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On Milnor classes of complex hypersurfaces

LAURENTIU MAXIM

ABSTRACT. We revisit known results about the Milnor class of a singular complex hypersurface, and rephrase some of them in a way that allows for a better comparison with the topological formula of Cappell and Shaneson for the L-class of such a hypersurface. Our approach is based on Verdier's specialization property for the Chern–MacPherson class, and simple constructible function calculus.

1. Introduction

It is well-known that for a compact complex hypersurface X with only isolated singularities the sum of the Milnor numbers at the singular points measures (up to a sign) the difference between the topological Euler characteristic of X and that of a nonsingular hypersurface linearly equivalent to X, provided such a hypersurface exists. This led Parusiński to a generalization of the notion of Milnor number to nonisolated hypersurface singularities [16], which in the case of isolated singularities reduces to the sum of Milnor numbers at the singular points.

For a (possibly singular) compact complex hypersurface X, the Euler characteristic $\chi(X)$ equals the degree of the zero-dimensional component of the Chern–MacPherson homology class $c_*(X)$; see [15]. On the other hand, the Euler characteristic of a nonsingular hypersurface linearly equivalent to X is just the degree of the Poincaré dual of the Chern class of the virtual tangent bundle of X, that is, the degree of the Fulton–Johnson class $c_*^{FJ}(X)$ [10; 11]. Thus, Parusiński's Milnor number equals (up to a sign) the degree of the homology class $c_*^{FJ}(X) - c_*(X)$. It is therefore natural to try to understand the

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higher-degree components of this difference class, which usually is called the *Milnor class* of X. The study of the Milnor class also comes up naturally while searching for a Verdier-type Riemann–Roch theorem for the Chern–MacPherson classes (see [20; 22; 23]); indeed, the Milnor class measures the defect of commutativity in a Verdier–Riemann–Roch diagram for MacPherson's Chern class transformation.

While the problem of understanding the Milnor class in terms of invariants of singularities can be formulated in more general contexts (e.g., for local complete intersections, or regular embeddings in arbitrary codimension, see [19; 20]), in this note we restrict ourselves, for simplicity, only to the case of hypersurfaces (i.e., regular embeddings in codimension 1) in complex manifolds. We recall known results about the Milnor class of a singular hypersurface, and rephrase some of these results in a way that, we believe, reflects better the geometry of the singular locus in terms of its stratification. For more comprehensive surveys on Milnor classes, the interested reader is advised to consult [2; 3; 17; 22].

The approach presented in this note is based on a well-known specialization argument [21], and simple calculus of constructible functions as developed in [9]. While this approach is not new (see [18; 19; 20] for similar considerations), the formulation of our main results (Theorem 4.3, Corollary 4.4 and Theorem 4.6) has the advantage of being conceptually very simple, and it allows for a better comparison with the topological formula of Cappell and Shaneson [7; 8] for the L-classes of singular hypersurfaces. Indeed, we also explore a Chern-class analogue of Goresky–MacPherson's homology L-class [12], defined via the constructible function associated to the intersection chain complex of a variety (see [9]). This class, which for a variety X is denoted by $Ic_*(X)$, encodes very detailed information about the geometry of a fixed Whitney stratification of X. In the case of hypersurfaces, we compare this class with the Fulton–Johnson class, and derive a formula for their difference in terms of invariants of the singular locus.

2. Canonical bases for the group of constructible functions

Let X be a topological space with a finite partition $\mathcal{V}$ into a disjoint union of finitely many connected subsets V satisfying the *frontier condition*:

$$W \cap \overline{V} \neq \varnothing \implies W \subset \overline{V}.$$

The main examples of such spaces are complex algebraic or compact complex analytic varieties with a fixed Whitney stratification. Consider on $\mathcal{V}$ the partial order given by

$$W \leq V \iff W \subset \overline{V}.$$

We also write $W < V$ if $W \leq V$ and $W \neq V$.

Let $F_{\mathcal{V}}(X)$ be the abelian group of $\mathcal{V}$-constructible functions on X, that is, functions $\alpha : X \to \mathbb{Z}$ such that $\alpha|_V$ is constant for all $V \in \mathcal{V}$. This is a free abelian group with basis

$$\mathcal{B}_1 := \{\, 1_V \mid V \in \mathcal{V} \,\},$$

so that any $\alpha \in F_{\mathcal{V}}(X)$ can be written as

$$\alpha = \sum_{V \in \mathcal{V}} \alpha(V) \cdot 1_V. \tag{2-1}$$

In what follows, we will discuss two more canonical bases on $F_{\mathcal{V}}(X)$, see [9] for complete details. First, the collection

$$\mathcal{B}_2 := \{\, 1_{\overline{V}} \mid V \in \mathcal{V} \,\}$$

is also a basis for $F_{\mathcal{V}}(X)$, since

$$1_{\overline{V}} = \sum_{W \leq V} 1_W$$

and the transition matrix $A = (a_{W,V})$, where $a_{W,V}$ is defined as 1 if $W \leq V$ and 0 otherwise, is upper triangular with respect to $\leq$, with all diagonal entries equal to 1 (so A is invertible). In this basis, a constructible function $\alpha \in F_{\mathcal{V}}(X)$ can be expressed by the identity

$$\alpha = \sum_{V} \alpha(V) \cdot \hat{1}_{\overline{V}}, \tag{2-2}$$

(see [9, Proposition 2.1]), where for each $V \in \mathcal{V}$, we define $\hat{1}_{\overline{V}}$ inductively by the formula

$$\hat{1}_{\overline{V}} = 1_{\overline{V}} - \sum_{W < V} \hat{1}_{\overline{W}}.$$

Note that if there is a stratum $S \in \mathcal{V}$ which is dense in X, i.e., $\overline{S} = X$, so $V \leq S$ for all $V \in \mathcal{V}$, then (2-2) can be rewritten as

$$\alpha = \alpha(S) \cdot 1_X + \sum_{V < S} (\alpha(V) - \alpha(S)) \cdot \hat{1}_{\overline{V}}. \tag{2-3}$$

If moreover $\alpha|_S = 0$, this reduces further to

$$\alpha = \sum_{V < S} \alpha(V) \cdot \hat{1}_{\overline{V}}. \tag{2-4}$$

In order to describe the third basis for the group of constructible functions, assume moreover that X is a topological pseudomanifold with a stratification $\mathcal{V}$ by finitely many oriented strata of *even* dimension. Then, by definition, the

strata of $\mathcal{V}$ satisfy the frontier condition, and $\mathcal{V}$ is locally topologically trivial along each stratum V, with fibers the cone on a compact pseudomanifold $L_{V,X}$, the *link* of V in X. Each stratum V, and also its closure $\overline{V}$, get an induced stratification of the same type. Important examples are provided by a complex algebraic (or analytic) Whitney stratification of a reduced complex algebraic (or compact complex analytic) variety.

For each $V \in \mathcal{V}$, let $IC_{\overline{V}}$ be the intersection cohomology complex [13] associated to the closure of V in X. This is a $\mathcal{V}$-constructible complex of sheaves (i.e., the restrictions of its cohomology sheaves to strata $W < V$ are locally constant), satisfying the normalization property that $IC_{\overline{V}}|_V = \mathbb{Q}_V$ (following Borel's indexing conventions). After extending by zero, we regard all these intersection chain sheaves as complexes on X. Let us fix for each $W \in \mathcal{V}$ a point $w \in W$ with inclusion $i_w : \{w\} \hookrightarrow X$. We now define a constructible function $ic_{\overline{V}} \in F_{\mathcal{V}}(X)$ by taking stalkwise the Euler characteristic for the complex $IC_{\overline{V}}$. That is, for $w \in W < V$ we let

$$ic_{\overline{V}}(w) := \chi(i_w^* IC_{\overline{V}}) = \chi(IH^*(c^\circ L_{W,V})) \stackrel{\text{def}}{=} I\chi(c^\circ L_{W,V}), \tag{2-5}$$

where $c^\circ L_{W,V}$ denotes the open cone on the link $L_{W,V}$ of W in $\overline{V}$, and $I\chi(-)$ stands for the intersection homology Euler characteristic. Moreover,

$$ic_{\overline{V}}|_V = 1_V. \tag{2-6}$$

Since clearly $\mathrm{supp}(ic_{\overline{V}}) = \overline{V}$, it is now easy to see that the collection

$$\mathcal{B}_3 := \{\, ic_{\overline{V}} \mid V \in \mathcal{V} \,\}$$

is another distinguished basis of $F_{\mathcal{V}}(X)$. Indeed, by (2-6), the transition matrix to the basis $\{1_V\}$ is upper triangular with respect to $\leq$, with all diagonal entries equal to 1, so it is invertible. The advantage of working with the latter basis is that it carries more information about the geometry of the chosen stratification.

Now assume that X has an open dense stratum $S \in \mathcal{V}$ so that $V \leq S$ for all $V \in \mathcal{V}$, e.g., X is an irreducible reduced complex algebraic (resp., compact complex analytic) variety. For each $V \in \mathcal{V} \setminus \{S\}$ define inductively

$$\widehat{ic}(\overline{V}) := ic_{\overline{V}} - \sum_{W<V} \widehat{ic}(\overline{W}) \cdot I\chi(c^\circ L_{W,V}) \in F_{\mathcal{V}}(X). \tag{2-7}$$

Then any $\mathcal{V}$-constructible function $\alpha \in F_{\mathcal{V}}(X)$ can be represented with respect to the basis $\{ic_{\overline{V}} | V \in \mathcal{V}\}$ by the following identity (see [9, Theorem 3.1]):

$$\alpha = \alpha(s) \cdot ic_X + \sum_{V<S} \big(\alpha(v) - \alpha(s) \cdot I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{ic}(\overline{V}). \tag{2-8}$$

In the particular case when $\alpha|_S = 0$, i.e., $\mathrm{supp}(\alpha) \subset X \setminus S$, this reduces to the identity

$$\alpha = \sum_{V<S} \alpha(V) \cdot \widehat{ic}(\overline{V}), \tag{2-9}$$

which will become very important in the context of computing Milnor classes of singular complex hypersurfaces. Also, if we plug $\alpha = 1_X$ in equation (2-8), we obtain under the assumptions in this paragraph the following comparison formula (also valid if we replace X by the closure of any given stratum of $\mathcal{V}$):

$$1_X = ic_X + \sum_{V<S} \left(1 - I\chi(c^\circ L_{V,Y})\right) \cdot \widehat{ic}(\overline{V}). \tag{2-10}$$

3. Chern classes of singular varieties

For the rest of the paper we specialize to the complex algebraic (respectively, compact complex analytic) context, with X a reduced complex algebraic (resp., compact complex analytic) variety. There are several generalizations of the (total) Chern class of complex manifolds to the context of such singular varieties. Among these we mention here the Chern–MacPherson class [15] and the Fulton–Johnson class [10; 11]. Both coincide with the Poincaré dual of the Chern class if the variety is smooth.

3.1. The Chern–MacPherson class. The group $F_c(X)$ of complex algebraically (resp., analytically) constructible functions is defined as the direct limit of groups $F_{\mathcal{V}}(X)$, with respect to the directed system $\{\mathcal{V}\}$ of Whitney stratifications of X. Moreover, there is a functorial pushdown transformation of constructible functions, namely, a proper complex algebraic (resp., analytic) map $f : X \to Y$ induces a group homomorphism

$$f_* : F_c(X) \to F_c(Y),$$

defined by

$$f_*(\alpha)(y) := \chi(\alpha|_{f^{-1}(y)}),$$

for $\chi : F_c(X) \to \mathbb{Z}$ the constructible function which for a closed algebraic (resp., analytic) subspace Z of X is given by

$$\chi(1_Z) := \chi(H^*(Z)) = \chi(Z).$$

In particular, for such a closed subset $Z \subset X$ we have that

$$f_*(1_Z)(y) = \chi(Z \cap f^{-1}(y)).$$

The fact that the pushdown f_* is well-defined requires a stratification of the morphism f (see [15]).

The Chern class transformation of MacPherson [15] is the group homomorphism

$$c_* : F_c(X) \to H_{2*}^{BM}(X;\mathbb{Z}),$$

which commutes with proper pushdowns, and is uniquely characterized by this property together with the normalization axiom asserting that

$$c_*(1_X) = c^*(TX) \cap [X]$$

if X is a complex algebraic (resp., analytic) manifold. Here $c^*(TX)$ is the Chern cohomology class of the tangent bundle TX. Also $H_{2*}^{BM}(-)$ stands for the even-dimensional Borel–Moore homology. The Chern–MacPherson class of X is then defined as

$$c_*(X) := c_*(1_X) \in H_{2*}^{BM}(X;\mathbb{Z}).$$

If X is compact, the degree of $c_*(X)$ is just $\chi(X)$, the topological Euler characteristic of X. Similarly, we set

$$Ic_*(X) := c_*(ic_X),$$

which is another possible extension of Chern classes of manifolds to the singular setting. Of course, if X is smooth then $c_*(X) = Ic_*(X)$, but in general they differ for singular varieties, their difference being a measure of the singular locus, which, moreover, is computable in terms of the geometry of the stratification. Indeed, by applying c_* to the identity (2-10), we obtain the following comparison formula:

$$c_*(X) - Ic_*(X) = \sum_{V<S} \big(1 - I\chi(c^\circ L_{V,Y})\big) \cdot \widehat{Ic}_*(\overline{V}). \tag{3-1}$$

If X is compact, the degree of $Ic_*(X)$ is just $I\chi(X)$, the intersection homology Euler characteristic of X.

3.2. The Fulton–Johnson class. Assume that X is a local complete intersection embedded in a complex manifold M with inclusion

$$X \overset{i}{\hookrightarrow} M.$$

If $N_X M$ denotes the normal cone of X in M, then the *virtual tangent bundle* of X, that is,

$$T_{\text{vir}}X := [i^*TM - N_X M] \in K^0(X), \tag{3-2}$$

is a well-defined element in the Grothendieck group of vector bundles on X (e.g., see [11][Ex.4.2.6]), so one can associate to the pair (M, X) an *intrinsic* homology class, $c_*^{FJ}(X) \in H_{2*}^{BM}(X;\mathbb{Z})$, called the Fulton–Johnson class and defined as follows (see [10; 11]):

$$c_*^{FJ}(X) := c^*(T_{\text{vir}}X) \cap [X]. \tag{3-3}$$

Of course, if X is also smooth, then $T_{\mathrm{vir}}X$ coincides with the (class of the) usual tangent bundle of X, and $c_*^{FJ}(X)$ is in this case just the Poincaré dual of $c^*(TX)$.

4. Milnor classes of hypersurfaces

This section is devoted to comparing the two notions of Chern classes mentioned in the previous section. For simplicity, we restrict to the case when X is a hypersurface in a complex manifold M. As already mentioned, the Chern–MacPherson class and the Fulton–Johnson class coincide if X is smooth. However, they differ in the singular case. For example, if X has only isolated singularities, the difference is (up to a sign) the sum of the Milnor numbers attached to the singular points. For this reason, the difference $c_*^{FJ}(X) - c_*(X)$ is usually called the *Milnor class* of X, and is denoted by $\mathcal{M}(X)$.[1] The Milnor class is a homology class supported on the singular locus of X, and it has been recently studied by many authors using quite different methods, e.g., see [1; 2; 3; 4; 5; 6; 18; 17; 19; 20; 22]. For example, it was computed in [18] (see also [17; 22]) as a weighted sum in the Chern–MacPherson classes of closures of singular strata of X, the weights depending only on the normal information to the strata. The approach we follow here is that of [19; 20], and relies only on the simple calculus of constructible functions, as outlined in Section 2, together with a well-known specialization argument due to Verdier [21].

Assume in what follows that X is a reduced complex analytic hypersurface, which is globally defined as the zero-set of a holomorphic function $f : M \to \mathbb{D}$ with a critical value at $0 \in \mathbb{D}$, for M a compact complex manifold and $\mathbb{D}$ the open unit disc about $0 \in \mathbb{C}$. For each point $x \in X$, we have a corresponding Milnor fibration with fiber

$$M_{f,x} := B_\delta(x) \cap f^{-1}(t)$$

for appropriate choices of $0 < |t| \ll \delta \ll 1$.

Denote by L the trivial line bundle on M, obtained by pulling back by f the tangent bundle of $\mathbb{C}$. Then the virtual tangent bundle of X can be identified with

$$T_{\mathrm{vir}}X = [TM|_X - L|_X]. \tag{4-1}$$

For each $t \neq 0$ small enough, each fiber $X_t := f^{-1}(t)$ is a compact complex manifold. Moreover, by compactness, given a regular neighborhood $\mathcal{U}$ of X in M, there is a sufficiently small t so that $X_t \subset \mathcal{U}$. Denote by i_t the corresponding

[1] The definition of the Milnor class usually includes a sign, but for simplicity we choose to ignore it here.

inclusion map. Also, let $r : \mathcal{U} \to X$ be the obvious deformation retract. *Verdier's specialization map* in homology is then defined as the composition

$$\psi_H = r_* \circ i_{t*} : H_*(X_t) \to H_*(X). \tag{4-2}$$

There is also a specialization map defined on the level of constructible functions [21],

$$\psi_{CF} : F_c(M) \to F_c(X), \tag{4-3}$$

which is just the constructible function version of Deligne's nearby cycle functor for constructible complexes of sheaves. This is defined by the formula

$$\psi_{CF}(\alpha)(x) = \chi(\alpha \cdot 1_{M_{f,x}}). \tag{4-4}$$

In particular,

$$\psi_{CF}(1_M) = \mu_X \in F_c(X), \tag{4-5}$$

where $\mu_X : X \to \mathbb{Z}$ is the constructible function defined by the rule:

$$\mu_X(x) := \chi(M_{f,x}), \tag{4-6}$$

for all $x \in X$. This definition justifies the analogy with the nearby cycle functor defined on the level of constructible complexes of sheaves.

Verdier's specialization property for the Chern–MacPherson classes [21] asserts that for any $\alpha \in F_c(M)$ we have:

$$\psi_H c_*(\alpha|_{X_t}) = c_*(\psi_{CF}(\alpha)). \tag{4-7}$$

In particular, by letting $\alpha = 1_M$ and using (4-5), we have that

$$\psi_H c_*(X_t) = c_*(\mu_X). \tag{4-8}$$

We can now state the following easy (known) consequence:

PROPOSITION 4.1.

$$\mathcal{M}(X) = c_*(\tilde{\mu}_X), \tag{4-9}$$

where $\tilde{\mu}_X \in F_c(X)$ is the constructible function supported on the singular locus of X, whose value at $x \in X$ is defined by the Euler characteristic of the reduced cohomology of the corresponding Milnor fiber, i.e.,

$$\tilde{\mu}_X(x) := \chi(\tilde{H}^*(M_{f,x})). \tag{4-10}$$

PROOF. First note that, since X_t is smooth,

$$\psi_H c_*(X_t) = \psi_H c_*^{FJ}(X_t) = c_*^{FJ}(X), \tag{4-11}$$

where the last equality follows from the fact that the homology specialization map ψ_H carries (the dual of) the Chern classes of $TM|_{X_t}$ and $L|_{X_t}$ into (the dual of) the Chern classes of $TM|_X$ and $L|_X$, respectively [21].

On the other hand,

$$c_*(X) = c_*(\mu_X) - c_*(\tilde{\mu}_X), \tag{4-12}$$

so the desired identity follows by combining (4-8) and (4-11). □

REMARK 4.2. Note that $\tilde{\mu}_X$ is the constructible function analogue of Deligne's vanishing cycle functor defined on constructible sheaves. Indeed,

$$\tilde{\mu}_X = \phi_{CF}(1_M), \tag{4-13}$$

where $\phi_{CF} := \psi_{CF} - i^*$, for $i^* : F_c(M) \to F_c(X)$ the pullback (restriction) of constructible functions defined by $i^*(\alpha) := \alpha \circ i$.

We are now ready to prove the main result of this note:

THEOREM 4.3. *Let M be a compact complex manifold, and X a reduced hypersurface defined by the zero-set of a holomorphic function $f : M \to \mathbb{D}$ with a critical value at the origin. Fix a Whitney stratification $\mathcal{V}$ on X, and for each stratum $V \in \mathcal{V}$ fix a point $v \in V$ with corresponding Milnor fiber $M_{f,v}$. Then the Milnor class of X, i.e., the class*

$$\mathcal{M}(X) := c_*^{FJ}(X) - c_*(X) \in H_*(X),$$

can be computed by the formula

$$\begin{aligned}\mathcal{M}(X) &= \sum_{\substack{V \in \mathcal{V} \\ V \subset \mathrm{Sing}(X)}} \chi(\tilde{H}^*(M_{f,v})) \cdot \big(c_*(\overline{V}) - c_*(\overline{V} \setminus V)\big) \\ &= \sum_{\substack{V \in \mathcal{V} \\ V \subset \mathrm{Sing}(X)}} \chi(\tilde{H}^*(M_{f,v})) \cdot \hat{c}_*(\overline{V}),\end{aligned}$$

where for a stratum $V \in \mathcal{V}$ we let $\hat{c}_*(\overline{V})$ be defined inductively as

$$\hat{c}_*(\overline{V}) := c_*(\overline{V}) - \sum_{W < V} \hat{c}_*(\overline{W}).$$

If, moreover, X is irreducible and we let S denote the dense open stratum in X, then:

$$\mathcal{M}(X) = \sum_{V < S} \chi(\tilde{H}^*(M_{f,v})) \cdot \widehat{Ic}_*(\overline{V}), \tag{4-14}$$

where for each $V \in \mathcal{V}$, $\widehat{Ic}_*(\overline{V})$ is defined inductively by

$$\widehat{Ic}_*(\overline{V}) := Ic_*(\overline{V}) - \sum_{W < V} I\chi(c^\circ L_{W,V}) \cdot \widehat{Ic}_*(\overline{W}),$$

for $L_{W,V}$ the link of W in V.[2]

PROOF. Recall that by Proposition 4.1 we have:

$$\mathcal{M}(X) = c_*(\tilde{\mu}_X). \tag{4-15}$$

Moreover, the function $\tilde{\mu}_X : X \to \mathbb{Z}$ is constructible with respect to the Whitney stratification $\mathcal{V}$. Therefore, as in (2-1) and (2-2), we can write

$$\tilde{\mu}_X = \sum_{V \in \mathcal{V}} \tilde{\mu}_X(v) \cdot 1_V = \sum_{V \in \mathcal{V}} \tilde{\mu}_X(v) \cdot (1_{\overline{V}} - 1_{\overline{V} \setminus V}) = \sum_{V \in \mathcal{V}} \tilde{\mu}_X(v) \cdot \hat{1}_{\overline{V}}.$$

Since smooth points have contractible Milnor fibers, only strata contained in the singular locus of X contribute to the above sums. The first part of the theorem follows from (4-15) by applying the Chern–MacPherson transformation c_* to the last two of the above equalities.

If X is irreducible with dense open stratum S, then as in (2-9) we can write

$$\tilde{\mu}_X = \sum_{V < S} \tilde{\mu}_X(v) \cdot \widehat{ic}(\overline{V}).$$

By applying c_*, we obtain the desired identity (4-14) from (4-15). □

As a consequence, the Chern–MacPherson class and the Fulton–Johnson class coincide in dimensions greater than the dimension of the singular locus. And it can be seen from any of the above formulae that if X has only isolated singularities, the Milnor class is (up to a sign) just the sum of the Milnor numbers at the singular points.

By combining (3-1) and (4-14) we also obtain a comparison formula for the Fulton–Johnson class $c_*^{FJ}(X)$ and the Chern class $Ic_*(X)$ defined via the intersection cohomology chain sheaf.

COROLLARY 4.4. *If X as above is a reduced irreducible hypersurface with dense open stratum S, then*

$$\mathcal{IM}(X) := c_*^{FJ}(X) - Ic_*(X) = \sum_{V < S} \widehat{Ic}_*(\overline{V}) \cdot \left(\chi(M_{f,v}) - I\chi(c^\circ L_{V,X})\right). \tag{4-16}$$

Note that by constructible function calculus, we have that

$$\mathcal{IM}(X) = c_*(\widetilde{I\mu}_X), \tag{4-17}$$

for $\widetilde{I\mu}_X : X \to \mathbb{Z}$ the $\mathcal{V}$-constructible function whose value at $v \in V$ is given by

$$\widetilde{I\mu}_X(v) = \chi(M_{f,v}) - I\chi(c^\circ L_{V,X}). \tag{4-18}$$

[2]By the functoriality of c_*, we can regard all classes $c_*(\overline{V})$, $\hat{c}_*(\overline{V})$ and $\hat{Ic}_*(\overline{V})$ associated to a stratum $V \in \mathcal{V}$ as homology classes in $H_*(X)$. This is the reason why we apply the Chern–MacPherson transformation c_* only to closed subvarieties of X.

By its definition, $\widetilde{I\mu}_X$ is supported on the singular locus of X, so (2-9) can be used directly to prove (4-16).

REMARK 4.5. Our formula (4-16) should be compared to the topological formula of Cappell and Shaneson [7; 8] for the Goresky–MacPherson L-class [12] of an irreducible reduced complex hypersurface $X \subset M$ as above, namely,

$$L_*(T_{\text{vir}}X) - L_*(X) = \sum_{V<S} L_*(\overline{V}) \cdot \sigma(\text{lk}(V)), \tag{4-19}$$

where $\sigma(\text{lk}(V)) \in \mathbb{Z}$ is a certain signature invariant associated to the link pair of the stratum V in (M, X). Here $L_*(T_{\text{vir}}X) := L^*(T_{\text{vir}}X) \cap [X]$, with L^* the L-polynomial of Hirzebruch [14] defined in terms of the power series $x/\tanh(x)$. The comparison is motivated by the fact that the L-class of a singular variety X is a topological invariant associated to the intersection cohomology complex of the variety. We should point out that the Cappell–Shaneson formula holds in much greater generality, namely for real codimension two PL embeddings with even codimension strata, and its proof relies on powerful algebraic cobordism decompositions of self-dual sheaves. However, we believe that in the context of complex algebraic/analytic geometry, a simpler proof could be given by using a specialization argument similar to the one presented here.

More generally, assume that $i : X \hookrightarrow M$ is a regular embedding in codimension one of complex algebraic (resp., compact complex analytic) spaces with M smooth. Then X is locally defined in M by one equation $\{f = 0\}$, and the specialization map $\psi_{CF} : F_c(M) \to F_c(X)$ is still well-defined, as it is independent of the chosen local equation for X. In particular, we still have that $\psi_{CF}(1_M) = \mu$, whose value at a point $x \in X$ is given by the Euler characteristic of a local Milnor fiber at x. In other words, if $\{f = 0\}$ is a defining equation for X near x, then

$$\tilde{\mu}_X(x) := \chi(\tilde{H}^*(M_{f,x})), \tag{4-20}$$

for $M_{f,x}$ the corresponding Milnor fiber. Then arguments similar to those used in this section apply to this more general situation, and yield the following result (see [20, Corollary 0.2] for equation (4-21) below):

THEOREM 4.6. *Let $i : X \hookrightarrow M$ be a regular embedding in codimension one of complex algebraic* (*resp., compact complex analytic*) *spaces with M smooth. Then,*

$$\mathcal{M}(X) = c^*(N_X M)^{-1} \cap c_*(\tilde{\mu}_X), \tag{4-21}$$

with $\tilde{\mu}$ the constructible function supported on the singular locus of X, whose value at a point $x \in X$ is given by the Euler characteristic of the reduced cohomology of a local Milnor fiber at x. So, if we assume X irreducible with dense

open stratum S, then in the notations of Theorem 4.3 *we get*

$$\begin{aligned}\mathcal{M}(X) &= \sum_{V<S} c^*(N_X M)^{-1} \cap \Big(c_*(\overline{V}) - c_*(\overline{V} \setminus V)\Big) \cdot \tilde{\mu}_X(v) \\ &= \sum_{V<S} c^*(N_X M)^{-1} \cap \hat{c}_*(\overline{V}) \cdot \tilde{\mu}_X(v) \\ &= \sum_{V<S} c^*(N_X M)^{-1} \cap \widehat{Ic}_*(\overline{V}) \cdot \tilde{\mu}_X(v).\end{aligned}$$

Similar considerations apply to $\mathcal{IM}(X)$. (Again, by functoriality, we regard all classes defined on the closure of a given stratum as homology classes in X.)

We conclude this note by recalling some functoriality results for the Milnor class of hypersurfaces (see [20; 24] for complete details). More precisely, we are concerned with the behavior of the Milnor class under a proper pushdown. Similar results were obtained in [9] for the Chern–MacPherson classes $c_*(-)$ and $Ic_*(-)$, respectively.

Let us consider the cartesian diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ j\ } & \tilde{M} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\ i\ } & M \end{array}$$

with M and $\tilde{M}$ compact analytic manifolds, and $\pi : \tilde{M} \to M$ a proper morphism. Also assume that i and j are regular closed embeddings of (local) codimension one, with M irreducible. Then it's easy to see that $N_{\tilde{X}}\tilde{M} \simeq f^*(N_X M)$. Therefore, by (4-21) and the projection formula, one has

$$\begin{aligned} f_*\mathcal{M}(\tilde{X}) &= f_*(c^*(N_{\tilde{X}}\tilde{M})^{-1} \cap c_*(\tilde{\mu}_{\tilde{X}})) \\ &= f_*(f^*c^*(N_X M)^{-1} \cap c_*(\tilde{\mu}_{\tilde{X}})) \\ &= c^*(N_X M)^{-1} \cap f_*c_*(\tilde{\mu}_{\tilde{X}}).\end{aligned}$$

Next, by the functoriality of c_* and the definition of $\tilde{\mu}_{\tilde{X}}$ in (4-13) we obtain

$$f_*c_*(\tilde{\mu}_{\tilde{X}}) = c_*f_*(\tilde{\mu}_{\tilde{X}}) = c_*f_*\phi_{CF}(1_{\tilde{M}}) = c_*\phi_{CF}(\pi_*(1_{\tilde{M}})),$$

where the last identity follows by proper base change. Assume now that π (hence also f) is an *Euler morphism*, i.e., the Euler characteristics of all its fibers are the same (e.g., π is smooth), and denote this value by χ_f. Then $\pi_*(1_{\tilde{M}}) = \chi_f \cdot 1_M$, and it follows in this case that

$$f_*\mathcal{M}(\tilde{X}) = \chi_f \cdot \mathcal{M}(X). \tag{4-22}$$

But in the case of a general morphism we have that

$$f_*\mathcal{M}(\tilde{X}) = \chi_f \cdot \mathcal{M}(X) + c^*(N_X M)^{-1} \cap c_*\phi_{CF}(\alpha), \qquad (4\text{-}23)$$

for $\alpha := \pi_*(1_{\tilde{M}}) - \chi_f \cdot 1_M$, with χ_f the Euler characteristic of the generic fiber of π. Note that α is supported on the critical locus of the morphism π.

To this end, we note that the above considerations can also be used to study the push-forward of the class $\mathcal{IM}(\tilde{X})$ in the case when $\tilde{X}$ is pure-dimensional and X is irreducible and reduced. Let us choose a stratification $\mathcal{V}$ on X with dense open stratum S, so that $f_*(1_{\tilde{X}}), f_*(ic_{\tilde{X}}) \in F_{\mathcal{V}}(X)$ (e.g., choose $\tilde{\mathcal{V}}$ and $\mathcal{V}$ complex Whitney stratifications on $\tilde{X}$ and X, respectively, so that f is a stratified submersion, and $1_X, ic_{\tilde{X}} \in F_{\tilde{\mathcal{V}}}(X)$). Then, since

$$\mathcal{IM}(\tilde{X}) = \mathcal{M}(\tilde{X}) + (c_*(\tilde{X}) - Ic_*(\tilde{X})),$$

a formula for $f_*\mathcal{IM}(\tilde{X})$ can be derived by using (4-23), together with the formulae from [9, Propositions 3.4 and 3.6] for the push-forward of the Chern classes $c_*(\tilde{X})$ and $Ic_*(\tilde{X})$, respectively. We leave the details as an exercise for the interested reader. We only want to point out that for an Euler morphism (with smooth generic fiber), we obtain

$$\begin{aligned} &f_*\mathcal{IM}(\tilde{X}) \\ &\quad = \chi_f \cdot \mathcal{IM}(X) + \sum_{V<S} \left(\chi_f\, I\chi(c^\circ L_{V,X}) - I\chi(f^{-1}(c^\circ L_{V,X}))\right) \cdot \widehat{ic}(\overline{V}), \end{aligned} \qquad (4\text{-}24)$$

where χ_f is the Euler characteristic of the generic fiber F of f (and π).

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LAURENTIU MAXIM
LAURENTIU MAXIM
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN-MADISON
480 LINCOLN DRIVE
MADISON, WI 53706-1388
UNITED STATES
maxim@math.wisc.edu