Hermitian form $i^{2p-k} Q(\,\cdot\,, \bar{\cdot}\,)$ is positive definite on $H^{p,k-p}(X)_{\mathrm{prim}}$.

It is a homogeneous complex manifold, quotient of a real Lie group by a compact subgroup. We already encountered the period domain $\mathcal{H}_g = \mathrm{Sp}(2g, \mathbf{R})/\,\mathrm{U}(g)$; however, the subgroup may be not maximal, so that $\mathfrak{D}$ is not in general Hermitian symmetric (see Examples 2.5 and 3.3).

Given a family $\mathcal{X} \to S$ of polarized varieties, we obtain as above a *holomorphic* map

$$\tilde{\wp} : \tilde{S} \longrightarrow \mathfrak{D}.$$

Note that the lattice $H^k(X, \mathbf{Z})$ has played no role here yet. It will however if we want to define a period map on S: one needs to quotient by the action of $\pi_1(S, 0)$ and this group acts via the monodromy representation

$$\pi_1(S, 0) \longrightarrow \mathrm{Aut}(H^k(X_0, \mathbf{Z})).$$

The discrete group $\Gamma = \mathrm{Aut}(H^k(X_0, \mathbf{Z}))$ acts on $H^k(X_0, \mathbf{C})_{\mathrm{prim}}$, and properly on $\mathcal{D}$, hence we obtain a diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\wp}"] \arrow[d] & \mathcal{D} \arrow[d] \\
S \arrow[r, "\wp"] & \mathcal{D}/\Gamma
\end{tikzcd}$$

where $\mathcal{D}/\Gamma$ is an analytic space (not algebraic in general; see Examples 2.5 and 2.6) and $\wp$ is holomorphic.

Example 2.1 (Quartic surfaces). If $B \subset \mathbf{P}^3$ is a quartic (hence K3) surface, we have

$$\begin{array}{lcccccc}
H^2(B, \mathbf{C}) & = & H^{0,2}(B) & \oplus & H^{1,1}(B) & \oplus & H^{2,0}(B) \\
\text{dimensions:} & & 1 & & 20 & & 1 \\
H^2(B, \mathbf{C})_{\mathrm{prim}} & = & H^{0,2}(B) & \oplus & H^{1,1}(B)_{\mathrm{prim}} & \oplus & H^{2,0}(B) \\
\text{dimensions:} & & 1 & & 19 & & 1
\end{array}$$

Because of its properties, explained above, relative to the intersection form Q, this decomposition is completely determined by the point of $\mathbf{P}(H^2(B, \mathbf{C})_{\mathrm{prim}})$ defined by the line $H^{2,0}(B)$. The period map takes values in the 19-dimensional period domain

$$\begin{aligned}
\mathcal{D}^{19} &= \{[\omega] \in \mathbf{P}^{20} \mid Q(\omega, \omega) = 0,\ Q(\omega, \bar{\omega}) > 0\} \\
&\simeq \mathrm{SO}(19, 2)^0 / \mathrm{SO}(19) \times \mathrm{SO}(2),
\end{aligned}$$

where Q is a quadratic form, integral on a lattice $H_{\mathbf{Z}}$, with signature $(19, 2)$ on $H_{\mathbf{R}}$. It is a bounded symmetric domain of type IV and the discrete group Γ^{19} can be explicitely described [Beauville et al. 1985].

Note that quartic surfaces are in one-to-one correspondence with quartic double solids (Example 1.2), so we may also associate to B the 5-dimensional intermediate Jacobian $J(X)$ of the double solid $X \to \mathbf{P}^3$ branched along B and get another kind of period map with values in $\mathcal{A}_5$.

Example 2.2 (Cubic fourfolds). If $X \subset \mathbf{P}^5$ is a smooth cubic fourfold, the situation is completely analogous: the decomposition

$$\begin{array}{lcccccc}
H^4(X, \mathbf{C})_{\mathrm{prim}} & = & H^{1,3}(X) & \oplus & H^{2,2}(X)_{\mathrm{prim}} & \oplus & H^{3,1}(X) \\
\text{dimensions:} & & 1 & & 20 & & 1
\end{array}$$

is completely determined by the point $[H^{3,1}(X)]$ of $\mathbf{P}(H^4(X, \mathbf{C})_{\mathrm{prim}})$, and

$$\begin{aligned}
\mathcal{D}^{20} &= \{[\omega] \in \mathbf{P}^{21} \mid Q(\omega, \omega) = 0,\ Q(\omega, \bar{\omega}) > 0\} \\
&\simeq \mathrm{SO}(20, 2)^0 / \mathrm{SO}(20) \times \mathrm{SO}(2).
\end{aligned}$$

Here the quadratic form Q, integral on a lattice $H_{\mathbf{Z}}$, has signature $(20, 2)$ on $H_{\mathbf{R}}$. The domain $\mathcal{D}^{20}$ is again is a bounded symmetric domain of type IV. The discrete group Γ^{20} can be explicitely described [Laza 2009].

Example 2.3 (Cubic surfaces). If $X \subset \mathbf{P}^3$ is a smooth cubic surface, with equation $F(x_0, x_1, x_2, x_3) = 0$, we have $H^2(X, \mathbf{C}) = H^{1,1}(X)$, so the period map is trivial.

Proceeding as in Example 2.1, where we associated to a quartic surface in $\mathbf{P}^3$ the double cover of $\mathbf{P}^3$ branched along this surface, we may consider the cyclic triple cover $\tilde{X} \to \mathbf{P}^3$ branched along X. It is isomorphic to the cubic threefold with equation $F(x_0, x_1, x_2, x_3) + x_4^3 = 0$ in $\mathbf{P}^4$, so its Hodge structure is as in Example 1.1 and the period domain is $\mathcal{H}_5$. On the other hand, the Hodge structure carries an action of the group μ_3 of cubic roots of unity. The eigenspace $H^3_\omega(\tilde{X})$ for the eigenvalue $e^{2i\pi/3}$ splits as

$$\begin{array}{rcccc} H^3_\omega(\tilde{X}) & = & H^{1,2}_\omega(\tilde{X}) & \oplus & H^{2,1}_\omega(\tilde{X}). \\ \text{dimensions:} & & 1 & & 4 \end{array}$$

Following Allcock, Carlson, and Toledo [2002], one can then define a period map with values in the 4-dimensional space

$$\{[\omega] \in \mathbf{P}^4 \mid Q(\omega, \bar{\omega}) < 0\},$$

where the quadratic form Q is integral on a lattice $H_{\mathbf{Z}}$, with signature $(4, 1)$ on $H_{\mathbf{R}}$. It is isomorphic to the complex hyperbolic space $\mathbf{B}^4$, which is much smaller than $\mathcal{H}_5$! The discrete group Γ^4 can be explicitely described.

Example 2.4 (Cubic threefolds, II). Similarly, if $X \subset \mathbf{P}^4$ is a smooth cubic threefold, we consider the cyclic triple cover $\tilde{X} \to \mathbf{P}^4$ branched along X. It is a cubic fourfold in $\mathbf{P}^5$, so its Hodge structure is as in Example 2.2, with an extra symmetry of order three. With analogous notation as above, $H^{3,1}_\omega(\tilde{X})$ has dimension 1 and $H^{2,2}_\omega(\tilde{X})$ has dimension 10. Allcock, Carlson, and Toledo [2011] then define a period map with values in the 10-dimensional space

$$\{\omega \in \mathbf{P}^{10} \mid Q(\omega, \bar{\omega}) < 0\} \simeq \mathbf{B}^{10},$$

where again the quadratic form Q is integral on a lattice $H_{\mathbf{Z}}$, with signature $(10, 1)$ on $H_{\mathbf{R}}$.

Example 2.5 (Calabi–Yau threefolds of mirror quintic type). Consider the quintic hypersurface $Q_\lambda \subset \mathbf{P}^4$ with equation

$$x_0^5 + \cdots + x_4^5 + \lambda x_0 \cdots x_4 = 0,$$

its quotient Q_λ / G by the diagonal action of the finite group

$$G = \{(\alpha_0, \ldots, \alpha_4) \in \mathbf{C}^5 \mid \alpha_0^5 = \cdots = \alpha_4^5 = \alpha_0 \cdots \alpha_4 = 1\},$$

and a minimal desingularization $\widetilde{Q_\lambda/G} \to Q_\lambda/G$. Its Hodge numbers are

$$h^{0,3}(\widetilde{Q_\lambda/G}) = h^{1,2}(\widetilde{Q_\lambda/G}) = h^{2,1}(\widetilde{Q_\lambda/G}) = h^{3,0}(\widetilde{Q_\lambda/G}) = 1.$$

The corresponding period domain $\mathfrak{D}^4$ has dimension 4. It is the first instance that we meet of what is called the "nonclassical" situation, where the analytic space $\mathfrak{D}^4/\Gamma^4$ is not quasiprojective in any way compatible with its analytic structure.

Example 2.6 (Hypersurfaces in the projective space). If X is a smooth hypersurface of degree d in $\mathbf{P}^{n+1}$, the only interesting Hodge structure is that of $H^n(X, \mathbf{C})$. If $F(x_0, \dots, x_{n+1}) = 0$ is an equation for X, and

$$R(F) := \mathbf{C}[x_0, \dots, x_{n+1}] \Big/ \Big\langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}} \Big\rangle$$

is the (graded) Jacobian quotient ring, we have, by [Griffiths 1969],

$$H^{p,n-p}(X)_{\mathrm{prim}} \simeq R(F)^{(n+1-p)d-n-2},$$

where $R(F)^e$ is the graded piece of degree e in $R(F)$. So again, except for small d and n (as in Examples 2.1, 2.2, 2.3, and 2.4), we are most of the times in a nonclassical situation.

3. Is the period map injective?

3.1. *Curves.* This is the famous *Torelli theorem* [Torelli 1913; Arbarello et al. 1985]: a smooth projective curve C is determined (up to isomorphism) by the pair $(J(C), \Theta)$. In fancy terms, the period map

$$\begin{tikzcd}
\mathcal{M}_g = \{\text{smooth projective curves of genus } g\}/\text{isomorphism} \arrow[d, "\wp_g"'] \\
\mathcal{A}_g = \{\text{ppavs of dimension } g\}/\text{isomorphism}
\end{tikzcd}$$

is injective.

More generally, it is customary to call *Torelli problem* the question of deciding whether an algebraic object is determined by a polarized abelian variety attached to it.

3.2. *Prym varieties.* The period map

$$\begin{tikzcd}
\mathcal{R}_g = \{\text{double étale covers of genus-}g\text{ curves}\}/\text{isom.} \arrow[d, "\wp_g"] \\
\mathcal{A}_{g-1}
\end{tikzcd}$$

cannot be injective in low genera for dimensional reasons. The following table sums up the situation (see [Donagi and Smith 1981; Friedman and Smith 1982; Welters 1987]):

g	$\dim(\mathscr{R}_g)$	$\dim(\mathscr{A}_{g-1})$	$\wp_g$
2	3	1	dominant, not injective
3	6	3	dominant, not injective
4	9	6	dominant, not injective
5	12	10	dominant, not injective
6	15	15	dominant, generically 27:1
$g \geq 7$	$3g-3$	$g(g-1)/2$	generically injective, *not* injective

The injectivity defect for $g \geq 7$ is not yet entirely understood (see [Donagi 1981; Debarre 1989b; Verra 2004; Izadi and Lange 2010]).

3.3. *Hypersurfaces in the projective space.* Donagi [1983] proved that the period map

$$\begin{tikzcd} \mathscr{M}_{d,n} = \left\{ \begin{matrix} \text{smooth hypersurfaces} \\ \text{of degree } d \text{ in } \mathbf{P}^n \end{matrix} \right\} \Big/ \text{isom.} \arrow[d, "\wp_{d,n}"] \\ \mathscr{D}_{d,n} / \Gamma_{d,n} \end{tikzcd}$$

for hypersurfaces is *generically injective,* except perhaps in the following cases (see also [Cox and Green 1990]):

- $n = 2$ and $d = 3$, i.e., cubic surfaces, where this is obviously false (see Example 2.3);
- d divides $n+2$ (the answer in these cases is unknown, except for $d = 5$ and $n = 3$; see [Voisin 1999]);
- $d = 4$ and $4 \mid n$.

The proof relies on Griffiths' theory of infinitesimal variation of Hodge structures and a clever argument in commutative algebra. Using analogous techniques, this result was extended later to hypersurfaces of more general homogeneous spaces [Konno 1989].

The period domain is in general much too big for the period map to be dominant (not to mention the fact that it is in general not even algebraic!). But for some small d and n, this can happen, as shown in the next examples.

3.4. ***Cubic threefolds.*** The period map

$$\begin{tikzcd} \mathcal{M}_{\mathrm{ct}}^{10} = \left\{\begin{matrix}\text{smooth cubic}\\ \text{threefolds}\end{matrix}\right\}\Big/\text{isom.} \arrow[d, "\wp_{\mathrm{ct}}"] \\ \mathcal{A}_5 \end{tikzcd}$$

for cubic threefolds is injective. This can be seen as follows [Beauville 1982]: if $X \subset \mathbf{P}^4$ is a cubic, we explained in Example 1.1 that the theta divisor $\Theta \subset J(X)$ has a unique singular point s, which has multiplicity 3. It turns out that the projectified tangent cone

$$\mathbf{P}(TC_{\Theta,s}) \subset \mathbf{P}(T_{J(X),s}) = \mathbf{P}^4$$

is isomorphic to X.

Of course, $\wp_{\mathrm{ct}}$ is not dominant, since it maps a 10-dimensional space to a 15-dimensional space. Its image was characterized geometrically in [Casalaina-Martin and Friedman 2005]: it is essentially the set of elements of $\mathcal{A}_5$ whose theta divisor has a point of multiplicity 3.

Recall (Example 2.4) that Allcock, Carlson, and Toledo defined [2011] another period map

$$\wp'_{ct} : \mathcal{M}_{\mathrm{ct}} \longrightarrow \mathbf{B}^{10}/\,\Gamma^{10}.$$

They prove (among other things) that $\wp'_{ct}$ induces an isomorphism onto an open subset whose complement is explicitly described.

3.5. ***Quartic double solids and quartic surfaces.*** The period map

$$\begin{tikzcd} \mathcal{M}_{\mathrm{qds}}^{19} = \left\{\begin{matrix}\text{smooth quartic}\\ \text{double solids}\end{matrix}\right\}\Big/\text{isom.} \arrow[d, "\wp_{\mathrm{qds}}"] \\ \mathcal{A}_{10} \end{tikzcd}$$

for quartic double solids is injective. This can be seen as follows: as mentioned in Example 1.2, if $X \to \mathbf{P}^3$ is a smooth quartic double solid, the singular locus of the theta divisor $\Theta \subset J(X)$ has a unique 5-dimensional component S. General points s of S are double points on Θ, and the projectified tangent cones $\mathbf{P}(TC_{\Theta,s})$ are, after translation, quadrics in $\mathbf{P}(T_{J(X),0}) = \mathbf{P}^9$. The intersection of these quadrics is isomorphic to the image of the branch quartic surface $B \subset \mathbf{P}^3$ by the Veronese morphism $v_2 : \mathbf{P}^3 \to \mathbf{P}^9$ [Clemens 1983].

Again, $\wp_{\mathrm{qds}}$ maps a 19-dimensional space to a 45-dimensional space, so it cannot be dominant. However, X is determined by the quartic surface $B \subset \mathbf{P}^3$, and we have another period map (Example 2.1)

$$\wp'_{\mathrm{qds}} : \mathcal{M}^{19}_{\mathrm{qds}} \longrightarrow \mathcal{D}^{19} / \Gamma^{19},$$

which is an isomorphism onto an explicitly described open subset [Piatetski-Shapiro and Shafarevich 1971].

3.6. ***Intersections of two quadrics.*** The period map

$$\begin{tikzcd} \mathcal{M}^{2n+1}_{\mathrm{i2q}} = \left\{ \begin{matrix} \text{smooth intersections of} \\ \text{two quadrics in } \mathbf{P}^{2n+3} \end{matrix} \right\} \Big/ \text{isom.} \arrow[d, "\wp_{\mathrm{i2q}}"] \\ \mathcal{A}_{n+1} \end{tikzcd}$$

for intersections of two quadrics is injective. This is because, by the Torelli theorem for curves (§3.1), one can reconstruct from the intermediate Jacobian $J(X)$ the hyperelliptic curve C (see Example 1.4), hence its Weierstrass points, hence the pencil of quadrics that defines X. The image of $\wp_{\mathrm{i2q}}$ is the set of hyperelliptic Jacobians, hence it is not dominant for $n \geq 2$.

3.7. ***Intersections of three quadrics.*** The period map

$$\begin{tikzcd} \mathcal{M}^{2n^2+13n+12}_{\mathrm{i3q}} = \left\{ \begin{matrix} \text{smooth intersections of} \\ \text{three quadrics in } \mathbf{P}^{2n+4} \end{matrix} \right\} \Big/ \text{isom.} \arrow[d, "\wp_{\mathrm{i3q}}"] \\ \mathcal{A}_{(n+1)(2n+5)} \end{tikzcd}$$

for intersections of three quadrics is injective: using *ad hoc* geometric constructions, we showed in [Debarre 1989a] how to recover, from the theta divisor $\Theta \subset J(X)$, the double cover $\tilde{C}$ of the discriminant curve C and from there, it was classically known how to reconstruct X.

Again, for dimensional reasons, $\wp_{\mathrm{i3q}}$ is not dominant.

3.8. ***Cubic surfaces.*** Allcock, Carlson, and Toledo [2002] proved that the modified period map

$$\begin{tikzcd} \mathcal{M}^{4}_{\mathrm{cs}} = \left\{ \begin{matrix} \text{smooth cubic} \\ \text{surfaces} \end{matrix} \right\} \Big/ \text{isom.} \arrow[d, "\wp'_{\mathrm{cs}}"] \\ \mathbf{B}^4 / \Gamma^4 \end{tikzcd}$$

constructed in Example 2.3 induces an isomorphism with an explicit open subset of $\mathbf{B}^4 / \Gamma^4$.

3.9. ***Cubic fourfolds, II.*** The period map (Example 2.2)

$$\begin{tikzcd} \mathcal{M}^{20}_{\mathrm{cf}} = \left\{ \begin{matrix} \text{smooth cubic} \\ \text{fourfolds} \end{matrix} \right\} \Big/ \text{isom.} \arrow[d, "\wp_{\mathrm{cf}}"] \\ \mathcal{D}^{20} / \Gamma^{20} \end{tikzcd}$$

for cubic fourfolds is injective [Voisin 1986; Looijenga 2009] and induces an isomorphism with an explicitly described open subset of $\mathcal{D}^{20}/\Gamma^{20}$ [Laza 2010].

3.10. ***Calabi–Yau threefolds of mirror quintic type, II.*** In the situation considered in Example 2.5, we have a period map $\wp : U \to \mathcal{D}^4/\Gamma^4$, where U is the open set of those $\lambda \in \mathbf{C}$ for which the quintic Q_λ is smooth. Using techniques from log-geometry, Usui [2008] showed that $\wp$ is generically injective.

3.11. ***Fano threefolds of degree*** **10.** I am referring here to the threefolds $X \subset \mathbf{P}^7$ considered in Example 1.3. Their moduli space $\mathcal{M}^{22}$ has dimension 22, so the period map

$$\wp : \mathcal{M}^{22} \longrightarrow \mathcal{A}_{10}$$

can certainly not be dominant. Furthermore, Debarre, Iliev, and Manivel proved that its fibers have everywhere dimension 2 [Debarre et al. 2011].

Here is a sketch of the construction. Conics $c \subset X$ are parametrized by a smooth connected projective surface $F(X)$ which is the blow-up at one point of a smooth minimal surface $F_m(X)$ of general type. Given such a smooth conic c, one can construct another smooth Fano threefold X_c of degree 10 and a birational map $X \dashrightarrow X_c$ which is an isomorphism in codimension 1. In particular, it induces an isomorphism $J(X_c) \simeq J(X)$. However, one shows that the surface $F(X_c)$ is isomorphic to the blow-up of $F_m(X)$ at the point corresponding to c. In particular, it is (in general) *not* isomorphic to $F(X)$, so X_c is also *not* isomorphic to X. We actually prove that this construction (and a variant thereof) produces *two smooth proper* 2*-dimensional connected components* of each general fiber.

4. Moduli spaces

Up to now, little care was taken to define the exact structure of the various "moduli spaces" we encountered. There are two main methods for constructing quasiprojective moduli spaces:

- Geometric invariant theory [Mumford et al. 1994]: roughly speaking, one "naturally" embeds the objects one wants to classify into some fixed projective space, then quotients the corresponding subset of the Hilbert scheme by the action of the special linear group using GIT.

- One constructs directly an ample line bundle “on the functor;” roughly speaking, one needs to construct, for every family $X \to S$ of objects, a “functorial” ample line bundle on the base S.

The advantage of the GIT method is that it also produces automatically a compactification of the moduli space. Its drawback is that it is difficult to apply in practice. The second method, pioneered by Kollár and Viehweg, is more general, but technically more difficult. It can also produce compactifications, but there, one needs to decide what kind of singular objects one needs to add to make the moduli space compact. This approach now seems to have had complete success for varieties with ample canonical bundle.

Once a compactification is constructed, one may then try to extend the various period maps constructed above to compactifications of the period domain $\mathcal{D}/\Gamma$. In the “classical” situation, i.e., when the period domain is an arithmetic quotient of a bounded symmetric domain, one can use the Baily–Borel theory [1966]. In general, this is much more difficult (see [Usui 2008]). These extensions turn out to be very useful, in some cases, in order to characterize the image of the original period map, or to prove that it is birational.

Here are a few examples.

4.1. ***Curves.*** It has been known for a long time that the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g and the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g are quasiprojective varieties (over $\mathbf{Z}$, this was established in [Mumford et al. 1994], Theorem 5.11 and Theorem 7.10, using GIT).

A compactification $\overline{\mathcal{M}_g}$ of $\mathcal{M}_g$ is obtained by adding certain singular curves called *stable curves* and the resulting moduli space was proved by Mumford, Knudsen, and Gieseker to be projective (see the discussion in [Mumford et al. 1994], Appendix D).

As explained in §2, $\mathcal{A}_g$ is an arithmetic quotient of the Siegel upper half-space, a bounded symmetric domain. A first compactification was constructed by Satake (this was the starting point of the Baily–Borel theory!). Set theoretically, it is simply the disjoint union of $\mathcal{A}_g, \mathcal{A}_{g-1}, \dots, \mathcal{A}_0$, but it is very singular. *Toroidal* compactifications were later constructed by Ash, Mumford, Rapoport, and Tai (see the references in [Mumford et al. 1994], Appendix E) and some of them are smooth. More recently, Alexeev [2004] constructed a compactification which is a moduli space.

The period map $\wp_g : \mathcal{M}_g \to \mathcal{A}_g$ defined in §3.1 does extend to a morphism from $\overline{\mathcal{M}_g}$ to the Satake compactification by sending a curve to the product of the Jacobians of the components of its normalization. It also extends to a morphism to some toroidal compactifications and to the Alexeev compactification. However, none of these extensions remain injective: points of $\overline{\mathcal{M}_g}$ which correspond to unions of two curves meeting in one point are sent to the product of the Jacobians of their components, regardless of the gluing points. The fibers of the extended period map are precisely described in [Caporaso and Viviani 2011].

4.2. ***Hypersurfaces in the projective space.*** Hypersurfaces of degree d in $\mathbf{P}^{n+1}$ are parametrized by the projective space

$$|dH| = \mathbf{P}(H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d))).$$

Let $|dH|^0$ be the dense open subset corresponding to *smooth* hypersurfaces. The complement $|dH| - |dH|^0$ is a hypersurface, because the condition that the equation F and its partial derivatives $\partial F/\partial x_0, \dots, \partial F/\partial x_{n+1}$ have a common zero is given by the vanishing of a single (homogeneous) polynomial in the coefficients of F. It follows that $|dH|^0$ is an affine open set, invariant by the action of the reductive group $\mathrm{SL}(n+2)$.

For $d \geq 3$, this action is *regular* in the sense of GIT (the dimensions of the stabilizers are locally constant) hence closed (the orbits are closed). Since $\mathcal{O}_{\mathbf{P}^{n+1}}(1)$ admits a $\mathrm{SL}(n+2)$-linearization, $|dH|^0$ is contained in the set $|dH|^s$ of stable points associated with these data. The GIT theory implies that the quotient $|dH|^0/\mathrm{SL}(n+2)$, which is the moduli space of smooth hypersurfaces of degree d in $\mathbf{P}^{n+1}$, can be realized as an open set in the GIT quotient $|dH|^{ss}/\!/\,\mathrm{SL}(n+2)$, which is a projective variety.

The precise description of the semistable points, i.e., of the kind of singularities one needs to add to obtain the GIT compactification of the moduli space, is a difficult task, impossible to achieve in general. Some cases are known: plane curves of degree ≤ 6 and cubic surfaces (Hilbert; [Mumford 1977; Shah 1976]), quartic surfaces [Shah 1976], cubic threefolds [Allcock 2003], cubic fourfolds [Laza 2009],...

Example 4.1 (Cubic surfaces, II). Stable points correspond to cubic surfaces that have at most ordinary double points ("type A_1"). Semistable points correspond to cubic surfaces whose singular points are all of type A_1 or A_2. GIT theory yields a compactification $\overline{\mathcal{M}^4_{\mathrm{cs}}}$ of the moduli space of smooth cubic surfaces (§3.8) and the modified period map extends to an isomorphism

$$\overline{\mathcal{M}^4_{\mathrm{cs}}} \xrightarrow{\sim} \overline{\mathbf{B}^4/\Gamma^4},$$

where the right side is the Baily–Borel compactification of $\mathbf{B}^4/\Gamma^4$ [Allcock et al. 2002; Doran 2004b].

Example 4.2 (Quartic surfaces, II). There is a list of all allowed singularities on quartic surfaces corresponding to semistable points [Shah 1976]. Again, the period map (§3.5) induces an isomorphism [Kulikov 1977]

$$\overline{\mathcal{M}^{19}} \xrightarrow{\sim} \overline{\mathcal{D}^{19}/\Gamma^{19}},$$

where the left side is the GIT-compactification and the right side is the Baily–Borel compactification.

Example 4.3 (Cubic threefolds, III). Stable points correspond to cubic threefolds whose singular points are of type A_n, with $1 \le n \le 4$. There is also a list of all possible singularities of cubic threefolds that correspond to semistable points [Allcock 2003]. The modified period map (§3.4) induces a morphism

$$\overline{\mathcal{M}^{10}_{\mathrm{ct}}} \longrightarrow \overline{\mathbf{B}^{10}/\Gamma^{10}}$$

which contracts a rational curve, where $\overline{\mathcal{M}^{10}_{\mathrm{ct}}}$ is an explicit blow-up of the GIT-compactification and $\overline{\mathbf{B}^{10}/\Gamma^{10}}$ is the Baily–Borel compactification [Allcock et al. 2011; Looijenga and Swierstra 2007].

Example 4.4 (Cubic fourfolds, II). There are complete lists of all possible singularities of cubic fourfolds that correspond to stable and semistable points [Laza 2009]. The period map (§3.9) induces an isomorphism

$$\overline{\mathcal{M}^{20}_{\mathrm{cf}}} \longrightarrow \overline{\mathcal{D}^{20}/\Gamma^{20}},$$

where $\overline{\mathcal{M}^{20}_{\mathrm{cf}}}$ is an explicit blow-up of the GIT-compactification and $\overline{\mathcal{D}^{20}/\Gamma^{20}}$ is the Looijenga compactification, a modification of the Baily–Borel compactification [Laza 2010; Looijenga 2009].

4.3. *Complete intersections.*

Along the same lines, some complete intersections have also been considered.

Example 4.5 (Intersections of two quadrics, II). The moduli space of smooth intersections of two quadrics in $\mathbf{P}^n$ can be constructed as the GIT quotient of an affine dense open set of the Grassmannian $G(2, H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(2)))$ by the reductive group $\mathrm{SL}(n+1)$. Using a slightly different presentation, Avritzer and Miranda [1999] proved that smooth intersections correspond exactly to stable points. Therefore, the moduli space is a quasiprojective variety.

Example 4.6 (Fano threefolds of degree 10, II). These threefolds $X \subset \mathbf{P}^7$ were considered in Example 1.3 and §3.11: they are obtained as intersections, in $\mathbf{P}^9$, of the Grassmannian $G(2, 5)$ in its Plücker embedding, two hyperplanes, and a smooth quadric, and their moduli space $\mathcal{M}^{22}$ can be seen as follows.

Let $G = G\big(8, \bigwedge^2 \mathbf{C}^5\big)$ be the 16-dimensional Grassmannian parametrizing pencils of skew-symmetric forms on 5, and let $\mathcal{T}$ be the tautological rank-8 vector bundle on G. The composition

$$\bigwedge^4 V_5^\vee \hookrightarrow \mathrm{Sym}^2\left(\bigwedge^2 V_5^\vee\right) \to \mathrm{Sym}^2 \mathcal{T}^\vee$$

is everywhere injective and its cokernel $\mathcal{E}$ is a vector bundle of rank 31 on G. To each point λ of $\mathbf{P}(\mathcal{E})$, one can associate a codimension-2 linear subspace of $\mathbf{P}\big(\bigwedge^2 V_5\big)$ and a quadric in that subspace, well-defined up to the space of quadrics that contain $G(2,\mathbf{C}^5)\subset\mathbf{P}\big(\bigwedge^2 V_5\big)$, hence a threefold $X_\lambda\subset\mathbf{P}^7$ of degree 10, which is in general smooth.

The group SL(5) acts on $\mathbf{P}(\mathcal{E})$ and one checks that the stabilizers, which correspond to the automorphisms group of X_λ, are finite when X_λ is smooth. So we expect that the moduli space should be an open subset of the GIT quotient $\mathbf{P}(\mathcal{E})/\!/\,\mathrm{SL}(5)$. However, the relationship between the smoothness of X_λ and the stability of λ (which of course involves the choice of a polarization, since $\mathbf{P}(\mathcal{E})$ has Picard number 2) is not clear at the moment.

Remark 4.7. These examples show that several GIT-moduli spaces admit, as ball quotients, complex hyperbolic structures. Another large class of examples of moduli spaces as ball quotients is due to Deligne and Mostow, in their exploration of moduli spaces of points on $\mathbf{P}^1$ and hypergeometric functions [Deligne and Mostow 1986]. “Our” examples are not directly of Deligne–Mostow type, since the corresponding discrete groups do not appear on the various lists in [Deligne and Mostow 1986; Thurston 1998]. However, Doran found, by taking a view of hypergeometric functions based on intersection cohomology valued in local systems, links between these two types of examples [Doran 2004a].

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