Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

Workshop on the Topology of Stratified Spaces Open Problems

The following open problems were suggested by the participants both during and following the Workshop.

1. L^2 Hodge and signature theorems; Signature theory on singular spaces

(a) (suggested by Eugénie Hunsicker)

Consider a pseudomanifold X as in Cheeger, [7]. Cheeger proves in this paper that if the smooth part of X, X_{reg}, is endowed with an iterated cone metric, and if X is a Witt space, then the L^2 cohomology of X_{reg} is isomorphic to the middle perversity intersection cohomology of X (which is also unique due to the Witt condition). This implies in turn that the space of L^2 harmonic forms for the maximal extension is isomorphic to the middle perversity intersection cohomology, and from this we get that the operator $d+\delta$ is essentially self-adjoint on X_{reg}, which in turn means that this operator has a unique closed extension to $L^2(X_{\text{reg}})$. Thus in the setting of Witt spaces and conical metrics, there is a clear and simple relationship between harmonic forms, L^2-cohomology and intersection cohomology.

If the Witt condition is dropped, then there is not generally a unique middle perversity intersection cohomology, and $d+\delta$ generally has different possible extensions, and in particular, can have different possible self-adjoint extensions. In the case of a pseudomanifold with only one singular stratum endowed again with a conical metric, the non-Witt case was studied in [14]. In this paper, it is shown that the operator $d+\delta$ on X_{reg} endowed with a cone metric has self-adjoint extensions whose kernels are isomorphic to the upper and to the lower middle perversity intersection cohomologies on X, and the kernel of the minimal extension of $d+\delta$ (for an appropriately chosen cone metric) is isomorphic to the image of lower middle in upper middle perversity intersection cohomology.

Versions of L^2 cohomology also can relate to more general perversities. For example, Nagase showed in [18] that for each standard perversity $\overline{p}$ greater than or equal to the upper middle perversity on a pseudomanifold X, there exists an incomplete metric on the regular set of X for which the L^2 cohomology associated to the maximal extension of d is isomorphic to $IH^{\overline{p}}(X)$. This in particular implies that there exists a self-dual L^2 extension of $d+\delta$ for this metric whose kernel is also isomorphic to $IH^{\overline{p}}(X)$.

It seems likely that these phenomena are part of a larger relationship among L^2 extensions of the geometric operator $d+\delta$ on the regular set of a pseudomanifold for various metrics, weighted L^2 cohomology for these metrics and intersection cohomologies on X with various perversities. It would be interesting to explore this further. In particular, consider the metrics constructed in [18]. What other closed L^2 extensions of $d+\delta$ exist for these metrics, and which perversity intersection cohomologies will their kernels be isomorphic to? Further, are there generalizations of intersection cohomology that are isomorphic to the kernel of some such extensions? Finally, can we understand the interesting extensions of $d+\delta$ using analytic approaches to singularities, such as the Melrose b-calculus or Schulze or Boutet de Monvel calculi?

(b) (suggested by Paolo Piazza)

Consider a non-Witt pseudomanifold X that has a Lagrangian structure à la Banagl. See for example Chapter 9 in the book [4]. For such spaces one can define a signature and an L-class.

(i) Is there an analytic description of the signature? More precisely: endow X with an iterated conic metric. Is there an extension of the signature operator which is Fredholm and such that its index is equal to the above signature?

For Witt spaces this is a well known result due to Cheeger and recently re-established by Albin, Leichtnam, Mazzeo and Piazza in the preprint [1]. In the latter preprint the extension of the full signature package from closed manifolds to Witt spaces, leading to the definition and the homotopy invariance of higher signatures on Witt spaces, is discussed. Notice that the higher signatures on Witt spaces involve the L-class of Goresky–MacPherson.

(ii) What part of the signature package on closed manifolds and Witt spaces can be extended to these non-Witt spaces?

(c) (suggested by Shmuel Weinberger)

What kind of elliptic operator theory "tracks" (i.e. contains a signature operator for G-manifolds) the cosheaf homology term in stratified surgery for M/G? Things are easy when G acts locally freely so the quotient is an orbifold but this looks interesting in general.

(d) (suggested by Shmuel Weinberger; clarifications by Les Saper and Paolo Piazza)

The entrance of sheaves with nontrivial local cohomology whose global vanishing is important for global self-duality in Saper's talk suggests that compactifications have global "index invariants" in $L(\mathbb{R}P)$ or $K(\mathbb{C}P)$ but do not localize, i.e. pullback to $K(B\Gamma)$ (when Γ has torsion). For Γ with torsion, a "pullback" probably would be accidental nonsense. This is like what happens (for a different reason) in Fowler's talk on uniform lattices with torsion.

(e) (suggested by David Trotman)

If P and P' are homeomorphic PL Witt spaces (i.e. you think of them as different triangulations of the same object), it is a consequence of Goresky–MacPherson II and Siegel (or other combinations) that these are cobordant in the Witt sense. How elementary is this fact? Is there a *direct* proof?

2. Topology of algebraic varieties

(a) (suggested by Anatoly Libgober)

Does there exist a cobordism theory of pairs (X, D) such that for D log-terminal, $\mathcal{E}ll(X, D)$ is invariant under such cobordisms? See [6] for a discussion of elliptic genus of pairs and results related to this question.

(b) (suggested by Clint McCrory)

(i) Define intersection homology for real algebraic varieties. This question appears on Goresky and MacPherson's 1994 problem list [9]. Interesting work has been done by van Hamel [21]; see also [3; 20].

(ii) Simplify Akbulut and King's conjectural topological characterization of real algebraic varieties [2], and compute the bordism ring of real algebraic varieties. Invariants are "Akbulut–King numbers" [15].

(iii) What is the topology of the weight filtration of a real algebraic variety [17]? How can the filtration vary within a homeomorphism type? Is the filtration trivial for $\mathbb{Z}_2$ homology manifolds? Is it a bi-Lipshitz invariant (Trotman)?

(iv) Prove that the Stiefel–Whitney homology classes of a real algebraic variety are topologically invariant (*cf.* [8]).

(v) Which real toric varieties X are maximal, that is, when is

$$\dim H_*(X(\mathbb{R}); \mathbb{Z}_2) = \dim H_*(X(\mathbb{C}); \mathbb{Z}_2)?$$

See Hower's counterexample [12].

(vi) If a complex algebraic variety is defined over $\mathbb{R}$, what is the relation between the Deligne weight filtration of the cohomology (or homology) of the complex points and the weight filtration defined by Totaro [20] and McCrory and Parusiński [17] for the real points? The weight filtration of the homology of a complex variety can be defined with arbitrary coefficients. What is the relation between the weight filtration of the homology of the complex points with $\mathbb{Z}_2$ coefficients and the weight filtration for the real points?

(vii) Are there motivic characteristic classes for real varieties analogous to those defined by Brasselet, Schürmann, and Yokura [22] for complex varieties? The virtual Betti numbers β_q of real algebraic varieties [16] satisfy the "scissor relations"

$$\beta_q(X) = \beta_q(Y) + \beta_q(X \setminus Y)$$

for Y a closed subvariety of X. Can the virtual Betti numbers be extended to characteristic classes of real varieties?

(c) (suggested by David Trotman)

(i) Is it true that every topologically conical complex stratification of a complex analytic variety is Whitney (A)-regular? (This is not true for real algebraic varieties.)

(ii) Does every Whitney C^k stratified set admit a C^k triangulation such that the open simplices are strata of a Whitney stratification? The same question replacing "Whitney" by "Bekka."

(iii) It is known that families of (germs of) complex hypersurfaces with an isolated singularity have constant Milnor number if an only if they have constant topological type (except for "only if" for surfaces where it is an open question). Could it be true that having constant topological type is equivalent to the family being Bekka C-regular over the parameter space?

(iv) Can Goresky–MacPherson's Morse theory be made to work for tame Bekka stratifications instead of tame Whitney stratifications?

(v) It is known (Noirel -1996) that every abstract stratified space (Thom–Mather space) can be embedded in some $\mathbb{R}^n$ as a semi-algebraic Whitney stratified set (even Verdier regular) with semi-algebriac control data without refining the original stratification. Is there such a Mostowski stratified embedding, or at least locally bi-Lipschitz trivial semi-algebraic stratification without refinement? (By theorems of Parusiński (1992) or Valette (2005), there are refinements with these properties.)

(vi) Suppose 2 germs of complex analytic functions on $\mathbb{C}^n$ with isolated singularities at 0 are topologically equivalent, i.e. there exists a homeomorphism $h:(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ over $f:(\mathbb{C}^n,0)\to\mathbb{C}$ and $g:(\mathbb{C}^n,0)\to\mathbb{C}$ such that $f=gh$. Can one find another such homeomorphism h_1 such that h_1 preserves distance to the origin, i.e. $\|h_1(z)\|=\|z\|$ for z near $0\in\mathbb{C}^n$? (A positive answer would solve Zariski's 1970 problem about the topological invariance of the multiplicity.)

3. Mixed Hodge theory and singularities

(suggested by Matt Kerr and Gregory Pearlstein)

The period domain classifying Hodge structures of type $(h^{n,0},h^{n-1,1},\dots,h^{0,n})$, (say all >0), $n>1$ odd, is a *non*-locally-symmetric homogeneous space. Understand the L^2-cohomology groups

$$H^q_{(2)}\big(\Gamma\backslash D,\otimes_k((\Lambda^{h^{k,n-k}})\mathcal{H}^{k,n-k})^{\otimes a_k}\big)$$

and the role played by these in algebraizing images of period maps. (Note that Γ is an arithmetic subgroup and $K_{\Gamma\backslash D}$ is a line bundle of the form shown.) Possible reference (somewhat outdated): [11].

4. Characteristic class theories for singular varieties

(a) (suggested by Jörg Schürmann)

We work in the algebraic context over $\mathbb{C}$. Find a pure-dimensional variety X such that the class of the intersection cohomology complex IC_X in the Grothendieck group of complex algebraically constructible sheaves

$$[IC_X]\in K_0(D^b_c(X))$$

is *not* in the subgroup generated by $[Rf_*\mathbb{Q}_Z]$ with Z smooth pure-dimensional and $f:Z\to X$ proper.

Note that it is important to take only the classes of the total direct images. If one asks the same question for the subgroup generated by *direct summands* of $[Rf_*\mathbb{Q}_Z]$ with Z smooth pure-dimensional and $f:Z\to X$ proper, then $[IC_X]$ belongs to this subgroup by the *decomposition theorem* (compare [19, Corollary 4.6], for example).

A positive answer to the question above, stated in the "topological context" of algebraically constructible sheaves, would also give an example such that the class

$$[IC^H_X]\in K_0(MHM(X))$$

of the corresponding (pure) intersection Hodge module IC^H_X in the Grothendieck group of algebraic mixed Hodge modules is *not* in the image of the natural group homomorphism

$$\chi_{\mathrm{Hdg}} : K_0(\mathrm{var}/X) \to K_0(MHM(X))$$

from the *motivic relative Grothendieck group* of complex algebraic varieties over X (compare [19, Section 4.2]).

This fact would further justify the study of characteristic classes of mixed Hodge modules in the works of Cappell, Libgober, Maxim, Schürmann, and Shaneson; see [19] and the references therein.

(b) (suggested by Shmuel Weinberger)

Are the elliptic genera, etc. part of an integral theory the way L-classes come back from $KO(M)$ in index theory? Schürmann and Yokura know something about this but with too few variables.

References

[1] Pierre Albin, Eric Leichtnam, Rafe Mazzeo, Paolo Piazza, *The signature package on Witt spaces, I. Index classes*, http://arxiv.org/abs/0906.1568

[2] S. Akbulut, H. King, *Topology of real algebraic sets*, MSRI Publ. **25**, Springer Verlag, New York, 1992.

[3] M. Banagl, *The signature of singular spaces and its refinements to generalized homology theories*, in this volume.

[4] M. Banagl, *Topological invariants of stratified spaces*, Springer Monographs in Mathematics, Springer, New York, 2006

[5] Jean-Paul Brasselet, Joerg Schürmann, Shoji Yokura, *Hirzebruch classes and motivic Chern classes for singular spaces*, J. Topol. Anal. 2:1 (2010), 1–55.

[6] Lev Borisov, Anatoly Libgober, *Elliptic genera of singular varieties*, Duke Math. J. 116 (2003), 319–351.

[7] Cheeger, Jeff, *On the Hodge theory of Riemannian pseudomanifolds*. Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 91–146, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., 1980.

[8] J. Fu, C. McCrory, *Stiefel–Whitney classes and the conormal cycle of a real analytic variety*, Trans. Amer. Math. Soc. **349** (1997), 809–835.

[9] M. Goresky, R. MacPherson, *Problems and bibliography on intersection homology*, in *Intersection cohomology*, A. Borel *et al.*, Birkhäuser 1994, 221–229.

[10] P. Griffiths, M. Green, M. Kerr, *Some enumerative global properties of variations of Hodge structure*, to appear in Moscow Math. J.

[11] P. Griffiths, W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. **123** (1969), 253-302.

[12] V. Hower, *A counterexample to the maximality of toric varieties*, Proc. Amer. Math. Soc., **136** (2008), 4139–4142.

[13] E. Hunsicker, *Hodge and signature theorems for a family of manifolds with fibre bundle boundary*, Geom. Topol. 11 (2007), 1581–1622.

[14] E. Hunsicker, R. Mazzeo, *Harmonic forms on manifolds with edges*, Int. Math. Res. Not. (2005) .

[15] C. McCrory, A. Parusiński, *The topology of real algebraic sets of dimension 4: necessary conditions*, Topology **39** (2000), 495–523.

[16] C. McCrory, A. Parusiński, *Virtual Betti numbers of real algebraic varieties*, Comptes Rendus Acad. Sci. Paris, Ser. I, **336** (2003), 763–768.

[17] C. McCrory, A. Parusiński, *The weight filtration for real algebraic varieties*, in this volume.

[18] M. Nagase, L^2*-cohomology and intersection homology of stratified spaces*, Duke Math. J. 50 (1983).

[19] J. Schürmann, *Characteristic classes of mixed Hodge modules*, in this volume.

[20] B. Totaro, *Topology of singular algebraic varieties*, Proc. Int. Cong. Math. Beijing (2002), 533-541.

[21] J. van Hamel, *Towards an intersection homology theory for real algebraic varieties*, Int. Math. Research Notices **25** (2003), 1395–1411.

[22] S. Yokura, *Motivic characteristic classes*, in this volume.