Characteristic classes of mixed Hodge modules

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ABSTRACT. This paper is an extended version of an expository talk given at the workshop "Topology of Stratified Spaces" at MSRI in September 2008. It gives an introduction and overview about recent developments on the interaction of the theories of characteristic classes and mixed Hodge theory for singular spaces in the complex algebraic context.

It uses M. Saito's deep theory of mixed Hodge modules as a black box, thinking about them as "constructible or perverse sheaves of Hodge structures", having the same functorial calculus of Grothendieck functors. For the "constant Hodge sheaf", one gets the "motivic characteristic classes" of Brasselet, Schürmann, and Yokura, whereas the classes of the "intersection homology Hodge sheaf" were studied by Cappell, Maxim, and Shaneson. The classes associated to "good" variation of mixed Hodge structures where studied in connection with understanding the monodromy action by these three authors together with Libgober, and also by the author.

There are two versions of these characteristic classes. The K-theoretical classes capture information about the graded pieces of the filtered de Rham complex of the filtered D-module underlying a mixed Hodge module. Application of a suitable Todd class transformation then gives classes in homology. These classes are functorial for proper pushdown and exterior products, together with some other properties one would expect for a satisfactory theory of characteristic classes for singular spaces.

For "good" variation of mixed Hodge structures they have an explicit classical description in terms of "logarithmic de Rham complexes". On a point space they correspond to a specialization of the Hodge polynomial of a mixed Hodge structure, which one gets by forgetting the weight filtration.

We also indicate some relations with other subjects of the conference, like index theorems, signature, L-classes, elliptic genera and motivic characteristic classes for singular spaces.

CONTENTS

1. Introduction

This paper gives an introduction and overview about recent developments on the interaction of the theories of characteristic classes and mixed Hodge theory for singular spaces in the complex algebraic context. The reader is not assumed to have a background on any of these subjects, and the paper can also be used as a bridge for communication between researchers on one of these subjects.

General references for the theory of characteristic classes of singular spaces are the survey [48] and the paper [55] in these proceedings. As references for mixed Hodge theory one can use [38; 52], as well as the nice paper [37] for explaining the motivic viewpoint to mixed Hodge theory. Finally as an introduction to M. Saito's deep theory of mixed Hodge modules one can use [38, Chapter 14], [41] as well as the introduction [45].

The theory of mixed Hodge modules is used here more or less as a black box; we think about them as constructible or perverse sheaves of Hodge structures, having the same functorial calculus of Grothendieck functors. The underlying theory of constructible and perverse sheaves can be found in [7; 30; 47].

For the "constant Hodge sheaf" $\mathbb{Q}_Z^H$ one gets the "motivic characteristic classes" of Brasselet, Schürmann, and Yokura [9], as explained in these proceedings [55]. The classes of the "intersection homology Hodge sheaf" IC_Z^H were studied by Cappell, Maxim, and Shaneson in [10; 11]. Also, the classes associated to "good" variation of mixed Hodge structures where studied via Atiyah–Meyer type formulae by Cappell, Libgober, Maxim, and Shaneson in [12; 13]. For a summary compare also with [35].

There are two versions of these characteristic classes, the *motivic Chern class transformation* MHC_y and the *motivic Hirzebruch class transformation* MHT_{y*}. The K-theoretical classes MHC_y capture information about the graded pieces of the filtered de Rham complex of the filtered D-module underlying a mixed Hodge module. Application of a suitable twisting $td_{(1+y)}$ of the Todd class transformation td_* of Baum, Fulton, and MacPherson [5; 22] then gives the classes $MHT_{y*} = td_{(1+y)} \circ MHC_y$ in homology. It is the *motivic Hirzebruch class transformation* MHT_{y*}, which unifies three concepts:

($y = -1$) the (rationalized) *Chern class transformation c_** of MacPherson [34];
($y = 0$) the *Todd class transformation td_** of Baum–Fulton–MacPherson [5];
($y = 1$) the *L-class transformation L_** of Cappell and Shaneson [14].

(Compare with [9; 48] and also with [55] in these proceedings.) But in this paper we focus on the K-theoretical classes MHC_y, because these imply then also the corresponding results for MHT_{y*} just by application of the (twisted) Todd class transformation. So the *motivic Chern class transformation MHC_y* studied here is really the basic one!

Here we explain the functorial calculus of these classes, first stating in a very precise form the key results used from Saito's theory of mixed Hodge modules, and then explaining how to get from this the basic results about the motivic Chern class transformation MHC_y. These results are illustrated by many interesting examples. For the convenience of the reader, the most general results are only stated near the end of the paper. In fact, while most of the paper is a detailed survey of the K-theoretical version of the theory as developed in [9; 12; 13; 35], it is this last section that contains new results on the important functorial properties of these characteristic classes. The first two sections do not use mixed Hodge modules and are formulated in the now classical language of (variation of) mixed Hodge structures. Here is the plan of the paper:

SECTION 2 introduces pure and mixed Hodge structures and the corresponding Hodge genera, such as the E-polynomial and the χ_y-genus. These are suitable generating functions of Hodge numbers with χ_y using only the Hodge filtration F, whereas the E-polynomial also uses the weight filtration. We also carefully explain why only the χ_y-genus can be further generalized to characteristic classes, i.e., why one has to forget the weight filtration for applications to characteristic classes.

SECTION 3 motivates and explains the notion of a variation (or family) of pure and mixed Hodge structures over a smooth (or maybe singular) base. Basic examples come from the cohomology of the fibers of a family of complex algebraic varieties. We also introduce the notion of a "good" variation of mixed Hodge structures on a complex algebraic manifold M, to shorten the notion for a graded polarizable variation of mixed Hodge structures on M that is *admissible* in the sense of Steenbrink and Zucker [50] and Kashiwara [28], with *quasi-unipotent monodromy* at infinity, i.e., with respect to a compactification $\overline{M}$ of M by a compact complex algebraic manifold $\overline{M}$, with complement $D := \overline{M} \setminus M$ a normal crossing divisor with smooth irreducible components. Later on these will give the basic example of so-called "smooth" mixed Hodge modules. And for these good variations we introduce a simple *cohomological* characteristic class transformation MHC^y, which behaves nicely with respect

to smooth pullback, duality and (exterior) products. As a first approximation to more general mixed Hodge modules and their characteristic classes, we also study in detail functorial properties of the canonical Deligne extension across a normal crossing divisor D at infinity (as above), leading to *cohomological* characteristic classes $MHC^y(j_*(\cdot))$ defined in terms of "logarithmic de Rham complexes". These classes of good variations have been studied in detail in [12; 13; 35], and most results described here are new functorial reformulations of the results from these sources.

SECTION 4 starts with an introduction to Saito's functorial theory of algebraic mixed Hodge modules, explaining its power in many examples, including how to get a pure Hodge structure on the global intersection cohomology $IH^*(Z)$ of a compact complex algebraic variety Z. From this we deduce the basic calculus of Grothendieck groups $K_0(MHM(\cdot))$ of mixed Hodge modules needed for our motivic Chern class transformation MHC_y. We also explain the relation to the motivic viewpoint coming from relative Grothendieck groups of complex algebraic varieties.

SECTION 5.1 is devoted to the definition of our motivic characteristic *homology class* transformations MHC_y and MHT_{y*} for mixed Hodge modules. By Saito's theory they commute with push down for proper morphisms, and on a compact space one gets back the corresponding χ_y-genus by pushing down to a point, i.e., by taking the degree of these characteristic homology classes.

SECTIONS 5.2 AND 5.3 finally explain other important functoriality properties:

(1) multiplicativity for exterior products;

(2) the behavior under smooth pullback given by a Verdier Riemann–Roch formula;

(3) a "going up and down" formula for proper smooth morphisms;

(4) multiplicativity between MHC^y and MHC_y for a suitable (co)homological pairing in the context of a morphism with smooth target (as special cases one gets interesting Atiyah and Atiyah–Meyer type formulae, as studied in [12; 13; 35]);

(5) the relation between MHC_y and duality, i.e., the Grothendieck duality transformation for coherent sheaves and Verdier duality for mixed Hodge modules;

(6) the identification of MHT_{-1*} with the (rationalized) Chern class transformation $c_* \otimes \mathbb{Q}$ of MacPherson for the underlying constructible sheaf complex or function.

Note that such a functorial calculus is expected for any good theory of functorial characteristic classes of singular spaces (compare [9; 48]):

- for MacPherson's Chern class transformation c_* compare with [9; 31; 34; 48];
- for the Baum–Fulton–MacPherson Todd class transformation td_* compare with [5; 6; 9; 22; 24; 48];
- for Cappell and Shaneson's L-class transformation L_* compare with [2; 3; 4; 9; 14; 48; 49; 54].

The counterpart of mixed Hodge modules in these theories are constructible functions and sheaves (for c_*), coherent sheaves (for td_*) and selfdual perverse or constructible sheaf complexes (for L_*). The cohomological counterpart of the smooth mixed Hodge modules (i.e., good variation of mixed Hodge structures) are locally constant functions and sheaves (for c^*), locally free coherent sheaves or vector bundles (for the Chern character ch^*) and selfdual local systems (for a twisted Chern character of the KO-classes of Meyer [36]).

In this paper we concentrate mainly on pointing out the relation and analogy to the L-class story related to important signature invariants, because these are the subject of many other talks from the conference given in more topological terms. Finally also some relations to other themes of the conference, like index theorems, L^2-cohomology, elliptic genera and motivic characteristic classes for singular spaces, will be indicated.

2. Hodge structures and genera

2A. Pure Hodge structures. Let M be a compact *Kähler manifold* (e.g., a complex projective manifold) of complex dimension m. By classical Hodge theory one gets the decomposition (for $0 \leq n \leq 2m$)

$$H^n(M, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(M) \qquad (2\text{-}1)$$

of the complex cohomology of M into the spaces $H^{p,q}(M)$ of harmonic forms of type (p, q). This decomposition doesn't depend on the choice of a Kähler form (or metric) on M, and for a complex algebraic manifold M it is of algebraic nature. Here it is more natural to work with the *Hodge filtration*

$$F^i(M) := \bigoplus_{p \geq i} H^{p,q}(M) \qquad (2\text{-}2)$$

so that $H^{p,q}(M) = F^p(M) \cap \overline{F^q(M)}$, with $\overline{F^q(M)}$ the complex conjugate of $F^q(M)$ with respect to the real structure $H^n(M, \mathbb{C}) = H^n(M, \mathbb{R}) \otimes \mathbb{C}$. If

$$\Omega_M^\bullet = [\mathcal{O}_M \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega_M^m]$$

denotes the usual holomorphic de Rham complex (with $\mathcal{O}_M$ in degree zero), then one gets

$$H^*(M, \mathbb{C}) = H^*(M, \Omega_M^\bullet)$$

by the holomorphic Poincaré lemma, and the Hodge filtration is induced from the "stupid" decreasing filtration

$$F^p \Omega_M^\bullet = [0 \longrightarrow \cdots 0 \longrightarrow \Omega_M^p \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega_M^m]. \quad (2\text{-}3)$$

More precisely, the corresponding *Hodge to de Rham spectral sequence* degenerates at E_1, with

$$E_1^{p,q} = H^q(M, \Omega_M^p) \simeq H^{p,q}(M). \quad (2\text{-}4)$$

The same results are true for a compact complex manifold M that is only *bimeromorphic to a Kähler manifold* (compare [38, Corollary 2.30], for example). This is especially true for a compact complex algebraic manifold M. Moreover in this case one can calculate by Serre's GAGA theorem $H^*(M, \Omega_M^\bullet)$ also with the algebraic (filtered) de Rham complex in the Zariski topology.

Abstracting these properties, one can say the $H^n(M, \mathbb{Q})$ gets an induced *pure Hodge structure of weight n* in the following sense:

DEFINITION 2.1. Let V be a finite-dimensional rational vector space. A (rational) Hodge structure of weight n on V is a decomposition

$$V_\mathbb{C} := V \otimes_\mathbb{Q} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}, \quad \text{with } V^{q,p} = \overline{V^{p,q}} \quad \text{(Hodge decomposition)}.$$

In terms of the (decreasing) *Hodge filtration* $F^i V_\mathbb{C} := \bigoplus_{p \geq i} V^{p,q}$, this is equivalent to the condition

$$F^p V \cap \overline{F^q V} = \{0\} \quad \text{whenever } p + q = n + 1 \quad (n\text{-opposed filtration}).$$

Then $V^{p,q} = F^p \cap \overline{F^q}$, with $h^{p,q}(V) := \dim(V^{p,q})$ the corresponding *Hodge number*.

If V, V' are rational vector spaces with Hodge structures of weight n and m, then $V \otimes V'$ gets an induced Hodge structure of weight $n + m$, with Hodge filtration

$$F^k(V \otimes V')_\mathbb{C} := \bigoplus_{i+j=k} F^i V_\mathbb{C} \otimes F^j V'_\mathbb{C}. \quad (2\text{-}5)$$

Similarly the dual vector space $V^\vee$ gets an induced Hodge structure of weight $-n$, with

$$F^k(V_\mathbb{C}^\vee) := (F^{-k} V_\mathbb{C})^\vee. \quad (2\text{-}6)$$

A basic example is the *Tate Hodge structure* of weight $-2n \in \mathbb{Z}$ given by the one-dimensional rational vector space

$$\mathbb{Q}(n) := (2\pi i)^n \cdot \mathbb{Q} \subset \mathbb{C}, \quad \text{with } \mathbb{Q}(n)_{\mathbb{C}} = (\mathbb{Q}(n)_{\mathbb{C}})^{-n,-n}.$$

Then integration defines an isomorphism

$$H^2(P^1(\mathbb{C}), \mathbb{Q}) \simeq \mathbb{Q}(-1),$$

with $\mathbb{Q}(-n) = \mathbb{Q}(-1)^{\otimes n}$, $\mathbb{Q}(1) = \mathbb{Q}(-1)^{\vee}$ and $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ for $n > 0$.

DEFINITION 2.2. A *polarization* of a rational Hodge structure V of weight n is a rational $(-1)^n$-symmetric bilinear form S on V such that

$$S(F^p, F^{n-p+1}) = 0 \quad \text{for all } p$$

and

$$i^{p-q} S(u, \bar{u}) > 0 \quad \text{for all nonzero } u \in V^{p,q}.$$

So for *n even* one gets in particular

$$(-1)^{p-n/2} S(u, \bar{u}) > 0 \quad \text{for all } q \text{ and all nonzero } u \in V^{p,q}. \qquad (2\text{-}7)$$

V is called *polarizable* if such a polarization exists.

For example, the cohomology $H^n(M, \mathbb{Q})$ of a projective manifold is polarizable by the choice of a suitable Kähler form! Also note that a polarization of a rational Hodge structure V of weight n induces an isomorphism of Hodge structures (of weight n):

$$V \simeq V^{\vee}(-n) := V^{\vee} \otimes_{\mathbb{Q}} \mathbb{Q}(-n).$$

So if we choose the isomorphism of rational vector spaces

$$\mathbb{Q}(-n) = (2\pi i)^{-n} \cdot \mathbb{Q} \simeq \mathbb{Q},$$

then a polarization induces a $(-1)^n$-symmetric duality isomorphism $V \simeq V^{\vee}$.

2B. Mixed Hodge structures. The cohomology (with compact support) of a singular or noncompact complex algebraic variety, denoted by $H^n_{(c)}(X, \mathbb{Q})$, can't have a pure Hodge structure in general, but by Deligne's work [20; 21] it carries a canonical functorial (graded polarizable) *mixed Hodge structure* in the following sense:

DEFINITION 2.3. A finite-dimensional rational vector space V has a mixed Hodge structure if there is a (finite) increasing *weight filtration* $W = W_{\bullet}$ on V (by rational subvector spaces), and a (finite) decreasing Hodge filtration $F = F^{\bullet}$ on $V_{\mathbb{C}}$, such that F induces a Hodge structure of weight n on $Gr^W_n V :=$ $W_n V / W_{n-1} V$ for all n. Such a mixed Hodge structure is called (*graded*) *polarizable* if each graded piece $Gr^W_n V$ is polarizable.

A morphism of mixed Hodge structures is just a homomorphism of rational vector spaces compatible with both filtrations. Such a morphism is then *strictly* compatible with both filtrations, so that the category $mHs^{(p)}$ of (graded polarizable) mixed Hodge structures is an abelian category, with Gr_*^W, Gr_F^* and $Gr_F^* Gr_*^W$ preserving short exact sequences. The category $mHs^{(p)}$ is also endowed with a tensor product $\otimes$ and a duality $(\cdot)^\vee$, where the corresponding Hodge and weight filtrations are defined as in (2-5) and (2-6). So for a complex algebraic variety X one can consider its cohomology class

$$[H_{(c)}^*(X)] := \sum_i (-1)^i \cdot [H_{(c)}^i(X, \mathbb{Q})] \in K_0(mHs^{(p)})$$

in the Grothendieck group $K_0(mHs^{(p)})$ of (graded polarizable) mixed Hodge structures. The functoriality of Deligne's mixed Hodge structure means, in particular, that for a closed complex algebraic subvariety $Y \subset X$, with open complement $U = X \setminus Y$, the corresponding long exact cohomology sequence

$$\cdots H_c^i(U, \mathbb{Q}) \to H_c^i(X, \mathbb{Q}) \to H_c^i(Y, \mathbb{Q}) \to \cdots \tag{2-8}$$

is an exact sequence of mixed Hodge structures. Similarly, for complex algebraic varieties X, Z, the Künneth isomorphism

$$H_c^*(X, \mathbb{Q}) \otimes H_c^*(Z, \mathbb{Q}) \simeq H_c^*(X \times Z, \mathbb{Q}) \tag{2-9}$$

is an isomorphism of mixed Hodge structures. Let us denote by $K_0(\text{var}/pt)$ the Grothendieck group of complex algebraic varieties, i.e., the free abelian group of isomorphism classes $[X]$ of such varieties divided out by the *additivity relation*

$$[X] = [Y] + [X \setminus Y]$$

for $Y \subset X$ a closed complex subvariety. This is then a commutative ring with addition resp. multiplication induced by the disjoint union resp. the product of varieties. So by (2-8) and (2-9) we get an induced ring homomorphism

$$\chi_{\text{Hdg}} : K_0(\text{var}/pt) \to K_0(mHs^{(p)}); \; [X] \mapsto [H_c^*(X)]. \tag{2-10}$$

2C. Hodge genera. The *E-polynomial*

$$E(V) := \sum_{p,q} h^{p,q}(V) \cdot u^p v^q \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}] \tag{2-11}$$

of a rational mixed Hodge structure V with *Hodge numbers*

$$h^{p,q}(V) := \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W(V_{\mathbb{C}}),$$

induces a *ring* homomorphism

$$E : K_0(mHs^{(p)}) \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}], \quad \text{with } E(\mathbb{Q}(-1)) = uv.$$

Note that $E(V)(u, v)$ is *symmetric* in u and v, since $h(V) = \sum_n h(W_n V)$ and $V^{q,p} = \overline{V^{p,q}}$ for a pure Hodge structure. With respect to *duality* one has in addition the relation

$$E(V^\vee)(u, v) = E(V)(u^{-1}, v^{-1}). \tag{2-12}$$

Later on we will be mainly interested in the specialized ring homomorphism

$$\chi_y := E(-y, 1) : K_0(mHs^{(p)}) \to \mathbb{Z}[y^{\pm 1}], \quad \text{with } \chi_y(\mathbb{Q}(-1)) = -y,$$

defined by

$$\chi_y(V) := \sum_p \dim_{\mathbb{C}}(Gr_F^p(V_{\mathbb{C}})) \cdot (-y)^p. \tag{2-13}$$

So here one uses only the Hodge and forgets the weight filtration of a mixed Hodge structure. With respect to *duality* one has then the relation

$$\chi_y(V^\vee) = \chi_{1/y}(V). \tag{2-14}$$

Note that $\chi_{-1}(V) = \dim(V)$ and for a pure polarized Hodge structure V of weight n one has by $\chi_1(V) = (-1)^n \chi_1(V^\vee) = (-1)^n \chi_1(V)$ and (2-7):

$$\chi_1(V) = \begin{cases} 0 & \text{for } n \text{ odd}, \\ \operatorname{sgn} V & \text{for } n \text{ even}, \end{cases}$$

where sgn V is the *signature* of the induced symmetric bilinear form $(-1)^{n/2} S$ on V. A similar but deeper result is the famous *Hodge index theorem* (compare [52, Theorem 6.3.3], for example):

$$\chi_1([H^*(M)]) = \operatorname{sgn}(H^m(M, \mathbb{Q}))$$

for M a compact Kähler manifold of even complex dimension $m = 2n$. Here the right side denotes the signature of the symmetric intersection pairing

$$H^m(M, \mathbb{Q}) \times H^m(M, \mathbb{Q}) \xrightarrow{\ \cup\ } H^{2m}(M, \mathbb{Q}) \simeq \mathbb{Q}.$$

The advantage of χ_y compared to E (and the use of $-y$ in the definition) comes from the following question:

Let $E(X) := E([H^*(X)])$ *for X a complex algebraic variety. For M a compact complex algebraic manifold one gets by* (2-4):

$$E(M) = \sum_{p,q \geq 0} (-1)^{p+q} \cdot \dim_{\mathbb{C}} H^q(M, \Omega_M^p) \cdot u^p v^q.$$

Is there a (*normalized multiplicative*) *characteristic class*

$$cl^* : \operatorname{Iso}(\mathbb{C} - VB(M)) \to H^*(M)[u^{\pm 1}, v^{\pm 1}]$$

of complex vector bundles such that the E-polynomial is a characteristic number in the sense that

$$E(M) = \sharp(M) := \deg(cl^*(TM) \cap [M]) \in H^*(pt)[u^{\pm 1}, v^{\pm 1}] \qquad \text{(2-15)}$$

for any compact complex algebraic manifold M with fundamental class $[M]$?

So the cohomology class $cl^*(V) \in H^*(M)[u^{\pm 1}, v^{\pm 1}]$ should only depend on the isomorphism class of the complex vector bundle V over M and commute with pullback. Multiplicativity says

$$cl^*(V) = cl^*(V') \cup cl^*(V'') \in H^*(M)[u^{\pm 1}, v^{\pm 1}]$$

for any short exact sequence $0 \to V' \to V \to V'' \to 0$ of complex vector bundles on M. Finally cl^* is normalized if $cl^*(\text{trivial}) = 1 \in H^*(M)$ for any trivial vector bundle. Then the answer to the question is *negative*, because there are unramified coverings $p : M' \to M$ of elliptic curves M, M' of (any) degree $d > 0$. Then $p^*TM \simeq TM'$ and $p_*([M']) = d \cdot [M]$, so the projection formula would give for the topological characteristic numbers the relation

$$\sharp(M') = d \cdot \sharp(M).$$

But one has

$$E(M) = (1-u)(1-v) = E(M') \neq 0,$$

so the equality $E(M) = \sharp(M)$ is not possible! Here we don't need to ask cl^* to be multiplicative or normalized. But if we use the invariant $\chi_y(X) := \chi_y([H^*(X)])$, then $\chi_y(M) = 0$ for an elliptic curve, and $\chi_y(M)$ is a characteristic number in the sense above by the famous *generalized Hirzebruch Riemann–Roch theorem* [27]:

THEOREM 2.4 (GHRR). *There is a unique normalized multiplicative characteristic class*

$$T_y^* : \text{Iso}(\mathbb{C} - VB(M)) \to H^*(M, \mathbb{Q})[y]$$

such that

$$\chi_y(M) = \deg(T_y^*(TM) \cap [M]) = \langle T_y^*(TM), [M] \rangle \in \mathbb{Z}[y] \subset \mathbb{Q}[y]$$

for any compact complex algebraic manifold M. Here $\langle \cdot, \cdot \rangle$ is the Kronecker pairing between cohomology and homology.

The *Hirzebruch class T_y^** and χ_y-genus unify the following (total) characteristic classes and numbers:

y	T_y^*	[name]	χ_y	[name]
-1	c^*	Chern class	χ	Euler characteristic
0	td^*	Todd class	χ_a	arithmetic genus
1	L^*	L-class	sgn	signature

In fact, gHRR is just a cohomological version of the following K-theoretical calculation. Let M be a compact complex algebraic manifold, so that

$$\chi_y(M) = \sum_{p,q \geq 0} (-1)^{p+q} \cdot \dim_{\mathbb{C}} H^q(M, \Omega_M^p) \cdot (-y)^p$$
$$= \sum_{p \geq 0} \chi(H^*(M, \Omega_M^p)) \cdot y^p. \tag{2-16}$$

Let us denote by $K_{\mathrm{an}}^0(Y)$ (or $G_0^{\mathrm{an}}(Y)$) the Grothendieck group of the exact (or abelian) category of holomorphic vector bundles (or coherent $\mathcal{O}_Y$-module sheaves) on the complex variety Y, i.e., the free abelian group of isomorphism classes V of such vector bundles (or sheaves), divided out by the relation

$$[V] = [V'] + [V''] \quad \text{for any short exact sequence } 0 \to V' \to V \to V'' \to 0.$$

Then $G_0^{\mathrm{an}}(Y)$ (or $K_{\mathrm{an}}^0(Y)$) is of (co)homological nature, with

$$f_* : G_0^{\mathrm{an}}(X) \to G_0^{\mathrm{an}}(Y), \quad [\mathcal{F}] \mapsto \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]$$

the functorial pushdown for a proper holomorphic map $f : X \to Y$. In particular, for X compact, the constant map $k : X \to pt$ is proper, with

$$\chi(H^*(X, \mathcal{F})) = k_*([\mathcal{F}]) \in G_0^{\mathrm{an}}(pt) \simeq K_{\mathrm{an}}^0(pt) \simeq \mathbb{Z}.$$

Moreover, the tensor product $\otimes_{\mathcal{O}_Y}$ induces a natural pairing

$$\cap = \otimes : K_{\mathrm{an}}^0(Y) \times G_0^{\mathrm{an}}(Y) \to G_0^{\mathrm{an}}(Y),$$

where we identify a holomorphic vector bundle V with its locally free coherent sheaf of sections $\mathcal{V}$. So for X compact we can define a *Kronecker pairing*

$$K_{\mathrm{an}}^0(X) \times G_0^{\mathrm{an}}(X) \to G_0^{\mathrm{an}}(pt) \simeq \mathbb{Z}; \ \langle [\mathcal{V}], [\mathcal{F}] \rangle := k_*([\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{F}]).$$

The *total λ-class* of the dual vector bundle

$$\lambda_y(V^\vee) := \sum_{i \geq 0} \Lambda^i(V^\vee) \cdot y^i$$

defines a multiplicative characteristic class

$$\lambda_y((\cdot)^\vee) : K_{\mathrm{an}}^0(Y) \to K_{\mathrm{an}}^0(Y)[y].$$

And for a compact complex algebraic manifold M one gets the equality

$$\chi_y(M) = \sum_{i \geq 0} k_*[\Omega_M^i] \cdot y^i$$
$$= \langle \lambda_y(T^*M), [\mathcal{O}_M] \rangle \in G_0^{\mathrm{an}}(pt)[y] \simeq \mathbb{Z}[y]. \tag{2-17}$$

3. Characteristic classes of variations of mixed Hodge structures

This section explains the definition of *cohomological* characteristic classes associated to good variations of mixed Hodge structures on complex algebraic and analytic manifolds. These were previously considered in [12; 13; 35] in connection with Atiyah–Meyer type formulae of Hodge-theoretic nature. Here we also consider important functorial properties of these classes.

3A. Variation of Hodge structures. Let $f : X \to Y$ be a *proper smooth* morphism of complex algebraic varieties or a *projective smooth* morphism of complex analytic varieties. Then the higher direct image sheaf $L = L^n := R^n f_* \mathbb{Q}_X$ is a *locally constant sheaf* on Y with finite-dimensional stalks

$$L_y = (R^n f_* \mathbb{Q}_X)_y = H^n(\{f = y\}, \mathbb{Q})$$

for $y \in Y$. Let $\mathcal{L} := L \otimes_{\mathbb{Q}_Y} \mathcal{O}_Y \simeq R^n f_*(\Omega^\bullet_{X/Y})$ denote the corresponding holomorphic vector bundle (or locally free sheaf), with $\Omega^\bullet_{X/Y}$ the *relative holomorphic de Rham complex*. Then the stupid filtration of $\Omega^\bullet_{X/Y}$ determines a decreasing filtration F of $\mathcal{L}$ by holomorphic subbundles $F^p \mathcal{L}$, with

$$Gr_F^p((R^{p+q} f_* \mathbb{Q}_X) \otimes_{\mathbb{Q}_Y} \mathcal{O}_Y) \simeq R^q f_*(\Omega^p_{X/Y}), \qquad (3\text{-}1)$$

inducing for all $y \in Y$ the Hodge filtration F on the cohomology

$$H^n(\{f = y\}, \mathbb{Q}) \otimes \mathbb{C} \simeq \mathcal{L}|_y$$

of the compact and smooth algebraic fiber $\{f = y\}$ (compare [38, Chapter 10]). If Y (and therefore also X) is smooth, then $\mathcal{L}$ gets an induced *integrable Gauss–Manin connection*

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega^1_Y, \quad \text{with } L \simeq \ker \nabla \text{ and } \nabla \circ \nabla = 0,$$

satisfying the *Griffiths transversality condition*

$$\nabla(F^p \mathcal{L}) \subset F^{p-1} \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega^1_Y \quad \text{for all } p. \qquad (3\text{-}2)$$

This motivates the following notion:

DEFINITION 3.1. A *holomorphic family* (L, F) of Hodge structures of weight n on the reduced complex space Y is a local system L with rational coefficients and finite-dimensional stalks on Y, and a decreasing filtration F of $\mathcal{L} = L \otimes_{\mathbb{Q}_Y} \mathcal{O}_Y$ by holomorphic subbbundles $F^p \mathcal{L}$ such that F determines by $L_y \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathcal{L}|_y$ a pure Hodge structure of weight n on each stalk L_y ($y \in Y$).

If Y is a smooth complex manifold, then such a holomorphic family (L, F) is called a *variation* of Hodge structures of weight n if, in addition, Griffiths transversality (3-2) holds for the induced connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega^1_Y$.

Finally a *polarization* of (L, F) is a pairing of local systems $S : L \otimes_{\mathbb{Q}_Y} L \to \mathbb{Q}_Y$ that induces a polarization of Hodge structures on each stalk L_y ($y \in Y$).

For example in the geometric case above, one can get such a polarization on $L = R^n f_* \mathbb{Q}_X$ for $f : X \to Y$ a *projective smooth* morphism of complex algebraic (or analytic) varieties. The existence of a polarization is needed for example for the following important result of Schmid [46, Theorem 7.22]:

THEOREM 3.2 (RIGIDITY). *Let Y be a connected complex manifold Zarisky open in a compact complex analytic manifold $\overline{Y}$, with (L, F) a polarizable variation of pure Hodge structures on Y. Then $H^0(Y, L)$ gets an induced Hodge structure such that the evaluation map $H^0(Y, L) \to L_y$ is an isomorphism of Hodge structures for all $y \in Y$. In particular the variation (L, F) is constant if the underlying local system L is constant.*

3B. Variation of mixed Hodge structures. If one considers a morphism $f : X \to Y$ of complex algebraic varieties with Y smooth, which is a topological fibration with possible singular or noncompact fiber, then the locally constant direct image sheaves $L = L^n := R^n f_* \mathbb{Q}_X$ ($n \geq 0$) are *variations of mixed Hodge structures* in the sense of the following definitions.

DEFINITION 3.3. Let Y be a reduced complex analytic space. A *holomorphic family of mixed Hodge structures* on Y consists of

(1) a local system L of rational vector spaces on Y with finite-dimensional stalks,
(2) a finite decreasing *Hodge filtration* F of $\mathcal{L} = L \otimes_{\mathbb{Q}_Y} \mathcal{O}_Y$ by holomorphic subbundles $F^p \mathcal{L}$,
(3) a finite increasing *weight filtration* W of L by local subsystems $W_n L$,

such that the induced filtrations on $\mathcal{L}_y \simeq L_y \otimes_{\mathbb{Q}} \mathbb{C}$ and L_y define a mixed Hodge structure on all stalks L_y ($y \in Y$).

If Y is a smooth complex manifold, such a holomorphic family (L, F, W) is called a *variation of mixed Hodge structures* if, in addition, Griffiths transversality (3-2) holds for the induced connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_Y} \Omega_Y^1$.

Finally, (L, F, W) is called *graded polarizable* if the induced family (or variation) of pure Hodge structures $Gr_n^W L$ (with the induced Hodge filtration F) is polarizable for all n.

With the obvious notion of morphisms, the two categories $FmHs^{(p)}(Y)$ and $VmHs^{(p)}(Y)$ of (graded polarizable) families and variations of mixed Hodge structures on Y become abelian categories with a tensor product $\otimes$ and duality $(\cdot)^\vee$. Again, any such morphism is strictly compatible with the Hodge and weight filtrations. Moreover, one has for a holomorphic map $f : X \to Y$ (of

complex manifolds) a functorial pullback

$$f^* : FmHs^{(p)}(Y) \to FmHs^{(p)}(X) \quad (\text{or } f^* : VmHs^{(p)}(Y) \to VmHs^{(p)}(X)),$$

commuting with tensor product $\otimes$ and duality $(\cdot)^\vee$. On a point space pt one just gets back the category

$$FmHs^{(p)}(pt) = VmHs^{(p)}(pt) = mHs^{(p)}$$

of (graded polarizable) mixed Hodge structures. Using the pullback under the constant map $k : Y \to pt$, we get the constant family (or variation) of Tate Hodge structures $\mathbb{Q}_Y(n) := k^*\mathbb{Q}(n)$ on Y.

3C. Cohomological characteristic classes. The Grothendieck group $K^0_{an}(Y)$ of holomorphic vector bundles on the complex variety Y is a commutative ring with multiplication induced by $\otimes$ and has a duality involution induced by $(\cdot)^\vee$. For a holomorphic map $f : X \to Y$ one has a functorial pullback f^* of rings with involutions. The situation is similar for $K^0_{an}(Y)[y^{\pm 1}]$, if we extend the duality involution by

$$([V] \cdot y^k)^\vee := [V^\vee] \cdot (1/y)^k.$$

For a family (or variation) of mixed Hodge structures (L, F, W) on Y let us introduce the characteristic class

$$MHC^y((L, F, W)) := \sum_p [Gr^p_F(\mathcal{L})] \cdot (-y)^p \in K^0_{an}(Y)[y^{\pm 1}]. \qquad (3\text{-}3)$$

Because morphisms of families (or variations) of mixed Hodge structures are strictly compatible with the Hodge filtrations, we get induced group homomorphisms of Grothendieck groups:

$$MHC^y : K_0(FmHs^{(p)}(Y)) \to K^0_{an}(Y)[y^{\pm 1}],$$
$$MHC^y : K_0(VmHs^{(p)}(Y)) \to K^0_{an}(Y)[y^{\pm 1}].$$

Note that $MHC^{-1}((L, F, W)) = [\mathcal{L}] \in K^0_{an}(Y)$ is just the class of the associated holomorphic vector bundle. And for $Y = pt$ a point, we get back the χ_y-genus:

$$\chi_y = MHC^y : K_0(mHs^{(p)}) = K_0(FmHs^{(p)}(pt)) \to K^0_{an}(pt)[y^{\pm 1}] = \mathbb{Z}[y^{\pm 1}].$$

THEOREM 3.4. *The transformations*

$$MHC^y : K_0(FmHs^{(p)}(Y)) \to K^0_{an}(Y)[y^{\pm 1}],$$
$$MHC^y : K_0(VmHs^{(p)}(Y)) \to K^0_{an}(Y)[y^{\pm 1}],$$

are contravariant functorial, and are transformations of commutative rings with unit, i.e., they commute with products and respect units: $MHC^y([\mathbb{Q}_Y(0)]) = [\mathcal{O}_Y]$. *Similarly they respect duality involutions*:

$$MHC^y([(L, F, W)^\vee]) = \sum_p [(Gr_F^{-p}(\mathcal{L}))^\vee] \cdot (-y)^p = (MHC^y([(L, F, W)]))^\vee.$$

EXAMPLE 3.5. Let $f : X \to Y$ be a *proper smooth* morphism of complex algebraic varieties or a *projective smooth* morphism of complex analytic varieties, so that the higher direct image sheaf $L^n := R^n f_* \mathbb{Q}_X$ ($n \geq 0$) with the induced Hodge filtration as in (3-1) defines a holomorphic family of pure Hodge structures on Y. If m is the complex dimension of the fibers, then $L_n = 0$ for $n > 2m$, so one can define

$$[Rf_* \mathbb{Q}_X] := \sum_{n=0}^{2m} (-1)^n \cdot [(R^n f_* \mathbb{Q}_X, F)] \in K_0(FmHs(Y)).$$

Then one gets, by (3-1),

$$\begin{aligned}
MHC^y([Rf_* \mathbb{Q}_X]) &= \sum_{p,q \geq 0} (-1)^{p+q} \cdot [R^q f_* \Omega_{X/Y}^p] \cdot (-y)^p \\
&= \sum_{p \geq 0} f_* [\Omega_{X/Y}^p] \cdot y^p \\
&=: f_* \left(\lambda_y (T_{X/Y}^*) \right) \in K_{an}^0(Y)[y].
\end{aligned} \tag{3-4}$$

Assume moreover that

(a) Y is a connected complex manifold Zarisky open in a compact complex analytic manifold $\overline{Y}$, and

(b) all direct images sheaves $L^n := R^n f_* \mathbb{Q}_X$ ($n \geq 0$) are *constant*.

Then one gets by the *rigidity theorem* 3.2 (for $z \in Y$):

$$f_* \left(\lambda_y (T_{X/Y}^*) \right) = \chi_y(\{f = z\}) \cdot [\mathcal{O}_Y] \in K_{an}^0(Y)[y].$$

COROLLARY 3.6 (MULTIPLICATIVITY). *Let $f : X \to Y$ be a smooth morphism of compact complex algebraic manifolds, with Y connected. Let $T_{X/Y}^*$ be the relative holomorphic cotangent bundle of the fibers, fitting into the short exact sequence*

$$0 \to f^* T^* Y \to T^* X \to T_{X/Y}^* \to 0.$$

Assume all direct images sheaves $L^n := R^n f_* \mathbb{Q}_X$ ($n \geq 0$) are constant, i.e., $\pi_1(Y)$ acts trivially on the cohomology $H^*(\{f = z\})$ of the fiber. Then one

gets the multiplicativity of the χ_y-genus (with $k : Y \to pt$ the constant map):

$$\begin{aligned} \chi_y(X) &= (k \circ f)_*[\lambda_y(T^*X)] \\ &= k_* f_*\big([\lambda_y(T^*_{X/Y})] \otimes f^*[\lambda_y(T^*Y)]\big) \\ &= k_* \big(\chi_y(\{f = z\}) \cdot [\lambda_y(T^*Y)]\big) \\ &= \chi_y(\{f = z\}) \cdot \chi_y(Y). \end{aligned} \tag{3-5}$$

REMARK 3.7. The multiplicativity relation (3-5) specializes for $y = 1$ to the classical multiplicativity formula

$$\mathrm{sgn}(X) = \mathrm{sgn}(\{f = z\}) \cdot \mathrm{sgn}(Y)$$

of Chern, Hirzebruch, and Serre [16] for the signature of an oriented fibration of smooth coherently oriented compact manifolds, if $\pi_1(Y)$ acts trivially on the cohomology $H^*(\{f = z\})$ of the fiber. So it is a Hodge theoretic counterpart of this. Moreover, the corresponding Euler characteristic formula for $y = -1$

$$\chi(X) = \chi(\{f = z\}) \cdot \chi(Y)$$

is even true *without* $\pi_1(Y)$ acting trivially on the cohomology $H^*(\{f = z\})$ of the fiber!

The Chern–Hirzebruch–Serre signature formula was motivational for many subsequent works which studied monodromy contributions to invariants (genera and characteristic classes). See, for exmaple, [1; 4; 10; 11; 12; 13; 14; 35; 36].

Instead of working with holomorphic vector bundles, we can of course also use only the underlying topological complex vector bundles, which gives the forgetful transformation

$$For : K^0_{\mathrm{an}}(Y) \to K^0_{\mathrm{top}}(Y).$$

Here the target can also be viewed as the even part of $\mathbb{Z}_2$-graded topological complex K-cohomology. Of course, the forgetful transformation is contravariant functorial and commutes with product $\otimes$ and with duality $(\cdot)^\vee$. This duality induces a $\mathbb{Z}_2$-grading on $K^0_{\mathrm{top}}(Y)\big[\frac{1}{2}\big]$ by splitting into the (anti-)invariant part, and similarly for $K^0_{\mathrm{an}}(Y)\big[\frac{1}{2}\big]$. Then the (anti-)invariant part of $K^0_{\mathrm{top}}(Y)\big[\frac{1}{2}\big]$ can be identified with the even part of $\mathbb{Z}_4$-graded topological real K-theory $KO^0_{\mathrm{top}}(Y)\big[\frac{1}{2}\big]$ (and $KO^2_{\mathrm{top}}(Y)\big[\frac{1}{2}\big]$).

Assume now that (L, F) is a holomorphic family of pure Hodge structures of weight n on the complex variety Y, with a polarization $S : L \otimes_{\mathbb{Q}_Y} L \to \mathbb{Q}_Y$. This induces an isomorphism of families of pure Hodge structures of weight n:

$$L \simeq L^\vee(-n) := L^\vee \otimes \mathbb{Q}_Y(-n).$$

So if we choose the isomorphism of rational local systems

$$\mathbb{Q}_Y(-n) = (2\pi i)^{-n} \cdot \mathbb{Q}_Y \simeq \mathbb{Q}_Y,$$

the polarization induces a $(-1)^n$-symmetric duality isomorphism $L \simeq L^\vee$ of the underlying local systems. And for such an (anti)symmetric selfdual local system L Meyer [36] has introduced a KO-characteristic class

$$[L]_{KO} \in KO^0_{\text{top}}(Y)\left[\tfrac{1}{2}\right] \oplus KO^2_{\text{top}}(Y)\left[\tfrac{1}{2}\right]) = K^0_{\text{top}}(Y)\left[\tfrac{1}{2}\right],$$

so that for Y a compact oriented manifold of even real dimension $2m$ the following *twisted signature formula* is true:

$$\operatorname{sgn}(H^m(Y, L)) = \langle ch^*(\Psi^2([L]_{KO})), L^*(TM) \cap [M]\rangle. \qquad (3\text{-}6)$$

Here $H^m(Y, L)$ gets an induced (anti)symmetric duality, with $\operatorname{sgn}(H^m(Y, L))$ defined as 0 in case of an antisymmetric pairing. Moreover ch^* is the Chern character, Ψ^2 the second Adams operation and L^* is the Hirzebruch–Thom L-class.

We now explain that $[L]_{KO}$ agrees up to some universal signs with

$$For(MHC^1((L, F))).$$

The underlying topological complex vector bundle of $\mathcal{L}$ has a natural real structure, so that, as a topological complex vector bundle, one gets an orthogonal decomposition

$$\mathcal{L} = \bigoplus_{p+q=n} \mathcal{H}^{p,q}, \quad \text{with } \mathcal{H}^{p,q} = F^p \mathcal{L} \cap \overline{F^q \mathcal{L}} = \overline{\mathcal{H}^{q,p}},$$

with

$$For(MHC^1((L, F))) = \sum_{\substack{p \text{ even} \\ q}} [\mathcal{H}^{p,q}] - \sum_{\substack{p \text{ odd} \\ q}} [\mathcal{H}^{p,q}]. \qquad (3\text{-}7)$$

If n is even, both sums on the right are invariant under conjugation. And, by (2-7), $(-1)^{-n/2} \cdot S$ is positive definite on the corresponding real vector bundle $\left(\bigoplus_{p \text{ even}, q} \mathcal{H}^{p,q}\right)_{\mathbb{R}}$, and negative definite on $\left(\bigoplus_{p \text{ odd}, q} \mathcal{H}^{p,q}\right)_{\mathbb{R}}$. So if we choose the pairing $(-1)^{n/2} \cdot S$ for the isomorphism $L \simeq L^\vee$, then this agrees with the splitting introduced by Meyer [36] in the definition of his KO-characteristic class $[L]_{KO}$ associated to this *symmetric* duality isomorphism of L:

$$For(MHC^1((L, F))) = [L]_{KO} \in KO^0_{\text{top}}(Y)\left[\tfrac{1}{2}\right].$$

Similarly, if n is odd, both sums of the right hand side in (3-7) are exchanged under conjugation. If we choose the pairing $(-1)^{(n+1)/2} \cdot S$ for the isomorphism $L \simeq L^\vee$, then this agrees by Definition 2.2 with the splitting introduced by

Meyer [36] in the definition of his KO-characteristic class $[L]_{KO}$ associated to this *antisymmetric* duality isomorphism of L:

$$For(MHC^1((L, F))) = [L]_{KO} \in KO^2_{\text{top}}(Y)\left[\tfrac{1}{2}\right].$$

COROLLARY 3.8. *Let (L, F) be a holomorphic family of pure Hodge structures of weight n on the complex variety Y, with a polarization S chosen. The class $[L]_{KO}$ introduced in [36] for the duality isomorphism coming from the pairing $(-1)^{n(n+1)/2} \cdot S$ is equal to*

$$For(MHC^1((L, F))) = [L]_{KO} \in KO^0_{\text{top}}(Y)\left[\tfrac{1}{2}\right] \oplus KO^2_{\text{top}}(Y)\left[\tfrac{1}{2}\right] = K^0_{\text{top}}(Y)\left[\tfrac{1}{2}\right].$$

It is therefore independent of the choice of the polarization S. Moreover, this identification is functorial under pullback and compatible with products (as defined in [36, p. 26] for (anti)symmetric selfdual local systems).

There are Hodge theoretic counterparts of the twisted signature formula (3-6). Here we formulate a corresponding K-theoretical result. Let (L, F, W) be a variation of mixed Hodge structures on the m-dimensional complex manifold M. Then

$$H^n(M, L) \simeq H^n(M, DR(\mathcal{L}))$$

gets an induced (decreasing) F filtration coming from the filtration of the holomorphic de Rham complex of the vector bundle $\mathcal{L}$ with its integrable connection ∇:

$$DR(\mathcal{L}) = [\mathcal{L} \xrightarrow{\ \nabla\ } \cdots \xrightarrow{\ \nabla\ } \mathcal{L} \otimes_{\mathcal{O}_M} \Omega^m_M]$$

(with $\mathcal{L}$ in degree zero), defined by

$$F^p DR(\mathcal{L}) = [F^p\mathcal{L} \xrightarrow{\ \nabla\ } \cdots \xrightarrow{\ \nabla\ } F^{p-m}\mathcal{L} \otimes_{\mathcal{O}_M} \Omega^m_M]. \qquad (3\text{-}8)$$

Note that here we are using Griffiths transversality (3-2)!

The following result is due to Deligne and Zucker [56, Theorem 2.9, Lemma 2.11] in the case of a compact Kähler manifold, whereas the case of a compact complex algebraic manifold follows from Saito's general results as explained in the next section.

THEOREM 3.9. *Assume M is a compact Kähler manifold or a compact complex algebraic manifold, with (L, F, W) a graded polarizable variation of mixed (or pure) Hodge structures on M. Then $H^n(M, L) \simeq H^n(M, DR(\mathcal{L}))$ gets an induced mixed (or pure) Hodge structure with F the Hodge filtration. Moreover, the corresponding Hodge to de Rham spectral sequence degenerates at E_1 so that*

$$Gr^p_F(H^n(M, L)) \simeq H^n(M, Gr^p_F DR(\mathcal{L})) \quad \text{for all } n, p.$$

Therefore one gets as a corollary (compare [12; 13; 35]):

$$
\begin{aligned}
\chi_y(H^*(M,L)) &= \sum_{n,p} (-1)^n \cdot \dim_{\mathbb{C}} \left(H^n(M, Gr_F^p DR(\mathcal{L}))\right) \cdot (-y)^p \\
&= \sum_p \chi\left(H^*(M, Gr_F^p DR(\mathcal{L}))\right) \cdot (-y)^p \\
&= \sum_{p,i} (-1)^i \cdot \chi\left(H^*(M, Gr_F^{p-i}(\mathcal{L}) \otimes_{\mathcal{O}_M} \Omega_M^i)\right) \cdot (-y)^p \\
&= k_*\left(MHC^y(L) \otimes \lambda_y(T^*M)\right) \\
&=: \langle MHC^y(L), \lambda_y(T^*M) \cap [\mathcal{O}_M]\rangle \in \mathbb{Z}[y^{\pm 1}].
\end{aligned}
\tag{3-9}
$$

3D. Good variation of mixed Hodge structures.

DEFINITION 3.10 (GOOD VARIATION). Let M be a complex algebraic manifold. A graded polarizable variation of mixed Hodge structures (L, F, W) on M is called good if it is *admissible* in the sense of Steenbrink and Zucker [50] and Kashiwara [28], with *quasi-unipotent monodromy* at infinity, i.e., with respect to a compactification $\overline{M}$ of M by a compact complex algebraic manifold $\overline{M}$, with complement $D := \overline{M} \setminus M$ a normal crossing divisor with smooth irreducible components.

EXAMPLE 3.11 (PURE AND GEOMETRIC VARIATIONS). Two important examples for such a good variation of mixed Hodge structures are the following:

(i) A polarizable variation of *pure* Hodge structures is always admissible by a deep theorem of Schmid [46, Theorem 6.16]. So it is good precisely when it has quasi-unipotent monodromy at infinity.

(ii) Consider a morphism $f : X \to Y$ of complex algebraic varieties with Y smooth, which is a topological fibration with possible singular or noncompact fiber. The locally constant direct image sheaves $R^n f_* \mathbb{Q}_X$ and $R^n f_! \mathbb{Q}_X$ ($n \geq 0$) are good variations of mixed Hodge structures (compare Remark 4.4).

This class of good variations on M is again an abelian category $VmHs^g(M)$ stable under tensor product $\otimes$, duality $(\cdot)^\vee$ and pullback f^* for f an algebraic morphism of complex algebraic manifolds. Moreover, in this case all vector bundles $F^p \mathcal{L}$ of the Hodge filtration carry the structure of a unique underlying complex algebraic vector bundle (in the Zariski topology), so that the characteristic class transformation MHC^y can be seen as a natural contravariant transformation of rings with involution

$$
MHC^y : K_0(VmHs^g(M)) \to K^0_{\mathrm{alg}}(M)[y^{\pm 1}].
$$

In fact, consider a (partial) compactification $\overline{M}$ of M as above, with $D :=$ $\overline{M} \setminus M$ a normal crossing divisor with smooth irreducible components and $j :$ $M \to \overline{M}$ the open inclusion. Then the holomorphic vector bundle $\mathcal{L}$ with integrable connection ∇ corresponding to L has a unique *canonical Deligne extension* $(\overline{\mathcal{L}}, \overline{\nabla})$ to a holomorphic vector bundle $\overline{\mathcal{L}}$ on $\overline{M}$, with *meromorphic* integrable connection

$$\overline{\nabla} : \overline{\mathcal{L}} \to \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \Omega^1_{\overline{M}}(\log(D)) \tag{3-10}$$

having *logarithmic poles* along D. Here the *residues* of $\overline{\nabla}$ along D have real eigenvalues, since L has *quasi-unipotent monodromy* along D. And the canonical extension is characterized by the property that all these eigenvalues are in the half-open interval $[0, 1)$. Moreover, also the Hodge filtration F of $\mathcal{L}$ extends uniquely to a filtration F of $\overline{\mathcal{L}}$ by holomorphic subvector bundles

$$F^p \overline{\mathcal{L}} := j_*(F^p \mathcal{L}) \cap \overline{\mathcal{L}} \subset j_* \mathcal{L},$$

since L is *admissible* along D. Finally, Griffiths transversality extends to

$$\overline{\nabla}(F^p \overline{\mathcal{L}}) \subset F^{p-1} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \Omega^1_{\overline{M}}(\log(D)) \quad \text{for all } p. \tag{3-11}$$

For more details see [19, Proposition 5.4] and [38, § 11.1, 14.4].

If we choose $\overline{M}$ as a compact algebraic manifold, then we can apply Serre's GAGA theorem to conclude that $\overline{\mathcal{L}}$ and all $F^p \overline{\mathcal{L}}$ are *algebraic* vector bundles, with $\overline{\nabla}$ an *algebraic* meromorphic connection.

REMARK 3.12. The canonical Deligne extension $\overline{\mathcal{L}}$ (as above) with its Hodge filtration F has the following compabilities (compare [19, Part II]):

SMOOTH PULLBACK: Let $f : \overline{M}' \to \overline{M}$ be a smooth morphism so that $D' :=$ $f^{-1}(D)$ is also a normal crossing divisor with smooth irreducible components on $\overline{M}'$ with complement M'. Then one has

$$f^*(\overline{\mathcal{L}}) \simeq \overline{f^* \mathcal{L}} \quad \text{and} \quad f^*(F^p \overline{\mathcal{L}}) \simeq F^p \overline{f^* \mathcal{L}} \text{ for all } p. \tag{3-12}$$

EXTERIOR PRODUCT: Let L and L' be two good variations on M and M'. Then their canonical Deligne extensions satisfy

$$\overline{\mathcal{L} \boxtimes_{\mathcal{O}_{M \times M'}} \mathcal{L}'} \simeq \overline{\mathcal{L}} \boxtimes_{\mathcal{O}_{\overline{M} \times \overline{M}'}} \overline{\mathcal{L}'},$$

since the residues of the corresponding meromorphic connections are compatible. Then one has for all p

$$F^p(\overline{\mathcal{L} \boxtimes_{\mathcal{O}_{M \times M'}} \mathcal{L}'}) \simeq \bigoplus_{i+k=p} (F^i \overline{\mathcal{L}}) \boxtimes_{\mathcal{O}_{\overline{M} \times \overline{M}'}} (F^k \overline{\mathcal{L}'}). \tag{3-13}$$

TENSOR PRODUCT: In general the canonical Deligne extensions of two good variations L and L' on M are *not* compatible with tensor products, because of the choice of different residues for the corresponding meromorphic connections. This problem doesn't appear if one of these variations, lets say L', is already defined on $\overline{M}$. Let L and L' be a good variation on M and $\overline{M}$, respectively. Then their canonical Deligne extensions satisfy

$$\overline{\mathcal{L} \otimes_{\mathcal{O}_M} (\mathcal{L}'|M)} \simeq \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \mathcal{L}',$$

and one has for all p:

$$F^p(\overline{\mathcal{L} \otimes_{\mathcal{O}_M} (\mathcal{L}'|M)}) \simeq \bigoplus_{i+k=p} (F^i \overline{\mathcal{L}}) \otimes_{\mathcal{O}_{\overline{M}}} (F^k \mathcal{L}'). \qquad (3\text{-}14)$$

Let $\overline{M}$ be a partial compactification of M as before, i.e., we don't assume that $\overline{M}$ is compact, with $m := \dim_{\mathbb{C}}(M)$. Then the *logarithmic de Rham complex*

$$DR_{\log}(\overline{\mathcal{L}}) := [\overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \Omega^m_{\overline{M}}(\log(D))]$$

(with $\overline{\mathcal{L}}$ in degree zero) is by [19] quasi-isomorphic to Rj_*L, so that

$$H^*(M, L) \simeq H^*\big(\overline{M}, DR_{\log}(\overline{\mathcal{L}})\big).$$

So these cohomology groups get an induced (decreasing) F-filtration coming from the filtration

$$F^p DR_{\log}(\overline{\mathcal{L}}) = [F^p \overline{\mathcal{L}} \xrightarrow{\overline{\nabla}} \cdots \xrightarrow{\overline{\nabla}} F^{p-m} \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \Omega^m_{\overline{M}}(\log(D))].$$
$$(3\text{-}15)$$

For $\overline{M}$ a compact algebraic manifold, this is again the Hodge filtration of an induced mixed Hodge structure on $H^*(M, L)$ (compare with Corollary 4.7).

THEOREM 3.13. *Assume $\overline{M}$ is a smooth algebraic compactification of the algebraic manifold M with the complement D a normal crossing divisor with smooth irreducible components. Let (L, F, W) be a good variation of mixed Hodge structures on M. Then $H^n(M, L) \simeq H^*\big(\overline{M}, DR_{\log}(\overline{\mathcal{L}})\big)$ gets an induced mixed Hodge structure with F the Hodge filtration. Moreover, the corresponding Hodge to de Rham spectral sequence degenerates at E_1 so that*

$$Gr_F^p(H^n(M, L)) \simeq H^n\big(M, Gr_F^p DR_{\log}(\overline{\mathcal{L}})\big) \quad \textit{for all } n, p.$$

Therefore one gets as a corollary (compare [12; 13; 35]):

$$\chi_y(H^*(M, L)) = \sum_{n,p}(-1)^n \cdot \dim_{\mathbb{C}}\left(H^n\left(M, Gr_F^p DR_{\log}(\overline{\mathcal{L}})\right)\right) \cdot (-y)^p$$

$$= \sum_p \chi\left(H^*\left(M, Gr_F^p DR_{\log}(\overline{\mathcal{L}})\right)\right) \cdot (-y)^p$$

$$= \sum_{p,i}(-1)^i \chi\left(H^*\left(M, Gr_F^{p-i}(\overline{\mathcal{L}}) \otimes_{\mathcal{O}_{\overline{M}}} \Omega_{\overline{M}}^i(\log(D))\right)\right)(-y)^p$$

$$=: \langle MHC^y(Rj_*L), \lambda_y\left(\Omega_{\overline{M}}^1(\log(D))\right) \cap [\mathcal{O}_{\overline{M}}]\rangle \in \mathbb{Z}[y^{\pm 1}].$$
(3-16)

Here we use the notation

$$MHC^y(Rj_*L) := \sum_p [Gr_F^p(\overline{\mathcal{L}})] \cdot (-y)^p \in K_{\mathrm{alg}}^0(\overline{M})[y^{\pm 1}]. \qquad (3\text{-}17)$$

Remark 3.12 then implies:

COROLLARY 3.14. *Let $\overline{M}$ be a smooth algebraic partial compactifiction of the algebraic manifold M with the complement D a normal crossing divisor with smooth irreducible components. Then $MHC^y(Rj_*(\cdot))$ induces a transformation*

$$MHC^y(j_*(\cdot)) : K_0(VmHs^g(M)) \to K_{\mathrm{alg}}^0(\overline{M})[y^{\pm 1}].$$

(1) *This is contravariant functorial for a smooth morphism $f : \overline{M}' \to \overline{M}$ of such partial compactifications, i.e.,*

$$f^*\left(MHC^y(j_*(\cdot))\right) \simeq MHC^y\left(j_*'(f^*(\cdot))\right).$$

(2) *It commutes with exterior products for two good variations L, L':*

$$MHC^y\left((j \times j')_*[(L \boxtimes_{\mathbb{Q}_{M \times M'}} L']\right) = MHC^y(j_*[L]) \boxtimes MHC^y(j_*'[(L']).$$

(3) *Let L be a good variation on M, and L' one on $\overline{M}$. Then $MHC^y(j_*[\cdot])$ is multiplicative in the sense that*

$$MHC^y\left(j_*[(L \otimes_{\mathbb{Q}_M}(L'|M)]\right) = MHC^y(j_*[L]) \otimes MHC^y([L']).$$

4. Calculus of mixed Hodge modules

4A. Mixed Hodge modules. Before discussing extensions of the characteristic cohomology classes MHC^y to the singular setting, we need to briefly recall some aspects of Saito's theory [39; 40; 41; 43; 44] of algebraic mixed Hodge modules, which play the role of singular extensions of good variations of mixed Hodge structures.

To each complex algebraic variety Z, Saito associated a category $MHM(Z)$ of *algebraic mixed Hodge modules* on Z (cf. [39; 40]). If Z is smooth, an object of this category consists of an algebraic (regular) holonomic D-module $(\mathcal{M}, F)$ with a good filtration F together with a perverse sheaf K of rational vector spaces, both endowed a finite increasing filtration W such that

$$\alpha : DR(\mathcal{M})^{\mathrm{an}} \simeq K \otimes_{\mathbb{Q}_Z} \mathbb{C}_Z \quad \text{is compatible with } W$$

under the Riemann–Hilbert correspondence coming from the (shifted) analytic de Rham complex (with α a chosen isomorphism). Here we use left D-modules, and the sheaf $\mathcal{D}_Z$ of algebraic differential operators on Z has the increasing filtration F with $F_i \mathcal{D}_Z$ given by the differential operators of order $\leq i$ ($i \in \mathbb{Z}$). Then a *good* filtration F of the algebraic holonomic D-module $\mathcal{M}$ is given by a bounded from below, increasing and exhaustive filtration $F_p \mathcal{M}$ by *coherent* algebraic $\mathcal{O}_Z$-modules such that

$$\begin{aligned} &F_i \mathcal{D}_Z (F_p \mathcal{M}) \subset F_{p+i} \mathcal{M} \quad \text{for all } i, p, \\ &\text{and this is an equality for } i \text{ big enough.} \end{aligned} \tag{4-1}$$

In general, for a singular variety Z one works with suitable local embeddings into manifolds and corresponding filtered D-modules supported on Z. In addition, these objects are required to satisfy a long list of complicated properties (not needed here). The *forgetful* functor rat is defined as

$$\mathrm{rat} : MHM(Z) \to \mathrm{Perv}(\mathbb{Q}_Z), \quad (\mathcal{M}(F), K, W) \mapsto K.$$

THEOREM 4.1 (M. SAITO). *$MHM(Z)$ is an abelian category with*

$$\mathrm{rat} : MHM(Z) \to \mathrm{Perv}(\mathbb{Q}_Z)$$

exact and faithful. It extends to a functor

$$\mathrm{rat} : D^b MHM(Z) \to D^b_c(\mathbb{Q}_Z)$$

to the derived category of complexes of $\mathbb{Q}$-sheaves with algebraically constructible cohomology. There are functors

$$f_*, \ f_!, \ f^*, \ f^!, \ \otimes, \ \boxtimes, \ \mathcal{D} \quad on \ D^b MHM(Z),$$

which are "lifts" via rat of the similar (derived) functors defined on $D^b_c(\mathbb{Q}_Z)$, with (f^*, f_*) and $(f_!, f^!)$ also pairs of adjoint functors. One has a natural map $f_! \to f_*$, which is an isomorphism for f proper. Here $\mathcal{D}$ is a duality involution $\mathcal{D}^2 \simeq \mathrm{id}$ "lifting" the Verdier duality functor, with

$$\mathcal{D} \circ f^* \simeq f^! \circ \mathcal{D} \quad and \quad \mathcal{D} \circ f_* \simeq f_! \circ \mathcal{D}.$$

Compare with [40, Theorem 0.1 and §4] for more details (as well as with [43] for a more general formal abstraction). The usual truncation $\tau_{\le}$ on $D^b MHM(Z)$ corresponds to the *perverse truncation* $^p\tau_{\le}$ on $D_c^b(Z)$ so that

$$\text{rat} \circ H = {}^p\mathcal{H} \circ \text{rat},$$

where H stands for the cohomological functor in $D^b MHM(Z)$ and $^p\mathcal{H}$ denotes the perverse cohomology (always with respect to the self-dual middle perversity).

EXAMPLE 4.2. Let M be a complex algebraic manifold of pure complex dimension m, with (L, F, W) a good variation of mixed Hodge structures on M. Then $\mathcal{L}$ with its integrable connection ∇ is a holonomic (left) D-module with $\alpha : DR(\mathcal{L})^{\text{an}} \simeq L[m]$, where this time we use the shifted de Rham complex

$$DR(\mathcal{L}) := [\mathcal{L} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{L} \otimes_{\mathcal{O}_M} \Omega_M^m]$$

with $\mathcal{L}$ in degree $-m$, so that $DR(\mathcal{L})^{\text{an}} \simeq L[m]$ is a perverse sheaf on M. The filtration F induces by Griffiths transversality (3-2) a good filtration $F_p(\mathcal{L}) := F^{-p}\mathcal{L}$ as a filtered D-module. As explained before, this comes from an underlying algebraic filtered D-module. Finally α is compatible with the induced filtration W defined by

$$W^i(L[m]) := W^{i-m}L[m] \quad \text{and} \quad W^i(\mathcal{L}) := (W^{i-m}L) \otimes_{\mathbb{Q}_M} \mathcal{O}_M.$$

And this defines a mixed Hodge module $\mathcal{M}$ on M, with $\text{rat}(\mathcal{M})[-m]$ a local system on M.

A mixed Hodge module $\mathcal{M}$ on the pure m-dimensional complex algebraic manifold M is called *smooth* if $\text{rat}(\mathcal{M})[-m]$ is a local system on M. Then this example corresponds to [40, Theorem 0.2], whereas the next theorem corresponds to [40, Theorem 3.27 and remark on p. 313]:

THEOREM 4.3 (M. SAITO). *Let M be a pure m-dimensional complex algebraic manifold. Associating to a good variation of mixed Hodge structures $\mathbb{V} = (L, F, W)$ on M the mixed Hodge module $\mathcal{M} := \mathbb{V}_H$ as in Example* 4.2 *defines an equivalence of categories*

$$MHM(M)_{sm} \simeq VmHs^g(M)$$

between the categories of smooth mixed Hodge modules $MHM(M)_{sm}$ and good variation of mixed Hodge structures on M. This commutes with exterior product $\boxtimes$ and with the pullbacks

$$f^* : VmHs^g(M) \to VmHs^g(M') \quad \text{and} \quad f^*[m'-m] : MHM(M) \to MHM(M')$$

for an algebraic morphism of smooth algebraic manifolds M, M' of dimension m, m'. For $M = pt$ a point, one gets in particular an equivalence

$$MHM(pt) \simeq mHs^p.$$

REMARK 4.4. These two theorems explain why a geometric variation of mixed Hodge structures as in Example 3.11(2) is good.

By the last identification of the theorem, there exists a unique Tate object

$$\mathbb{Q}^H(n) \in MHM(pt)$$

such that $\text{rat}(\mathbb{Q}^H(n)) = \mathbb{Q}(n)$ and $\mathbb{Q}^H(n)$ is of type $(-n, -n)$:

$$MHM(pt) \ni \mathbb{Q}^H(n) \simeq \mathbb{Q}(n) \in mHs^p.$$

For a complex variety Z with constant map $k : Z \to pt$, define

$$\mathbb{Q}_Z^H(n) := k^* \mathbb{Q}^H(n) \in D^b MHM(Z), \quad \text{with } \text{rat}(\mathbb{Q}_Z^H(n)) = \mathbb{Q}_Z(n).$$

So tensoring with $\mathbb{Q}_Z^H(n)$ defines the Tate twist $\cdot(n)$ of mixed Hodge modules. To simplify the notation, let $\mathbb{Q}_Z^H := \mathbb{Q}_Z^H(0)$. If Z is *smooth* of complex dimension n then $\mathbb{Q}_Z[n]$ is perverse on Z, and $\mathbb{Q}_Z^H[n] \in MHM(Z)$ is a single mixed Hodge module, explicitly described by

$$\mathbb{Q}_Z^H[n] = ((\mathcal{O}_Z, F), \mathbb{Q}_Z[n], W), \quad \text{with } gr_i^F = 0 = gr_{i+n}^W \text{ for all } i \neq 0.$$

It follows from the definition that every $\mathcal{M} \in MHM(Z)$ has a finite increasing *weight filtration* W so that the functor $M \to Gr_k^W M$ is exact. We say that $\mathcal{M} \in D^b MHM(Z)$ has *weights* $\leq n$ (*resp.* $\geq n$) if $Gr_j^W H^i M = 0$ for all $j > n+i$ (resp. $j < n + i$). $\mathcal{M}$ is called *pure of weight n* if it has weights both $\leq n$ and $\geq n$. For the following results compare with [40, Proposition 2.26 and (4.5.2)]:

PROPOSITION 4.5. *If f is a map of algebraic varieties, then $f_!$ and f^* preserve weight $\leq n$, and f_* and $f^!$ preserve weight $\geq n$. If f is smooth of pure complex fiber dimension m, then $f^! \simeq f^*[2m](m)$ so that f^*, $f^!$ preserve pure objects for f smooth. Moreover, if $\mathcal{M} \in D^b MHM(X)$ is pure and $f : X \to Y$ is proper, then $f_* \mathcal{M} \in D^b MHM(Y)$ is pure of the same weight as $\mathcal{M}$.*

Similarly the duality functor $\mathcal{D}$ exchanges "weight $\leq n$" and "weight $\geq -n$", in particular it preserves pure objects. Finally let $j : U \to Z$ be the inclusion of a Zariski open subset. Then the intermediate extension *functor*

$$j_{!*} : MHM(U) \to MHM(Z) : \mathcal{M} \mapsto \text{Im}\big(H^0(j_! \mathcal{M}) \to H^0(j_*(\mathcal{M}))\big) \quad (4\text{-}2)$$

preserves weight $\leq n$ and $\geq n$, and so preserves pure objects (of weight n).

We say that $\mathcal{M} \in D^b MHM(Z)$ is supported on $S \subset Z$ if and only if $\mathrm{rat}(\mathcal{M})$ is supported on S. There are the abelian subcategories $MH(Z,k)^p \subset MHM(Z)$ of pure Hodge modules of weight k, which in the algebraic context are assumed to be polarizable (and extendable at infinity).

For each $k \in \mathbb{Z}$, the abelian category $MH(Z,k)^p$ is semisimple, in the sense that every pure Hodge module on Z can be uniquely written as a finite direct sum of pure Hodge modules with strict support in irreducible closed subvarieties of Z. Let $MH_S(Z,k)^p$ denote the subcategory of *pure Hodge modules of weight k with strict support in S*. Then every $\mathcal{M} \in MH_S(Z,k)^p$ is generically a good variation of Hodge structures $\mathbb{V}_U$ of weight $k - d$ (where $d = \dim S$) on a Zariski dense smooth open subset $U \subset S$; i.e., $\mathbb{V}_U$ is polarizable with quasi-unipotent monodromy at infinity. This follows from Theorem 4.3 and the fact that a perverse sheaf is generically a shifted local system on a smooth dense Zariski open subset $U \subset S$. Conversely, every such good variation of Hodge structures $\mathbb{V}$ on such an U corresponds by Theorem 4.3 to a pure Hodge module $\mathbb{V}_H$ on U, which can be extended in an unique way to a pure Hodge module $j_{!*}\mathbb{V}_H$ on S with strict support (here $j : U \to S$ is the inclusion). Under this correspondence, for $M \in MH_S(Z,k)^p$ we have that

$$\mathrm{rat}(\mathcal{M}) = IC_S(\mathbb{V})$$

is the *twisted intersection cohomology complex* for $\mathbb{V}$ the corresponding variation of Hodge structures. Similarly

$$\mathcal{D}(j_{!*}\mathbb{V}_H) \simeq j_{!*}(\mathbb{V}_H^\vee)(d). \tag{4-3}$$

Moreover, a *polarization* of $\mathcal{M} \in MH_S(Z,k)^p$ corresponds to an isomorphism of Hodge modules (compare [38, Definition 14.35, Remark 14.36])

$$S : \mathcal{M} \simeq \mathcal{D}(\mathcal{M})(-k), \tag{4-4}$$

whose restriction to U gives a polarization of $\mathbb{V}$. In particular it induces a self-duality isomorphism

$$S : \mathrm{rat}(\mathcal{M}) \simeq \mathcal{D}(\mathrm{rat}(\mathcal{M}))(-k) \simeq \mathcal{D}(\mathrm{rat}(\mathcal{M}))$$

of the underlying twisted intersection cohomology complex, if an isomorphism $\mathbb{Q}_U(-k) \simeq \mathbb{Q}_U$ is chosen.

So if U is smooth of pure complex dimension n, then $\mathbb{Q}_U^H[n]$ is a pure Hodge module of weight n. If moreover $j : U \hookrightarrow Z$ is a Zariski-open dense subset in Z, then the *intermediate extension* $j_{!*}$ for mixed Hodge modules (cf. also with [7]) preserves the weights. This shows that if Z is a complex algebraic variety of pure dimension n and $j : U \hookrightarrow Z$ is the inclusion of a smooth Zariski-open dense subset then the intersection cohomology module $IC_Z^H := j_{!*}(\mathbb{Q}_U^H[n])$ is pure of weight n, with underlying perverse sheaf $\mathrm{rat}(IC_Z^H) = IC_Z$.

Note that the stability of a pure object $\mathcal{M} \in MHM(X)$ under a proper morphism $f : X \to Y$ implies the famous *decomposition theorem* of [7] in the context of pure Hodge modules [40, (4.5.4) on p. 324]:

$$f_* \mathcal{M} \simeq \bigoplus_i H^i f_* \mathcal{M}[-i], \quad \text{with } H^i f_* \mathcal{M} \text{ semisimple for all } i. \qquad \text{(4-5)}$$

Assume Y is pure-dimensional, with $f : X \to Y$ a *resolution of singularities*, i.e., X is smooth with f a proper morphism, which generically is an isomorphism on some Zariski dense open subset U. Then $\mathbb{Q}_X^H$ is pure, since X is smooth, and IC_Y^H has to be the direct summand of $H^0 f_* \mathbb{Q}_X^H$ which corresponds to $\mathbb{Q}_U^H$.

COROLLARY 4.6. *Assume Y is pure-dimensional, with $f : X \to Y$ a* resolution *of singularities. Then IC_Y^H is a direct summand of $f_* \mathbb{Q}_X^H \in D^b MHM(Y)$.*

Finally we get the following results about the existence of a mixed Hodge structure on the cohomology (with compact support) $H_{(c)}^i(Z, \mathcal{M})$ for $\mathcal{M} \in D^b MHM(Z)$.

COROLLARY 4.7. *Let Z be a complex algebraic variety with constant map $k : Z \to pt$. Then the cohomology (with compact support) $H_{(c)}^i(Z, \mathcal{M})$ of $\mathcal{M} \in D^b MHM(Z)$ gets an induced graded polarizable mixed Hodge structure*:

$$H_{(c)}^i(Z, \mathcal{M}) = H^i(k_{*(!)} \mathcal{M}) \in MHM(pt) \simeq mHs^p.$$

In particular:

(1) *The rational cohomology (with compact support) $H_{(c)}^i(Z, \mathbb{Q})$ of Z gets an induced graded polarizable mixed Hodge structure by*

$$H^i(Z, \mathbb{Q}) = \text{rat}(H^i(k_* k^* \mathbb{Q}^H)) \quad \text{and} \quad H_c^i(Z, \mathbb{Q}) = \text{rat}(H^i(k_! k^* \mathbb{Q}^H)).$$

(2) *Let $\mathbb{V}_U$ be a good variation of mixed Hodge structures on a smooth pure n-dimensional complex variety U, which is Zariski open and dense in a variety Z, with $j : U \to Z$ the open inclusion. Then the global twisted intersection cohomology (with compact support)*

$$IH_{(c)}^i(Z, \mathbb{V}) := H_{(c)}^i\big(Z, IC_Z(\mathbb{V})[-n]\big)$$

gets a mixed Hodge structure by

$$IH_{(c)}^i(Z, \mathbb{V}) = H^i\big(k_{*(!)} IC_Z(\mathbb{V})[-n]\big) = H^i\big(k_{*(!)} j_{!*}(\mathbb{V})[-n]\big).$$

If Z is compact, with $\mathbb{V}$ a polarizable variation of pure Hodge structures of weight w, then also $IH^i(Z, \mathbb{V})$ has a (polarizable) pure Hodge structure of weight $w + i$.

(3) *Let $\mathbb{V}$ be a good variation of mixed Hodge structures on a smooth (pure-dimensional) complex manifold M, which is Zariski open and dense in complex algebraic manifold $\overline{M}$, with complement D a normal crossing divisor with smooth irreducible components. Then $H^i(M, \mathbb{V})$ gets a mixed Hodge structure by*

$$H^i(M, \mathbb{V}) \simeq H^i(\overline{M}, j_*\mathbb{V}) \simeq H^i(k_*j_*\mathbb{V}),$$

with $j : U \to Z$ the open inclusion.

REMARK 4.8. Here are important properties of these mixed Hodge structures:

(1) By a deep theorem of Saito [44, Theorem 0.2, Corollary 4.3], the mixed Hodge structure on $H^i_{(c)}(Z, \mathbb{Q})$ defined as above coincides with the classical mixed Hodge structure constructed by Deligne ([20; 21]).

(2) Assume we are in the context of Corollary 4.7(3) with $Z = \overline{M}$ projective and $\mathbb{V}$ a good variation of pure Hodge structures on $U = M$. Then the pure Hodge structure of (2) on the global intersection cohomology $IH^i(Z, \mathbb{V})$ agrees with that of [15; 29] defined in terms of L^2-cohomology with respect to a Kähler metric with Poincaré singularities along D (compare [40, Remark 3.15]). The case of a 1-dimensional complex algebraic curve $Z = \overline{M}$ due to Zucker [56, Theorem 7.12] is used in the work of Saito [39, (5.3.8.2)] in the proof of the stability of pure Hodge modules under projective morphisms [39, Theorem 5.3.1] (compare also with the detailed discussion of this 1-dimensional case in [45]).

(3) Assume we are in the context of Corollary 4.7(3) with $\overline{M}$ compact. Then the mixed Hodge structure on $H^i(M, \mathbb{V})$ is the one of Theorem 3.13, whose Hodge filtration F comes from the filtered logarithmic de Rham complex (compare [40, § 3.10, Proposition 3.11]).

4B. Grothendieck groups of algebraic mixed Hodge modules. In this section, we describe the functorial calculus of Grothendieck groups of algebraic mixed Hodge modules. Let Z be a complex algebraic variety. By associating to (the class of) a complex the alternating sum of (the classes of) its cohomology objects, we obtain the following identification (compare, for example, [30, p. 77] and [47, Lemma 3.3.1])

$$K_0(D^b MHM(Z)) = K_0(MHM(Z)). \tag{4-6}$$

In particular, if Z is a point, then

$$K_0(D^b MHM(pt)) = K_0(mHs^p), \tag{4-7}$$

and the latter is a commutative ring with respect to the tensor product, with unit $[\mathbb{Q}^H]$. Then we have, for any complex $\mathcal{M}^\bullet \in D^b MHM(Z)$, the identification

$$[\mathcal{M}^\bullet] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(\mathcal{M}^\bullet)] \in K_0(D^b MHM(Z)) \cong K_0(MHM(Z)). \quad (4\text{-}8)$$

In particular, if for any $\mathcal{M} \in MHM(Z)$ and $k \in \mathbb{Z}$ we regard $\mathcal{M}[-k]$ as a complex concentrated in degree k, then

$$[\mathcal{M}[-k]] = (-1)^k [\mathcal{M}] \in K_0(MHM(Z)). \quad (4\text{-}9)$$

All the functors f_*, $f_!$, f^*, $f^!$, $\otimes$, $\boxtimes$, $\mathcal{D}$ induce corresponding functors on $K_0(MHM(\cdot))$. Moreover, $K_0(MHM(Z))$ becomes a $K_0(MHM(pt))$-module, with the multiplication induced by the exact exterior product with a point space:

$$\boxtimes : MHM(Z) \times MHM(pt) \to MHM(Z \times \{pt\}) \simeq MHM(Z).$$

Also note that

$$\mathcal{M} \otimes \mathbb{Q}_Z^H \simeq \mathcal{M} \boxtimes \mathbb{Q}_{pt}^H \simeq \mathcal{M}$$

for all $\mathcal{M} \in MHM(Z)$. Therefore, $K_0(MHM(Z))$ is a unitary $K_0(MHM(pt))$-module. The functors f_*, $f_!$, f^*, $f^!$ commute with exterior products (and f^* also commutes with the tensor product $\otimes$), so that the induced maps at the level of Grothendieck groups $K_0(MHM(\cdot))$ are $K_0(MHM(pt))$-linear. Similarly $\mathcal{D}$ defines an involution on $K_0(MHM(\cdot))$. Moreover, by the functor

$$\mathrm{rat} : K_0(MHM(Z)) \to K_0(D_c^b(\mathbb{Q}_Z)) \simeq K_0(\mathrm{Perv}(\mathbb{Q}_Z)),$$

all these transformations lift the corresponding transformations from the (topological) level of Grothendieck groups of constructible (or perverse) sheaves.

REMARK 4.9. The Grothendieck group $K_0(MHM(Z))$ has two different types of generators:

(1) It is generated by the classes of pure Hodge modules $[IC_S(\mathbb{V})]$ with strict support in an irreducible complex algebraic subset $S \subset Z$, with $\mathbb{V}$ a good variation of (pure) Hodge structures on a dense Zariski open smooth subset U of S. These generators behave well under duality.

(2) It is generated by the classes $f_*[j_* \mathbb{V}]$, with $f : \overline{M} \to Z$ a proper morphism from the smooth complex algebraic manifold $\overline{M}$, $j : M \to \overline{M}$ the inclusion of a Zariski open and dense subset M, with complement D a normal crossing divisor with smooth irreducible components, and $\mathbb{V}$ a good variation of mixed (or if one wants also pure) Hodge structures on M. These generators will be used in the next section about characteristic classes of mixed Hodge modules.

Here (1) follows from the fact that a mixed Hodge module has a finite weight filtration, whose graded pieces are pure Hodge modules, i.e., are finite direct sums of pure Hodge modules $IC_S(\mathbb{V})$ with strict support S as above. The claim in (2) follows by induction from resolution of singularities and from the existence of a "standard" distinguished triangle associated to a closed inclusion.

Let $i : Y \to Z$ be a closed inclusion of complex algebraic varieties with open complement $j : U = Z \backslash Y \to Z$. Then one has by Saito's work [40, (4.4.1)] the following functorial distinguished triangle in $D^b MHM(Z)$:

$$j_! j^* \xrightarrow{\ ad_j\ } \mathrm{id} \xrightarrow{\ ad_i\ } i_* i^* \xrightarrow{\ [1]\ } . \tag{4-10}$$

Here the maps ad are the adjunction maps, with $i_* = i_!$ since i is proper. If $f : Z \to X$ is a complex algebraic morphism, then we can apply $f_!$ to get another distinguished triangle

$$f_! j_! j^* \mathbb{Q}_Z^H \xrightarrow{\ ad_j\ } f_! \mathbb{Q}_Z^H \xrightarrow{\ ad_i\ } f_! i_! i^* \mathbb{Q}_Z^H \xrightarrow{\ [1]\ } . \tag{4-11}$$

On the level of Grothendieck groups, we get the important *additivity relation*

$$f_! [\mathbb{Q}_Z^H] = (f \circ j)_! [\mathbb{Q}_U^H] + (f \circ i)_! [\mathbb{Q}_Y^H]$$
$$\in K_0(D^b MHM(X)) = K_0(MHM(X)). \tag{4-12}$$

COROLLARY 4.10. *One has a natural group homomorphism*

$$\chi_{\mathrm{Hdg}} : K_0(\mathrm{var}\,/X) \to K_0(MHM(X)); [f : Z \to X] \mapsto [f_! \mathbb{Q}_Z^H],$$

which commutes with pushdown $f_!$, exterior product $\boxtimes$ and pullback g^*. For $X = pt$ this corresponds to the ring homomorphism (2-10) *under the identification $MHM(pt) \simeq mHs^p$.*

Here $K_0(\mathrm{var}\,/X)$ is the motivic *relative Grothendieck group* of complex algebraic varieties over X, i.e., the free abelian group generated by isomorphism classes $[f] = [f : Z \to X]$ of morphisms f to X, divided out be the *additivity relation*

$$[f] = [f \circ i] + [f \circ j]$$

for a closed inclusion $i : Y \to Z$ with open complement $j : U = Z \backslash Y \to Z$. The pushdown $f_!$, exterior product $\boxtimes$ and pullback g^* for these relative Grothendieck groups are defined by composition, exterior product and pullback of arrows. The fact that χ_{Hdg} commutes with exterior product $\boxtimes$ (or pullback g^*) follows then from the corresponding Künneth (or base change) theorem for the functor

$$f_! : D^b MHM(Z) \to D^b MHM(X)$$

(contained in Saito's work [43] and [40, (4.4.3)]).

Let $\mathbb{L} := [\mathbb{A}^1_{\mathbb{C}}] \in K_0(\text{var}/pt)$ be the class of the affine line, so that

$$\chi_{\text{Hdg}}(\mathbb{L}) = [H^2(P^1(\mathbb{C}), \mathbb{Q})] = [\mathbb{Q}(-1)] \in K_0(MHM(pt)) = K_0(mHs^p)$$

is the Lefschetz class $[\mathbb{Q}(-1)]$. This class is invertible in $K_0(MHM(pt)) = K_0(mHs^p)$ so that the transformation χ_{Hdg} of Corollary 4.10 factorizes over the localization

$$M_0(\text{var}/X) := K_0(\text{var}/X)[\mathbb{L}^{-1}].$$

Altogether we get the following diagram of natural transformations commuting with $f_!$, $\boxtimes$ and g^*:

$$
\begin{array}{ccccc}
F(X) & \xleftarrow{\text{can}} & M_0(\text{var}/X) & \longleftarrow & K_0(\text{var}/X) \\
{\scriptstyle\chi_{\text{stalk}}}\Big\uparrow & & \Big\downarrow{\scriptstyle\chi_{\text{Hdg}}} & & \\
K_0(D^b_c(X)) & \underset{\text{rat}}{\longleftarrow} & K_0(MHM(X)). & &
\end{array}
\qquad (4\text{-}13)
$$

Here $F(X)$ is the group of algebraically constructible functions on X, which is generated by the collection $\{1_Z\}$, for $Z \subset X$ a closed complex algebraic subset, with χ_{stalk} given by the Euler characteristic of the stalk complexes (compare [47, §2.3]). The pushdown $f_!$ for algebraically constructible functions is defined for a morphism $f : Y \to X$ by

$$f_!(1_Z)(x) := \chi\left(H^*_c(Z \cap \{f = x\}, \mathbb{Q})\right) \quad \text{for } x \in X,$$

so that the horizontal arrow marked "can" is given by

$$\text{can} : [f : Y \to X] \mapsto f_!(1_Y), \quad \text{with can}(\mathbb{L}) = 1_{pt}.$$

The advantage of $M_0(\text{var}/X)$ compared to $K_0(\text{var}/X)$ is that it has an induced *duality* involution $\mathcal{D} : M_0(\text{var}/X) \to M_0(\text{var}/X)$ characterized uniquely by the equality

$$\mathcal{D}([f : M \to X]) = \mathbb{L}^{-m} \cdot [f : M \to X]$$

for $f : M \to X$ a proper morphism with M smooth and pure m-dimensional (compare [8]). This "motivic duality" $\mathcal{D}$ commutes with pushdown $f_!$ for proper f, so that χ_{Hdg} also commutes with duality by

$$\chi_{\text{Hdg}}\left(\mathcal{D}[\text{id}_M]\right) = \chi_{\text{Hdg}}\left(\mathbb{L}^{-m} \cdot [\text{id}_M]\right) = [\mathbb{Q}^H_M(m)]$$
$$= [\mathbb{Q}^H_M[2m](m)] = [\mathcal{D}(\mathbb{Q}^H_M)] = \mathcal{D}\left(\chi_{\text{Hdg}}([\text{id}_M])\right) \qquad (4\text{-}14)$$

for M smooth and pure m-dimensional. In fact by resolution of singularities and "additivity", $K_0(\text{var}/X)$ is generated by such classes $f_![\text{id}_M] = [f : M \to X]$.

Then all the transformations in the diagram (4-13) *commute with duality*, were $K_0(D^b_c(X))$ gets this involution from Verdier duality, and $\mathcal{D} = \text{id}$ for algebraically constructible functions by can $([\mathbb{Q}(-1)]) = 1_{pt}$ (compare also with

[47, §6.0.6]). Similarly they commute with f_* and $g^!$ defined by the relations (compare [8]):

$$\mathcal{D} \circ g^* = g^! \circ \mathcal{D} \quad \text{and} \quad \mathcal{D} \circ f_* = f_! \circ \mathcal{D}.$$

For example for an open inclusion $j : M \to \overline{M}$, one gets

$$\chi_{\text{Hdg}}(j_*[\text{id}_M]) = j_*[\mathbb{Q}_M^H]. \tag{4-15}$$

5. Characteristic classes of mixed Hodge modules

5A. Homological characteristic classes. In this section we explain the theory of K-theoretical characteristic homology classes of mixed Hodge modules based on the following result of Saito (compare with [39, §2.3] and [44, §1] for the first part, and with [40, §3.10, Proposition 3.11]) for part (2)):

THEOREM 5.1 (M. SAITO). *Let Z be a complex algebraic variety. Then there is a functor of triangulated categories*

$$Gr_p^F DR : D^b MHM(Z) \to D^b_{\text{coh}}(Z) \tag{5-1}$$

commuting with proper push-down, with $Gr_p^F DR(\mathcal{M}) = 0$ for almost all p and $\mathcal{M}$ fixed, where $D^b_{\text{coh}}(Z)$ is the bounded derived category of sheaves of algebraic $\mathcal{O}_Z$-modules with coherent cohomology sheaves. If M is a (pure m-dimensional) complex algebraic manifold, then one has in addition:

(1) *Let $\mathcal{M} \in MHM(M)$ be a single mixed Hodge module. Then $Gr_p^F DR(\mathcal{M})$ is the corresponding complex associated to the de Rham complex of the underlying algebraic left D-module $\mathcal{M}$ with its integrable connection ∇*:

$$DR(\mathcal{M}) = [\mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_M} \Omega_M^m]$$

with $\mathcal{M}$ in degree $-m$, filtered by

$$F_p DR(\mathcal{M}) = [F_p \mathcal{M} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} F_{p+m} \mathcal{M} \otimes_{\mathcal{O}_M} \Omega_M^m].$$

(2) *Let $\overline{M}$ be a smooth partial compactification of the complex algebraic manifold M with complement D a normal crossing divisor with smooth irreducible components, with $j : M \to \overline{M}$ the open inclusion. Let $\mathbb{V} = (L, F, W)$ be a good variation of mixed Hodge structures on M. Then the filtered de Rham complex*

$$(DR(j_*\mathbb{V}), F) \quad \text{of} \quad j_*\mathbb{V} \in MHM(\overline{M})[-m] \subset D^b MHM(\overline{M})$$

is filtered quasi-isomorphic to the logarithmic de Rham complex $DR_{\log}(\mathcal{L})$ with the increasing filtration $F_{-p} := F^p$ ($p \in \mathbb{Z}$) associated to the decreasing

F-filtration (3-15). *In particular* $Gr^F_{-p} DR(j_*\mathbb{V})$ $(p \in \mathbb{Z})$ *is quasi-isomorphic to*

$$Gr^p_F DR_{\log}(\overline{\mathcal{L}}) = [Gr^p_F \overline{\mathcal{L}} \xrightarrow{Gr\,\overline{\nabla}} \cdots \xrightarrow{Gr\,\overline{\nabla}} Gr^{p-m}_F \overline{\mathcal{L}} \otimes_{\mathcal{O}_{\overline{M}}} \Omega^m_{\overline{M}}(\log(D))].$$

Here the filtration $F_p DR(\mathcal{M})$ of the de Rham complex is well defined, since the action of the integrable connection ∇ is given in local coordinates $(z_1, \ldots, z_m)$ by

$$\nabla(\cdot) = \sum_{i=1}^{m} \frac{\partial}{\partial z_i}(\cdot) \otimes dz_i, \quad \text{with } \frac{\partial}{\partial z_i} \in F_1 \mathcal{D}_M,$$

so that $\nabla(F_p \mathcal{M}) \subset F_{p+1}\mathcal{M}$ for all p by (4-1). For later use, let us point that the maps $Gr\,\nabla$ and $Gr\,\overline{\nabla}$ in the complexes

$$Gr^F_p DR(\mathcal{M}) \quad \text{and} \quad Gr^p_F DR_{\log}(\overline{\mathcal{L}})$$

are $\mathcal{O}$-linear!

EXAMPLE 5.2. Let M be a pure m-dimensional complex algebraic manifold. Then

$$Gr^F_{-p} DR(\mathbb{Q}^H_M) \simeq \Omega^p_M[-p] \in D^b_{\mathrm{coh}}(M)$$

if $0 \leq p \leq m$, and $Gr^F_{-p} DR(\mathbb{Q}^H_M) \simeq 0$ otherwise. Assume in addition that $f : M \to Y$ is a resolution of singularities of the pure-dimensional complex algebraic variety Y. Then IC^H_Y is a direct summand of $f_*\mathbb{Q}^H_M \in D^b MHM(Y)$ so that by functoriality $gr^F_{-p} DR(IC^H_Y)$ is a direct summand of $Rf_*\Omega^p_M[-p] \in D^b_{\mathrm{coh}}(Y)$. In particular

$$Gr^F_{-p} DR(IC^H_Y) \simeq 0 \quad \text{for } p < 0 \text{ or } p > m.$$

The transformations $Gr^F_p DR$ $(p \in \mathbb{Z})$ induce functors on the level of Grothendieck groups. Therefore, if $G_0(Z) \simeq K_0(D^b_{\mathrm{coh}}(Z))$ denotes the Grothendieck group of coherent *algebraic* $\mathcal{O}_Z$-sheaves on Z, we get group homomorphisms

$$Gr^F_p DR : K_0(MHM(Z)) = K_0(D^b MHM(Z)) \to K_0(D^b_{\mathrm{coh}}(Z)) \simeq G_0(Z).$$

DEFINITION 5.3. The *motivic Hodge Chern class transformation*

$$MHC_y : K_0(MHM(Z)) \to G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}]$$

is defined by

$$[\mathcal{M}] \mapsto \sum_{i,p} (-1)^i [\mathcal{H}^i(Gr^F_{-p} DR(\mathcal{M}))] \cdot (-y)^p. \tag{5-2}$$

So this characteristic class captures information from the graded pieces of the filtered de Rham complex of the filtered D-module underlying a mixed Hodge module $\mathcal{M} \in MHM(Z)$, instead of the graded pieces of the filtered D-module itself (as more often studied). Let $p' = \min\{p \mid F_p\mathcal{M} \neq 0\}$. Using Theorem 5.1(1) for a local embedding $Z \hookrightarrow M$ of Z into a complex algebraic manifold M of dimension m, one gets

$$Gr_p^F DR(\mathcal{M}) = 0 \quad \text{for } p < p' - m,$$

and

$$Gr_{p'-m}^F DR(\mathcal{M}) \simeq (F_{p'}\mathcal{M}) \otimes_{\mathcal{O}_M} \omega_M$$

is a coherent $\mathcal{O}_Z$-sheaf independent of the local embedding. Here we are using left D-modules (related to variation of Hodge structures), whereas for this question the corresponding filtered right D-module (as used in [42])

$$\mathcal{M}^r := \mathcal{M} \otimes_{\mathcal{O}_M} \omega_M \quad \text{with} \quad F_p\mathcal{M}^r := (F_{p+m}\mathcal{M}) \otimes_{\mathcal{O}_M} \omega_M$$

would work better. Then the coefficient of the "top-dimensional" power of y in

$$MHC_y([\mathcal{M}]) = [F_{p'}\mathcal{M} \otimes_{\mathcal{O}_M} \omega_M] \otimes (-y)^{m-p'} + \sum_{i < m-p'} (\cdots) \cdot y^i \in G_0(Z)[y^{\pm 1}] \tag{5-3}$$

is given by the class $[F_{p'}\mathcal{M} \otimes_{\mathcal{O}_M} \omega_M] \in G_0(Z)$ of this coherent $\mathcal{O}_Z$-sheaf (up to a sign). Using resolution of singularities, one gets for example for an m-dimensional complex algebraic variety Z that

$$MHC_y([\mathbb{Q}_Z^H]) = [\pi_*\omega_M] \cdot y^m + \sum_{i < m} (\cdots) \cdot y^i \in G_0(Z)[y^{\pm 1}],$$

with $\pi : M \to Z$ any resolution of singularities of Z (compare [44, Corollary 0.3]). More generally, for an irreducible complex variety Z and $\mathcal{M} = IC_Z^H(\mathcal{L})$ a pure Hodge module with strict support Z, the corresponding coherent $\mathcal{O}_Z$-sheaf

$$S_Z(\mathcal{L}) := F_{p'}IC_Z^H(\mathcal{L}) \otimes_{\mathcal{O}_M} \omega_M$$

only depends on Z and the good variation of Hodge structures $\mathcal{L}$ on a Zariski open smooth subset of Z, and it behaves much like a dualizing sheaf. Its formal properties are studied in Saito's proof given in [42] of a conjecture of Kollár. So the "top-dimensional" power of y in $MHC_y([IC_Z^H(\mathcal{L})])$ exactly picks out (up to a sign) the class $[S_Z(\mathcal{L})] \in G_0(Z)$ of this interesting coherent sheaf $S_Z(\mathcal{L})$ on Z.

Let $td_{(1+y)}$ be the *twisted Todd transformation*

$$td_{(1+y)} : G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}] \to H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}];$$

$$[\mathcal{F}] \mapsto \sum_{k \geq 0} td_k([\mathcal{F}]) \cdot (1+y)^{-k}, \tag{5-4}$$

where $H_*(\cdot)$ stands either for the Chow homology groups $CH_*(\cdot)$ or for the Borel–Moore homology groups $H_{2*}^{BM}(\cdot)$ (in even degrees), and td_k is the degree k component in $H_k(Z)$ of the *Todd class transformation* $td_* : G_0(Z) \to H_*(Z) \otimes \mathbb{Q}$ of Baum, Fulton, and MacPherson [5], which is linearly extended over $\mathbb{Z}[y^{\pm 1}]$. Compare also with [22, Chapter 18] and [24, Part II].

DEFINITION 5.4. The (un)normalized *motivic Hirzebruch class transformations* MHT_{y*} (and $\widehat{MHT}_{y*}$) are defined by the composition

$$MHT_{y*} := td_{(1+y)} \circ MHC_y : K_0(MHM(Z)) \to H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}] \tag{5-5}$$

and

$$\widehat{MHT}_{y*} := td_* \circ MHC_y : K_0(MHM(Z)) \to H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}]. \tag{5-6}$$

REMARK 5.5. By precomposing with the transformation χ_{Hdg} from Corollary 4.10 one gets similar transformations

$$mC_y := MHC_y \circ \chi_{\text{Hdg}}, \quad T_{y*} := MHT_{y*} \circ \chi_{\text{Hdg}}, \quad \tilde{T}_{y*} := \widehat{MHT}_{y*} \circ \chi_{\text{Hdg}}$$

defined on the relative Grothendieck group of complex algebraic varieties as studied in [9]. Then it is the (normalized) motivic Hirzebruch class transformation T_{y*}, which, as mentioned in the Introduction, "unifies" in a functorial way

$(y = -1)$ the (rationalized) Chern class transformation c_* of MacPherson [34];
$(y = 0)$ the Todd class transformation td_* of Baum–Fulton–MacPherson [5];
$(y = 1)$ the L-class transformation L_* of Cappell and Shaneson [14].

(Compare with [9; 48] and also with [55] in these proceedings.)

In this paper we work most the time only with the more important K-theoretical transformation MHC_y. The corresponding results for MHT_{y*} follow from this by the known properties of the Todd class transformation td_* (compare [5; 22; 24]).

EXAMPLE 5.6. Let $\mathbb{V} = (V, F, W) \in MHM(pt) = mHs^p$ be a (graded polarizable) mixed Hodge structure. Then:

$$MHC_y([\mathbb{V}]) = \sum_p \dim_{\mathbb{C}}(Gr_F^p V_{\mathbb{C}}) \cdot (-y)^p = \chi_y([\mathbb{V}]) \in \mathbb{Z}[y^{\pm 1}]$$
$$= G_0(pt) \otimes \mathbb{Z}[y^{\pm 1}]. \tag{5-7}$$

So over a point the transformation MHC_y coincides with the χ_y-genus ring homomorphism $\chi_y : K_0(mHs^p) \to \mathbb{Z}[y^{\pm 1}]$ (and similarly for $\widehat{MHT}_{y*}$ and MHT_{y*}).

The *motivic Chern class* $C_y(Z)$ and the *motivic Hirzebruch class* $T_{y*}(Z)$ of a complex algebraic variety Z are defined by

$$C_y(Z) := MHC_y([\mathbb{Q}_Z^H]) \quad \text{and} \quad T_{y*}(Z) := MHT_{y*}([\mathbb{Q}_Z^H]). \qquad \text{(5-8)}$$

Similarly, if U is a pure n-dimensional complex algebraic manifold and L is a local system on U underlying a good variation of mixed Hodge structures $\mathcal{L}$, we define the *twisted motivic Chern and Hirzebruch characteristic classes* by (compare [12; 13; 35])

$$C_y(U;\mathcal{L}) := MHC_y([\mathcal{L}^H]) \quad \text{and} \quad T_{y*}(U;\mathcal{L}) := MHT_{y*}([\mathcal{L}^H]), \qquad \text{(5-9)}$$

where $\mathcal{L}^H[n]$ is the smooth mixed Hodge module on U with underlying perverse sheaf $L[n]$. Assume, in addition, that U is dense and Zariski open in the complex algebraic variety Z. Let $IC_Z^H, IC_Z^H(L) \in MHM(Z)$ be the (twisted) intersection homology (mixed) Hodge module on Z, whose underlying perverse sheaf is IC_Z or $IC_Z(L)$, as the case may be. Then we define *intersection characteristic classes* as follows (compare [9; 11; 13; 35]):

$$\begin{aligned} IC_y(Z) &:= MHC_y\big([IC_Z^H[-n]]\big), \\ IT_{y*}(Z) &:= MHT_{y*}\big([IC_Z^H[-n]]\big), \end{aligned} \qquad \text{(5-10)}$$

and, similarly,

$$\begin{aligned} IC_y(Z;\mathcal{L}) &:= MHC_y\big([IC_Z^H(\mathcal{L})[-n]]\big), \\ IT_{y*}(Z;\mathcal{L}) &:= MHT_{y*}\big([IC_Z^H(\mathcal{L})[-n]]\big). \end{aligned} \qquad \text{(5-11)}$$

By definition and Theorem 5.1, the transformations MHC_y and MHT_{y*} *commute with proper push-forward*. The following *normalization* property holds (compare [9]): If M is smooth, then

$$C_y(Z) = \lambda_y(T^*M) \cap [\mathcal{O}_M] \quad \text{and} \quad T_{y*}(Z) = T_y^*(TM) \cap [M], \qquad \text{(5-12)}$$

where $T_y^*(TM)$ is the cohomology Hirzebruch class of M as in Theorem 2.4.

EXAMPLE 5.7. Let Z be a compact (possibly singular) complex algebraic variety, with $k : Z \to pt$ the proper constant map to a point. Then for $\mathcal{M} \in D^b MHM(Z)$ the pushdown

$$k_*(MHC_y(\mathcal{M})) = MHC_y(k_*\mathcal{M}) = \chi_y\big([H^*(Z,\mathcal{M})]\big)$$

is the Hodge genus

$$\chi_y([H^*(Z,\mathcal{M})]) = \sum_{i,p} (-1)^i \dim_{\mathbb{C}}(Gr_F^p H^i(Z,\mathcal{M})) \cdot (-y)^p. \qquad \text{(5-13)}$$

In particular:

(1) If Z is smooth, then

$$k_* C_y(Z) = \chi_y(Z) := \chi_y \left([H^*(Z, \mathbb{Q})] \right)$$
$$k_* C_y(Z; \mathcal{L}) = \chi_y(Z; \mathcal{L}) := \chi_y \left([H^*(Z, \mathcal{L})] \right).$$

(2) If Z is pure-dimensional, then

$$k_* IC_y(Z) = I\chi_y(Z) := \chi_y \left([IH^*(Z, \mathbb{Q})] \right)$$
$$k_* IC_y(Z; \mathcal{L}) = I\chi_y(Z; \mathcal{L}) := \chi_y \left([IH^*(Z, \mathcal{L})] \right).$$

Note that, for Z compact,

$$I\chi_{-1}(Z) = \chi([IH^*(Z; \mathbb{Q})]$$

is the *intersection (co)homology Euler characteristic* of Z, whereas, for Z projective,

$$I\chi_1(Z) = \text{sgn}\left(IH^*(Z, \mathbb{Q})\right)$$

is the *intersection (co)homology signature* of Z, introduced by Goresky and MacPherson [25]. In fact this follows as in the smooth context from Saito's relative version of the Hodge index theorem for intersection cohomology [39, Theorem 5.3.2]. Finally $\chi_0(Z)$ and $I\chi_0(Z)$ are two possible extensions to singular varieties of the *arithmetic genus*. Here it makes sense to take $y = 0$, since one has, by Example 5.2,

$$k_* IC_y(Z) = I\chi_y(Z) \in \mathbb{Z}[y].$$

It is conjectured that, for a pure n-dimensional compact variety Z,

$$IT_{1*}(Z) \stackrel{?}{=} L_*(Z) \in H_{2*}(Z, \mathbb{Q})$$

is the Goresky–MacPherson homology L-class [25] of the Witt space Z; see [9, Remark 5.4]. Similarly one should expect for a pure-dimensional compact variety Z that

$$\alpha(IC_1(Z)) \stackrel{?}{=} \triangle(Z) \in KO_0^{\text{top}}(Z)\left[\tfrac{1}{2}\right] \oplus KO_2^{\text{top}}(Z)\left[\tfrac{1}{2}\right] \simeq K_0^{\text{top}}(Z)\left[\tfrac{1}{2}\right], \quad (5\text{-}14)$$

where $\alpha : G_0(Z) \to K_0^{\text{top}}(Z)$ is the K-theoretical Riemann–Roch transformation of Baum, Fulton, and MacPherson [6], and $\triangle(Z)$ is the *Sullivan class* of the Witt space Z (compare with [3] in these proceedings). These conjectured equalities are true for a smooth Z, or more generally for a pure n-dimensional compact complex algebraic variety Z with a *small resolution* of singularities $f : M \to Z$, in which case one has $f_*(\mathbb{Q}_M^H) = IC_Z^H[-n]$, so that

$$IT_{1*}(Z) = f_* T_{1*}(M) = f_* L_*(M) = L_*(Z)$$

and

$$\alpha\left(IC_1(Z)\right) = f_*\left(\alpha(C_1(M))\right) = f_* \triangle(M) = \triangle(Z).$$

Here the functoriality $f_* L_*(M) = L_*(Z)$ and $f_* \triangle(M) = \triangle(Z)$ for a small resolution follows, for instance, from [54], which allows one to think of the characteristic classes L_* and $\triangle$ as covariant functors for suitable Witt groups of selfdual constructible sheaf complexes.

In particular, the classes $f_* C_1(M)$ and $f_* T_{1*}(M)$ do not depend on the choice of a small resolution. In fact the same functoriality argument applies to

$$IC_y(Z) = f_* C_y(M) \in G_0(Z) \otimes \mathbb{Z}[y],$$
$$IT_{y*}(Z) = f_* T_{y*}(M) \in H_{2*}(Z) \otimes \mathbb{Q}[y, (1+y)^{-1}];$$

compare [11; 35]. Note that in general a complex variety Z doesn't have a small resolution, and even if it exists, it is in general not unique. This type of independence question were discussed by Totaro [51], pointing out the relation to the famous *elliptic genus and classes* (compare also with [32; 53] in these proceedings). Note that we get such a result for the K-theoretical class

$$IC_y(Z) = f_* C_y(M) \in G_0(Z) \otimes \mathbb{Z}[y]!$$

5B. Calculus of characteristic classes. So far we only discussed the functoriality of MHC_y with respect to proper push down, and the corresponding relation to Hodge genera for compact Z coming from the push down for the proper constant map $k : Z \to pt$. Now we explain some other important functoriality properties. Their proof is based on the following (see [35, (4.6)], for instance):

EXAMPLE 5.8. Let $\overline{M}$ be a smooth partial compactification of the complex algebraic manifold M with complement D a normal crossing divisor with smooth irreducible components, with $j : M \to \overline{M}$ the open inclusion. Let $\mathbb{V} = (L, F, W)$ be a good variation of mixed Hodge structures on M. Then the filtered de Rham complex

$$(DR(j_* \mathbb{V}), F) \quad \text{of} \quad j_* \mathbb{V} \in MHM(\overline{M})[-m] \subset D^b MHM(\overline{M})$$

is by Theorem 5.1(2) filtered quasi-isomorphic to the logarithmic de Rham complex $DR_{\log}(\mathcal{L})$ with the increasing filtration $F_{-p} := F^p$ ($p \in \mathbb{Z}$) associated to the decreasing F-filtration (3-15). Then

$$
\begin{aligned}
MHC_y(j_* \mathbb{V}) &= \sum_{i,p} (-1)^i [\mathcal{H}^i(Gr_F^p DR_{\log}(\mathcal{L}))] \cdot (-y)^p \\
&= \sum_p [Gr_F^p DR_{\log}(\mathcal{L})] \cdot (-y)^p \\
&\overset{(*)}{=} \sum_{i,p} (-1)^i [Gr_F^{p-i}(\mathcal{L}) \otimes_{\mathcal{O}_{\overline{M}}} \Omega_{\overline{M}}^i(\log(D))] \cdot (-y)^p \\
&= MHC^y(Rj_* L) \cap \big(\lambda_y\big(\Omega_{\overline{M}}^1(\log(D))\big) \cap [\mathcal{O}_{\overline{M}}]\big). \quad\quad (5\text{-}15)
\end{aligned}
$$

In particular for $j = \mathrm{id} : M \to M$ we get the following *Atiyah–Meyer type formula* (compare [12; 13; 35]):

$$MHC_y(\mathbb{V}) = MHC^y(\mathcal{L}) \cap \left(\lambda_y(T^*M) \cap [\mathcal{O}_M]\right). \qquad (5\text{-}16)$$

REMARK 5.9. The formula (5-15) is a class version of the formula (3-16) of Theorem 3.13, which one gets back from (5-15) by pushing down to a point for the proper constant map $k : \overline{M} \to pt$ on the compactification $\overline{M}$ of M.

Also note that in the equality $(*)$ in (5-15) we use the fact that the complex $Gr_F^p DR_{\log}(\mathcal{L})$ has coherent (locally free) objects, with $\mathcal{O}_{\overline{M}}$-linear maps between them.

The formula (5-15) describes a *splitting* of the characteristic class $MHC_y(j_*\mathbb{V})$ into two terms:

(coh) a cohomological term $MHC^y(Rj_*\mathcal{L})$, capturing the information of the good variation of mixed Hodge structures $\mathcal{L}$, and

(hom) the homological term $\lambda_y\left(\Omega^1_{\overline{M}}(\log(D))\right) \cap [\mathcal{O}_{\overline{M}}] = MHC_y(j_*\mathbb{Q}_M^H)$, capturing the information of the underlying space or embedding $j : M \to \overline{M}$.

By Corollary 3.14, the term $MHC^y(Rj_*L)$ has good functorial behavior with respect to exterior and suitable tensor products, as well as for smooth pullbacks. For the exterior products one gets similarly (compare [19, Proposition 3.2]):

$$\Omega^1_{\overline{M} \times \overline{M}'}(\log(D \times M' \cup M \times D')) \simeq \left(\Omega^1_{\overline{M}}(\log(D))\right) \boxtimes \left(\Omega^1_{\overline{M}'}(\log(D'))\right)$$

so that

$$\lambda_y\left(\Omega^1_{\overline{M} \times \overline{M}'}(\log(D \times M' \cup M \times D'))\right) \cap [\mathcal{O}_{\overline{M} \times \overline{M}'}]$$
$$= \left(\lambda_y\left(\Omega^1_{\overline{M}}(\log(D))\right) \cap [\mathcal{O}_{\overline{M}}]\right) \boxtimes \left(\lambda_y\left(\Omega^1_{\overline{M}'}(\log(D'))\right) \cap [\mathcal{O}_{\overline{M}'}]\right)$$

for the product of two partial compactifications as in example 5.8. But the Grothendieck group $K_0(MHM(Z))$ of mixed Hodge modules on the complex variety Z is generated by classes of the form $f_*(j_*[\mathbb{V}])$, with $f : \overline{M} \to Z$ proper and $M, \overline{M}, \mathbb{V}$ as before. Finally one also has the multiplicativity

$$(f \times f')_* = f_* \boxtimes f'_*$$

for the push down for proper maps $f : \overline{M} \to Z$ and $f' : \overline{M}' \to Z'$ on the level of Grothendieck groups $K_0(MHM(\cdot))$ as well as for $G_0(\cdot) \otimes \mathbb{Z}[y^{\pm 1}]$. Then one gets the following result from Corollary 3.14 and Example 5.8 (as in [9, Proof of Corollary 2.1(3)]):

COROLLARY 5.10 (MULTIPLICATIVITY FOR EXTERIOR PRODUCTS). *The motivic Chern class transformation MHC_y commutes with exterior products*:

$$MHC_y([M \boxtimes M']) = MHC_y([M] \boxtimes [M'])$$
$$= MHC_y([M]) \boxtimes MHC_y([M']) \qquad (5\text{-}17)$$

for $M \in D^b MHM(Z)$ and $M' \in D^b MHM(Z')$.

Next we explain the behavior of MHC_y for smooth pullbacks. Consider a cartesian diagram of morphisms of complex algebraic varieties

$$
\begin{array}{ccc}
\overline{M}' & \xrightarrow{\ g'\ } & \overline{M} \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Z' & \xrightarrow[\ g\ ]{} & Z,
\end{array}
$$

with g smooth, f proper and $M, \overline{M}, \mathbb{V}$ as before. Then g' too is smooth and f' is proper, and one has the *base change isomorphism*

$$g^* f_* = f'_* g'^*$$

on the level of Grothendieck groups $K_0(MHM(\cdot))$ as well as for $G_0(\cdot) \otimes \mathbb{Z}[y^{\pm 1}]$. Finally for the induced partial compactification $\overline{M}'$ of $M' := g'^{-1}(M)$, with complement D' the induced normal crossing divisor with smooth irreducible components, one has a short exact sequence of vector bundles on $\overline{M}'$:

$$0 \to g'^* \big(\Omega^1_{\overline{M}}(\log(D))\big) \to \Omega^1_{\overline{M}'}(\log(D')) \to T^*_{g'} \to 0,$$

with $T^*_{g'}$ the relative cotangent bundle along the fibers of the smooth morphism g'. And by base change one has $T^*_{g'} = f'^*(T^*_g)$. So for the corresponding lambda classes we get

$$\lambda_y\big(\Omega^1_{\overline{M}'}(\log(D'))\big) = \big(g'^* \lambda_y\big(\Omega^1_{\overline{M}}(\log(D))\big)\big) \otimes \lambda_y(T^*_{g'})$$
$$= \big(g'^* \lambda_y\big(\Omega^1_{\overline{M}}(\log(D))\big)\big) \otimes f'^* \lambda_y(T^*_g). \qquad (5\text{-}18)$$

Finally (compare also with [9, Proof of Corollary 2.1(4)]), by using the *projection formula*

$$\lambda_y(T^*_g) \otimes f'_*(\cdot) = f'_* \big(f'^* \lambda_y(T^*_g) \otimes (\cdot)\big):$$
$$G_0(\overline{M}') \otimes \mathbb{Z}[y^{\pm 1}] \to G_0(Z') \otimes \mathbb{Z}[y^{\pm 1}]$$

one gets from Corollary 3.14 and Example 5.8 the following consequence:

COROLLARY 5.11 (VRR FOR SMOOTH PULLBACKS). *For a smooth morphism $g : Z' \to Z$ of complex algebraic varieties one has for the motivic Chern class transformation the following Verdier Riemann–Roch formula*:

$$\lambda_y(T_g^*) \cap g^* MHC_y([\mathcal{M}]) = MHC_y(g^*[\mathcal{M}]) = MHC_y([g^*\mathcal{M}]) \qquad (5\text{-}19)$$

for $\mathcal{M} \in D^b MHM(Z)$. In particular

$$g^* MHC_y([\mathcal{M}]) = MHC_y(g^*[\mathcal{M}]) = MHC_y([g^*\mathcal{M}]) \qquad (5\text{-}20)$$

for g an étale morphism (i.e., a smooth morphism with zero dimensional fibers), or in more topological terms, for g an unramified covering. The most important special case is that of an open embedding.

If moreover g is also proper, then one gets from Corollary 5.11 and the projection formula the following result:

COROLLARY 5.12 (GOING UP AND DOWN). *Let $g : Z' \to Z$ be a smooth and proper morphism of complex algebraic varieties. Then one has for the motivic Chern class transformation the following going up und down formula*:

$$
\begin{aligned}
MHC_y(g_* g^*[\mathcal{M}]) &= g_* MHC_y(g^*[\mathcal{M}]) \\
&= g_* \left(\lambda_y(T_g^*) \cap g^* MHC_y([\mathcal{M}]) \right) \\
&= \left(g_* \lambda_y(T_g^*) \right) \cap MHC_y([\mathcal{M}]) \qquad (5\text{-}21)
\end{aligned}
$$

for $\mathcal{M} \in D^b MHM(Z)$, with

$$g_* \left(\lambda_y(T_g^*) \right) := \sum_{p,q \geq 0} (-1)^q \cdot [R^q g_*(\Omega_{Z'/Z}^p)] \cdot y^p \in K_{\text{alg}}^0(Z)[y]$$

the algebraic cohomology class being given (as in Example 3.5) by

$$MHC^y([Rg_* \mathbb{Q}_{Z'}]) = \sum_{p,q \geq 0} (-1)^q \cdot [R^q g_*(\Omega_{Z'/Z}^p)] \cdot y^p.$$

Note that all higher direct image sheaves $R^q g_*(\Omega_{Z'/Z}^p)$ are locally free in this case, since g is a smooth and proper morphism of complex algebraic varieties (compare with [18]). In particular

$$g_* C_y(Z') = \left(g_* \lambda_y(T_g^*) \right) \cap C_y(Z),$$

and

$$g_* IC_y(Z') = \left(g_* \lambda_y(T_g^*) \right) \cap IC_y(Z)$$

for Z and Z' pure-dimensional. If, in addition, Z and Z' are compact, with $k : Z \to pt$ the constant proper map, then

$$\chi_y(g^*[\mathcal{M}]) = k_* g_* MHC_y(g^*[\mathcal{M}]) = \langle g_* \lambda_y(T_g^*), MHC_y([\mathcal{M}]) \rangle. \qquad (5\text{-}22)$$

In particular,

$$\chi_y(Z') = \langle g_* \lambda_y(T_g^*), C_y(Z) \rangle \quad and \quad I\chi_y(Z') = \langle g_* \lambda_y(T_g^*), IC_y(Z) \rangle.$$

The result of this corollary can also be seen form a different viewpoint, by making the "going up and down" calculation already on the level of Grothendieck groups of mixed Hodge modules, where this time one only needs the assumption that $f : Z' \to Z$ is proper (to get the projection formula):

$$f_* f^*[\mathcal{M}] = [f_* f^* \mathcal{M}] = [f_*(\mathbb{Q}_{Z'}^H \otimes f^* \mathcal{M})] = [f_* \mathbb{Q}_{Z'}^H] \otimes [\mathcal{M}] \in K_0(MHM(Z))$$

for $\mathcal{M} \in D^b MHM(Z)$. The problem for a singular Z is then that we do not have a precise relation between

$$[f_* \mathbb{Q}_{Z'}^H] \in K_0(MHM(Z)) \quad and \quad [Rf_* \mathbb{Q}_{Z'}] \in K_0(FmHs^p(Z)).$$

REMARK 5.13. What is missing up to now is the right notion of a good variation (or family) of mixed Hodge structures on a *singular* complex algebraic variety Z! This class should contain at least

(1) the higher direct image local systems $R^i f_* \mathbb{Q}_{Z'}$ ($i \in \mathbb{Z}$) for a smooth and proper morphism $f : Z' \to Z$ of complex algebraic varieties, and
(2) the pullback $g^* \mathcal{L}$ of a good variation of mixed Hodge structures $\mathcal{L}$ on a smooth complex algebraic manifold M under an algebraic morphism $g : Z \to M$.

At the moment we have to assume that Z is smooth (and pure-dimensional), so as to use Theorem 4.3.

Nevertheless, in case (2) above we can already prove the following interesting result (compare with [35, §4.1] for a similar result for MHT_{y*} in the case when f is a closed embedding):

COROLLARY 5.14 (MULTIPLICATIVITY). *Let $f : Z \to N$ be a morphism of complex algebraic varieties, with N smooth and pure n-dimensional. Then one has a natural pairing*

$$f^*(\cdot) \cap (\cdot) : K_0(VmHs^g(N)) \times K_0(MHM(Z)) \to K_0(MHM(Z)),$$
$$([\mathcal{L}], [\mathcal{M}]) \mapsto [f^*(\mathcal{L}^H) \otimes \mathcal{M}].$$

Here $\mathcal{L}^H[m]$ is the smooth mixed Hodge module on N with underlying perverse sheaf $L[m]$. One also has a similar pairing on (co)homological level:

$$f^*(\cdot) \cap (\cdot) : K_{\mathrm{alg}}^0(N) \otimes \mathbb{Z}[y^{\pm 1}] \times G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}] \to G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}],$$
$$([\mathcal{V}] \cdot y^i, [\mathcal{F}] \cdot y^j) \mapsto [f^*(\mathcal{V}) \otimes \mathcal{F}] \cdot y^{i+j}.$$

And the motivic Chern class transformations MHC^y and MHC_y commute with these natural pairings:

$$MHC_y\big([f^*(\mathcal{L}^H) \otimes \mathcal{M}]\big) = MHC^y([f^*\mathcal{L}]) \cap MHC_y([\mathcal{M}])$$
$$= f^*\big(MHC^y([\mathcal{L}])\big) \cap MHC_y([\mathcal{M}]) \qquad (5\text{-}23)$$

for $\mathcal{L} \in VmHs^g(N)$ and $\mathcal{M} \in D^b MHM(Z)$.

For the proof we can once more assume $\mathcal{M} = g_* j_* \mathbb{V}$ for $g : \overline{M} \to Z$ proper, with $\overline{M}$ a pure-dimensional smooth complex algebraic manifold, $j : M \to \overline{M}$ a Zariski open inclusion with complement D a normal crossing divisor with smooth irreducible components, and finally $\mathbb{V}$ a good variation of mixed Hodge structures on M. Using the projection formula, it is then enough to prove

$$MHC_y\big([g^*f^*(\mathcal{L}^H) \otimes j_*\mathbb{V}]\big) = MHC^y\big([g^*f^*\mathcal{L}]\big) \cap MHC_y([j_*\mathbb{V}]).$$

But $g^*f^*\mathcal{L}$ is a good variation of mixed Hodge structures on $\overline{M}$. Therefore, by Example 5.8 and Corollary 3.14(3), both sides are equal to

$$\big(MHC^y(g^*f^*\mathcal{L}) \otimes MHC^y(j_*\mathbb{V})\big) \cap \big(\lambda_y\big(\Omega^1_{\overline{M}}(\log(D))\big)\big) \cap [\mathcal{O}_{\overline{M}}]\big).$$

As an application of the very special case where $f = \mathrm{id} : Z \to N$ is the identity of a complex algebraic manifold Z, with

$$MHC_y([\mathbb{Q}^H_Z]) = \lambda_y(T^*Z) \cap [\mathcal{O}_Z],$$

one gets the Atiyah–Meyer type formula (5-16) as well as the following result (cf. [12; 13; 35]):

EXAMPLE 5.15 (ATIYAH TYPE FORMULA). Let $g : Z' \to Z$ be a proper morphism of complex algebraic varieties, with Z smooth and connected. Assume that for a given $\mathcal{M} \in D^b MHM(Z')$ all direct image sheaves

$$R^i g_* \mathrm{rat}(\mathcal{M}) \quad (i \in \mathbb{Z}) \quad \text{are locally constant}:$$

for instance, g may be a locally trivial fibration and $\mathcal{M} = \mathbb{Q}^H_{Z'}$ or $\mathcal{M} = IC^H_{Z'}$ (for Z' pure-dimensional), so that they all underlie a good variation of mixed Hodge structures. Then one can define

$$[Rg_* \mathrm{rat}(\mathcal{M})] := \sum_{i \in \mathbb{Z}} (-1)^i \cdot [R^i g_* \mathrm{rat}(\mathcal{M})] \in K_0(VmHs^g(Z)),$$

with

$$g_* MHC_y([\mathcal{M}]) = MHC_y(g_*[\mathcal{M}])$$
$$= MHC^y([Rg_* \mathrm{rat}(\mathcal{M})]) \otimes \big(\lambda_y(T^*Z) \cap [\mathcal{O}_Z]\big). \qquad (5\text{-}24)$$

Here is a final application:

EXAMPLE 5.16 (FORMULA OF ATIYAH–MEYER TYPE FOR INTERSECTION COHOMOLOGY). Let $f : Z \to N$ be a morphism of complex algebraic varieties, with N smooth and pure n-dimensional (e.g., a closed embedding). Assume also Z is pure m-dimensional. Then one has for a good variation of mixed Hodge structures $\mathcal{L}$ on N the equality

$$IC_Z^H(f^*\mathcal{L})[-m] \simeq f^*\mathcal{L}^H \otimes IC_Z^H[-m] \in MHM(Z)[-m] \subset D^b MHM(Z),$$

so that

$$IC_y(Z; f^*\mathcal{L}) = MHC^y(f^*\mathcal{L}) \cap IC_y(Z) = f^*\big(MHC^y(\mathcal{L})\big) \cap IC_y(Z). \quad (5\text{-}25)$$

If in addition Z is also compact, then one gets by pushing down to a point:

$$I\chi_y(Z; f^*\mathcal{L}) = \langle MHC^y(f^*\mathcal{L}), IC_y(Z) \rangle. \qquad (5\text{-}26)$$

REMARK 5.17. This example should be seen as a Hodge-theoretical version of the corresponding result of Banagl, Cappell, and Shaneson [4] for the L-classes $L_*(IC_Z(L))$ of a selfdual *Poincaré local system* L on all of Z. The special case of Example 5.16 for f a closed inclusion was already explained in [35, §4.1].

Finally note that all the results of this section can easily be applied to the (un)normalized *motivic Hirzebruch class transformation* MHT_{y*} (and $\widetilde{MHT}_{y*}$), because the *Todd class transformation* $td_* : G_0(\cdot) \to H_*(\cdot) \otimes \mathbb{Q}$ of Baum, Fulton, and MacPherson [5] has the following properties (compare also with [22, Chapter 18] and [24, Part II]):

FUNCTORIALITY: The Todd class transformation td_* commutes with push-down f_* for a proper morphism $f : Z \to X$:

$$td_*(f_*([\mathcal{F}])) = f_*(td_*([\mathcal{F}])) \quad \text{for } [\mathcal{F}] \in G_0(Z).$$

MULTIPLICATIVITY FOR EXTERIOR PRODUCTS: The Todd class transformation td_* commutes with exterior products:

$$td_*([\mathcal{F} \boxtimes \mathcal{F}']) = td_*([\mathcal{F}]) \boxtimes td_*([\mathcal{F}']) \quad \text{for } [\mathcal{F}] \in G_0(Z) \text{ and } [\mathcal{F}'] \in G_0(Z').$$

VRR FOR SMOOTH PULLBACKS: For a smooth morphism $g : Z' \to Z$ of complex algebraic varieties one has for the Todd class transformation td_* the following Verdier Riemann–Roch formula:

$$td^*(T_g) \cap g^* td_*([\mathcal{F}]) = td_*(g^*[\mathcal{F}]) = td_*([g^*\mathcal{F}]) \quad \text{for } [\mathcal{F}] \in G_0(Z).$$

MULTIPLICATIVITY: Let $ch^* : K_{\mathrm{alg}}^0(\cdot) \to H^*(\cdot) \otimes \mathbb{Q}$ be the cohomological *Chern character* to the cohomology $H^*(\cdot)$ given by the operational Chow

ring $CH^*(\,\cdot\,)$ or the usual cohomology $H^{2*}(\,\cdot\,,\mathbb{Z})$ in even degrees. Then one has the multiplicativity relation

$$td_*([\mathcal{V}\otimes\mathcal{F}]) = ch^*([\mathcal{V}])\cap td_*([\mathcal{F}])$$

for $[\mathcal{V}]\in K^0_{\mathrm{alg}}(Z)$ and $[\mathcal{F}]\in G_0(Z)$, with Z a (possible singular) complex algebraic variety.

5C. Characteristic classes and duality. In this final section we explain the characteristic class version of the duality formula (2-14) for the χ_y-genus. We also show that the specialization of MHT_{y*} for $y=-1$ exists and is equal to the rationalized MacPherson Chern class c_* of the underlying constructible sheaf complex. The starting point is the following result [39, §2.4.4]:

THEOREM 5.18 (M. SAITO). *Let M be a pure m-dimensional complex algebraic manifold. Then one has for $\mathcal{M}\in D^b MHM(M)$ the duality result (for $j\in\mathbb{Z}$)*

$$Gr_j^F(DR(\mathcal{D}\mathcal{M})) \simeq \mathcal{D}\big(Gr_{-j}^F DR(\mathcal{M})\big)\in D^b_{\mathrm{coh}}(M). \qquad (5\text{-}27)$$

Here $\mathcal{D}$ on the left side is the duality of mixed Hodge modules, wheres $\mathcal{D}$ on the right is the Grothendieck duality

$$\mathcal{D} = \mathrm{Rhom}(\,\cdot\,,\omega_M[m]) : D^b_{\mathrm{coh}}(M)\to D^b_{\mathrm{coh}}(M),$$

with $\omega_M = \Omega_M^m$ the canonical sheaf of M.

A priori this is a duality for the corresponding analytic (cohomology) sheaves. Since $\mathcal{M}$ and $DR(\mathcal{M})$ can be extended to smooth complex algebraic compactification $\overline{M}$, one can apply Serre's GAGA theorem to get the same result also for the underlying algebraic (cohomology) sheaves.

COROLLARY 5.19 (CHARACTERISTIC CLASSES AND DUALITY). *Let Z be a complex algebraic variety with dualizing complex $\omega_Z^\bullet\in D^b_{\mathrm{coh}}(Z)$, so that the Grothendieck duality transformation $\mathcal{D} = \mathrm{Rhom}(\,\cdot\,,\omega_Z^\bullet)$ induces a duality involution*

$$\mathcal{D} : G_0(Z)\to G_0(Z).$$

Extend this to $G_0(Z)\otimes\mathbb{Z}[y^{\pm 1}]$ by $y\mapsto 1/y$. Then the motivic Hodge Chern class transformation MHC_y commutes with duality $\mathcal{D}$:

$$MHC_y(\mathcal{D}(\cdot)) = \mathcal{D}(MHC_y(\cdot)) : K_0(MHM(Z))\to G_0(Z)\otimes\mathbb{Z}[y^{\pm 1}]. \quad (5\text{-}28)$$

Note that for $Z = pt$ a point this reduces to the duality formula (2-14) for the χ_y-genus. For dualizing complexes and (relative) Grothendieck duality we refer to [26; 17; 33] as well as [24, Part I, §7]). Note that for M smooth of pure dimension m, one has

$$\omega_M[m]\simeq\omega_M^\bullet\in D^b_{\mathrm{coh}}(M).$$

Moreover, for a proper morphism $f : X \to Z$ of complex algebraic varieties one has the relative Grothendieck duality isomorphism

$$Rf_* \left(\mathrm{Rhom}(\mathcal{F}, \omega_X^\bullet) \right) \simeq \mathrm{Rhom}(Rf_* \mathcal{F}, \omega_Z^\bullet) \quad \text{for } \mathcal{F} \in D_{\mathrm{coh}}^b(X),$$

so that the duality involution

$$\mathcal{D} : G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}] \to G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}]$$

commutes with proper push down. Since $K_0(MHM(Z))$ is generated by classes $f_*[\mathcal{M}]$, with $f : M \to Z$ proper morphism from a pure dimensional complex algebraic manifold M (and $\mathcal{M} \in MHM(M)$), it is enough to prove (5-28) in the case $Z = M$ a pure dimensional complex algebraic manifold, in which case it directly follows from Saito's result (5-27).

For a systematic study of the behavior of the Grothendieck duality transformation $\mathcal{D} : G_0(Z) \to G_0(Z)$ with respect to exterior products and smooth pullback, we refer to [23] and [24, Part I, §7], where a corresponding "bivariant" result is stated. Here we only point out that the dualities $(\cdot)^\vee$ and $\mathcal{D}$ commute with the *pairings* of Corollary 5.14:

$$f^* \left((\cdot)^\vee \right) \cap (\mathcal{D}(\cdot)) = \mathcal{D} \left(f^*(\cdot) \cap (\cdot) \right) :$$
$$K_{\mathrm{alg}}^0(N) \otimes \mathbb{Z}[y^{\pm 1}] \times G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}] \to G_0(Z) \otimes \mathbb{Z}[y^{\pm 1}], \tag{5-29}$$

and similarly

$$f^* \left((\cdot)^\vee \right) \cap (\mathcal{D}(\cdot)) = \mathcal{D} \left(f^*(\cdot) \cap (\cdot) \right) :$$
$$K_0(VmHs^g(N)) \times K_0(MHM(Z)) \to K_0(MHM(Z)). \tag{5-30}$$

Here the last equality needs only be checked for classes $[IC_S(\mathcal{L})]$, with $S \subset Z$ irreducible of dimension d and $\mathcal{L}$ a good variation of pure Hodge structures on a Zariski dense open smooth subset U of S, and $\mathbb{V}$ a good variation of pure Hodge structures on N. But then the claim follows from

$$f^*(\mathbb{V}) \otimes IC_S(\mathcal{L}) \simeq IC_S(f^*(\mathbb{V})|U \otimes \mathcal{L})$$

and (4-3) in the form

$$\mathcal{D}\big(IC_S(f^*(\mathbb{V})|U \otimes \mathcal{L})\big) \simeq IC_S\big((f^*(\mathbb{V})|U \otimes \mathcal{L})^\vee\big)(d)$$
$$\simeq IC_S\big(f^*(\mathbb{V}^\vee)|U \otimes \mathcal{L}^\vee\big)(d).$$

REMARK 5.20. The Todd class transformation $td_* : G_0(\cdot) \to H_*(\cdot) \otimes \mathbb{Q}$, too, commutes with duality (compare with [22, Example 18.3.19] and [24, Part I, Corollary 7.2.3]) if the duality involution $\mathcal{D} : H_*(\cdot) \otimes \mathbb{Q} \to H_*(\cdot) \otimes \mathbb{Q}$ in homology is defined as $\mathcal{D} := (-1)^i \cdot \mathrm{id}$ on $H_i(\cdot) \otimes \mathbb{Q}$. So also the unnormalized Hirzebruch class transformation $\widetilde{MHT}_{y*}$ commutes with duality, if this duality in homology is extended to $H_*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}]$ by $y \mapsto 1/y$.

As a final result of this paper, we have:

PROPOSITION 5.21. *Let Z be a complex algebraic variety, and consider $[\mathcal{M}] \in K_0(MHM(Z))$. Then*

$$MHT_{y*}([\mathcal{M}]) \in H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}] \subset H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}],$$

so that the specialization $MHT_{-1*}([\mathcal{M}]) \in H_*(Z) \otimes \mathbb{Q}$ for $y = -1$ is defined. Then

$$MHT_{-1*}([\mathcal{M}]) = c_*([\mathrm{rat}(\mathcal{M})]) =: c_*(\chi_{\mathrm{stalk}}([\mathrm{rat}(\mathcal{M})])) \in H_*(Z) \otimes \mathbb{Q} \quad \text{(5-31)}$$

is the rationalized MacPherson Chern class of the underlying constructible sheaf complex $\mathrm{rat}(\mathcal{M})$ *(or the constructible function $\chi_{\mathrm{stalk}}([\mathrm{rat}(\mathcal{M})])$). In particular*

$$MHT_{-1*}(\mathcal{D}[\mathcal{M}]) = MHT_{-1*}([\mathcal{D}\mathcal{M}]) = MHT_{-1*}([\mathcal{M}]). \quad \text{(5-32)}$$

Here χ_{stalk} is the transformation form the diagram (4-13). Similarly, all the transformations from this diagram (4-13), like χ_{stalk} and rat, commute with duality $\mathcal{D}$. This implies already the last claim, since $\mathcal{D} = \mathrm{id}$ for algebraically constructible functions (compare [47, §6.0.6]). So we only need to prove the first part of the proposition. Since MHT_{-1*} and c_* both commute with proper push down, we can assume $[\mathcal{M}] = [j_*\mathbb{V}]$, with $Z = \overline{M}$ a smooth pure-dimensional complex algebraic manifold, $j : M \to \overline{M}$ a Zariski open inclusion with complement D a normal crossing divisor with smooth irreducible components, and $\mathbb{V}$ a good variation of mixed Hodge structures on M. So

$$\widetilde{MHT}_{y*}([j_*\mathbb{V}]) = ch^* \left(MHC^y(Rj_*\mathcal{L}) \right) \cap \widetilde{MHT}_{y*}([j_*\mathbb{Q}_M^H]) \in H_*(\overline{M}) \otimes \mathbb{Q}[y^{\pm 1}]$$

by (5-15) and the *multiplicativity* of the Todd class transformation td_*. Introduce the *twisted Chern character*

$$ch^{(1+y)} : K_{\mathrm{alg}}^0(\,\cdot\,) \otimes \mathbb{Q}[y^{\pm 1}] \to H^*(\,\cdot\,) \otimes \mathbb{Q}[y^{\pm 1}],$$
$$[\mathcal{V}] \cdot y^j \mapsto \sum_{i \geq 0} ch^i([\mathcal{V}]) \cdot (1+y)^i \cdot y^j, \quad \text{(5-33)}$$

with $ch^i([\mathcal{V}]) \in H^i(\,\cdot\,) \otimes \mathbb{Q}$ the i-th component of ch^*. Then one easily gets

$$MHT_{y*}([j_*\mathbb{V}]) = ch^{(1+y)}\left(MHC^y(Rj_*\mathcal{L})\right) \cap MHT_{y*}([j_*\mathbb{Q}_M^H])$$
$$\in H_*(\overline{M}) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}].$$

But $[j_*\mathbb{Q}_M^H] = \chi_{\mathrm{Hdg}}(j_*[\mathrm{id}_M])$ is by (4-15) in the image of

$$\chi_{\mathrm{Hdg}} : M_0(\mathrm{var}\,/\overline{M}) = K_0(\mathrm{var}\,/\overline{M})[\mathbb{L}^{-1}] \to K_0(MHM(\overline{M})).$$

So for $MHT_{y*}([j_*\mathbb{Q}_M^H])$ we can apply the following special case of Proposition 5.21:

LEMMA 5.22. *The transformation*

$$T_{y*} = MHT_{y*} \circ \chi_{\text{Hdg}} : M_0(\text{var}/Z) \to H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]$$

takes values in $H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}] \subset H_*(Z) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]$, *with*

$$T_{-1*} = T_{-1*} \circ \mathcal{D} = c_* \circ \text{can} : M_0(\text{var}/Z) \to H_*(Z) \otimes \mathbb{Q}.$$

Assuming this lemma, we can derive from the following commutative diagram that the specialization $MHT_{-1*}([j_*\mathbb{V}])$ for $y = -1$ exists:

$$
\begin{array}{ccc}
H^*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}] \times H_*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}] & \xrightarrow{\ \cap\ } & H_*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}] \\
\text{\small incl.} \uparrow & & \uparrow \text{\small incl.} \\
H^*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}] \times H_*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}] & \xrightarrow{\ \cap\ } & H_*(\cdot) \otimes \mathbb{Q}[y^{\pm 1}] \\
\text{\small } y=-1 \downarrow & & \downarrow y=-1 \\
H^*(\cdot) \otimes \mathbb{Q} \times H_*(\cdot) \otimes \mathbb{Q} & \xrightarrow{\ \cap\ } & H_*(\cdot) \otimes \mathbb{Q}.
\end{array}
$$

Moreover $ch^{(1+y)}(MHC^y(Rj_*\mathcal{L}))$ specializes for $y = -1$ just to

$$\text{rk}(L) = ch^0([\bar{\mathcal{L}}]) \in H^0(\overline{M}) \otimes \mathbb{Q},$$

with $\text{rk}(L)$ the rank of the local system L on M. So we get

$$MHT_{-1*}([j_*\mathbb{V}]) = \text{rk}(L) \cdot c_*(j_*1_M) = c_*(\text{rk}(L) \cdot j_*1_M) \in H_*(\overline{M}) \otimes \mathbb{Q},$$

with $\text{rk}(L) \cdot j_*1_M = \chi_{\text{stalk}}(\text{rat}([j_*\mathbb{V}]))$.

It remains to prove Lemma 5.22. But all transformations — T_{y*}, $\mathcal{D}$, c_* and can — commute with pushdown for proper maps. Moreover, by resolution of singularities and additivity, $M_0(\text{var}/Z)$ is generated by classes $[f : N \to Z] \cdot \mathbb{L}^k$ ($k \in \mathbb{Z}$), with N smooth pure n-dimensional and f proper. So it is enough to prove that $T_{y*}([\text{id}_N] \cdot \mathbb{L}^k) \in H_*(N) \otimes \mathbb{Q}[y^{\pm 1}]$, with

$$T_{y*}([\text{id}_N] \cdot \mathbb{L}^k) = T_{y*}(\mathcal{D}([\text{id}_N] \cdot \mathbb{L}^k)) = c_*(\text{can}([\text{id}_N] \cdot \mathbb{L}^k)).$$

But by the *normalization condition* for our characteristic class transformations one has (compare [9]):

$$T_{y*}([\text{id}_N]) = T_y^*(TN) \cap [N] \in H_*(N) \otimes \mathbb{Q}[y],$$

with $T_{-1*}([\text{id}_N]) = c^*(TN) \cap [N] = c_*(1_N)$. Similarly

$$T_{y*}([\mathbb{L}]) = \chi_y([\mathbb{Q}(-1)]) = -y \quad \text{and} \quad \text{can}([\mathbb{L}]) = 1_{pt},$$

so the multiplicativity of T_{y*} for exterior products (with a point space) yields

$$T_{y*}([\text{id}_N] \cdot \mathbb{L}^k) \in H_*(N) \otimes \mathbb{Q}[y^{\pm 1}].$$

Moreover

$$T_{-1*}([\mathrm{id}_N] \cdot \mathbb{L}^k) = c_*(1_N) = c_*\big(\mathrm{can}([\mathrm{id}_N] \cdot \mathbb{L}^k)\big).$$

Finally $\mathcal{D}([\mathrm{id}_N] \cdot \mathbb{L}^k) = [\mathrm{id}_N] \cdot \mathbb{L}^{k-n}$ by the definition of $\mathcal{D}$, so that

$$T_{-1*}([\mathrm{id}_N] \cdot \mathbb{L}^k) = T_{-1*}\big(\mathcal{D}([\mathrm{id}_N] \cdot \mathbb{L}^k)\big).$$

Acknowledgements

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