Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

The signature of singular spaces and its refinements to generalized homology theories

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ABSTRACT. These notes are based on an expository lecture that I gave at the workshop "Topology of Stratified Spaces" at MSRI Berkeley in September 2008. We will first explain the definition of a bordism invariant signature for a singular space, proceeding along a progression from less singular to more and more singular spaces, starting out from spaces that have no odd codimensional strata and, after having discussed Goresky–Siegel spaces and Witt spaces, ending up with general (non-Witt) stratified spaces. We will moreover discuss various refinements of the signature to orientation classes in suitable bordism theories based on singular cycles. For instance, we will indicate how one may define a symmetric $\mathbb{L}^\bullet$-homology orientation for Goresky–Siegel spaces or a Sullivan orientation for those non-Witt spaces that still possess generalized Poincaré duality. These classes can be thought of as refining the L-class of a singular space. Along the way, we will also see how to compute twisted versions of the signature and L-class.

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The author was in part supported by a research grant of the Deutsche Forschungsgemeinschaft. He thanks Andrew Ranicki for suggestions and clarifying comments.

1. Introduction

Let M be a closed smooth n-dimensional manifold. The Hirzebruch L-classes $L^i(M) \in H^{4i}(M;\mathbb{Q})$ of its tangent bundle are powerful tools in the classification of such M, particularly in the high dimensional situation where $n \geq 5$. To make this plausible, we observe first that the $L^i(M)$, with the exception of the top class $L^{n/4}(M)$ if n is divisible by 4, are not generally homotopy invariants of M, and are therefore capable of distinguishing manifolds in a given homotopy type, contrary to the ability of homology and other homotopy invariants. For example, there exist infinitely many manifolds $M_i, i = 1, 2, \ldots$ in the homotopy type of $S^2 \times S^4$, distinguished by the first Pontrjagin class of their tangent bundle $p_1(TM_i) \in H^4(S^2 \times S^4) = \mathbb{Z}$, namely $p_1(TM_i) = Ki$, K a fixed nonzero integer. The first L-class L^1 is proportional to the first Pontrjagin class p_1, in fact they are related by the formula $L^1 = \frac{1}{3}p_1$.

Suppose that M^n, $n \geq 5$, is simply connected, as in the example. The classification of manifolds breaks up into two very different tasks: Classify Poincaré complexes up to homotopy equivalence and, given a Poincaré complex, determine all manifolds homotopy equivalent to it.

In dimension 3, one has a relatively complete answer to the former problem. One can associate purely algebraic data to a Poincaré complex such that two such complexes are homotopy equivalent if, and only if, their algebraic data are isomorphic, see the classification result in [Hen77]. Furthermore, every given algebraic data is realizable as the data of a Poincaré complex; see [Tur90]. In higher dimensions, the problem becomes harder. While one can still associate classifying data to a Poincaré complex, this data is not purely algebraic anymore, though at least in dimension 4, one can endow Poincaré duality chain complexes with an additional structure that allows for classification, [BB08].

The latter problem is the realm of surgery theory. Elements of the *structure set* $S(M)$ of M are represented by homotopy equivalences $N \to M$, where N is another closed smooth manifold, necessarily simply connected, since M is. Two such homotopy equivalences represent the same element of $S(M)$ if there is a diffeomorphism between the domains that commutes with the homotopy equivalences. The goal of surgery theory is to compute $S(M)$. The central tool provided by the theory is the surgery exact sequence

$$L_{n+1} \longrightarrow S(M) \xrightarrow{\eta} N(M) \longrightarrow L_n,$$

an exact sequence of pointed sets. The L_n are the 4-periodic simply connected surgery obstruction groups, $L_n = \mathbb{Z}, 0, \mathbb{Z}/2, 0$ for $n \equiv 0, 1, 2, 3 \mod 4$. The term $N(M)$ is the *normal invariant set*, investigated by Sullivan. It is a generalized cohomology theory and a Pontrjagin–Thom type construction yields $N(M) \cong [M, G/O]$, where $[M, G/O]$ denotes homotopy classes of maps from

M into a certain universal space G/O, which does not depend on M. Since $[M, G/O]$ is a cohomology theory, it is particularly important to know its coefficients $\pi_*(G/O)$. While the torsion is complicated, one has modulo torsion

$$\pi_i(G/O) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & i = 4j, \\ 0, & \text{otherwise.} \end{cases}$$

One obtains an isomorphism

$$[M, G/O] \otimes \mathbb{Q} \cong \bigoplus_{j \geq 0} H^{4j}(M; \mathbb{Q}).$$

The group L_{n+1} acts on $S(M)$ so that the point-inverses of η are the orbits of the action, i.e. for all $f, h \in S(M)$ one has $\eta(f) = \eta(h)$ if, and only if, there is a $g \in L_{n+1}$ which moves f to h, $g \cdot f = h$.

Suppose our manifold M is even dimensional. Then L_{n+1} vanishes and thus $\eta(f) = \eta(h)$ implies $f = g \cdot f = h$, so that η is injective. In particular, we obtain an injection

$$S(M) \otimes \mathbb{Q} \hookrightarrow N(M) \otimes \mathbb{Q}.$$

Composing this with $N(M) \otimes \mathbb{Q} \cong \bigoplus H^{4j}(M; \mathbb{Q})$, we obtain an injective map

$$S(M) \otimes \mathbb{Q} \xrightarrow{L} \bigoplus H^{4j}(M; \mathbb{Q}).$$

This map sends a homotopy equivalence $h : N \to M$ to the cohomology class $L^*(h)$ uniquely determined by $h^*(L^*(M) + L^*(h)) = L^*(N)$. Thus M is determined, up to finite ambiguity, by *its homotopy type and its L-classes*. This demonstrates impressively the power of the L-classes as a tool to classify manifolds.

The L-classes are closely related to the signature invariant, and indeed the classes can be defined, following Thom [Tho58], by the signatures of submanifolds, as we shall now outline. The link between the L-classes and the signature is the Hirzebruch signature theorem. It asserts that the evaluation of the top L-class $L^j(M) \in H^n(M; \mathbb{Q})$ of an $n = 4j$-dimensional oriented manifold M on the fundamental class of M equals the signature $\sigma(M)$ of M. Once we know this, we can define $L^*(M)$ as follows. A theorem of Serre states that the Hurewicz map is an isomorphism

$$\pi^k(M) \otimes \mathbb{Q} \cong H^k(M; \mathbb{Q})$$

in the range $n < 2k-1$, where $\pi^*(M)$ denotes the cohomotopy sets of M, whose elements are homotopy classes of maps from M to spheres. Thus, in this range, we may think of a rational cohomology class as a (smooth) map $f : M \to S^k$. The preimage $f^{-1}(p)$ of a regular value $p \in S^k$ is a submanifold and has a signature $\sigma(f^{-1}(p))$. Use the bordism invariance of the signature to conclude that

this signature depends only on the homotopy class of f. Assigning $\sigma(f^{-1}(p))$ to the homotopy class of f yields a map $H^k(M;\mathbb{Q}) \to \mathbb{Q}$, that is, a *homology* class $L_k(M) \in H_k(M;\mathbb{Q})$. By Poincaré duality, this class can be dualized back into cohomology, where it agrees with the Hirzebruch classes $L^*(M)$. Note that all you need for this procedure is transversality for maps to spheres in order to get suitable subspaces and a bordism invariant signature defined on these subspaces. Thus, whenever these ingredients are present for a singular space X, we will obtain an L-class $L_*(X) \in H_*(X;\mathbb{Q})$ in the rational homology of X. (This class cannot necessarily be dualized back into cohomology, due to the lack of classical Poincaré duality for singular X.) Therefore, we only need to discuss which classes of singular spaces have a bordism invariant signature. The required transversality results are available for Whitney stratified spaces, for example. The notion of a Whitney stratified space incorporates smoothness in a particularly amenable way into the world of singular spaces. A Whitney stratification of a space X consists of a (locally finite) partition of X into locally closed smooth manifolds of various dimensions, called the *pure strata*. If one stratum intersects the closure of another one, then it must already be completely contained in it. Connected components S of strata have tubular neighborhoods T_S that possess locally trivial projections $\pi_S : T_S \to S$ whose fiber $\pi_S^{-1}(p)$, $p \in S$, is the cone on a compact space $L(p)$ (also Whitney stratified), called the *link* of S at p. It follows that every point p has a neighborhood homeomorphic to $\mathbb{R}^{\dim S} \times \operatorname{cone} L(p)$. Real and complex algebraic varieties possess a natural Whitney stratification, as do orbit spaces of smooth group actions. The pseudomanifold condition means that the singular strata have codimension at least two and the complement of the singular set (the *top stratum*) is dense in X. The figure eight space, for instance, can be Whitney stratified but is not a pseudomanifold. The pinched 2-torus is a Whitney stratifiable pseudomanifold. If we attach a whisker to the pinched 2-torus, then it loses its pseudomanifold property, while retaining its Whitney stratifiability. By [Gor78], a Whitney stratified pseudomanifold X can be triangulated so that the Whitney stratification defines a PL stratification of X.

Inspired by the success of L-classes in manifold theory sketched above, one would like to have L-classes for stratified pseudomanifolds as well. In [CW91], see also [Wei94], Cappell and Weinberger indicate the following result, analogous to the manifold classification result sketched above. Suppose X is a stratified pseudomanifold that has no strata of odd dimension. Assume that all strata S have dimension at least 5, and that all fundamental groups in sight are trivial, that is, all strata are simply connected and all links are simply connected. (A pseudomanifold whose links are all simply connected is called *supernormal*. This is compatible with the notion of a *normal* pseudomanifold, meaning that

all links are connected.) Then differences of L-classes give an injection

$$S(X)\otimes\mathbb{Q}\hookrightarrow\bigoplus_{S\subset X}\bigoplus_{j}H_j(\overline{S};\mathbb{Q}),$$

where S ranges over the strata of X, $\overline{S}$ denotes the closure of S in X, and $S(X)$ is an appropriately[1] defined structure set for X. This would suggest that L-classes are as powerful in classifying stratified spaces as in classifying manifolds. Since, as we have seen, the definition of L-classes is intimately related to, and can be given in terms of, the signature, we shall primarily investigate the possibility of defining a bordism invariant signature for an oriented stratified pseudomanifold X.

2. Pseudomanifolds without odd codimensional strata

In order to define a signature, one needs an intersection form. But singular spaces do not possess Poincaré duality, in particular no intersection form, in ordinary homology. The solution is to change to a different kind of homology. Motivated by a question of D. Sullivan [Sul70], Goresky and MacPherson define (in [GM80] for PL pseudomanifolds and in [GM83] for topological pseudomanifolds) a collection of groups $IH^{\bar{p}}_*(X)$, called *intersection homology groups of* X, depending on a multi-index $\bar{p}$, called a *perversity*. For these groups, a Poincaré–Lefschetz-type intersection theory can be defined, and a generalized form of Poincaré duality holds, but only between groups with "complementary perversities." More precisely, with $\bar{t}(k)=k-2$ denoting the top perversity, there are intersection pairings

$$IH^{\bar{p}}_i(X)\otimes IH^{\bar{q}}_j(X)\longrightarrow\mathbb{Z} \tag{2-1}$$

for an oriented closed pseudomanifold X, $\bar{p}+\bar{q}=\bar{t}$ and $i+j=\dim X$, which are nondegenerate when tensored with the rationals. Jeff Cheeger discovered, working independently of Goresky and MacPherson and not being aware of their intersection homology, that Poincaré duality on triangulated pseudomanifolds equipped with a suitable (locally conical) Riemannian metric on the top stratum, can be recovered by using the complex of L^2 differential forms on the top stratum, see [Che80], [Che79] and [Che83]. The connection between his and the work of Goresky and MacPherson was pointed out by Sullivan in 1976. For an introduction to intersection homology see [B$^+$84], [KW06] or [Ban07]. A third method, introduced in [Ban09] and implemented there for pseudomanifolds

[1] In [CW91], the structure sets $S(X)$ are defined as the homotopy groups of the homotopy fiber of the assembly map $X\wedge\mathbb{L}_\bullet(\mathbb{Z})_0\to\mathbb{L}_\bullet(\mathbb{Z}\pi_1(X))$, constructed in [Ran79]. This can be defined for any space, but under the stated assumptions on X, [CW91] interprets $S(X)$ geometrically in terms of classical structure sets of the strata of X.

with isolated singularities and two-strata spaces with untwisted link bundle, associates to a singular pseudomanifold X an *intersection space* $I^{\bar{p}}X$, whose *ordinary* rational homology has a nondegenerate intersection pairing

$$\widetilde{H}_i(I^{\bar{p}}X;\mathbb{Q}) \otimes \widetilde{H}_j(I^{\bar{q}}X;\mathbb{Q}) \longrightarrow \mathbb{Q}.$$

This theory is not isomorphic, albeit related, to intersection homology. It solves a problem posed in string theory, related to the presence of massless D-branes in the course of conifold transitions.

In sheaf-theoretic language, the groups $IH_*^{\bar{p}}(X)$ are given as the hypercohomology groups of a sheaf complex $\mathbf{IC}^{\bullet}_{\bar{p}}(X)$ over X. If we view this complex as an object of the derived category (that is, we invert quasi-isomorphisms), then $\mathbf{IC}^{\bullet}_{\bar{p}}(X)$ is characterized by certain stalk/costalk vanishing conditions. The rationalization of the above intersection pairing (2-1) is induced on hypercohomology by a duality isomorphism $\mathcal{D}\mathbf{IC}^{\bullet}_{\bar{p}}(X;\mathbb{Q})[n] \cong \mathbf{IC}^{\bullet}_{\bar{q}}(X;\mathbb{Q})$ in the derived category, where $\mathcal{D}$ denotes the Verdier dualizing functor. This means roughly that one does not just have a global chain equivalence to the dual (intersection) chain complex, but a chain equivalence on every open set.

Let X^n be an oriented closed topological stratified pseudomanifold which has only even dimensional strata. A wide class of examples is given by complex algebraic varieties. In this case, the intersection pairing (2-1) allows us to define a signature $\sigma(X)$ by using the two complementary middle perversities $\bar{m}$ and $\bar{n}$:

k	2	3	4	5	6	7	8	9	...
$\bar{m}(k)$	0	0	1	1	2	2	3	3	...
$\bar{n}(k)$	0	1	1	2	2	3	3	4	...

Since $\bar{m}(k) = \bar{n}(k)$ for even values of k, and only these values are relevant for our present X, we have $IH^{\bar{m}}_{n/2}(X) = IH^{\bar{n}}_{n/2}(X)$. Therefore, the pairing (2-1) becomes

$$IH^{\bar{m}}_{n/2}(X;\mathbb{Q}) \otimes IH^{\bar{m}}_{n/2}(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

(symmetric if $n/2$ is even), that is, defines a quadratic form on the vector space $IH^{\bar{m}}_{n/2}(X;\mathbb{Q})$. Let $\sigma(X)$ be the signature of this quadratic form. Goresky and MacPherson show that this is a bordism invariant for bordisms that have only strata of even codimension. Since $\mathbf{IC}^{\bullet}_{\bar{m}}(X) = \mathbf{IC}^{\bullet}_{\bar{n}}(X)$, the intersection pairing is induced by a self-duality isomorphism $\mathcal{D}\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathbb{Q})[n] \cong \mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathbb{Q})$. This is an example of a *self-dual sheaf*.

3. Witt spaces

To form a bordism theory based on pseudomanifold cycles, one could consider bordism based on all (say topological, or PL) closed pseudomanifolds,

$$\Omega_*^{\text{all pseudomfds}}(Y) = \left\{ [X \xrightarrow{f} Y] \mid X \text{ a pseudomanifold} \right\},$$

where the admissible bordisms consist of compact pseudomanifolds with collared boundary, without further restrictions. Now it is immediately clear that the associated coefficient groups vanish, $\Omega_*^{\text{all pseudomfds}}(\text{pt}) = 0$, $* > 0$, since any pseudomanifold X is the boundary of the cone on X, which is an admissible bordism. Thus this naive definition does not lead to an interesting and useful new theory, and we conclude that a subclass of pseudomanifolds has to be selected to define such theories. Given the results on middle perversity intersection homology presented so far, our next approach would be to select the class of all closed pseudomanifolds with only even codimensional strata,

$$\Omega_*^{\text{ev}}(Y) = \left\{ [X \xrightarrow{f} Y] \mid X \text{ has only even codim strata} \right\}$$

(and the same condition is imposed on all admissible bordisms). While we do know that the signature is well-defined on $\Omega_*^{\text{ev}}(\text{pt})$, this is however still not a good theory as this definition leads to a large number of geometrically insignificant generators. Many operations (such as coning or refinement of the stratification) do introduce strata of odd codimension, so we need to allow some strata of this kind, but so as not to destroy Poincaré duality. In [Sie83], Paul Siegel introduced a class of oriented stratified PL pseudomanifolds called *Witt spaces*, by imposing the condition that $IH^{\bar{m}}_{\text{middle}}(\text{Link}(x);\mathbb{Q}) = 0$ for all points x in odd codimensional strata of X. The suspension $X^7 = \Sigma\mathbb{CP}^3$ has two singular points which form a stratum of odd codimension 7. The link is $\mathbb{CP}^3$ with middle homology $H_3(\mathbb{CP}^3) = 0$. Hence X^7 is a Witt space. The suspension $X^3 = \Sigma T^2$ has two singular points which form a stratum of odd codimension 3. The space X^3 is not Witt, since the middle Betti number of the link T^2 is 2. In sheaf-theoretic language, a pseudomanifold X is Witt if and only if the canonical morphism $\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathbb{Q}) \to \mathbf{IC}^{\bullet}_{\bar{n}}(X;\mathbb{Q})$ is an isomorphism (in the derived category). Thus, $\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathbb{Q})$ is self-dual on a Witt space, and if X is compact, we have a nonsingular intersection pairing $IH_i^{\bar{m}}(X;\mathbb{Q}) \otimes IH^{\bar{m}}_{n-i}(X;\mathbb{Q}) \to \mathbb{Q}$. Let $\Omega_*^{\text{Witt}}(Y)$ denote Witt space bordism, that is, bordism of closed oriented Witt spaces X mapping continuously into Y. Admissible bordisms are compact pseudomanifolds with collared boundary that satisfy the Witt condition, together with a map into Y.

When is a Witt space X^n a boundary? Suppose the dimension n is odd. Then $X = \partial Y$ with $Y = \text{cone}\, X$. The cone Y is a Witt space, since the cone-point is

a stratum of even codimension in Y. This shows that $\Omega^{\mathrm{Witt}}_{2k+1}(\mathrm{pt}) = 0$ for all k. In particular, the de Rham invariant does not survive in $\Omega^{\mathrm{Witt}}_*(\mathrm{pt})$.

Let $W(\mathbb{Q})$ denote the Witt group of the rationals. Its structure is known and given by

$$W(\mathbb{Q}) \cong W(\mathbb{Z}) \oplus \bigoplus_{p \text{ prime}} W(\mathbb{Z}/_p),$$

where $W(\mathbb{Z}) \cong \mathbb{Z}$ via the signature, $W(\mathbb{Z}/_2) \cong \mathbb{Z}/_2$, and for $p \neq 2$, $W(\mathbb{Z}/_p) \cong \mathbb{Z}/_4$ or $\mathbb{Z}/_2 \oplus \mathbb{Z}/_2$. Sending a Witt space X^{4k} to its intersection form on $IH^{\bar{m}}_{2k}(X;\mathbb{Q})$ defines a bordism-invariant element $w(X) \in W(\mathbb{Q})$. Siegel shows that the induced map $w : \Omega^{\mathrm{Witt}}_{4k}(\mathrm{pt}) \to W(\mathbb{Q})$ is an isomorphism for $k > 0$. In dimension zero we get $\Omega^{\mathrm{Witt}}_0 = \mathbb{Z}$. If X has dimension congruent 2 modulo 4, then X bounds a Witt space by singular surgery on a symplectic basis for the antisymmetric intersection form. Thus $\Omega^{\mathrm{Witt}}_n = 0$ for n not congruent 0 mod 4. Since $W(\mathbb{Q})$ is just another name for the L-group $L^{4k}(\mathbb{Q})$ and $L^n(\mathbb{Q}) = 0$ for n not a multiple of 4, we can summarize Siegel's result succinctly as saying that $\Omega^{\mathrm{Witt}}_*(\mathrm{pt}) \cong L^*(\mathbb{Q})$ in positive degrees. By the Brown representability theorem, Witt space bordism theory is given by a spectrum MWITT, which is in fact an MSO module spectrum, see [Cur92]. (Regard a manifold as a Witt space with one stratum.) By [TW79], any MSO module spectrum becomes a product of Eilenberg–Mac Lane spectra after localizing at 2. Thus,

$$\mathrm{MWITT}_{(2)} \simeq K(\mathbb{Z}_{(2)}, 0) \times \prod_{j>0} K(L^j(\mathbb{Q})_{(2)}, j)$$

and we conclude that

$$\Omega^{\mathrm{Witt}}_n(Y)_{(2)} \cong H_n(Y;\mathbb{Z})_{(2)} \oplus \bigoplus_{j>0} H_{n-j}(Y; L^j(\mathbb{Q})_{(2)}).$$

(As $\mathbb{Z}_{(2)}$ is flat over $\mathbb{Z}$, we have $S_*(X)_{(2)} = (S_{(2)})_*(X)$ for any spectrum S.) Let us focus on the odd-primary situation. Regard $\mathbb{Z}[\frac{1}{2}, t]$ as a graded ring with $\deg(t) = 4$. Let $\Omega^{\mathrm{SO}}_*(Y)$ denote bordism of smooth oriented manifolds. Considering the signature as a map $\sigma : \Omega^{\mathrm{SO}}_*(\mathrm{pt}) \to \mathbb{Z}[\frac{1}{2}, t]$, $[M^{4k}] \mapsto \sigma(M)t^k$, makes $\mathbb{Z}[\frac{1}{2}, t]$ into an $\Omega^{\mathrm{SO}}_*(\mathrm{pt})$-module and we can form the homology theory

$$\Omega^{\mathrm{SO}}_*(Y) \otimes_{\Omega^{\mathrm{SO}}_*(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t].$$

On a point, this is

$$\Omega^{\mathrm{SO}}_*(\mathrm{pt}) \otimes_{\Omega^{\mathrm{SO}}_*(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t] \cong \mathbb{Z}[\tfrac{1}{2}, t],$$

the isomorphism being given by $[M^{4l}] \otimes at^k \mapsto a\sigma(M^{4l})t^{k+l}$. Let $ko_*(Y)$ denote connective KO homology, regarded as a $\mathbb{Z}$-graded, not $\mathbb{Z}/_4$-graded, theory.

It is given by a spectrum bo whose homotopy groups vanish in negative degrees and are given by

$$ko_*(\mathrm{pt}) = \pi_*(bo) = \mathbb{Z} \oplus \Sigma^1\mathbb{Z}/_2 \oplus \Sigma^2\mathbb{Z}/_2 \oplus \Sigma^4\mathbb{Z} \oplus \Sigma^8\mathbb{Z} \oplus \Sigma^9\mathbb{Z}/_2 \oplus \cdots,$$

repeating with 8-fold periodicity in nonnegative degrees. Inverting 2 kills the torsion in degrees 1 and 2 mod 8 so that $ko_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}, t]$. In his MIT-notes [Sul05], Sullivan constructs a natural Conner–Floyd-type isomorphism of homology theories

$$\Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t] \xrightarrow{\cong} ko_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}].$$

Siegel [Sie83] shows that Witt spaces provide a geometric description of connective KO homology at odd primes: He constructs a natural isomorphism of homology theories

$$\Omega_*^{\mathrm{Witt}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} ko_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \tag{3-1}$$

(which we shall return to later). It reduces to the signature homomorphism on coefficients, i.e. an element $[X^{4k}] \otimes a \in \Omega_{4k}^{\mathrm{Witt}}(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}]$ maps to $a\sigma(X)t^k \in ko_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] = \mathbb{Z}[\frac{1}{2}, t]$. This is an isomorphism, since inverting 2 kills the torsion components of the invariant $w(X)$, $W(\mathbb{Q}) \otimes \mathbb{Z}[\frac{1}{2}] \cong W(\mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}]$. Now $\Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\frac{1}{2}, t]$ being a quotient of $\Omega_*^{\mathrm{SO}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}, t]$, yields a natural surjection

$$\Omega_*^{\mathrm{SO}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}, t] \twoheadrightarrow \Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t].$$

Let us consider the diagram of natural transformations

$$\begin{array}{ccc} \Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\frac{1}{2}, t] & \xrightarrow{\cong} & ko_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \\ \uparrow & & \uparrow \cong \\ \Omega_*^{\mathrm{SO}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}, t] & \longrightarrow & \Omega_*^{\mathrm{Witt}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}], \end{array}$$

where the lower horizontal arrow maps an element $[M \xrightarrow{f} Y] \otimes at^k$ to $[M \times \mathbb{CP}^{2k} \xrightarrow{f\pi_1} Y] \otimes a$. On a point, this arrow thus maps an element $[M^{4l}] \otimes at^k$ to $[M^{4l} \times \mathbb{CP}^{2k}] \otimes a$. Mapping an element $[M^{4l}] \otimes at^k$ clockwise yields $a\sigma(M)t^{k+l} \in ko_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}, t]$. Mapping the same element counterclockwise gives

$$a\sigma(M^{4l} \times \mathbb{CP}^{2k})t^{(4l+4k)/4} = a\sigma(M)t^{k+l}$$

also. The diagram commutes and shows that away from 2, the canonical map from manifold bordism to Witt bordism is a surjection. This is a key observation of [BCS03] and frequently allows bordism invariant calculations for Witt spaces

to be pulled back to calculations on smooth manifolds. This principle may be viewed as a topological counterpart of resolution of singularities in complex algebraic and analytic geometry (though one should point out that there are complex 2-dimensional singular projective toric varieties $X(\Delta)$ such that no nonzero multiple of $X(\Delta)$ is bordant to any toric resolution of $X(\Delta)$). It is applied in [BCS03] to prove that the twisted signature $\sigma(X;\mathcal{S})$ of a closed oriented Whitney stratified Witt space X^n of even dimension with coefficients in a (Poincaré-) local system $\mathcal{S}$ on X can be computed as the product of the (untwisted) L-class of X and a modified Chern character of the K-theory signature $[\mathcal{S}]_{\boldsymbol{K}}$ of $\mathcal{S}$,

$$\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_{\boldsymbol{K}}, L_*(X)\rangle \in \mathbb{Z}$$

where $L_*(X) \in H_*(X;\mathbb{Q})$ is the total L-class of X. The higher components of the product $\widetilde{\mathrm{ch}}[\mathcal{S}]_{\boldsymbol{K}} \cap L_*(X)$ in fact compute the rest of the twisted L-class $L_*(X;\mathcal{S})$. Such twisted classes come up naturally if one wants to understand the pushforward under a stratified map of characteristic classes of the domain, see [CS91] and [Ban06c].

4. IP spaces: integral duality

Witt spaces satisfy generalized Poincaré duality rationally. Is there a class of pseudomanifolds whose members satisfy Poincaré duality integrally? This requires restrictions more severe than those imposed on Witt spaces. An *intersection homology Poincaré space* ("IP space"), introduced in [GS83], is an oriented stratified PL pseudomanifold such that the middle perversity, middle dimensional intersection homology of even dimensional links vanishes and the torsion subgroup of the middle perversity, lower middle dimensional intersection homology of odd dimensional links vanishes. This condition characterizes spaces for which the integral intersection chain sheaf $\mathbf{IC}^{\bullet}_{\bar{m}}(X;\mathbb{Z})$ is self-dual. Goresky and Siegel show that for such spaces X^n there are nonsingular pairings

$$IH_i^{\bar{m}}(X)/\operatorname{Tors} \otimes IH_{n-i}^{\bar{m}}(X)/\operatorname{Tors} \longrightarrow \mathbb{Z}$$

and

$$\operatorname{Tors} IH_i^{\bar{m}}(X) \otimes \operatorname{Tors} IH_{n-i-1}^{\bar{m}}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Let $\Omega_*^{\mathrm{IP}}(\mathrm{pt})$ denote the bordism groups of IP spaces. The signature $\sigma(X)$ of the above intersection pairing is a bordism invariant and induces a homomorphism $\Omega_{4k}^{\mathrm{IP}}(\mathrm{pt}) \to \mathbb{Z}$. If $\dim X = n = 4k+1$, then the number mod 2 of $\mathbb{Z}/2$-summands in $\operatorname{Tors} IH_{2k}^{\bar{m}}(X)$ is a bordism invariant, the *de Rham invariant* $\mathrm{dR}(X) \in \mathbb{Z}/2$ of X. It induces a homomorphism $\mathrm{dR} : \Omega_{4k+1}^{\mathrm{IP}}(\mathrm{pt}) \to \mathbb{Z}/2$. Pardon shows in [Par90] that these maps are both isomorphisms for $k \geq 1$ and that all other groups

$\Omega_n^{\mathrm{IP}}(\mathrm{pt}) = 0$, $n > 0$. In summary, one obtains

$$\Omega_n^{\mathrm{IP}}(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ \mathbb{Z}/_2, & n \geq 5, n \equiv 1(4), \\ 0 & \text{otherwise.} \end{cases}$$

Let $L^*(\mathbb{Z}G)$ denote the symmetric L-groups, as defined by Ranicki, of the group ring $\mathbb{Z}G$ of a group G. For the trivial group $G = e$, these are the homotopy groups $\pi_*(\mathbb{L}^\bullet) = \mathbb{L}^\bullet_*(\mathrm{pt})$ of the symmetric L-spectrum $\mathbb{L}^\bullet$ and are given by

$$L^n(\mathbb{Z}e) = \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ \mathbb{Z}/_2, & n \equiv 1(4), \\ 0 & \text{otherwise.} \end{cases}$$

We notice that this is extremely close to the IP bordism groups, the only difference being a $\mathbb{Z}/_2$ in dimension 1. A comparison of their respective coefficient groups thus leads us to expect that the difference between the generalized homology theory $\Omega_*^{\mathrm{IP}}(Y)$ given by mapping IP pseudomanifolds X continuously into a space Y and symmetric $\mathbb{L}^\bullet$-homology $\mathbb{L}^\bullet_*(Y)$ is very small. Indeed, according to [Epp07], there exists a map $\phi : \mathrm{MIP} \to \mathbb{L}^\bullet$, where MIP is the spectrum giving rise to IP bordism theory, whose homotopy cofiber is an Eilenberg–Mac Lane spectrum $K(\mathbb{Z}/_2, 1)$. The map is obtained by using a description of $\mathbb{L}^\bullet$ as a simplicial Ω-spectrum, whose k-th space has its n-simplices given by homotopy classes of $(n-k)$-dimensional n-ads of symmetric algebraic Poincaré complexes (pairs). Similarly, MIP can be described as a simplicial Ω-spectrum, whose k-th space has its n-simplices given by $(n-k)$-dimensional n-ads of compact IP pseudomanifolds. Given these simplicial models, one has to map n-ads of IP spaces to n-ads of symmetric Poincaré complexes. On a suitable incarnation of the middle perversity integral intersection chain sheaf on a compact IP space, a Poincaré symmetric structure can be constructed by copying Goresky's symmetric construction of [Gor84]. Taking global sections and resolving by finitely generated projectives (observing that the cohomology of the section complex is finitely generated by compactness), one obtains a symmetric algebraic Poincaré complex. This assignment can also be done for pairs and behaves well under gluing. The symmetric structure is uniquely determined by its restriction to the top stratum. On the top stratum, which is a manifold, the construction agrees sheaf-theoretically with the construction used classically for manifolds, see e.g. [Bre97]. In particular, if we start with a smooth oriented closed manifold and view it as an IP space with one stratum, then the top stratum is the *entire* space and the constructed symmetric structure agrees with Ranicki's symmetric structure. Modelling MSO and MSTOP as simplicial Ω-spectra consisting of n-ads of smooth oriented manifolds and n-ads of topological oriented manifolds,

respectively, we thus see that the diagram

$$\begin{array}{ccc} \mathrm{MSO} & \longrightarrow & \mathrm{MIP} \\ \downarrow & & \downarrow \phi \\ \mathrm{MSTOP} & \longrightarrow & \mathbb{L}^\bullet \end{array}$$

homotopy commutes, where the "symmetric signature map" $\mathrm{MSTOP} \to \mathbb{L}^\bullet$ of ring spectra has been constructed by Ranicki. (Technically, IP spaces are PL pseudomanifolds, so to obtain the canonical map $\mathrm{MSO} \to \mathrm{MIP}$, it is necessary to find a canonical PL structure on a given smooth manifold. This is possible by J. H. C. Whitehead's triangulation results of [Whi40], where it is shown that every smooth manifold admits a compatible triangulation as a PL manifold and this PL manifold is unique to within a PL homeomorphism; see also [WJ66].) It follows that

$$\begin{array}{ccc} \Omega_*^{\mathrm{SO}}(Y) & \longrightarrow & \Omega_*^{\mathrm{IP}}(Y) \\ \downarrow & & \downarrow \phi_*(Y) \\ \Omega_*^{\mathrm{STOP}}(Y) & \longrightarrow & \mathbb{L}_*^\bullet(Y) \end{array} \tag{4-1}$$

commutes. Let us verify the commutativity for $Y = \mathrm{pt}$ by hand. If $* = 4k+2$ or $4k+3$, then $\mathbb{L}_*^\bullet(\mathrm{pt}) = L^*(\mathbb{Z}e) = 0$, so the two transformations agree in these dimensions. Commutativity in dimension 1 follows from $\Omega_1^{\mathrm{SO}}(\mathrm{pt}) = 0$. For $* = 4k+1$, $k > 0$, the homotopy cofiber sequence of spectra

$$\mathrm{MIP} \xrightarrow{\phi} \mathbb{L}^\bullet \longrightarrow K(\mathbb{Z}/2, 1)$$

induces on homotopy groups an exact sequence and hence an isomorphism

$$\pi_{4k+1}(\mathrm{MIP}) \xrightarrow{\cong} \pi_{4k+1}(\mathbb{L}^\bullet).$$

But both of these groups are $\mathbb{Z}/2$, whence the isomorphism is the identity map. Thus if M^{4k+1} is a smooth oriented manifold, then $[M^{4k+1}] \in \Omega_{4k+1}^{\mathrm{IP}}(\mathrm{pt})$ maps under ϕ to the de Rham invariant $\mathrm{dR}(M) \in L^{4k+1}(\mathbb{Z}e) = \mathbb{Z}/2$. Hence the two transformations agree on a point in dimensions $4k+1$. Again using the exact sequence of homotopy groups determined by the above cofibration sequence, ϕ induces isomorphisms $\pi_{4k}(\mathrm{MIP}) \xrightarrow{\cong} \pi_{4k}(\mathbb{L}^\bullet)$. Both of these groups are $\mathbb{Z}$, so this isomorphism is ± 1. Consequently, a smooth oriented manifold M^{4k}, defining an element $[M^{4k}] \in \Omega_{4k}^{\mathrm{IP}}(\mathrm{pt})$, maps under ϕ to $\pm\sigma(M) \in L^{4k}(\mathbb{Z}e) = \mathbb{Z}$, and it is $+\sigma(M)$ when the signs in the two symmetric structures are correctly matched.

For an n-dimensional Poincaré space which is either a topological manifold or a combinatorial homology manifold (i.e. a polyhedron whose links of

simplices are homology spheres), Ranicki defines a canonical $\mathbb{L}^\bullet$-*orientation* $[M]_{\mathbb{L}} \in \mathbb{L}^\bullet_n(M)$, see [Ran92]. Its image under the assembly map

$$\mathbb{L}^\bullet_n(M) \xrightarrow{A} L^n(\mathbb{Z}\pi_1(M))$$

is the *symmetric signature* $\sigma^*(M)$, which is a homotopy invariant. The class $[M]_{\mathbb{L}}$ itself is a topological invariant. The geometric meaning of the $\mathbb{L}^\bullet$-orientation class is that its existence for a geometric Poincaré complex X^n, $n \geq 5$, assembling to the symmetric signature (which any Poincaré complex possesses), implies up to 2-torsion that X is homotopy equivalent to a compact topological manifold. (More precisely, X is homotopy equivalent to a compact manifold if it has an $\mathbb{L}^\bullet$-orientation class, which assembles to the *visible* symmetric signature of X.) Cap product with $[M]_{\mathbb{L}}$ induces an $\mathbb{L}^\bullet$-homology Poincaré duality isomorphism $(\mathbb{L}^\bullet)^i(M) \xrightarrow{\cong} \mathbb{L}^\bullet_{n-i}(M)$. Rationally, $[M]_{\mathbb{L}}$ is given by the homology L-class of M,

$$[M]_{\mathbb{L}} \otimes 1 = L_*(M) \in \mathbb{L}^\bullet_n(M) \otimes \mathbb{Q} \cong \bigoplus_{j \geq 0} H_{n-4j}(M; \mathbb{Q}).$$

Thus, we may view $[M]_{\mathbb{L}}$ as an integral refinement of the L-class of M. Another integral refinement of the L-class is the signature homology orientation class $[M]_{\mathrm{Sig}} \in \mathrm{Sig}_n(M)$, to be defined below. The identity $A[M]_{\mathbb{L}} = \sigma^*(M)$ may then be interpreted as a non-simply connected generalization of the Hirzebruch signature formula. The localization of $[M]_{\mathbb{L}}$ at odd primes is the Sullivan orientation $\Delta(M) \in KO_n(M) \otimes \mathbb{Z}[\frac{1}{2}]$, which we shall return to later. Under the map $\Omega^{\mathrm{STOP}}_n(M) \to \mathbb{L}^\bullet_n(M)$, $[M]_{\mathbb{L}}$ is the image of the identity map $[M \xrightarrow{\mathrm{id}} M] \in \Omega^{\mathrm{STOP}}_n(M)$.

We shall now apply ϕ in defining an $\mathbb{L}^\bullet$-orientation $[X]_{\mathbb{L}} \in \mathbb{L}^\bullet_n(X)$ for an oriented closed n-dimensional IP pseudomanifold X. (For Witt spaces, an $\mathbb{L}^\bullet$-orientation and a symmetric signature has been defined in [CSW91].) The identity map $X \to X$ defines an orientation class $[X]_{\mathrm{IP}} \in \Omega^{\mathrm{IP}}_n(X)$.

DEFINITION 4.1. The $\mathbb{L}^\bullet$-orientation $[X]_{\mathbb{L}} \in \mathbb{L}^\bullet_n(X)$ of an oriented closed n-dimensional IP pseudomanifold X is defined to be the image of $[X]_{\mathrm{IP}} \in \Omega^{\mathrm{IP}}_n(X)$ under the map

$$\Omega^{\mathrm{IP}}_n(X) \xrightarrow{\phi_*(X)} \mathbb{L}^\bullet_n(X).$$

If $X = M^n$ is a smooth oriented manifold, then the identity map $M \to M$ defines an orientation class $[M]_{\mathrm{SO}} \in \Omega^{\mathrm{SO}}_n(M)$, which maps to $[M]_{\mathbb{L}}$ under the map

$$\Omega^{\mathrm{SO}}_n(M) \longrightarrow \Omega^{\mathrm{STOP}}_n(M) \longrightarrow \mathbb{L}^\bullet_n(M).$$

Thus, the above definition of $[X]_{\mathbb{L}}$ for an IP pseudomanifold X is compatible with manifold theory in view of the commutativity of diagram (4-1). Applying Ranicki's assembly map, it is then straightforward to define the symmetric signature of an IP pseudomanifold.

DEFINITION 4.2. The symmetric signature $\sigma^*(X) \in L^n(\mathbb{Z}\pi_1(X))$ of an oriented closed n-dimensional IP pseudomanifold X is defined to be the image of $[X]_{\mathbb{L}}$ under the assembly map

$$\mathbb{L}^{\bullet}_n(X) \xrightarrow{A} L^n(\mathbb{Z}\pi_1(X)).$$

This then agrees with the definition of the Mishchenko–Ranicki symmetric signature $\sigma^*(M)$ of a manifold $X = M$ because $A[M]_{\mathbb{L}} = \sigma^*(M)$.

5. Non-Witt spaces

All pseudomanifolds previously considered had to satisfy a vanishing condition for the middle dimensional intersection homology of the links of odd codimensional strata. Can a bordism invariant signature be defined for an even larger class of spaces? As pointed out above, taking the cone on a pseudomanifold immediately proves the futility of such an attempt on the full class of all pseudomanifolds. What, then, are the obstructions for an oriented pseudomanifold to possess Poincaré duality compatible with intersection homology?

Let $\mathcal{LK}$ be a collection of closed oriented pseudomanifolds. We might envision forming a bordism group $\Omega_n^{\mathcal{LK}}$, whose elements are represented by closed oriented n-dimensional stratified pseudomanifolds whose links are all homeomorphic to (finite disjoint unions of) elements of $\mathcal{LK}$. Two spaces X and X' represent the same bordism class, $[X] = [X']$, if there exists an $(n+1)$-dimensional oriented compact pseudomanifold-with-boundary Y^{n+1} such that all links of the interior of Y are in $\mathcal{LK}$ and $\partial Y \cong X \sqcup -X'$ under an orientation-preserving homeomorphism. (The boundary is, as always, to be collared in a stratum-preserving way.) If, for instance,

$$\mathcal{LK} = \{S^1, S^2, S^3, \ldots\},$$

then $\Omega_*^{\mathcal{LK}}$ is bordism of manifolds. If

$$\mathcal{LK} = \text{Odd} \cup \{L^{2l} \mid IH_l^{\bar{m}}(L;\mathbb{Q}) = 0\},$$

where Odd is the collection of all odd dimensional oriented closed pseudomanifolds, then $\Omega_*^{\mathcal{LK}} = \Omega_*^{\text{Witt}}$. The question is: Which other spaces can one throw into this $\mathcal{LK}$, yielding an enlarged collection $\mathcal{LK}' \supset \mathcal{LK}$, such that one can still

define a bordism invariant signature $\sigma : \Omega_*^{\mathcal{LK}'} \longrightarrow \mathbb{Z}$ so that the diagram

$$\begin{array}{ccc} \Omega_*^{\text{Witt}} & \xrightarrow{\sigma} & \mathbb{Z} \\ \downarrow & \nearrow_{\sigma} & \\ \Omega_*^{\mathcal{LK}'} & & \end{array}$$

commutes, where $\Omega_*^{\text{Witt}} \longrightarrow \Omega_*^{\mathcal{LK}'}$ is the canonical map induced by the inclusion $\mathcal{LK} \subset \mathcal{LK}'$? Note that $\sigma(L) = 0$ for every $L \in \mathcal{LK}$. Suppose we took an $\mathcal{LK}'$ that contains a manifold P with $\sigma(P) \neq 0$, e.g. $P = \mathbb{CP}^{2k}$. Then $[P] = 0 \in \Omega_*^{\mathcal{LK}'}$, since P is the boundary of the cone on P, and the cone on P is an admissible bordism in $\Omega_*^{\mathcal{LK}'}$, as the link of the cone-point is P and $P \in \mathcal{LK}'$. Thus, in the above diagram,

$$\begin{array}{ccc} [P] & \xmapsto{\sigma} & \sigma(P) \neq 0. \\ \downarrow & \nearrow_{\sigma} & \\ 0 & & \end{array}$$

This argument shows that the desired diagonal arrow cannot exist for any collection $\mathcal{LK}'$ that contains any manifolds with nonzero signature. Thus we are naturally led to consider only links with zero signature, that is, links whose intersection form on middle dimensional homology possesses a Lagrangian subspace. As you move along a stratum of odd codimension, these Lagrangian subspaces should fit together, forming a subsheaf of the middle dimensional cohomology sheaf **H** associated to the link-bundle over the stratum. (Actually, no bundle neighborhood structure is required to do this.) So a natural language in which to phrase and solve the problem is sheaf theory.

From the sheaf-theoretic vantage point, the statement that a space X^n does not satisfy the Witt condition means precisely that the canonical morphism $\mathbf{IC}^{\bullet}_{\bar{m}}(X) \to \mathbf{IC}^{\bullet}_{\bar{n}}(X)$ from lower to upper middle perversity is not an isomorphism in the derived category. (We are using sheaves of real vector spaces now and shall not indicate this further in our notation.) Thus there is no way to introduce a quadratic form whose signature one could take, using intersection chain sheaves. But one may ask how close to such sheaves one might get by using self-dual sheaves on X. In [Ban02], we define a full subcategory $SD(X)$ of the derived category on X, whose objects $\mathbf{S}^{\bullet}$ satisfy all the axioms that $\mathbf{IC}^{\bullet}_{\bar{n}}(X)$ satisfies, with the exception of the last axiom, the costalk vanishing axiom. This axiom is replaced with the requirement that $\mathbf{S}^{\bullet}$ be self-dual, that is, there is an isomorphism $\mathcal{D}\mathbf{S}^{\bullet}[n] \cong \mathbf{S}^{\bullet}$, just as there is for $\mathbf{IC}^{\bullet}_{\bar{m}}$ on a Witt space. Naturally, this category may be empty, depending on the geometry of X. So we need to develop a structure theorem for $SD(X)$, and this is done in [Ban02]. It turns out that every such object $\mathbf{S}^{\bullet}$ interpolates between $\mathbf{IC}^{\bullet}_{\bar{m}}$ and $\mathbf{IC}^{\bullet}_{\bar{n}}$, i.e. possesses a

factorization $\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{S}^\bullet \to \mathbf{IC}^\bullet_{\bar{n}}$ of the canonical morphism. The two morphisms of the factorization are dual to each other. Note that in the basic two strata case, the mapping cone of the canonical morphism is the middle cohomology sheaf $\mathbf{H}$ of the link-bundle. We prove that the mapping cone of $\mathbf{IC}^\bullet_{\bar{m}} \to \mathbf{S}^\bullet$, restricted to the stratum of odd codimension, is a Lagrangian subsheaf of $\mathbf{H}$, so that the circle to the above geometric ideas closes. The main result of [Ban02] is an equivalence of categories between $SD(X)$ and a fibered product of categories of Lagrangian structures, one such category for each stratum of odd codimension. This then is a kind of Postnikov system for $SD(X)$, encoding both the obstruction theory and the constructive technology to manufacture objects in $SD(X)$.

Suppose X is such that $SD(X)$ is not empty. An object $\mathbf{S}^\bullet$ in $SD(X)$ defines a signature $\sigma(\mathbf{S}^\bullet) \in \mathbb{Z}$ by taking the signature of the quadratic form that the self-duality isomorphism $\mathcal{D}\mathbf{S}^\bullet[n] \cong \mathbf{S}^\bullet$ induces on the middle dimensional hypercohomology group of $\mathbf{S}^\bullet$. Since restricting a self-dual sheaf to a transverse (to the stratification) subvariety again yields a self-dual sheaf on the subvariety, we get a signature for all transverse subvarieties and thus an L-class $L_*(\mathbf{S}^\bullet) \in H_*(X;\mathbb{Q})$, using maps to spheres and Serre's theorem as indicated in the beginning. We prove in [Ban06b] that $L_*(\mathbf{S}^\bullet)$, in particular $\sigma(\mathbf{S}^\bullet) = L_0(\mathbf{S}^\bullet)$, is independent of the choice of $\mathbf{S}^\bullet$ in $SD(X)$. Consequently, a non-Witt space has a well-defined L-class $L_*(X)$ and signature $\sigma(X)$, provided $SD(X)$ is not empty.

Let $\mathrm{Sig}_n(\mathrm{pt})$ be the bordism group of pairs $(X, \mathbf{S}^\bullet)$, where X is a closed oriented topological or PL n-dimensional pseudomanifold and $\mathbf{S}^\bullet$ is an object of $SD(X)$. Admissible bordisms are oriented compact pseudomanifolds-with-boundary Y^{n+1}, whose interior $\operatorname{int} Y$ is covered with an object of $SD(\operatorname{int} Y)$ which pushes to the given sheaf complexes on the boundary. These groups have been introduced in [Ban02] under the name Ω^{SD}_*. Let us compute these groups. The signature $(X, \mathbf{S}^\bullet) \mapsto \sigma(\mathbf{S}^\bullet)$ is a bordism invariant and hence induces a map $\sigma : \mathrm{Sig}_{4k}(\mathrm{pt}) \to \mathbb{Z}$. This map is onto, since e.g. $(\mathbb{CP}^{2k}, \mathbb{R}_{\mathbb{CP}^{2k}}[4k])$ (and disjoint copies of it) is in $\mathrm{Sig}_{4k}(\mathrm{pt})$. However, contrary for example to Witt bordism, σ is also injective: Suppose $\sigma(X, \mathbf{S}^\bullet) = 0$. Let Y^{4k+1} be the closed cone on X. Define a self-dual sheaf on the interior of the punctured cone by pulling back $\mathbf{S}^\bullet$ from X under the projection from the interior of the punctured cone, $X \times (0,1)$, to X. According to the Postnikov system of Lagrangian structures for $SD(\operatorname{int} Y)$, the self-dual sheaf on the interior of the punctured cone will have a self-dual extension in $SD(\operatorname{int} Y)$ if, and only if, there exists a Lagrangian structure at the cone-point (which has odd codimension $4k+1$ in Y). That Lagrangian structure exists because $\sigma(X, \mathbf{S}^\bullet) = 0$. Let $\mathbf{T}^\bullet \in SD(\operatorname{int} Y)$ be any self-dual extension given by a choice of Lagrangian structure. Then $\partial(Y, \mathbf{T}^\bullet) = (X, \mathbf{S}^\bullet)$ and thus $[(X, \mathbf{S}^\bullet)] = 0$ in $\mathrm{Sig}_{4k}(\mathrm{pt})$. Clearly, $\mathrm{Sig}_n(\mathrm{pt}) = 0$ for $n \not\equiv 0(4)$

because an anti-symmetric form always has a Lagrangian subspace and the cone on an odd dimensional space is even dimensional, so in these cases there are no extension problems at the cone point — just perform a one-step Goresky–MacPherson–Deligne extension. In summary then, one has

$$\mathrm{Sig}_n(\mathrm{pt}) \cong \begin{cases} \mathbb{Z}, & n \equiv 0(4), \\ 0, & n \not\equiv 0(4). \end{cases}$$

(Note that in particular the de Rham invariant has been disabled and the signature is a complete invariant for these bordism groups.) Minatta [Min04], [Min06] takes this as his starting point and constructs a bordism theory $\mathrm{Sig}_*(-)$, called *signature homology*, whose coefficients are the above groups $\mathrm{Sig}_*(\mathrm{pt})$. Elements of $\mathrm{Sig}_n(Y)$ are represented by pairs $(X, \mathbf{S}^\bullet)$ as above together with a continuous map $X \to Y$. For a detailed proof that $\mathrm{Sig}_*(-)$ is a generalized homology theory when PL pseudomanifolds are used, consult the appendix of [Ban06a]. Signature homology is represented by an MSO module spectrum MSIG, which is also a ring spectrum. Regarding a smooth manifold as a pseudomanifold with one stratum covered by the constant sheaf of rank 1 concentrated in one dimension defines a natural transformation of homology theories $\Omega_*^{\mathrm{SO}}(-) \to \mathrm{Sig}_*(-)$. Thus, MSIG is 2-integrally a product of Eilenberg–Mac Lane spectra,

$$\mathrm{MSIG}_{(2)} \simeq \prod_{j \geq 0} K(\mathbb{Z}_{(2)}, 4j).$$

As for the odd-primary situation, the isomorphism $\mathrm{Sig}_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \to \mathbb{Z}[\frac{1}{2}, t]$ given by $[(X^{4k}, \mathbf{S}^\bullet)] \otimes a \mapsto a\sigma(\mathbf{S}^\bullet)t^k$, determines an identification

$$\Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathrm{Sig}_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} \Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t].$$

A natural isomorphism of homology theories

$$\Omega_*^{\mathrm{SO}}(Y) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathrm{Sig}_*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} \mathrm{Sig}_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

is induced by sending $[M \xrightarrow{f} Y] \otimes [(X, \mathbf{S}^\bullet)]$ to $[(M \times X, \mathbf{P}^\bullet, M \times X \to M \xrightarrow{f} Y)]$, where $\mathbf{P}^\bullet$ is the pullback sheaf of $\mathbf{S}^\bullet$ under the second-factor projection. Composing, we obtain a natural isomorphism

$$\Omega_*^{\mathrm{SO}}(-) \otimes_{\Omega_*^{\mathrm{SO}}(\mathrm{pt})} \mathbb{Z}[\tfrac{1}{2}, t] \xrightarrow{\cong} \mathrm{Sig}_*(-) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}],$$

describing signature homology at odd primes in terms of manifold bordism.

Again, it follows in particular that the natural map

$$\Omega_*^{\mathrm{SO}}(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}, t] \to \mathrm{Sig}_*(Y) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{2}]$$

is a surjection, which frequently allows one to reduce bordism invariant calculations on non-Witt spaces to the manifold case. We observed this in [Ban06a] and apply it there to establish a multiplicative characteristic class formula for the twisted signature and L-class of non-Witt spaces. Let X^n be a closed oriented Whitney stratified pseudomanifold and let $\mathcal{S}$ be a nondegenerate symmetric local system on X. If $SD(X)$ is not empty, that is, X possesses Lagrangian structures along its strata of odd codimension so that $L_*(X) \in H_*(X;\mathbb{Q})$ is defined, then

$$L_*(X;\mathcal{S}) = \widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L_*(X).$$

For the special case of the twisted signature $\sigma(X;\mathcal{S}) = L_0(X;\mathcal{S})$, one has therefore

$$\sigma(X;\mathcal{S}) = \langle \widetilde{\mathrm{ch}}[\mathcal{S}]_K, L(X)\rangle.$$

We shall apply the preceding ideas in defining a *Sullivan orientation* $\Delta(X) \in ko_*(X)\otimes\mathbb{Z}[\frac{1}{2}]$ for a pseudomanifold X that possesses generalized Poincaré duality (that is, its self-dual perverse category $SD(X)$ is not empty), but need not satisfy the Witt condition. In [Sul05], Sullivan defined for an oriented rational PL homology manifold M an orientation class $\Delta(M) \in ko_*(M)\otimes\mathbb{Z}[\frac{1}{2}]$, whose Pontrjagin character is the L-class $L_*(M)$. For a Witt space X^n, a Sullivan class $\Delta(X) \in ko_*(X)\otimes\mathbb{Z}[\frac{1}{2}]$ was constructed by Siegel [Sie83], using the intersection homology signature of a Witt space and transversality to produce the requisite Sullivan periodicity squares that represent elements of $KO^{4k}(N,\partial N)\otimes\mathbb{Z}[\frac{1}{2}]$, where N is a regular neighborhood of a codimension $4k$ PL-embedding of X in a high dimensional Euclidean space. An element in $ko^{4k}(N,\partial N)\otimes\mathbb{Z}[\frac{1}{2}]$ corresponds to a unique element in $ko_n(X)\otimes\mathbb{Z}[\frac{1}{2}]$ by Alexander duality. Siegel's isomorphism (3-1) is then given by the Hurewicz-type map

$$\begin{aligned}\Omega_*^{\mathrm{Witt}}(Y)\otimes\mathbb{Z}[\tfrac{1}{2}] &\longrightarrow ko_*(Y)\otimes\mathbb{Z}[\tfrac{1}{2}]\\ [X \xrightarrow{f} Y]\otimes 1 &\mapsto f_*\Delta(X),\end{aligned}$$

where $f_* : ko_*(X)\otimes\mathbb{Z}[\frac{1}{2}] \to ko_*(Y)\otimes\mathbb{Z}[\frac{1}{2}]$. In particular, the transformation (3-1) maps the Witt orientation class $[X]_{\mathrm{Witt}} \in \Omega_n^{\mathrm{Witt}}(X)$, given by the identity map $f = \mathrm{id}_X : X \to X$, to $\Delta(X)$.

REMARK 5.1. In [CSW91], there is indicated an extension to continuous actions of a finite group G on a Witt space X. If the action satisfies a weak condition on the fixed point sets, then there is a homeomorphism invariant class $\Delta^G(X)$ in the equivariant KO-homology of X away from 2, which is the Atiyah–Singer G-signature invariant for smooth actions on smooth manifolds.

Let P be a compact polyhedron. Using Balmer's 4-periodic Witt groups of triangulated categories with duality, Woolf [Woo08] defines groups $W_*^c(P)$, called constructible Witt groups of P because the underlying triangulated categories

are the derived categories of sheaf complexes that are constructible with respect to the simplicial stratifications of admissible triangulations of P. (The duality is given by Verdier duality.) Elements of $W_n^c(P)$ are represented by symmetric self-dual isomorphisms $d : \mathbf{S}^\bullet \to (\mathcal{D}\mathbf{S}^\bullet)[n]$. The periodicity isomorphism $W_n^c(P) \cong W_{n+4}^c(P)$ is induced by shifting such a d twice:

$$d[2] : \mathbf{S}^\bullet[2] \longrightarrow (\mathcal{D}\mathbf{S}^\bullet)[n][2] = (\mathcal{D}\mathbf{S}^\bullet)[-2][n+4] = \mathcal{D}(\mathbf{S}^\bullet[2])[n+\mathbf{4}].$$

(Shifting only once does not yield a correct symmetric isomorphism with respect to the duality fixed for $W_*^c(P)$.) Woolf shows that for commutative regular Noetherian rings R of finite Krull dimension in which 2 is invertible, for example $R = \mathbb{Q}$, the assignment $P \mapsto W_*^c(P)$ is a generalized homology theory on compact polyhedra and continuous maps. Let K be a simplicial complex triangulating P. Relating both Ranicki's (R, K)-modules on the one hand and constructible sheaves on the other hand to combinatorial sheaves on K, Woolf obtains a natural transformation

$$\mathbb{L}^\bullet(R)_*(K) \longrightarrow W_*^c(|K|)$$

($|K| = P$), which he shows to be an isomorphism when every finitely generated R-module can be resolved by a finite complex of finitely generated free R-modules. Again, this applies to $R = \mathbb{Q}$. Given a map $f : X^n \to P$ from a compact oriented Witt space X^n into P, the pushforward $Rf_*(d)$ of the symmetric self-duality isomorphism $d : \mathbf{IC}_{\bar{m}}^\bullet(X) \cong \mathcal{D}\mathbf{IC}_{\bar{m}}^\bullet(X)[n]$ defines an element $[Rf_*(d)] \in W_n^c(P)$. This induces a natural map

$$\Omega_n^{\mathrm{Witt}}(P) \longrightarrow W_n^c(P),$$

which is an isomorphism when $n > \dim P$. Given any $n \geq 0$, we can iterate the 4-periodicity until $n + 4* > \dim P$ and obtain

$$W_n^c(P) \cong W_{n+4}^c(P) \cong \cdots \cong W_{n+4k}^c(P) \cong \Omega_{n+4k}^{\mathrm{Witt}}(P),$$

where $n + 4k > \dim P$. Thus, as Woolf points out, the Witt class of any symmetric self-dual sheaf on P is given, after a suitable even number of shifts, by the pushforward of an intersection chain sheaf on some Witt space. This viewpoint also allows for the interpretation of L-classes as homology operations $W_*^c(-) \to H_*(-)$ or $\Omega_*^{\mathrm{Witt}}(-) \to H_*(-)$. Other characteristic classes arising in complex algebraic geometry can be interpreted through natural transformations as well. MacPherson's Chern class of a variety can be defined as the image $c_*^{\mathrm{M}}(1_X)$ of the function 1_X under a natural transformation $c_*^{\mathrm{M}} : F(-) \to H_*(-)$, where $F(X)$ is the abelian group of constructible functions on X. The Baum–Fulton–MacPherson Todd class can be defined as the image $\mathrm{td}_*^{\mathrm{BMF}}(\mathcal{O}_X)$ of $\mathcal{O}_X$

under a natural transformation

$$\mathrm{td}_*^{\mathrm{BMF}} : G_0(-) \to H_*(-) \otimes \mathbb{Q},$$

where $G_0(X)$ is the Grothendieck group of coherent sheaves on X. In [BSY], Brasselet, Schürmann and Yokura realized two important facts: First, there exists a source $K_0(\mathcal{VAR}/X)$ which possesses natural transformations to all three domains of the characteristic class transformations mentioned. That is, there exist natural transformations

$$\begin{array}{ccccc} & & K_0(\mathcal{VAR}/-) & & \\ & \swarrow & \downarrow & \searrow & \\ F(-) & & G_0(-) & & \Omega^{\mathrm{Y}}(-), \end{array} \tag{5-1}$$

where $\Omega^{\mathrm{Y}}(X)$ is the abelian group of Youssin's bordism classes of self-dual constructible sheaf complexes on X. That source $K_0(\mathcal{VAR}/X)$ is the free abelian group generated by algebraic morphisms $f : V \to X$ modulo the relation

$$[V \xrightarrow{f} X] = [V - Z \xrightarrow{f|} X] + [Z \xrightarrow{f|} X]$$

for every closed subvariety $Z \subset V$. Second, there exists a unique natural transformation, the *motivic characteristic class transformation*,

$$T_{y*} : K_0(\mathcal{VAR}/X) \longrightarrow H_*(X) \otimes \mathbb{Q}[y]$$

such that

$$T_{y*}[\mathrm{id}_X] = T_y(TX) \cap [X]$$

for nonsingular X, where $T_y(TX)$ is Hirzebruch's generalized Todd class of the tangent bundle TX of X. Characteristic classes for singular varieties are of course obtained by taking $T_{y*}[\mathrm{id}_X]$. Under the above three transformations (5-1), $[\mathrm{id}_X]$ is mapped to 1_X, $[\mathcal{O}_X]$, and $[\mathbb{Q}_X[2\dim X]]$ (when X is nonsingular), respectively. Following these three transformations with c_*^{M}, $\mathrm{td}_*^{\mathrm{BMF}}$, and the L-class transformation

$$\Omega^{\mathrm{Y}}(-) \to H_*(-) \otimes \mathbb{Q},$$

one obtains T_{y*} for $y = -1, 0, 1$, respectively. This, then, is an attractive unification of Chern-, Todd- and L-classes of singular complex algebraic varieties, see also Yokura's paper in this volume, as well as [SY07].

The natural transformation

$$\Omega_*^{\mathrm{Witt}}(-) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow \mathrm{Sig}_*(-) \otimes \mathbb{Z}[\tfrac{1}{2}],$$

given by covering a Witt space X with the middle perversity intersection chain sheaf $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(X)$, which is an object of $SD(X)$, is an isomorphism because on a point, it is given by the signature

$$\Omega^{\text{Witt}}_{4k}(\text{pt}) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong L^{4k}(\mathbb{Q}) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}] \cong \text{Sig}_{4k}(\text{pt}) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

(the infinitely generated torsion of $L^{4k}(\mathbb{Q})$ is killed by inverting 2), and

$$\Omega^{\text{Witt}}_{j}(\text{pt}) = 0 = \text{Sig}_j(\text{pt})$$

for j not divisible by 4. Inverting this isomorphism and composing with Siegel's isomorphism (3-1), we obtain a natural isomorphism of homology theories

$$D : \text{Sig}_*(-) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\cong} ko_*(-) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Let X^n be a closed pseudomanifold, not necessarily a Witt space, but still supporting self-duality, i.e. $SD(X)$ is not empty. Choose a sheaf $\mathbf{S}^\bullet \in SD(X)$. Then the pair $(X, \mathbf{S}^\bullet)$, together with the identity map $X \to X$, defines an element $[X]_{\text{Sig}} \in \text{Sig}_n(X)$.

DEFINITION 5.2. The *signature homology orientation class* of an n-dimensional closed pseudomanifold X with $SD(X) \neq \varnothing$, but not necessarily a Witt space, is the element $[X]_{\text{Sig}} \in \text{Sig}_n(X)$.

PROPOSITION 5.3. The orientation class $[X]_{\text{Sig}}$ is well-defined, that is, independent of the choice of sheaf $\mathbf{S}^\bullet \in SD(X)$.

PROOF. Let $\mathbf{T}^\bullet \in SD(X)$ be another choice. In [Ban06b], a bordism $(Y, \mathbf{U}^\bullet)$, $\mathbf{U}^\bullet \in SD(\text{int}\, Y)$, is constructed between $(X, \mathbf{S}^\bullet)$ and $(X, \mathbf{T}^\bullet)$. Topologically, Y is a cylinder $Y \cong X \times I$, but equipped with a nonstandard stratification, of course. The identity map $X \to X$ thus extends over this bordism by taking

$$Y \to X, \quad (x, t) \mapsto x. \qquad \square$$

DEFINITION 5.4. The *Sullivan orientation* of an n-dimensional closed pseudomanifold X with $SD(X) \neq \varnothing$, but not necessarily a Witt space, is defined as

$$\Delta(X) = D([X]_{\text{Sig}} \otimes 1) \in ko_*(X) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Let us compare signature homology and $\mathbb{L}^\bullet$-homology away from 2, at 2, and rationally, following [Epp07] and drawing on work of Taylor and Williams, [TW79]. For a spectrum S, let $S_{(\text{odd})}$ denote its localization at odd primes. We have observed above that

$$\text{MSIG}_{(2)} \simeq \prod_{j \geq 0} K(\mathbb{Z}_{(2)}, 4j)$$

and, according to [Epp07] and [Min04],

$$\mathrm{MSIG}_{(\mathrm{odd})} \simeq bo_{(\mathrm{odd})}.$$

Rationally, we have the decomposition

$$\mathrm{MSIG} \otimes \mathbb{Q} \simeq \prod_{j \geq 0} K(\mathbb{Q}, 4j).$$

Thus MSIG fits into a localization pullback square

$$\begin{array}{ccc} \mathrm{MSIG} & \xrightarrow{\mathrm{loc}_{(\mathrm{odd})}} & bo_{(\mathrm{odd})} \\ {\scriptstyle \mathrm{loc}_{(2)}}\downarrow & & \downarrow \\ \prod K(\mathbb{Z}_{(2)}, 4j) & \xrightarrow{\lambda} & \prod K(\mathbb{Q}, 4j). \end{array}$$

The symmetric L-spectrum $\mathbb{L}^\bullet$ is an MSO module spectrum, so it is 2-integrally a product of Eilenberg–Mac Lane spectra,

$$\mathbb{L}^\bullet_{(2)} \simeq \prod_{j \geq 0} K(\mathbb{Z}_{(2)}, 4j) \times K(\mathbb{Z}/_2, 4j+1).$$

Comparing this to $\mathrm{MSIG}_{(2)}$, we thus see the de Rham invariants coming in. Away from 2, $\mathbb{L}^\bullet$ coincides with bo,

$$\mathbb{L}^\bullet_{(\mathrm{odd})} \simeq bo_{(\mathrm{odd})},$$

as does MSIG. Rationally, $\mathbb{L}^\bullet$ is again

$$\mathbb{L}^\bullet \otimes \mathbb{Q} \simeq \prod_{j \geq 0} K(\mathbb{Q}, 4j).$$

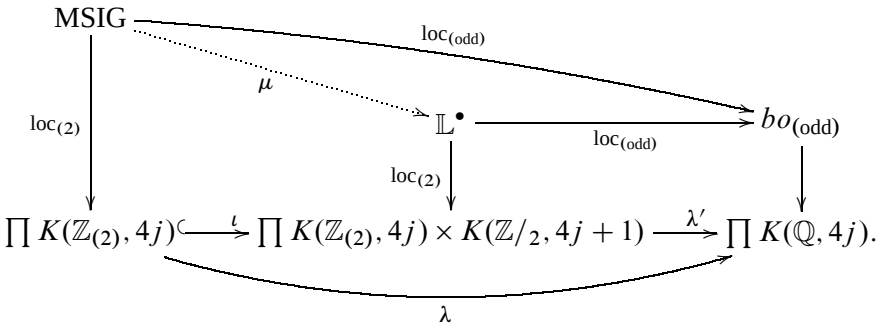
Thus $\mathbb{L}^\bullet$ fits into a localization pullback square

$$\begin{array}{ccc} \mathbb{L}^\bullet & \xrightarrow{\qquad\mathrm{loc}_{(\mathrm{odd})}\qquad} & bo_{(\mathrm{odd})} \\ {\scriptstyle \mathrm{loc}_{(2)}}\downarrow & & \downarrow \\ \prod K(\mathbb{Z}_{(2)}, 4j) \times K(\mathbb{Z}/_2, 4j+1) & \xrightarrow{\lambda'} & \prod K(\mathbb{Q}, 4j). \end{array}$$

The map λ factors as

$$\prod_{j \geq 0} K(\mathbb{Z}_{(2)}, 4j) \xhookrightarrow{\iota} \prod_{j \geq 0} K(\mathbb{Z}_{(2)}, 4j) \times K(\mathbb{Z}/_2, 4j+1) \xrightarrow{\lambda'} \prod_{j \geq 0} K(\mathbb{Q}, 4j),$$

where ι is the obvious inclusion, not touching the 2-torsion nontrivially. Hence, by the universal property of a pullback, we get a map μ from signature homology to $\mathbb{L}^\bullet$-homology,



On the other hand, λ' factors as

$$\prod_{j\geq 0} K(\mathbb{Z}_{(2)}, 4j) \times K(\mathbb{Z}/_2, 4j+1) \xrightarrow{\text{proj}} \prod_{j\geq 0} K(\mathbb{Z}_{(2)}, 4j) \xrightarrow{\lambda} \prod_{j\geq 0} K(\mathbb{Q}, 4j),$$

where proj is the obvious projection. Again using the universal property of a pullback, we obtain a map $\nu : \mathbb{L}^\bullet \to \text{MSIG}$. The map μ is a homotopy splitting for ν, $\nu\mu \simeq \text{id}$, since $\text{proj} \circ \iota = \text{id}$. It follows that via μ, signature homology is a direct summand in symmetric $\mathbb{L}^\bullet$-homology. We should like to point out that the diagram

$$\begin{array}{ccc} \text{MSO} & \longrightarrow & \text{MSIG} \\ \downarrow & & \downarrow \mu \\ \text{MSTOP} & \longrightarrow & \mathbb{L}^\bullet \end{array}$$

does *not* commute. This is essentially due to the fact that the de Rham invariant is lost in Sig_*, but is still captured in $\mathbb{L}^\bullet_*$. In more detail, consider the induced diagram on π_5,

$$\begin{array}{ccc} \Omega_5^{\text{SO}}(\text{pt}) & \longrightarrow & \text{Sig}_5(\text{pt}) = 0 \\ \downarrow & & \downarrow \mu \\ \Omega_5^{\text{STOP}}(\text{pt}) & \longrightarrow & \mathbb{L}^\bullet_5(\text{pt}). \end{array}$$

The clockwise composition in the diagram is zero, but the counterclockwise composition is not. Indeed, let M^5 be the Dold manifold $P(1,2) = (S^1 \times \mathbb{CP}^2)/(x,z) \sim (-x,\bar{z})$. Its cohomology ring with $\mathbb{Z}/_2$-coefficients is the same as the one of the untwisted product, that is, the truncated polynomial ring

$$\mathbb{Z}/_2[c,d]/(c^2 = 0, d^3 = 0),$$

where c has degree one and d has degree two. The total Stiefel–Whitney class of M^5 is

$$w(M) = (1+c)(1+c+d)^3,$$

so that the de Rham invariant $\mathrm{dR}(M)$ is given by $\mathrm{dR}(M) = w_2 w_3(M) = cd^2$, which is the generator. We also see that $w_1(M) = 0$, so that M is orientable. The counterclockwise composition maps the bordism class of a smooth 5-manifold to its de Rham invariant in $\mathbb{L}^\bullet_5(\mathrm{pt}) = L^5(\mathbb{Z}e) = \mathbb{Z}/_2$. The Dold manifold M^5 represents the generator $[M^5] \in \Omega^{\mathrm{SO}}_5(\mathrm{pt}) = \mathbb{Z}/_2$. Thus the counterclockwise composition is the identity map $\mathbb{Z}/_2 \to \mathbb{Z}/_2$ and the diagram does not commute. Mapping M^5 to a point and using the naturality of the assembly map induces a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Sig}_5(M) & \xrightarrow{\mu(M)} & \mathbb{L}^\bullet_5(M) & \xrightarrow{A} & L^5(\mathbb{Z}\pi_1 M) \\
\downarrow & & \downarrow & & \downarrow \varepsilon \\
0 = \mathrm{Sig}_5(\mathrm{pt}) & \xrightarrow{\mu(\mathrm{pt})} & \mathbb{L}^\bullet_5(\mathrm{pt}) & \xrightarrow[A]{\cong} & L^5(\mathbb{Z}e) = \mathbb{Z}/_2,
\end{array}$$

which shows that the signature homology orientation class of M, $[M]_{\mathrm{Sig}} \in \mathrm{Sig}_5(M)$ does not hit the $\mathbb{L}^\bullet$-orientation of M, $[M]_{\mathbb{L}} \in \mathbb{L}^\bullet_5(M)$ under μ, for otherwise

$$0 = \varepsilon A \mu [M]_{\mathrm{Sig}} = \varepsilon A [M]_{\mathbb{L}} = \varepsilon \sigma^*(M) = \mathrm{dR}(M) \neq 0.$$

Thus one may take the viewpoint that it is perhaps not prudent to call $\mu[X]_{\mathrm{Sig}}$ an "$\mathbb{L}^\bullet$-orientation" of a pseudomanifold X with $SD(X)$ not empty. Nor might even its image under assembly deserve the title "symmetric signature" of X. On the other hand, one may wish to attach higher priority to the bordism invariance (in the singular world) of a concept such as the symmetric signature than to its compatibility with manifold invariants and nonsingular bordism invariance, and therefore deem such terminology justified.

We conclude with a brief remark on integral Novikov problems. Let π be a discrete group and let $K(\pi,1)$ be the associated Eilenberg–Mac Lane space. The composition of the split inclusion $\mathrm{Sig}_n(K(\pi,1)) \hookrightarrow \mathbb{L}^\bullet_n(K(\pi,1))$ with the assembly map

$$A : \mathbb{L}^\bullet_n(K(\pi,1)) \to L^n(\mathbb{Z}\pi)$$

yields what one may call a "signature homology assembly" map

$$A_{\mathrm{Sig}} : \mathrm{Sig}_n(K(\pi,1)) \to L^n(\mathbb{Z}\pi),$$

which may be helpful in studying an integral refinement of the Novikov conjecture, as suggested by Matthias Kreck: When is the integral orientation class

$$\alpha_*[M]_{\mathrm{Sig}} \in \mathrm{Sig}_n(K(\pi,1))$$

homotopy invariant? Here M^n is a closed smooth oriented manifold with fundamental group $\pi = \pi_1(M)$; the map $\alpha : M \to K(\pi,1)$ classifies the universal cover of M. Note that when tensored with the rationals, one obtains the classical Novikov conjecture because rationally the signature homology orientation class $[M]_{\mathrm{Sig}}$ is the L-class $L_*(M)$. One usually refers to integral refinements such as this one as "Novikov *problems*" because there are groups π for which they are known to be false.

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