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Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities

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ABSTRACT. We survey some recent developments in the study of singularities related to the classification theory of algebraic varieties. In particular, the definition and basic properties of Du Bois singularities and their connections to the more commonly known singularities of the minimal model program are reviewed and discussed.

1. Introduction

The primary goal of this note is to survey some recent developments in the study of singularities related to the minimal model program. In particular, we review the definition and basic properties of *Du Bois singularities* and explain how these singularities fit into the minimal model program and moduli theory.

Since we can resolve singularities [Hir64], one might ask why we care about them at all. It turns out that in various situations we are forced to work with singularities even if we are only interested in understanding smooth objects.

One reason we are led to study singular varieties is provided by the minimal model program [KM98]. The main goal is the classification of algebraic varieties and the plan is to find reasonably simple representatives of all birational classes and then classify these representatives. It turns out that the simplest objects in a birational class tend to be singular. What this really means is that when choosing a birational representative, we aim to have simple *global* properties and this is often achieved by a singular variety. Being singular means that

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there are points where the *local* structure is more complicated than on a smooth variety, but that allows for the possibility of still having a somewhat simpler global structure and along with it, good local properties at most points.

Another reason to study singularities is that to understand smooth objects we should also understand how smooth objects may deform and degenerate. This leads to the need to construct and understand moduli spaces. And not just moduli for the smooth objects: degenerations provide important information as well. In other words, it is always useful to work with complete moduli problems, i.e., extend our moduli functor so it admits a compact (and preferably projective) coarse moduli space. This also leads to having to consider singular varieties.

On the other hand, we have to be careful to limit the kinds of singularities that we allow in order to be able to handle them. One might view this survey as a list of the singularities that we must deal with to achieve the goals stated above. Fortunately, it is also a class of singularities with which we have a reasonable chance to be able to work.

In particular, we will review Du Bois singularities and related notions, including some very recent important results. We will also review a family of singularities defined via characteristic-p methods, the Frobenius morphism, and their connections to the other set of singularities we are discussing.

Definitions and notation. Let k be an algebraically closed field. Unless otherwise stated, all objects will be assumed to be defined over k. A *scheme* will refer to a scheme of finite type over k and unless stated otherwise, a *point* refers to a closed point.

For a morphism $Y \to S$ and another morphism $T \to S$, the symbol Y_T will denote $Y \times_S T$. In particular, for $t \in S$ we write $X_t = f^{-1}(t)$. In addition, if $T = \operatorname{Spec} F$, then Y_T will also be denoted by Y_F.

Let X be a scheme and $\mathscr{F}$ an $\mathscr{O}_X$-module. The *m-th reflexive power* of $\mathscr{F}$ is the double dual (or reflexive hull) of the m-th tensor power of $\mathscr{F}$:

$$\mathscr{F}^{[m]} := (\mathscr{F}^{\otimes m})^{**}.$$

A *line bundle* on X is an invertible $\mathscr{O}_X$-module. A *$\mathbb{Q}$-line bundle* $\mathscr{L}$ on X is a reflexive $\mathscr{O}_X$-module of rank 1 that possesses a reflexive power which is a line bundle, i.e., there exists an $m \in \mathbb{N}_+$ such that $\mathscr{L}^{[m]}$ is a line bundle. The smallest such m is called the *index* of $\mathscr{L}$.

- For the advanced reader: whenever we mention Weil divisors, assume that X is S_2 [Har77, Theorem 8.22A(2)] and think of a *Weil divisorial sheaf*, that is, a rank 1 reflexive $\mathscr{O}_X$-module which is locally free in codimension 1. For flatness issues consult [Kol08a, Theorem 2].
- For the novice: whenever we mention Weil divisors, assume that X is normal and adopt the definition [Har77, p. 130].

For a Weil divisor D on X, its associated *Weil divisorial sheaf* is the $\mathscr{O}_X$-module $\mathscr{O}_X(D)$ defined on the open set $U \subseteq X$ by the formula

$$\Gamma(U, \mathscr{O}_X(D)) = \left\{ \frac{a}{b} \;\middle|\; \begin{array}{l} a, b \in \Gamma(U, \mathscr{O}_X),\ b \text{ is not a zero divisor anywhere} \\ \text{on } U, \text{ and } D|_U + \operatorname{div}_U(a) - \operatorname{div}_U(b) \geq 0. \end{array} \right\}$$

and made into a sheaf by the natural restriction maps.

A Weil divisor D on X is a *Cartier divisor*, if its associated Weil divisorial sheaf, $\mathscr{O}_X(D)$ is a line bundle. If the associated Weil divisorial sheaf, $\mathscr{O}_X(D)$ is a $\mathbb{Q}$-line bundle, then D is a $\mathbb{Q}$-*Cartier divisor*. The latter is equivalent to the property that there exists an $m \in \mathbb{N}_+$ such that mD is a Cartier divisor. Weil divisors form an abelian group. Tensoring this group with $\mathbb{Q}$ (over $\mathbb{Z}$) one obtains the group of $\mathbb{Q}$-*divisors* on X. (If X is not normal, some unexpected things can happen in this process; see [Kol92, Chapter 16].)

The symbol $\sim$ stands for *linear* and $\equiv$ for *numerical equivalence* of divisors.

Let $\mathscr{L}$ be a line bundle on a scheme X. It is said to be *generated by global sections* if for every point $x \in X$ there exists a global section $\sigma_x \in H^0(X, \mathscr{L})$ such that the germ σ_x generates the stalk $\mathscr{L}_x$ as an $\mathscr{O}_X$-module. If $\mathscr{L}$ is generated by global sections, then the global sections define a morphism

$$\phi_{\mathscr{L}} : X \to \mathbb{P}^N = \mathbb{P}\big(H^0(X, \mathscr{L})^*\big).$$

$\mathscr{L}$ is called *semi-ample* if $\mathscr{L}^m$ is generated by global sections for $m \gg 0$. $\mathscr{L}$ is called *ample* if it is semi-ample and $\phi_{\mathscr{L}^m}$ is an embedding for $m \gg 0$. A line bundle $\mathscr{L}$ on X is called *big* if the global sections of $\mathscr{L}^m$ define a rational map $\phi_{\mathscr{L}^m} : X \dashrightarrow \mathbb{P}^N$ such that X is birational to $\phi_{\mathscr{L}^m}(X)$ for $m \gg 0$. Note that in this case $\mathscr{L}^m$ need not be generated by global sections, so $\phi_{\mathscr{L}^m}$ is not necessarily defined everywhere. We leave it for the reader the make the obvious adaptation of these notions for the case of $\mathbb{Q}$-line bundles.

The *canonical divisor* of a scheme X is denoted by K_X and the *canonical sheaf* of X is denoted by ω_X.

A smooth projective variety X is of *general type* if ω_X is big. It is easy to see that this condition is invariant under birational equivalence between smooth projective varieties. An arbitrary projective variety is of *general type* if so is a desingularization of it.

A projective variety is *canonically polarized* if ω_X is ample. Notice that if a smooth projective variety is canonically polarized, then it is of general type.

2. Pairs and resolutions

For the reader's convenience, we recall a few definitions regarding pairs.

DEFINITION 2.1. A *pair* (X, Δ) consists of a normal[1] quasiprojective variety or complex space X and an effective $\mathbb{Q}$-divisor $\Delta \subset X$. A *morphism of pairs* $\gamma : (\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$ is a morphism $\gamma : \widetilde{X} \to X$ such that $\gamma(\mathrm{Supp}(\mathrm{wt}\,\Delta)) \subseteq \mathrm{Supp}(\Delta)$. A morphism of pairs $\gamma : (\widetilde{X}, \widetilde{\Delta}) \to (X, \Delta)$ is called *birational* if it induces a birational morphism $\gamma : \mathrm{wt}\,X \dashrightarrow X$ and $\gamma(\mathrm{wt}\,\Delta) = \Delta$. It is an *isomorphism* if it is birational and it induces an isomorphism $\gamma : \mathrm{wt}\,X \xrightarrow{\cong} X$.

DEFINITION 2.2. Let (X, Δ) be a pair, and $x \in X$ a point. We say that (X, Δ) is *snc at* x, if there exists a Zariski-open neighborhood U of x such that U is smooth and $\Delta \cap U$ is reduced and has only simple normal crossings (see Section 3B for additional discussion). The pair (X, Δ) is *snc* if it is snc at all $x \in X$.

Given a pair (X, Δ), let $(X, \Delta)_{\mathrm{reg}}$ be the maximal open set of X where (X, Δ) is snc, and let $(X, \Delta)_{\mathrm{Sing}}$ be its complement, with the induced reduced subscheme structure.

REMARK 2.2.1. If a pair (X, Δ) is snc at a point x, this implies that all components of Δ are smooth at x. If instead of the condition that U is Zariski-open one would only require this analytically locally, then Definition 2.2 would define normal crossing pairs rather than pairs with simple normal crossing.

DEFINITION 2.3. A *log resolution* of (X, Δ) is a proper birational morphism of pairs $\pi : (\widetilde{X}, \mathrm{wt}\,\Delta) \to (X, \Delta)$ that satisfies the following four conditions:

- $\widetilde{X}$ is smooth.
- $\mathrm{wt}\,\Delta = \pi_*^{-1}\Delta$ is the strict transform of Δ.
- $\mathrm{Exc}(\pi)$ is of pure codimension 1.
- $\mathrm{Supp}(\widetilde{\Delta} \cup \mathrm{Exc}(\pi))$ is a simple normal crossings divisor.

If, in addition,

- the strict transform $\widetilde{\Delta}$ of Δ has smooth support,

then we call π an *embedded resolution* of $\Delta \subset X$.

In many cases, it is also useful to require that π is an isomorphism over $(X, \Delta)_{\mathrm{reg}}$.

3. Introduction to the singularities of the mmp

Even though we have introduced pairs and most of these singularities make sense for pairs, to make the introduction easier to digest we will mostly discuss the case when $\Delta = \varnothing$. As mentioned in the introduction, one of our goals is to show why we are forced to work with singular varieties even if our primary interest lies with smooth varieties.

[1] Occasionally, we will discuss pairs in the nonnormal setting. See Section 3F for more details.

3A. Canonical singularities. For an excellent introduction to this topic the reader is urged to take a thorough look at Miles Reid's Young Person's Guide [Rei87]. Here we will only touch on the subject.

Let us suppose that we would like to get a handle on some varieties. Perhaps we want to classify them or make some computations. In any case, a useful thing to do is to embed the object in question into a projective space (if we can). Doing so requires a (very) ample line bundle. It turns out that in practice these can be difficult to find. In fact, it is not easy to find any nontrivial line bundle on an abstract variety.

One possibility, when X is smooth, is to try a line bundle that is "handed" to us, namely some (positive or negative) power of the *canonical line bundle*; $\omega_X = \det T^*_X$. If X is not smooth but instead normal, we can construct ω_X on the smooth locus and then push it forward to obtain a rank one reflexive sheaf on all of X (which sometimes is still a line bundle). Next we will explore how we might "force" this line bundle to be ample in some (actually many) cases.

Let X be a minimal surface of general type that contains a (-2)-curve (a smooth rational curve with self-intersection -2). For an example of such a surface consider the following.

EXAMPLE 3.1. $\widetilde{X} = (x^5 + y^5 + z^5 + w^5 = 0) \subseteq \mathbb{P}^3$ with the $\mathbb{Z}_2$-action that interchanges $x \leftrightarrow y$ and $z \leftrightarrow w$. This action has five fixed points, $[1:1:-\varepsilon^i:-\varepsilon^i]$ for $i = 1, \dots, 5$, where ε is a primitive fifth root of unity. Hence the quotient $\widetilde{X}/\mathbb{Z}_2$ has five singular points, each a simple double point of type A_1. Let $X \to \widetilde{X}/\mathbb{Z}_2$ be the minimal resolution of singularities. Then X contains five (-2)-curves, the exceptional divisors over the singularities.

Let us return to the general case, that is, X is a minimal surface of general type that contains a (-2)-curve, $C \subseteq X$. As $C \simeq \mathbb{P}^1$, and X is smooth, the adjunction formula gives us that $K_X \cdot C = 0$. Therefore K_X is not ample.

On the other hand, since X is a minimal surface of general type, it follows that K_X is semi-ample, that is, some multiple of it is base-point free. In other words, there exists a morphism,

$$|mK_X| : X \to X_{\text{can}} \subseteq \mathbb{P}\big(H^0(X, \mathscr{O}_X(mK_X))^*\big).$$

This may be shown in several ways. For example, it follows from Bombieri's classification of pluricanonical maps, but perhaps the simplest proof is provided by Miles Reid [Rei97, E.3].

It is then relatively easy to see that this morphism onto its image is independent of m (as long as mK_X is base point free). This constant image is called the *canonical model* of X, and will be denoted by X_{can}.

The good news is that the canonical line bundle of X_{can} is indeed ample, but the trouble is that X_{can} is singular. We might consider this as the first sign of

the necessity of working with singular varieties. Fortunately the singularities are not too bad, so we still have a good chance to work with this model. In fact, the singularities that can occur on the canonical model of a surface of general type belong to a much studied class. This class goes by several names; they are called *du Val singularities*, or *rational double points*, or *Gorenstein, canonical singularities*. For more on these singularities, refer to [Dur79; Rei87].

3B. Normal crossings. These singularities already appear in the construction of the moduli space of stable curves (or if the reader prefers, the construction of a compactificaton of the moduli space of smooth projective curves). If we want to understand degenerations of smooth families, we have to allow normal crossings.

A *normal crossing* singularity is one that is locally analytically (or formally) isomorphic to the intersection of coordinate hyperplanes in a linear space. In other words, it is a singularity locally analytically defined as $(x_1x_2\cdots x_r = 0) \subseteq \mathbb{A}^n$ for some $r \leq n$. In particular, as opposed to the curve case, for surfaces it allows for triple intersections. However, triple intersections may be "resolved": Let $X = (xyz = 0) \subseteq \mathbb{A}^3$. Blow up the origin $O \in \mathbb{A}^3$ to obtain $\sigma : \mathrm{Bl}_O\mathbb{A}^3 \to \mathbb{A}^3$, and consider the proper transform of X, $\sigma : \widetilde{X} \to X$. Observe that $\widetilde{X}$ has only double normal crossings.

Another important point to remember about normal crossings is that they are *not* normal. In particular they do not belong to the previous category. For some interesting and perhaps surprising examples of surfaces with normal crossings see [Kol07].

3C. Pinch points. Another nonnormal singularity that can occur as the limit of smooth varieties is the pinch point. It is locally analytically defined as the locus $(x_1^2 = x_2x_3^2) \subseteq \mathbb{A}^n$. This singularity is a double normal crossing away from the pinch point. Its normalization is smooth, but blowing up the pinch point (i.e., the origin) does not make it any better. (Try it for yourself!)

3D. Cones. Let $C \subseteq \mathbb{P}^2$ be a curve of degree d and $X \subseteq \mathbb{P}^3$ the projectivized cone over C. As X is a degree d hypersurface, it admits a smoothing.

EXAMPLE 3.2. Let $\mathcal{Z} = (x^d + y^d + z^d + tw^d = 0) \subseteq \mathbb{P}^3_{x:y:z:w} \times \mathbb{A}^1_t$. The special fiber $\mathcal{Z}_0$ is a cone over a smooth plane curve of degree d and the general fiber $\mathcal{Z}_t$, for $t \neq 0$, is a smooth surface of degree d in $\mathbb{P}^3$.

This, again, suggests that we must allow some singularities. The question is whether we can limit the type of singularities we must deal with. More particularly to this case, can we limit the type of cones we need to allow?

First we need an auxiliary computation. By the nature of the computation it is easier to use *divisors* instead of *line bundles*.

COMMENTARY 3.3. One of our ultimate goals is to construct a moduli space for canonical models of varieties. We are already aware that the minimal model program has to deal with singularities and so we must allow some singularities on canonical models. We would also like to understand what constraints are imposed if our goal is to construct a moduli space. The point is that in order to construct our moduli space, the objects must have an ample canonical class. It is possible that a family of canonical models degenerates to a singular fiber that has singularities worse than the original canonical models. An important question then is whether we may resolve the singularities of this special fiber and retain ampleness of the canonical class. The next example shows that this is not always possible.

EXAMPLE 3.4. Let W be a smooth variety and $X = X_1 \cup X_2 \subseteq W$ such that X_1 and X_2 are Cartier divisors in W. Then by the adjunction formula we have

$$\begin{aligned} K_X &= (K_W + X)|_X, \\ K_{X_1} &= (K_W + X_1)|_{X_1}, \\ K_{X_2} &= (K_W + X_2)|_{X_2}. \end{aligned}$$

Therefore

$$K_X|_{X_i} = K_{X_i} + X_{3-i}|_{X_i} \tag{3.4.1}$$

for $i = 1, 2$, so we have

$$K_X \text{ is ample} \iff K_X|_{X_i} = K_{X_i} + X_{3-i}|_{X_i} \text{ is ample for } i = 1, 2. \tag{3.4.2}$$

Next, let X be a normal projective surface with K_X ample and an isolated singular point $P \in \operatorname{Sing} X$. Assume that X is isomorphic to a cone $\Xi_0 \subseteq \mathbb{P}^3$ as in Example 3.2, locally analytically near P. Further assume that X is the special fiber of a family Ξ that itself is smooth. In particular, we may assume that all fibers other than X are smooth. As explained in (3.3), we would like to see whether we may resolve the singular point $P \in X$ and still be able to construct our desired moduli space, i.e., that K of the resolved fiber would remain ample. For this purpose we may assume that P is the only singular point of X.

Let $\Upsilon \to \Xi$ be the blowing up of $P \in \Xi$ and let $\widetilde{X}$ denote the proper transform of X. Then $\Upsilon_0 = \widetilde{X} \cup E$, where $E \simeq \mathbb{P}^2$ is the exceptional divisor of the blowup. Clearly, $\sigma : \widetilde{X} \to X$ is the blowup of P on X, so it is a smooth surface and $\widetilde{X} \cap E$ is isomorphic to the degree d curve over which X is locally analytically a cone.

We would like to determine the condition on d that ensures that the canonical divisor of Υ_0 is still ample. According to (3.4.2) this means that we need that $K_E + \widetilde{X}|_E$ and $K_{\widetilde{X}} + E|_{\widetilde{X}}$ be ample.

As $E \simeq \mathbb{P}^2$, $\omega_E \simeq \mathscr{O}_{\mathbb{P}^2}(-3)$, so $\mathscr{O}_E(K_E + \widetilde{X}|_E) \simeq \mathscr{O}_{\mathbb{P}^2}(d-3)$. This is ample if and only if $d > 3$.

As this computation is local near P the only relevant issue about the ampleness of $K_{\widetilde{X}} + E|_{\widetilde{X}}$ is whether it is ample in a neighborhood of $E_X := E|_{\widetilde{X}}$. By the next claim this is equivalent to asking when $(K_{\widetilde{X}} + E_X) \cdot E_X$ is positive.

CLAIM. *Let Z be a smooth projective surface with nonnegative Kodaira dimension and $\Gamma \subset Z$ an effective divisor. If $(K_Z + \Gamma) \cdot C > 0$ for every proper curve $C \subset Z$, then $K_Z + \Gamma$ is ample.*

PROOF. By the assumption on the Kodaira dimension there exists an $m > 0$ such that mK_Z is effective, hence so is $m(K_Z + \Gamma)$. Then by the assumption on the intersection number, $(K_Z + \Gamma)^2 > 0$, so the statement follows by the Nakai–Moishezon criterium. □

Observe that, by the adjunction formula,

$$(K_{\widetilde{X}} + E_X) \cdot E_X = \deg K_{E_X} = d(d-3),$$

as E_X is isomorphic to a plane curve of degree d. Again, we obtain the same condition as above and thus conclude that K_{Υ_0} may be ample only if $d > 3$.

Now, if we are interested in constructing moduli spaces, one of the requirements of being stable is that the canonical bundle be ample. This means that in order to obtain a compact moduli space we have to allow cone singularities over curves of degree $d \leq 3$. The singularity we obtain for $d = 2$ is a rational double point, but the singularity for $d = 3$ is not even rational. This does not fit any of the earlier classes we discussed. It belongs to the one discussed in the next section.

3E. Log canonical singularities. Let us investigate the previous situation under more general assumptions.

COMPUTATION 3.5. Let $D = \sum_{i=0}^{r} \lambda_i D_i$ ($\lambda_i \in \mathbb{N}$), be a divisor with only normal crossing singularities in a smooth ambient variety such that $\lambda_0 = 1$. Using a generalized version of the adjunction formula shows that in this situation (3.4.1) remains true.

$$K_D|_{D_0} = K_{D_0} + \sum_{i=1}^{r} \lambda_i D_i|_{D_0} \tag{3.5.1}$$

Let $f : \Xi \to B$ a projective family with $\dim B = 1$, Ξ smooth and K_{Ξ_b} ample for all $b \in B$. Further let $X = \Xi_{b_0}$ for some $b_0 \in B$ a singular fiber and let $\sigma : \Upsilon \to \Xi$ be an embedded resolution of $X \subseteq \Xi$. Finally let $Y = \sigma^* X = \widetilde{X} + \sum_{i=1}^{r} \lambda_i F_i$ where $\widetilde{X}$ is the proper transform of X and F_i are exceptional divisors for σ. We are interested in finding conditions that are necessary for K_Y to remain ample.

Let $E_i := F_i|_{\widetilde{X}}$ be the exceptional divisors for $\sigma : \widetilde{X} \to X$ and for the simplicity of computation, assume that the E_i are irreducible. For K_Y to be ample we need $K_Y|_{\mathrm{wt}\,X}$ as well as $K_Y|_{F_i}$ for all i to be ample. Clearly, the important one of these for our purposes is $K_Y|_{\mathrm{wt}\,X}$, for which 3.5.1) gives

$$K_Y|_{\widetilde{X}} = K_{\widetilde{X}} + \sum_{i=1}^{r} \lambda_i E_i.$$

As usual, we may write $K_{\widetilde{X}} = \sigma^* K_X + \sum_{i=1}^{r} a_i E_i$, so we are looking for conditions to guarantee that $\sigma^* K_X + \sum (a_i + \lambda_i) E_i$ be ample. In particular, its restriction to any of the E_i has to be ample. To further simplify our computation let us assume that $\dim X = 2$. Then the condition that we want satisfied is that, for all j,

$$\left(\sum_{i=1}^{r} (a_i + \lambda_i) E_i \right) \cdot E_j > 0. \tag{3.5.2}$$

Write the sum in parentheses as $E_+ - E_-$, where

$$E_+ = \sum_{a_i + \lambda_i \geq 0} |a_i + \lambda_i| E_i \quad \text{and} \quad E_- = \sum_{a_i + \lambda_i < 0} |a_i + \lambda_i| E_i.$$

Choose a j such that $E_j \subseteq \operatorname{Supp} E_+$. Then $E_- \cdot E_j \geq 0$ since $E_j \not\subseteq E_-$ and (3.5.2) implies that $(E_+ - E_-) \cdot E_j > 0$. These together imply that $E_+ \cdot E_j > 0$ and then that $E_+^2 > 0$. However, the E_i are exceptional divisors of a birational morphism, so their intersection matrix, $(E_i \cdot E_j)$ is negative definite.

The only way this can happen is if $E_+ = 0$. In other words, $a_i + \lambda_i < 0$ for all i. However, the λ_i are positive integers, so this implies that K_Y may remain ample only if $a_i < -1$ for all $i = 1, \ldots, r$.

The definition of a *log canonical singularity* is the exact opposite of this condition. It requires that X be normal and admit a resolution of singularities, say $Y \to X$, such that all the $a_i \geq -1$. This means that the above argument shows that we may stand a fighting chance if we resolve singularities that are *worse* than log canonical, but have no hope to do so with log canonical singularities. In other words, this is another class of singularities that we have to allow. As we remarked above, the class of singularities we obtained for the cones in the previous subsection belong to this class. In fact, all the normal singularities that we have considered so far belong to this class.

The good news is that by now we have covered most of the ways that something can go wrong and found the class of singularities we must allow. Since we already know that we have to deal with some nonnormal singularities and in fact in this example we have not really needed that X be normal, we conclude that we will have to allow the nonnormal cousins of log canonical singularities. These are called *semi-log canonical singularities*, and we turn to them now.

3F. Semi-log canonical singularities. Semi-log canonical singularities are very important in moduli theory. These are exactly the singularities that appear on stable varieties, the higher dimensional analogs of stable curves. However, their definition is rather technical, so the reader might want to skip this section at the first reading.

As a warm-up, let us first define the normal and more traditional singularities that are relevant in the minimal model program.

DEFINITION 3.6. A pair (X, Δ) is called *log* $\mathbb{Q}$*-Gorenstein* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier, i.e., some integer multiple of $K_X + \Delta$ is a Cartier divisor. Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair and $f : \widetilde{X} \to X$ a log resolution of singularities with exceptional divisor $E = \bigcup E_i$. Express the log canonical divisor of $\widetilde{X}$ in terms of $K_X + \Delta$ and the exceptional divisors:

$$K_{\widetilde{X}} + \operatorname{wt} \Delta \equiv f^*(K_X + \Delta) + \sum a_i E_i$$

where $\operatorname{wt} \Delta = f_*^{-1}\Delta$, the strict transform of Δ on $\operatorname{wt} X$ and $a_i \in \mathbb{Q}$. Then the pair (X, Δ) has

$$\left.\begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{plt} \\ \textit{klt} \\ \textit{log canonical} \end{array}\right\} \text{ singularities if } \begin{cases} a_i > 0 \\ a_i \geq 0 \\ a_i > -1 \\ a_i > -1 \text{ and } \lfloor \Delta \rfloor \leq 0 \\ a_i \geq -1 \end{cases}$$

for all log resolutions f and all i. The corresponding definitions for nonnormal varieties are somewhat more cumbersome. We include them here for completeness, but the reader should feel free to skip them and assume that for instance "semi-log canonical" means something that can be reasonably considered a nonnormal version of log canonical.

Suppose that X is a reduced equidimensional scheme that

(i) satisfies Serre's condition S2 (see [Har77, Theorem 8.22A(2)]), and
(ii) has only simple normal double crossings in codimension 1 (in particular X is Gorenstein in codimension 1).[2]

The conditions imply that we can treat the canonical module of X as a divisorial sheaf even though X is not normal. Further suppose that D is a $\mathbb{Q}$-Weil divisor on X (again, following [Kol92, Chapter 16], we assume that X is regular at the generic point of each component in Supp D).

REMARK 3.7. Conditions (i) and (ii) in Definition 3.6 imply that X is seminormal since it is seminormal in codimension 1; see [GT80, Corollary 2.7].

[2] Sometimes a ring that is S2 and Gorenstein in codimension 1 is called quasinormal.

Set $\rho : X^N \to X$ to be the normalization of X and suppose that B is the divisor of the conductor ideal on X^N. We denote by $\rho^{-1}(D)$ the pullback of D to X^N.

DEFINITION 3.8. We say that (X, D) is *semi-log canonical* if

(i) $K_X + D$ is $\mathbb{Q}$-Cartier, and
(ii) the pair $(X^N, B + \rho^{-1} D)$ is log canonical.

Actually, the original definition of semi-log canonical singularities (which is equivalent to this one) uses the theory of semi-resolutions. For details, see [KSB88], [Kol92, Chapter 12], and [Kol08b].

4. Hyperresolutions and Du Bois's original definition

A very important construction is Du Bois's generalized De Rham complex. The original construction of Du Bois's complex, $\underline{\Omega}^{\cdot}_X$, is based on simplicial resolutions. The reader interested in the details is referred to the original article [DB81]. Note also that a simplified construction was later obtained in [Car85] and [GNPP88] via the general theory of polyhedral and cubic resolutions. At the end of the paper, we include an appendix in which we explain how to construct, and give examples of cubical hyperresolutions. An easily accessible introduction can be found in [Ste85]. Another useful reference is the recent book [PS08].

In [Sch07] one of us found a simpler alternative construction of (part of) the Du Bois complex, not involving a simplicial resolution; see also Section 6 below. However we will discuss the original construction because it is important to keep in mind the way these singularities appeared, as that explains their usefulness. For more on applications of Du Bois's complex and Du Bois singularities see [Ste83], [Kol95, Chapter 12], [Kov99], and [Kov00b].

The word "hyperresolution" will refer to either simplicial, polyhedral, or cubic resolution. Formally, the construction of $\underline{\Omega}^{\cdot}_X$ is the same regardless the type of resolution used and no specific aspects of either types will be used.

The following definition is included to make sense of the statements of some of the forthcoming theorems. It can be safely ignored if the reader is not interested in the detailed properties of Du Bois's complex and is willing to accept that it is a very close analog of the De Rham complex of smooth varieties.

DEFINITION 4.1. Let X be a complex scheme (i.e., a scheme of finite type over $\mathbb{C}$) of dimension n. Let $D_{\mathrm{filt}}(X)$ denote the derived category of filtered complexes of $\mathscr{O}_X$-modules with differentials of order ≤ 1 and $D_{\mathrm{filt,coh}}(X)$ the subcategory of $D_{\mathrm{filt}}(X)$ of complexes $K^{\bullet}$ such that for all i, the cohomology sheaves of $\mathrm{Gr}^i_{\mathrm{filt}} K^{\cdot}$ are coherent; see [DB81], [GNPP88]. Let $D(X)$ and $D_{\mathrm{coh}}(X)$ denote the derived categories with the same definition except that the complexes are assumed to have the trivial filtration. The superscripts $+, -, b$ carry the usual

meaning (bounded below, bounded above, bounded). Isomorphism in these categories is denoted by $\simeq_{\text{qis}}$. A sheaf $\mathscr{F}$ is also considered a complex $\mathscr{F}^\bullet$ with $\mathscr{F}^0 = \mathscr{F}$ and $\mathscr{F}^i = 0$ for $i \neq 0$. If $K^\bullet$ is a complex in any of the above categories, then $h^i(K^\bullet)$ denotes the i-th cohomology sheaf of $K^\bullet$.

The right derived functor of an additive functor F, if it exists, is denoted by RF and R^iF is short for $h^i \circ RF$. Furthermore, $\mathbb{H}^i$, $\mathbb{H}^i_Z$, and $\mathscr{H}^i_Z$ will denote $R^i\Gamma$, $R^i\Gamma_Z$, and $R^i\mathscr{H}_Z$ respectively, where Γ is the functor of global sections, Γ_Z is the functor of global sections with support in the closed subset Z, and $\mathscr{H}_Z$ is the functor of the sheaf of local sections with support in the closed subset Z. Note that according to this terminology, if $\phi : Y \to X$ is a morphism and $\mathscr{F}$ is a coherent sheaf on Y, then $R\phi_*\mathscr{F}$ is the complex whose cohomology sheaves give rise to the usual higher direct images of $\mathscr{F}$.

THEOREM 4.2 [DB81, 6.3, 6.5]. *Let X be a proper complex scheme of finite type and D a closed subscheme whose complement is dense in X. Then there exists a unique object $\underline{\Omega}^\bullet_X \in \operatorname{Ob} D_{\text{filt}}(X)$ such that, using the notation*

$$\underline{\Omega}^p_X := \operatorname{Gr}^p_{\text{filt}} \underline{\Omega}^\bullet_X[p],$$

the following properties are satisfied:

(a) $\underline{\Omega}^\bullet_X \simeq_{\text{qis}} \mathbb{C}_X$; *i.e., $\underline{\Omega}^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}$ on X.*

(b) $\underline{\Omega}^\bullet_{(\_)}$ *is functorial; i.e., if $\phi : Y \to X$ is a morphism of proper complex schemes of finite type, there exists a natural map ϕ^* of filtered complexes*

$$\phi^* : \underline{\Omega}^\bullet_X \to R\phi_*\underline{\Omega}^\bullet_Y.$$

Furthermore, $\underline{\Omega}^\bullet_X \in \operatorname{Ob}\big(D^b_{\text{filt,coh}}(X)\big)$, and if ϕ is proper, ϕ^* is a morphism in $D^b_{\text{filt,coh}}(X)$.

(c) *Let $U \subseteq X$ be an open subscheme of X. Then*

$$\underline{\Omega}^\bullet_X|_U \simeq_{\text{qis}} \underline{\Omega}^\bullet_U.$$

(d) *If X is proper, there exists a spectral sequence degenerating at E_1 and abutting to the singular cohomology of X*:

$$E_1^{pq} = \mathbb{H}^q\left(X, \underline{\Omega}^p_X\right) \Rightarrow H^{p+q}(X^{\text{an}}, \mathbb{C}).$$

(e) *If $\varepsilon_\bullet : X_\bullet \to X$ is a hyperresolution, then*

$$\underline{\Omega}^\bullet_X \simeq_{\text{qis}} R\varepsilon_{\bullet *}\Omega^\bullet_{X_\bullet}.$$

In particular, $h^i\left(\underline{\Omega}^p_X\right) = 0$ for $i < 0$.

(f) *There exists a natural map, $\mathscr{O}_X \to \underline{\Omega}^0_X$, compatible with* (4.2.(b)).

(g) *If X is smooth, then*

$$\underline{\Omega}^{\cdot}_X \simeq_{\text{qis}} \Omega^{\cdot}_X.$$

In particular,

$$\underline{\Omega}^{p}_X \simeq_{\text{qis}} \Omega^{p}_X.$$

(h) *If $\phi : Y \to X$ is a resolution of singularities, then*

$$\underline{\Omega}^{\dim X}_X \simeq_{\text{qis}} R\phi_* \omega_Y.$$

(i) *Suppose that $\pi : \widetilde{Y} \to Y$ is a projective morphism and $X \subset Y$ a reduced closed subscheme such that π is an isomorphism outside of X. Let E denote the reduced subscheme of $\widetilde{Y}$ with support equal to $\pi^{-1}(X)$ and $\pi' : E \to X$ the induced map. Then for each p one has an exact triangle of objects in the derived category,*

$$\underline{\Omega}^{p}_Y \longrightarrow \underline{\Omega}^{p}_X \oplus R\pi_* \underline{\Omega}^{p}_{\widetilde{Y}} \xrightarrow{-} R\pi'_* \underline{\Omega}^{p}_E \xrightarrow{+1}.$$

It turns out that Du Bois's complex behaves very much like the de Rham complex for smooth varieties. Observe that condition (d) says that the Hodge-to-de Rham spectral sequence works for singular varieties if one uses the Du Bois complex in place of the de Rham complex. This has far reaching consequences and if the associated graded pieces, $\underline{\Omega}^{p}_X$ turn out to be computable, then this single property leads to many applications.

Notice that condition (f) gives a natural map $\mathcal{O}_X \to \underline{\Omega}^0_X$, and we will be interested in situations when this map is a quasi-isomorphism. When X is proper over $\mathbb{C}$, such a quasi-isomorphism will imply that the natural map

$$H^i(X^{\text{an}}, \mathbb{C}) \to H^i(X, \mathcal{O}_X) = \mathbb{H}^i(X, \underline{\Omega}^0_X)$$

is surjective because of the degeneration at E_1 of the spectral sequence in condition (d).

Following Du Bois, Steenbrink was the first to study this condition and he christened this property after Du Bois.

DEFINITION 4.3. A scheme X is said to have *Du Bois singularities* (or *DB singularities* for short) if the natural map $\mathcal{O}_X \to \underline{\Omega}^0_X$ from condition (f) in Theorem 4.2 is a quasi-isomorphism.

REMARK 4.4. If $\varepsilon : X_{\cdot} \to X$ is a hyperresolution of X (see the Appendix for a how to construct cubical hyperresolutions) then X has Du Bois singularities if and only if the natural map $\mathcal{O}_X \to R\varepsilon_{\cdot *}\mathcal{O}_{X_{\cdot}}$ is a quasi-isomorphism.

EXAMPLE 4.5. It is easy to see that smooth points are Du Bois. Deligne proved that normal crossing singularities are Du Bois as well [DJ74, Lemme 2(b)].

We will see more examples of Du Bois singularities in later sections.

5. An injectivity theorem and splitting the Du Bois complex

In this section, we state an injectivity theorem involving the dualizing sheaf that plays a role for Du Bois singularities similar to the role that Grauert–Riemenschneider plays for rational singularities. As an application, we state a criterion for Du Bois singularities related to a "splitting" of the Du Bois complex.

THEOREM 5.1 [Kov99, Lemma 2.2; Sch09, Proposition 5.11]. *Let X be a reduced scheme of finite type over $\mathbb{C}$, $x \in X$ a (possibly nonclosed) point, and $Z = \overline{\{x\}}$ its closure. Assume that $X \setminus Z$ has Du Bois singularities in a neighborhood of x (for example, x may correspond to an irreducible component of the non-Du Bois locus of X). Then the natural map*

$$H^i\left(R\mathcal{H}om^{\bullet}_X(\underline{\Omega}^0_X, \omega^{\bullet}_X)\right)_x \to H^i(\omega^{\bullet}_X)_x$$

is injective for every i.

The proof uses the fact that for a projective X, $H^i(X^{\mathrm{an}}, \mathbb{C}) \to \mathbb{H}^i(X, \underline{\Omega}^0_X)$ is surjective for every $i > 0$, which follows from Theorem 4.2.

It would also be interesting and useful if the following generalization of this injectivity were true.

QUESTION 5.2. Suppose that X is a reduced scheme essentially of finite type over $\mathbb{C}$. Is it true that the natural map of sheaves

$$H^i\left(R\mathcal{H}om^{\bullet}_X(\underline{\Omega}^0_X, \omega^{\bullet}_X)\right) \to H^i(\omega^{\bullet}_X)$$

is injective for every i?

Even though Theorem 5.1 does not answer Question 5.2, it has the following extremely useful corollary.

THEOREM 5.3 [Kov99, Theorem 2.3; Kol95, Theorem 12.8]. *Suppose that the natural map $\mathcal{O}_X \to \underline{\Omega}^0_X$ has a left inverse in the derived category (that is, a map $\rho : \underline{\Omega}^0_X \to \mathcal{O}_X$ such that the composition $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X \xrightarrow{\rho} \mathcal{O}_X$ is an isomorphism). Then X has Du Bois singularities.*

PROOF. Apply the functor $R\mathcal{H}om_X(\_, \omega^{\bullet}_X)$ to the maps $\mathcal{O}_X \longrightarrow \underline{\Omega}^0_X \xrightarrow{\rho} \mathcal{O}_X$. Then by the assumption, the composition

$$\omega^{\bullet}_X \xrightarrow{\delta} R\mathcal{H}om_X(\underline{\Omega}^0_X, \omega^{\bullet}_X) \longrightarrow \omega^{\bullet}_X$$

is an isomorphism. Let $x \in X$ be a possibly nonclosed point corresponding to an irreducible component of the non-Du Bois locus of X and consider the stalks at x of the cohomology sheaves of the complexes above. We obtain that the natural map

$$H^i\left(R\mathcal{H}om_X(\underline{\Omega}^0_X, \omega^{\bullet}_X)\right)_x \to H^i(\omega^{\bullet}_X)_x$$

is surjective for every i. But it is also injective by Theorem 5.1. This proves that $\delta : (\omega_X^{\bullet})_x \to R\mathcal{H}om_X(\underline{\Omega}^0_X, \omega_X^{\bullet})_x$ is a quasi-isomorphism. Finally, applying the functor $R\mathcal{H}om_{\mathcal{O}_{X,x}}(\_, (\omega_X^{\bullet})_x)$ one more time proves that X is Du Bois at x, contradicting our choice of $x \in X$ □

This also gives the following Boutot-like theorem for Du Bois singularities (cf. [Bou87]).

COROLLARY 5.4 [Kov99, Theorem 2.3; Kol95, Theorem 12.8]. *Suppose that $f : Y \to X$ is a morphism, Y has Du Bois singularities and the natural map $\mathcal{O}_X \to Rf_*\mathcal{O}_Y$ has a left inverse in the derived category. Then X also has Du Bois singularities.*

PROOF. Observe that the composition is an isomorphism

$$\mathcal{O}_X \to \underline{\Omega}^0_X \to Rf_*\underline{\Omega}^0_Y \simeq Rf_*\mathcal{O}_Y \to \mathcal{O}_X.$$

Then apply Theorem 5.3. □

As an easy corollary, we see that rational singularities are Du Bois (which was first observed in the isolated case by Steenbrink in [Ste83, Proposition 3.7]).

COROLLARY 5.5 [Kov99; Sai00]. *If X has rational singularities, then X has Du Bois singularities.*

PROOF. Let $\pi : \widetilde{X} \to X$ be a log resolution. One has the composition $\mathcal{O}_X \to \underline{\Omega}^0_X \to R\pi_*\mathcal{O}_{\widetilde{X}}$. Since X has rational singularities, this composition is a quasi-isomorphism. Apply Corollary 5.4. □

6. Hyperresolution-free characterizations of Du Bois singularities

The definition of Du Bois singularities given via hyperresolution is relatively complicated (hyperresolutions themselves can be rather complicated to compute; see 2). In this section we state several hyperresolution free characterizations of Du Bois singularities. The first such characterization was given by Steenbrink in the isolated case. Another, more analytic characterization was given by Ishii and improved by Watanabe in the isolated quasi-Gorenstein[3] case. Finally the second named author gave a characterization that works for any reduced scheme.

A relatively simple characterization of an affine cone over a projective variety being Du Bois is given in [DB81]. Steenbrink generalized this criterion to all normal isolated singularities. It is this criterion that Steenbrink, Ishii, Watanabe, and others used extensively to study isolated Du Bois singularities.

[3] A variety X is quasi-Gorenstein if K_X is a Cartier divisor. It is not required that X is Cohen–Macaulay.

THEOREM 6.1 [DB81, Proposition 4.13; Ste83, 3.6]. *Let (X, x) be a normal isolated Du Bois singularity, and $\pi : \widetilde{X} \to X$ a log resolution of (X, x) such that π is an isomorphism outside of $X \setminus \{x\}$. Let E denote the reduced preimage of x. Then (X, x) is a Du Bois singularity if and only if the natural map*

$$R^i\pi_*\mathscr{O}_{\widetilde{X}} \to R^i\pi_*\mathscr{O}_E$$

is an isomorphism for all $i > 0$.

PROOF. Using Theorem 4.2, we have an exact triangle

$$\underline{\Omega}^0_X \longrightarrow \underline{\Omega}^0_{\{x\}} \oplus R\pi_*\underline{\Omega}^0_{\widetilde{X}} \xrightarrow{-} R\pi_*\underline{\Omega}^0_E \xrightarrow{+1} .$$

Since $\{x\}$, $\widetilde{X}$ and E are all Du Bois (the first two are smooth, and E is snc), we have the following exact triangle

$$\underline{\Omega}^0_X \longrightarrow \mathscr{O}_{\{x\}} \oplus R\pi_*\mathscr{O}_{\widetilde{X}} \xrightarrow{-} R\pi_*\mathscr{O}_E \xrightarrow{+1} .$$

Suppose first that X has Du Bois singularities (that is, $\underline{\Omega}^0_X \simeq_{\text{qis}} \mathscr{O}_X$). By taking cohomology and examining the long exact sequence, we see that $R^i\pi_*\mathscr{O}_{\widetilde{X}} \to R^i\pi_*\mathscr{O}_E$ is an isomorphism for all $i > 0$.

So now suppose that $R^i\pi_*\mathscr{O}_{\widetilde{X}} \to R^i\pi_*\mathscr{O}_E$ is an isomorphism for all $i > 0$. By considering the long exact sequence of cohomology, we see that $H^i(\underline{\Omega}^0_X)$ is zero for all $i > 0$. On the other hand, $H^0(\underline{\Omega}^0_X)$ is naturally identified with the seminormalization of $\mathscr{O}_X$; see Proposition 7.8 below. Thus if X is normal, then $\mathscr{O}_X \to H^0(\underline{\Omega}^0_X)$ is an isomorphism. □

We now state a more analytic characterization, due to Ishii and slightly improved by Watanabe. First we recall the definition of the plurigenera of a singularity.

DEFINITION 6.2. For a singularity (X, x), we define the plurigenera $\{\delta_m\}_{m\in\mathbb{N}}$;

$$\delta_m(X, x) = \dim_{\mathbb{C}} \Gamma(X \setminus x, \mathscr{O}_X(mK_X))/L^{2/m}(X \setminus \{x\}),$$

where $L^{2/m}(X \setminus \{x\})$ denotes the set of all $L^{2/m}$-integrable m-uple holomorphic n-forms on $X \setminus \{x\}$.

THEOREM 6.3 [Ish85, Theorem 2.3; Wat87, Theorem 4.2]. *Let $f : \widetilde{X} \to X$ be a log resolution of a normal isolated Gorenstein singularity (X, x) of dimension $n \geq 2$. Set E to be the reduced exceptional divisor (the preimage of x). Then (X, x) is a Du Bois singularity if and only if $\delta_m(X, x) \leq 1$ for any $m \in \mathbb{N}$.*

In [Sch07], a characterization of arbitrary Du Bois singularities is given that did not rely on hyperresolutions, but instead used a single resolution of singularities. An improvement of this was also obtained in [ST08, Proposition 2.20]. We provide a proof for the convenience of the reader.

THEOREM 6.4 [Sch07; ST08, Proposition 2.20]. *Let X be a reduced separated scheme of finite type over a field of characteristic zero. Suppose that $X \subseteq Y$ where Y is smooth and suppose that $\pi : \widetilde{Y} \to Y$ is a proper birational map with $\widetilde{Y}$ smooth and where $\overline{X} = \pi^{-1}(X)_{\mathrm{red}}$, the reduced preimage of X, is a simple normal crossings divisor* (*or in fact any scheme with Du Bois singularities*). *Then X has Du Bois singularities if and only if the natural map $\mathscr{O}_X \to R\pi_*\mathscr{O}_{\overline{X}}$ is a quasi-isomorphism.*

In fact, we can say more. There is an isomorphism

$$R\pi_*\mathscr{O}_{\overline{X}} \xrightarrow{\sim} \underline{\Omega}^0_X$$

such that the natural map $\mathscr{O}_X \to \underline{\Omega}^0_X$ can be identified with the natural map $\mathscr{O}_X \to R\pi_*\mathscr{O}_{\overline{X}}$.

PROOF. We first assume that π is an isomorphism outside of X. Then using Theorem 4.2, we have an exact triangle

$$\underline{\Omega}^0_Y \longrightarrow \underline{\Omega}^0_X \oplus R\pi_*\underline{\Omega}^0_{\widetilde{Y}} \xrightarrow{-} R\pi_*\underline{\Omega}^0_{\overline{X}} \xrightarrow{+1}.$$

Using the octahedral axiom, we obtain the diagram

$$\begin{array}{ccccccc}
C^{\cdot} & \longrightarrow & \underline{\Omega}^0_Y & \longrightarrow & \underline{\Omega}^0_X & \xrightarrow{+1} & \\
\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\
C^{\cdot} & \longrightarrow & R\pi_*\underline{\Omega}^0_{\widetilde{Y}} & \longrightarrow & R\pi_*\underline{\Omega}^0_{\overline{X}} & \xrightarrow{+1} &.
\end{array}$$

where $C^{\cdot}$ is simply the object in the derived category that completes the triangles. But notice that the vertical arrow α is an isomorphism since Y has rational singularities (in which case each term in the middle column is isomorphic to $\mathscr{O}_Y$). Thus the vertical arrow β is also an isomorphism.

One always has a commutative diagram

$$\begin{array}{ccc}
\mathscr{O}_X & \longrightarrow & \underline{\Omega}^0_X \\
\downarrow & & \downarrow{\scriptstyle \beta} \\
R\pi_*\mathscr{O}_{\overline{X}} & \xrightarrow[\delta]{} & R\pi_*\underline{\Omega}^0_{\overline{X}}
\end{array}$$

(where the arrows are the natural ones). Observe that $\overline{X}$ has Du Bois singularities since it has normal crossings, thus δ is a quasi-isomorphism. But then the theorem is proven at least in the case that π is an isomorphism outside of X.

For the general case, it is sufficient to show that $R\pi_*\mathscr{O}_{\overline{X}}$ is independent of the choice of resolution. Since any two log resolutions can be dominated by a

third, it is sufficient to consider two log resolutions $\pi_1 : Y_1 \to Y$ and $\pi_2 : Y_2 \to Y$ and a map between them $\rho : Y_2 \to Y_1$ over Y. Let $F_1 = (\pi_1^{-1}(X))_{\text{red}}$ and $F_2 = (\pi_2^{-1}(X))_{\text{red}} = (\rho^{-1}(F_1))_{\text{red}}$. Dualizing the map and applying Grothendieck duality implies that it is sufficient to prove that $\omega_{Y_1}(F_1) \leftarrow R\rho_*(\omega_{Y_2}(F_2))$ is a quasi-isomorphism.

We now apply the projection formula while twisting by $\omega_{Y_1}^{-1}(-F_1)$. Thus it is sufficient to prove that

$$R\rho_*(\omega_{Y_2/Y_1}(F_2 - \rho^* F_1)) \to \mathscr{O}_{Y_1}$$

is a quasi-isomorphism. But note that $F_2 - \rho^* F_1 = -\lfloor \rho^*(1-\varepsilon)F_1 \rfloor$ for sufficiently small $\varepsilon > 0$. Thus it is sufficient to prove that the pair $(Y_1, (1-\varepsilon)F_1)$ has klt singularities by Kawamata–Viehweg vanishing in the form of local vanishing for multiplier ideals; see [Laz04, 9.4]. But this is true since Y_1 is smooth and F_1 is a reduced integral divisor with simple normal crossings. □

It seems that in this characterization the condition that the ambient variety Y is smooth is asking for too much. We propose that the following may be a more natural characterization. For some motivation and for a statement that may be viewed as a sort of converse; see Conjecture 12.5 and the discussion preceding it.

CONJECTURE 6.5. *Theorem* 6.4 *should remain true if the hypothesis that Y is smooth is replaced by the condition that Y has rational singularities.*

Having Du Bois singularities is a local condition, so even if X is not embeddable in a smooth scheme, one can still use Theorem 6.4 by passing to an affine open covering.

To illustrate the utility and meaning of Theorem 6.4, we will explore the situation when X is a hypersurface inside a smooth scheme Y. In the notation of Theorem 6.4, we have the diagram of exact triangles

$$\begin{array}{ccccccccc}
 & & R\pi_*\mathscr{O}_{\widetilde{Y}}(-\overline{X}) & \longrightarrow & R\pi_*\mathscr{O}_{\widetilde{Y}} & \longrightarrow & R\pi_*\mathscr{O}_{\overline{X}} & \xrightarrow{+1} & \\
 & & \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathscr{O}_Y(-X) & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0
\end{array}$$

Since Y is smooth, β is a quasi-isomorphism (as then Y has at worst rational singularities). Therefore, X has Du Bois singularities if and only if the map α is a quasi-isomorphism. However, α is a quasi-isomorphism if and only if the dual map

$$R\pi_*\omega_{\widetilde{Y}}^{\bullet}(\overline{X}) \to \omega_Y^{\bullet}(X) \tag{6.5.1}$$

is a quasi-isomorphism. The projection formula tells us that Equation 6.5.1 is a quasi-isomorphism if and only

$$R\pi_*\mathscr{O}_{\widetilde{Y}}(K_{\widetilde{Y}/Y} - \pi^*X + \overline{X}) \to \mathscr{O}_X \tag{6.5.2}$$

is a quasi-isomorphism. Note however that $-\pi^*X + \overline{X} = \lceil -(1-\varepsilon)\pi^*X \rceil$ for $\varepsilon > 0$ and sufficiently close to zero. Thus the left side of Equation 6.5.2 can be viewed as $R\pi_*\mathscr{O}_{\widetilde{Y}}(\lceil K_{\widetilde{Y}/Y} - (1-\varepsilon)\pi^*X \rceil)$ for $\varepsilon > 0$ sufficiently small. Note that Kawamata–Viehweg vanishing in the form of local vanishing for multiplier ideals implies that $\mathscr{J}(Y, (1-\varepsilon)X) \simeq_{\text{qis}} R\pi_*\mathscr{O}_{\widetilde{Y}}(\lceil K_{\widetilde{Y}/Y} - (1-\varepsilon)\pi^*X \rceil)$. Therefore X has Du Bois singularities if and only if $\mathscr{J}(Y, (1-\varepsilon)X) \simeq \mathscr{O}_X$.

COROLLARY 6.6. *If X is a hypersurface in a smooth Y, then X has Du Bois singularities if and only if the pair (Y, X) is log canonical.*

Du Bois hypersurfaces have also been characterized via the Bernstein–Sato polynomial; see [Sai09, Theorem 0.5].

7. Seminormality of Du Bois singularities

In this section we show that Du Bois singularities are partially characterized by seminormality. First we remind the reader what it means for a scheme to be seminormal.

DEFINITION 7.1 [Swa80; GT80]. Suppose that R is a reduced excellent ring and that $S \supseteq R$ is a reduced R-algebra which is finite as an R-module. We say that the extension $i : R \hookrightarrow S$ is *subintegral* if

(i) i induces a bijection on spectra, Spec $S \to$ Spec R, and
(ii) i induces an isomorphism of residue fields over every (possibly nonclosed) point of Spec R.

REMARK 7.2. In [GT80], subintegral extensions are called quasi-isomorphisms.

DEFINITION 7.3 [Swa80; GT80]. Suppose that R is a reduced excellent ring. We say that R is *seminormal* if every subintegral extension $R \hookrightarrow S$ is an isomorphism. We say that a scheme X is *seminormal* if all of its local rings are seminormal.

REMARK 7.4. In [GT80], the authors call R seminormal if there is no proper subintegral extension $R \hookrightarrow S$ such that S is contained in the integral closure of R (in its total field of fractions). However, it follows from [Swa80, Corollary 3.4] that the definition above is equivalent.

REMARK 7.5. Seminormality is a local property. In particular, a ring is seminormal if and only if it is seminormal after localization at each of its prime (equivalently, maximal) ideals.

REMARK 7.6. The easiest example of seminormal schemes are schemes with snc singularities. In fact, a one dimensional variety over an algebraically closed field is seminormal if and only if its singularities are locally analytically isomorphic to a union of coordinate axes in affine space.

We will use the following well known fact about seminormality.

LEMMA 7.7. *If X is a seminormal scheme and $U \subseteq X$ is any open set, then $\Gamma(U, \mathscr{O}_X)$ is a seminormal ring.*

PROOF. We leave it as an exercise to the reader. □

It is relatively easy to see, using the original definition via hyperresolutions, that if X has Du Bois singularities, then it is seminormal. Du Bois certainly knew this fact (see [DB81, Proposition 4.9]) although he didn't use the word seminormal. Later Saito proved that seminormality in fact partially characterizes Du Bois singularities. We give a different proof of this fact, from [Sch09].

PROPOSITION 7.8 [Sai00, Proposition 5.2; Sch09, Lemma 5.6]. *Suppose that X is a reduced separated scheme of finite type over $\mathbb{C}$. Then $h^0(\underline{\Omega}^0_X) = \mathscr{O}_{X^{\mathrm{sn}}}$ where $\mathscr{O}_{X^{\mathrm{sn}}}$ is the structure sheaf of the seminormalization of X.*

PROOF. Without loss of generality we may assume that X is affine. We need only consider $\pi_*\mathscr{O}_E$ by Theorem 6.4. By Lemma 7.7, $\pi_*\mathscr{O}_E$ is a sheaf of seminormal rings. Now let $X' = \operatorname{Spec}(\pi_*\mathscr{O}_E)$ and consider the factorization

$$E \to X' \to X.$$

Note $E \to X'$ must be surjective since it is dominant by construction and is proper by [Har77, II.4.8(e)]. Since the composition has connected fibers, so must have $\rho : X' \to X$. On the other hand, ρ is a finite map since π is proper. Therefore ρ is a bijection on points. Because these maps and schemes are of finite type over an algebraically closed field of characteristic zero, we see that $\Gamma(X, \mathscr{O}_X) \to \Gamma(X', \mathscr{O}_{X'})$ is a subintegral extension of rings. Since X' is seminormal, so is $\Gamma(X', \mathscr{O}_{X'})$, which completes the proof. □

8. A multiplier-ideal-like characterization of Cohen–Macaulay Du Bois singularities

In this section we state a characterization of Cohen–Macaulay Du Bois singularities that explains why Du Bois singularities are so closely linked to rational and log canonical singularities.

We first do a suggestive computation. Suppose that X embeds into a smooth scheme Y and that $\pi : \widetilde{Y} \to Y$ is an embedded resolution of X in Y that is an isomorphism outside of X. Set $\widetilde{X}$ to be the strict transform of X and set $\overline{X}$ to

be the reduced preimage of X. We further assume that $\overline{X} = \widetilde{X} \cup E$ where E is a reduced simple normal crossings divisor that intersects $\widetilde{X}$ transversally in another reduced simple normal crossing divisor. Note that E is the exceptional divisor of π (with reduced scheme structure). Set $\Sigma \subseteq X$ be the image of E. We have the short exact sequence

$$0 \to \mathscr{O}_{\widetilde{X}}(-E) \to \mathscr{O}_{\overline{X}} \to \mathscr{O}_E \to 0$$

We apply $R\mathcal{H}om_{\mathscr{O}_Y}(\_, \omega^{\bullet}_{\widetilde{Y}})$ followed by $R\pi_* \_$ and obtain the exact triangle

$$R\pi_*\omega^{\bullet}_E \longrightarrow R\pi_*\omega^{\bullet}_{\overline{X}} \longrightarrow R\pi_*\omega_{\widetilde{X}}(E)[\dim X] \xrightarrow{+1}$$

Using condition (i) in Theorem 4.2, the leftmost object can be identified with $R\mathcal{H}om_{\mathscr{O}_\Sigma}(\underline{\Omega}^0_\Sigma, \omega^{\bullet}_\Sigma)$ and the middle object, $R\pi_*\omega^{\bullet}_{\overline{X}}$, can be identified with $R\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}^0_X, \omega^{\bullet}_X)$. Recall that X has Du Bois singularities if and only if the natural map $R\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}^0_X, \omega^{\bullet}_X) \to \omega^{\bullet}_X$ is an isomorphism. Therefore, the object $\pi_*\omega_{\widetilde{X}}(E)$ is closely related to whether or not X has Du Bois singularities. This inspired the following result, which we do not prove.

THEOREM 8.1 [KSS10, Theorem 3.1] . *Suppose that X is normal and Cohen–Macaulay. Let $\pi : X' \to X$ be a log resolution, and denote the reduced exceptional divisor of π by G. Then X has Du Bois singularities if and only if $\pi_*\omega_{X'}(G) \simeq \omega_X$.*

We mention that the main idea in the proof is to show that

$$\pi_*\omega_{X'}(G) \simeq H^{-\dim X}\big(R\mathcal{H}om_{\mathscr{O}_X}(\underline{\Omega}^0_X, \omega^{\bullet}_X)\big).$$

Related results can also be obtained in the nonnormal Cohen–Macaulay case; see [KSS10] for details.

REMARK 8.2. The submodule $\pi_*\omega_{X'}(G) \subseteq \omega_X$ is independent of the choice of log resolution. Thus this submodule may be viewed as an invariant which partially measures how far a scheme is from being Du Bois (compare with [Fuj08]).

As an easy corollary, we obtain another proof that rational singularities are Du Bois (this time via the Kempf-criterion for rational singularities).

COROLLARY 8.3. *If X has rational singularities, then X has Du Bois singularities.*

PROOF. Since X has rational singularities, it is Cohen–Macaulay and normal. Then $\pi_*\omega_{X'} = \omega_X$ but we also have $\pi_*\omega_{X'} \subseteq \pi_*\omega_{X'}(G) \subseteq \omega_X$, and thus $\pi_*\omega_{X'}(G) = \omega_X$ as well. Then use Theorem 8.1. □

We also see immediately that log canonical singularities coincide with Du Bois singularities in the Gorenstein case.

COROLLARY 8.4. *Suppose that X is Gorenstein and normal. Then X is Du Bois if and only if X is log canonical.*

PROOF. X is easily seen to be log canonical if and only if $\pi_*\omega_{X'/X}(G) \simeq \mathscr{O}_X$. The projection formula then completes the proof. □

In fact, a slightly improved version of this argument can be used to show that every Cohen–Macaulay log canonical pair is Du Bois; see [KSS10, Theorem 3.16].

9. The Kollár–Kovács splitting criterion

The proof of the following, rather flexible, criterion for Du Bois singularities can be found in the original paper.

THEOREM 9.1 [KK10]. *Let $f : Y \to X$ be a proper morphism between reduced schemes of finite type over $\mathbb{C}$, $W \subseteq X$ an arbitrary subscheme, and $F := f^{-1}(W)$, equipped with the induced reduced subscheme structure. Let $\mathscr{I}_{W\subseteq X}$ denote the ideal sheaf of W in X and $\mathscr{I}_{F\subseteq Y}$ the ideal sheaf of F in Y. Assume that the natural map ϱ*

$$\begin{array}{ccc} & & \varrho' \\ & \swarrow & \cdots \\ \mathscr{I}_{W\subseteq X} & \xrightarrow[\varrho]{} & Rf_*\mathscr{I}_{F\subseteq Y} \end{array}$$

admits a left inverse ϱ', that is, $\rho' \circ \rho = \mathrm{id}_{\mathscr{I}_{W\subseteq X}}$. Then if Y, F, and W all have DB singularities, so does X.

REMARK 9.1.1. Notice that it is not required that f be birational. On the other hand the assumptions of the theorem and [Kov00a, Theorem 1] imply that if $Y \setminus F$ has rational singularities, e.g., if Y is smooth, then $X \setminus W$ has rational singularities as well.

This theorem is used to derive various consequences in [KK10], some of which are formally unrelated to Du Bois singularities. We will mention some of these in the sequel, but the interested reader should look at the original article to obtain the full picture.

10. Log canonical singularities are Du Bois

Log canonical and Du Bois singularities are very closely related as we have seen in the previous sections. This was first observed in [Ish85]; see also [Wat87] and [Ish87].

Recently, Kollár and the first named author gave a proof that log canonical singularities are Du Bois using Theorem 9.1. We will sketch some ideas of the

proof here. There are two main steps. First, one shows that the non-klt locus of a log canonical singularity is Du Bois (this generalizes [Amb98] and [Sch08, Corollary 7.3]). Then one uses Theorem 9.1 to show that this property is enough to conclude that X itself is Du Bois. For the first part we refer the reader to the original paper. The key point of the second part is contained in the following Lemma. Here we give a different proof than in [KK10].

LEMMA 10.1. *Suppose (X, Δ) is a log canonical pair and that the reduced non-klt locus of (X, Δ) has Du Bois singularities. Then X has Du Bois singularities.*

PROOF. First recall that the multiplier ideal $\mathcal{J}(X, \Delta)$ is precisely the defining ideal of the non-klt locus of (X, Δ) and since (X, Δ) is log canonical, it is a radical ideal. We set $\Sigma \subseteq X$ to be the reduced subscheme of X defined by this ideal. Since the statement is local, we may assume that X is affine and thus that X is embedded in a smooth scheme Y. We let $\pi : \widetilde{Y} \to Y$ be an embedded resolution of (X, Δ) in Y and we assume that π is an isomorphism outside the singular locus of X. Set $\overline{\Sigma}$ to be the reduced-preimage of Σ (which we may assume is a divisor in $\widetilde{Y}$) and let $\widetilde{X}$ denote the strict transform of X. We consider the diagram of exact triangles

$$\begin{array}{ccccccccc}
 & & A^{\cdot} & \longrightarrow & B^{\cdot} & \longrightarrow & C^{\cdot} & \xrightarrow{+1} & \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \mathcal{J}(X, \Delta) & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{O}_{\Sigma} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \delta} & & \downarrow & & \downarrow{\scriptstyle \varepsilon} & & \\
 & & R\pi_*\mathscr{O}_{\widetilde{X}}(-\overline{\Sigma}) & \longrightarrow & R\pi_*\mathscr{O}_{\overline{\Sigma}\cup\widetilde{X}} & \longrightarrow & R\pi_*\mathscr{O}_{\overline{\Sigma}} & \xrightarrow{+1} &
\end{array}$$

Here the first row is made up of objects in $D^b_{\mathrm{coh}}(X)$ needed to make the columns into exact triangles. Since Σ has Du Bois singularities, the map ε is an isomorphism and so $C^{\cdot} \simeq 0$. On the other hand, there is a natural map $R\pi_*\mathscr{O}_{\widetilde{X}}(-\overline{\Sigma}) \to R\pi_*\mathscr{O}_{\widetilde{X}}(K_{\widetilde{X}} - \pi^*(K_X + \Delta)) \simeq \mathcal{J}(X, \Delta)$ since (X, Δ) is log canonical. This implies that the map α is the zero map in the derived category. However, we then see that β is also zero in the derived category which implies that $\mathscr{O}_X \to R\pi_*\mathscr{O}_{\overline{\Sigma}\cup\widetilde{X}}$ has a left inverse. Therefore, X has Du Bois singularities (since $\overline{\Sigma}\cup\widetilde{X}$ has simple normal crossing singularities) by Theorems 5.3 and 6.4. □

11. Applications to moduli spaces and vanishing theorems

The connection between log canonical and Du Bois singularities have many useful applications in moduli theory. We will list a few without proof.

SETUP 11.1. Let $\phi : X \to B$ be a flat projective morphism of complex varieties with B connected. Assume that for all $b \in B$ there exists a $\mathbb{Q}$-divisor D_b on X_b such that (X_b, D_b) is log canonical.

REMARK 11.2. Notice that it is not required that the divisors D_b form a family.

THEOREM 11.3 [KK10]. *Under the assumptions* 11.1, $h^i(X_b, \mathscr{O}_{X_b})$ *is independent of* $b \in B$ *for all* i.

THEOREM 11.4 [KK10]. *Under the assumptions* 11.1, *if one fiber of* ϕ *is Cohen–Macaulay* (*resp.* S_k *for some* k), *so are all the fibers.*

THEOREM 11.5 [KK10]. *Under the assumptions* 11.1, *the cohomology sheaves* $h^i(\omega^{\bullet}_{\phi})$ *are flat over* B, *where* $\omega^{\bullet}_{\phi}$ *denotes the relative dualizing complex of* ϕ.

Du Bois singularities also appear naturally in vanishing theorems. As a culmination of the work of Tankeev, Ramanujam, Miyaoka, Kawamata, Viehweg, Kollár, and Esnault–Viehweg, Kollár proved a rather general form of a Kodaira-type vanishing theorem in [Kol95, 9.12]. Using the same ideas this was slightly generalized to the following theorem in [KSS10].

THEOREM 11.6 [Kol95, 9.12; KSS08, 6.2]. *Let* X *be a proper variety and* $\mathscr{L}$ *a line bundle on* X. *Let* $\mathscr{L}^m \simeq \mathscr{O}_X(D)$, *where* $D = \sum d_i D_i$ *is an effective divisor, and let* s *be a global section whose zero divisor is* D. *Assume that* $0 < d_i < m$ *for every* i. *Let* Z *be the scheme obtained by taking the* m*-th root of* s (*that is,* $Z = X[\sqrt{s}]$ *using the notation from* [Kol95, 9.4]). *Assume further that*

$$H^j(Z, \mathbb{C}_Z) \to H^j(Z, \mathscr{O}_Z)$$

is surjective. Then, for any collection of $b_i \geq 0$, *the natural map*

$$H^j\left(X, \mathscr{L}^{-1}\left(-\textstyle\sum b_i D_i\right)\right) \to H^j(X, \mathscr{L}^{-1})$$

is surjective.

This, combined with the fact that log canonical singularities are Du Bois, yields that Kodaira vanishing holds for log canonical pairs:

THEOREM 11.7 [KSS10, 6.6]. *Kodaira vanishing holds for Cohen–Macaulay semi-log canonical varieties*: *Let* (X, Δ) *be a projective Cohen–Macaulay semi-log canonical pair and* $\mathscr{L}$ *an ample line bundle on* X. *Then* $H^i(X, \mathscr{L}^{-1}) = 0$ *for* $i < \dim X$.

It turns out that Du Bois singularities appear naturally in other kinds of vanishing theorems. We cite one here.

THEOREM 11.8 [GKKP10, 9.3]. *Let (X, D) be a log canonical reduced pair of dimension $n \geq 2$, $\pi : \widetilde{X} \to X$ a log resolution with π-exceptional set E, and $\widetilde{D} = \operatorname{Supp}\big(E + \pi^{-1} D\big)$. Then*

$$R^{n-1}\pi_* \mathscr{O}_{\widetilde{X}}(-\widetilde{D}) = 0.$$

12. Deformations of Du Bois singularities

Given the importance of Du Bois singularities in moduli theory it is an important obvious question whether they are invariant under small deformation.

It is relatively easy to see from the construction of the Du Bois complex that a general hyperplane section (or more generally, the general member of a base point free linear system) on a variety with Du Bois singularities again has Du Bois singularities. Therefore the question of deformation follows from the following.

CONJECTURE 12.1. *(cf.* [Ste83]) *Let $D \subset X$ be a reduced Cartier divisor and assume that D has only Du Bois singularities in a neighborhood of a point $x \in D$. Then X has only Du Bois singularities in a neighborhood of the point x.*

This conjecture was proved for isolated Gorenstein singularities by Ishii [Ish86]. Also note that rational singularities satisfy this property; see [Elk78].

We also have the following easy corollary of the results presented earlier:

THEOREM 12.2. *Assume that X is Gorenstein and D is normal.*[4] *Then the statement of Conjecture* 12.1 *is true.*

PROOF. The question is local so we may restrict to a neighborhood of x. If X is Gorenstein, then so is D as it is a Cartier divisor. Then D is log canonical by (8.4), and then the pair (X, D) is also log canonical by inversion of adjunction [Kaw07]. (Recall that if D is normal, then so is X along D). This implies that X is also log canonical and thus Du Bois. □

It is also stated in [Kov00b, 3.2] that the conjecture holds in full generality. Unfortunately, the proof is not complete. The proof published there works if one assumes that the non-Du Bois locus of X is contained in D. For instance, one may assume that this is the case if the non-Du Bois locus is isolated.

The problem with the proof is the following: it is stated that by taking hyperplane sections one may assume that the non-Du Bois locus is isolated. However, this is incorrect. One may only assume that the *intersection* of the non-Du Bois locus of X with D is isolated. If one takes a further general section then it will

[4] This condition is actually not necessary, but the proof becomes rather involved without it.

miss the intersection point and then it is not possible to make any conclusions about that case.

Therefore currently the best known result with regard to this conjecture is the following:

THEOREM 12.3 [Kov00b, 3.2]. *Let $D \subset X$ be a reduced Cartier divisor and assume that D has only Du Bois singularities in a neighborhood of a point $x \in D$ and that $X \setminus D$ has only Du Bois singularities. Then X has only Du Bois singularities in a neighborhood of x.*

Experience shows that divisors not in general position tend to have worse singularities than the ambient space in which they reside. Therefore one would in fact expect that if $X \setminus D$ is reasonably nice, and D has Du Bois singularities, then perhaps X has even better ones.

We have also seen that rational singularities are Du Bois and at least Cohen–Macaulay Du Bois singularities are not so far from being rational cf. 8.1. The following result of the second named author supports this philosophical point.

THEOREM 12.4 [Sch07, Theorem 5.1]. *Let X be a reduced scheme of finite type over a field of characteristic zero, D a Cartier divisor that has Du Bois singularities and assume that $X \setminus D$ is smooth. Then X has rational singularities* (*in particular, it is Cohen–Macaulay*).

Let us conclude with a conjectural generalization of this statement:

CONJECTURE 12.5. *Let X be a reduced scheme of finite type over a field of characteristic zero, D a Cartier divisor that has Du Bois singularities and assume that $X \setminus D$ has rational singularities. Then X has rational singularities* (*in particular, it is Cohen–Macaulay*).

Essentially the same proof as in (12.2) shows that this is also true under the same additional hypotheses.

THEOREM 12.6. *Assume that X is Gorenstein and D is normal.*[5] *Then the statement of Conjecture* 12.5 *is true.*

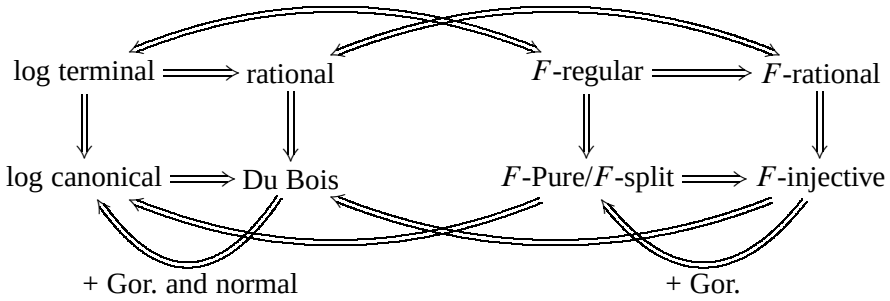
PROOF. If X is Gorenstein, then so is D as it is a Cartier divisor. Then by (8.4) D is log canonical. Then by inversion of adjunction [Kaw07] the pair (X, D) is also log canonical near D. (Recall that if D is normal, then so is X along D).

As X is Gorenstein and $X \setminus D$ has rational singularities, it follows that $X \setminus D$ has canonical singularities. Then X has only canonical singularities everywhere. This can be seen by observing that D is a Cartier divisor and examining the discrepancies that lie over D for (X, D) as well as for X. Therefore, by [Elk81], X has only rational singularities along D. □

[5] Again, this condition is not necessary, but makes the proof simpler.

13. Analogs of Du Bois singularities in characteristic $p > 0$

Starting in the early 1980s, the connections between singularities defined by the action of the Frobenius morphism in characteristic $p > 0$ and singularities defined by resolutions of singularities started to be investigated, cf. [Fed83]. After the introduction of tight closure in [HH90], a precise correspondence between several classes of singularities was established. See, for example, [FW89; MS91; HW02; Smi97; Har98; MS97; Smi00; Har05; HY03; Tak04; TW04; Tak08]. The second named author partially extended this correspondence in his doctoral dissertation by linking Du Bois singularities with F-injective singularities, a class of singularities defined in [Fed83]. The currently known implications are summarized below.



We will give a short proof that normal Cohen–Macaulay singularities of dense F-injective type are Du Bois, based on the characterization of Du Bois singularities given in Section 8.

Note that Du Bois and F-injective singularities also share many common properties. For example F-injective singularities are also seminormal [Sch09, Theorem 4.7].

First however, we will define F-injective singularities (as well as some necessary prerequisites).

DEFINITION 13.1. Suppose that X is a scheme of characteristic $p > 0$ with absolute Frobenius map $F : X \to X$. We say that X is *F-finite* if $F_*\mathcal{O}_X$ is a coherent $\mathcal{O}_X$-module. A ring R is called *F-finite* if the associated scheme Spec R is F-finite.

REMARK 13.2. Any scheme of finite type over a perfect field is F-finite; see for example [Fed83].

DEFINITION 13.3. Suppose that $(R, \mathfrak{m})$ is an F-finite local ring. We say that R is *F-injective* if the induced Frobenius map $F : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for every $i > 0$. We say that an F-finite scheme is *F-injective* if all of its stalks are F-injective local rings.

REMARK 13.4. If $(R, \mathfrak{m})$ is F-finite, F-injective and has a dualizing complex, then R_Q is also F-injective for any $Q \in \operatorname{Spec} R$. This follows from local duality; see [Sch09, Proposition 4.3] for details.

LEMMA 13.5. *Suppose X is a Cohen–Macaulay scheme of finite type over a perfect field k. Then X is F-injective if and only if the natural map $F_*\omega_X \to \omega_X$ is surjective.*

PROOF. Without loss of generality (since X is Cohen–Macaulay) we can assume that X is equidimensional. Set $f : X \to \operatorname{Spec} k$ to be the structural morphism. Since X is finite type over a perfect field, it has a dualizing complex $\omega_X^\bullet = f^! k$ and we set $\omega_X = h^{-\dim X}(\omega_X^\bullet)$. Since X is Cohen–Macaulay, X is F-injective if and only if the Frobenius map $H^{\dim X}_x(\mathscr{O}_{X,x}) \longrightarrow H^{\dim X}_x(F_*\mathscr{O}_{X,x})$ is injective for every closed point $x \in X$. By local duality (see [Har66, Theorem 6.2] or [BH93, Section 3.5]) such a map is injective if and only if the dual map $F_*\omega_{X,x} \to \omega_{X,x}$ is surjective. But that map is surjective, if and only if the map of sheaves $F_*\omega_X \to \omega_X$ is surjective. □

We now briefly describe reduction to characteristic $p > 0$. Excellent and far more complete references include [HH09, Section 2.1] and [Kol96, II.5.10]. Also see [Smi01] for a more elementary introduction.

Let R be a finitely generated algebra over a field k of characteristic zero. Write $R = k[x_1, \dots, x_n]/I$ for some ideal I and let S denote $k[x_1, \dots, x_n]$. Let $X = \operatorname{Spec} R$ and $\pi : \widetilde{X} \to X$ a log resolution of X corresponding to the blow-up of an ideal J. Let E denote the reduced exceptional divisor of π. Then E is the subscheme defined by the radical of the ideal $J \cdot \mathscr{O}_{\widetilde{X}}$.

There exists a finitely generated $\mathbb{Z}$-algebra $A \subset k$ that includes all the coefficients of the generators of I and J, a finitely generated A algebra $R_A \subset R$, an ideal $J_A \subset R_A$, and schemes $\widetilde{X}_A$ and E_A of finite type over A such that $R_A \otimes_A k = R$, $J_A R = J$, $\widetilde{X}_A \times_{\operatorname{Spec} A} \operatorname{Spec} k = X$ and $E_A \times_{\operatorname{Spec} A} \operatorname{Spec} k = E$ with E_A an effective divisor with support defined by the ideal $J_A \cdot \mathscr{O}_{\widetilde{X}_A}$. We may localize A at a single element so that Y_A is smooth over A and E_A is a reduced simple normal crossings divisor over A. By further localizing A (at a single element), we may assume any finite set of finitely generated R_A modules is A-free (see [Hun96, 3.4] or [HR76, 2.3]) and we may assume that A itself is regular. We may also assume that a fixed affine cover of E_A and a fixed affine cover of $\widetilde{X}_A$ are also A-free.

We will now form a family of positive characteristic models of X by looking at all the rings $R_t = R_A \otimes_A k(t)$ where $k(t)$ is the residue field of a maximal ideal $t \in T = \operatorname{Spec} A$. Note that $k(t)$ is a finite, and thus perfect, field of characteristic p. We may also tensor the various schemes X_A, E_A, etc. with $k(t)$ to produce a characteristic p model of an entire situation.

By making various cokernels of maps free A-modules, we may also assume that maps between modules that are surjective (respectively injective) over k correspond to surjective (respectively injective) maps over A, and thus surjective (respectively injective) in our characteristic p model as well; see [HH09] for details.

DEFINITION 13.6. A ring R of characteristic zero is said to have dense F-injective type if for every family of characteristic $p \gg 0$ models with A chosen sufficiently large, a Zariski dense set of those models (over Spec A) have F-injective singularities.

THEOREM 13.7 [Sch09]. *Let X be a reduced scheme of finite type over $\mathbb{C}$ and assume that it has dense F-injective type. Then X has Du Bois singularities.*

PROOF. We only provide a proof in the case that X is normal and Cohen–Macaulay. For a complete proof, see [Sch09]. Let $\pi : \widetilde{X} \to X$ be a log resolution of X with exceptional divisor E. We reduce this entire setup to characteristic $p \gg 0$ such that the corresponding X is F-injective. Let $F^e : X \to X$ be the e-iterated Frobenius map.

We have the commutative diagram

$$\begin{array}{ccc} F^e_* \pi_* \omega_{\widetilde{X}}(p^e E) & \longrightarrow & \pi_* \omega_{\widetilde{X}}(E) \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \beta} \\ F^e_* \omega_X & \xrightarrow{\phi} & \omega_X \end{array}$$

where the horizontal arrows are induced by the dual of the Frobenius map, $\mathcal{O}_X \to F^e_* \mathcal{O}_X$, and the vertical arrows are the natural maps induced by π. By hypothesis, ϕ is surjective. On the other hand, for $e > 0$ sufficiently large, the map labeled ρ is an isomorphism. Therefore the map $\phi \circ \rho$ is surjective which implies that the map β is also surjective. But as this holds for a dense set of primes, it must be surjective in characteristic zero as well, and in particular, as a consequence X has Du Bois singularities. □

It is not known whether the converse of this statement is true:

OPEN PROBLEM 13.8. If X has Du Bois singularities, does it have dense F-injective type?

Since F-injective singularities are known to be closely related to Du Bois singularities, it is also natural to ask how F-injective singularities deform cf. Conjecture 12.1. In general, this problem is also open.

OPEN PROBLEM 13.9. If a Cartier divisor D in X has F-injective singularities, does X have F-injective singularities near D?

In the case that X (equivalently D) is Cohen–Macaulay, the answer is affirmative, see [Fed83]. In fact, Fedder defined F-injective singularities partly because they seemed to deform better than F-pure singularities (the conjectured analog of log canonical singularities).

Appendix A. Connections with Buchsbaum rings

In this section we discuss the links between Du Bois singularities and Buchsbaum rings. Du Bois singularities are not necessarily Cohen–Macaulay, but in many cases, they are Buchsbaum (a weakening of Cohen–Macaulay).

Recall that a local ring $(R, \mathfrak{m}, k)$ has *quasi-Buchsbaum* singularities if

$$\mathfrak{m} H^i_{\mathfrak{m}}(R) = 0$$

for all $i < \dim R$. Further recall that a ring is called *Buchsbaum* if $\tau^{\dim R} R\Gamma_{\mathfrak{m}}(R)$ is quasi-isomorphic to a complex of k-vector spaces. Here $\tau^{\dim R}$ is the brutal truncation of the complex at the $\dim R$ location. Note that this is not the usual definition of Buchsbaum singularities, rather it is the so-called Schenzel's criterion; see [Sch82]. Notice that Cohen–Macaulay singularities are Buchsbaum (after truncation, one obtains the zero-object in the derived category).

It was proved by Tomari that isolated Du Bois singularities are quasi-Buchsbaum (a proof can be found in [Ish85, Proposition 1.9]), and then by Ishida that isolated Du Bois singularities were in fact Buchsbaum. Here we briefly review the argument to show that isolated Du Bois singularities are quasi-Buchsbaum since this statement is substantially easier.

PROPOSITION A.1. *Suppose that (X, x) is an isolated Du Bois singularity with $R = \mathscr{O}_{X,x}$. Then R is quasi-Buchsbaum.*

PROOF. Note that we may assume that X is affine. Since $\operatorname{Spec} R$ is regular outside its the maximal ideal $\mathfrak{m}$, it is clear that some power of $\mathfrak{m}$ annihilates $H^i_{\mathfrak{m}}(R)$ for all $i < \dim R$. We need to show that the smallest power for which this happens is 1. We let $\pi : \tilde{X} \to X$ be a log resolution with exceptional divisor E as in Theorem 6.1. Since X is affine, we see that $H^i_{\mathfrak{m}}(R) \simeq H^{i-1}(X \setminus \{\mathfrak{m}\}, \mathscr{O}_X) \simeq H^{i-1}(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}})$ for all $i > 0$. Therefore, it is enough to show that $\mathfrak{m} H^{i-1}(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}}) = 0$ for all $i < \dim X$. In other words, we need to show that $\mathfrak{m} H^i(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}}) = 0$ for all $i < \dim X - 1$.

We examine the long exact sequence

$$\cdots \longrightarrow H^{i-1}(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}}) \longrightarrow H^i_E(\tilde{X}, \mathscr{O}_{\tilde{X}}) \longrightarrow H^i(\tilde{X}, \mathscr{O}_{\tilde{X}}) \longrightarrow H^i(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}}) \longrightarrow \cdots$$

Now, $H^i_E(\tilde{X}, \mathscr{O}_{\tilde{X}}) = R^i(\Gamma_{\mathfrak{m}} \circ \pi_*)(\mathscr{O}_X)$ which vanishes for $i < \dim X$ by the Matlis dual of Grauert-Riemenschneider vanishing. Therefore

$$H^i(\tilde{X} \setminus E, \mathscr{O}_{\tilde{X}}) \simeq H^i(\tilde{X}, \mathscr{O}_{\tilde{X}})$$

for $i < \dim X - 1$. Finally, since X is Du Bois, $H^i(\tilde{X}, \mathscr{O}_{\tilde{X}}) = H^i(E, \mathscr{O}_E)$ by Theorem 6.1. But it is obvious that $\mathfrak{m}H^i(E, \mathscr{O}_E) = 0$ since E is a *reduced* divisor whose image in X is the point corresponding to $\mathfrak{m}$. The result then follows. □

It is easy to see that isolated F-injective singularities are also quasi-Buchsbaum.

PROPOSITION A.2. *Suppose that* $(R, \mathfrak{m})$ *is a local ring that is* F*-injective. Further suppose that* Spec $R \setminus \{\mathfrak{m}\}$ *is Cohen–Macaulay. Then* $(R, \mathfrak{m})$ *is quasi-Buchsbaum.*

PROOF. Since the punctured spectrum of R is Cohen–Macaulay, $H^i_{\mathfrak{m}}(R)$ is annihilated by some power of $\mathfrak{m}$ for $i < \dim R$. We will show that the smallest such power is 1. Choose $c \in \mathfrak{m}$. Since R is F-injective, $F^e : H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ is injective for all $e > 0$. Choose e large enough so that $c^{p^e} H^i_{\mathfrak{m}}(R)$ is zero for all $i < e$. However, for any element $z \in H^i_{\mathfrak{m}}(R)$, $F^e(cz) = c^{p^e} F^e(z) \in c^{p^e} H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$. This implies that $cz = 0$ and so $\mathfrak{m}H^i_{\mathfrak{m}}(R) = 0$ for $i < \dim R$. □

Perhaps the most interesting open question in this area is the following:

OPEN PROBLEM 1.3 (TAKAGI). Are F-injective singularities with isolated non-CM locus Buchsbaum?

Given the close connection between F-injective and Du Bois singularities, this question naturally leads to the next one:

OPEN PROBLEM 1.4. Are Du Bois singularities with isolated non-CM locus Buchsbaum?

2. Cubical hyperresolutions

For the convenience of the reader we include a short appendix explaining the construction of cubical hyperresolutions, as well as several examples. We follow [GNPP88] and mostly use their notation.

First let us fix a small universe to work in. Let $\underline{\mathrm{Sch}}$ denote the category of reduced schemes. Note that the usual fibred product of schemes $X \times_S Y$ need not be reduced, even when X and Y are reduced. We wish to construct the fibred product in the category of reduced schemes. Given any scheme W (reduced or not) with maps to X and Y over S, there is always a unique morphism $W \to X \times_S Y$, which induces a natural unique morphism $W_{\mathrm{red}} \to (X \times_S Y)_{\mathrm{red}}$.

It is easy to see that $(X \times_S Y)_{\mathrm{red}}$ is the fibred product in the category of reduced schemes.

Let us denote by $\underline{1}$ the category $\{0\}$ and by $\underline{2}$ the category $\{0 \to 1\}$. Let n be an integer ≥ -1. We denote by $\square_n^+$ the product of $n+1$ copies of the category $\underline{2} = \{0 \to 1\}$ [GNPP88, I, 1.15]. The objects of $\square_n^+$ are identified with the sequences $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ such that $\alpha_i \in \{0,1\}$ for $0 \leq i \leq n$. For $n = -1$, we set $\square_{-1}^+ = \{0\}$ and for $n = 0$ we have $\square_0^+ = \{0 \to 1\}$. We denote by $\square_n$ the full subcategory consisting of all objects of $\square_n^+$ except the initial object $(0, \ldots, 0)$. Clearly, the category $\square_n^+$ can be identified with the category of $\square_n$ with an augmentation map to $\{0\}$.

DEFINITION 2.1. A *diagram of schemes* is a functor Φ from a category C^{op} to the category of schemes. A *finite diagram of schemes* is a diagram of schemes such that the aforementioned category C has finitely many objects and morphisms; in this case such a functor will be called a C-*scheme*. A morphism of diagrams of schemes $\Phi : \mathsf{C}^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ to $\Psi : \mathsf{D}^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ is the combined data of a functor $\Gamma : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$ together with a natural transformation of functors $\eta : \Phi \to \Psi \circ \Gamma$.

REMARK 2.2. With these definitions, the class of (finite) diagrams of schemes can be made into a category. Likewise the set of C-schemes can also be made into a category (where the functor $\Gamma : \mathsf{C}^{\mathrm{op}} \to \mathsf{C}^{\mathrm{op}}$ is always chosen to be the identity functor).

REMARK 2.3. Let I be a category. If instead of a functor to the category of reduced schemes, one considers a functor to the category of topological spaces, or the category of categories, one can define I-topological spaces, and I-categories in the obvious way.

If $X_\cdot : I^{\mathrm{op}} \to \underline{\mathrm{Sch}}$ is an I-scheme, and $i \in \mathrm{Ob}\, I$, then X_i will denote the scheme corresponding to i. Likewise if $\phi \in \mathrm{Mor}\, I$ is a morphism $\phi : j \to i$, then X_ϕ will denote the corresponding morphism $X_\phi : X_i \to X_j$. If $f : Y_\cdot \to X_\cdot$ is a morphism of I-schemes, we denote by f_i the induced morphism $Y_i \to X_i$. If $X_\cdot$ is an I-scheme, a closed sub-I-scheme is a morphism of I-schemes $g : Z_\cdot \to X_\cdot$ such that for each $i \in I$, the map $g_i : Z_i \to X_i$ is a closed immersion. We will often suppress the g of the notation if no confusion is likely to arise. More generally, any property of a morphism of schemes (projective, proper, separated, closed immersion, etc...) can be generalized to the notion of a morphism of I-schemes by requiring that for each object i of I, g_i has the desired property (projective, proper, separated, closed immersion, etc...)

DEFINITION 2.4 [GNPP88, I, 2.2]. Suppose that $f : Y_\cdot \to X_\cdot$ is a morphism of I-schemes. Define the *discriminant of* f to be the smallest closed sub-I-scheme

$Z_\cdot$ of $X_\cdot$ such that $f_i : (Y_i - (f_i^{-1}(Z_i))) \to (X_i - Z_i)$ is an isomorphism for all i.

DEFINITION 2.5 [GNPP88, I, 2.5]. Let $S_\cdot$ be an I-scheme, $f : X_\cdot \to S_\cdot$ a proper morphism of I-schemes, and $D_\cdot$ the discriminant of f. We say that f is a *resolution*[6] *of* $S_\cdot$ if $X_\cdot$ is a smooth I-scheme (meaning that each X_i is smooth) and $\dim f_i^{-1}(D_i) < \dim S_i$, for all $i \in \operatorname{Ob} I$.

REMARK 2.6. This is the definition found in [GNPP88]. Note that the maps are not required to be surjective (of course, the ones one constructs in practice are usually surjective).

Consider the following example: the map $k[x, y]/(xy) \to k[x]$ which sends y to 0. We claim that the associated map of schemes is a "resolution" of the $*$-scheme $\operatorname{Spec} k[x, y]/(xy)$. The discriminant is $\operatorname{Spec} k[x, y]/(x)$. However, the preimage is simply the origin on $k[x]$, which has lower dimension than "1". Resolutions like this one are sometimes convenient to consider.

On the other hand, this definition seems to allow something it perhaps should not. Choose any variety X of dimension greater than zero and a closed point $z \in X$. Consider the map $z \to X$ and consider the $*$-scheme X. The discriminant is all of X. However, the preimage of X is still just a point, which has lower dimension than X itself, by hypothesis.

In view of these remarks, sometimes it is convenient to assume also that $\dim D_i < \dim S_i$ for each $i \in \operatorname{Ob} I$. In the resolutions of I-schemes that we construct (in particular, in the ones that are used to that prove cubic hyperresolutions exist), this always happens.

Let I be a category. The set of objects of I are given the preorder relation defined by $i \leq j$ if and only if $\operatorname{Hom}_I(i, j)$ is nonempty. We will say that a category I is ordered if this preorder is a partial order and, for each $i \in \operatorname{Ob} I$, the only endomorphism of i is the identity [GNPP88, I, C, 1.9]. Note that a category I is ordered if and only if all isomorphisms and endomorphisms of I are the identity.

It turns out of that resolutions of I-schemes always exist under reasonable hypotheses.

THEOREM 2.7 [GNPP88, I, Theorem 2.6]. *Let S be an I-scheme of finite type over a field k. Suppose that k is a field of characteristic zero and that I is a finite ordered category. Then there exists a resolution of S.*

In order to construct a resolution $Y_\cdot$ of an I-scheme $X_\cdot$, it might be tempting to simply resolve each X_i, set Y_i equal to that resolution, and somehow combine

[6] A resolution is a distinct notion from a cubic hyperresolution.

this data together. Unfortunately this cannot work, as shown by the example below.

EXAMPLE 2.8. Consider the pinch point singularity,

$$X = \operatorname{Spec} k[x, y, z]/(x^2 y - z^2) = \operatorname{Spec} k[s, t^2, st],$$

and let Z be the closed subscheme defined by the ideal (s, st) (this is the singular set). Let I be the category $\{0 \to 1\}$. Consider the I-scheme defined by $X_0 = X$ and $X_1 = Z$ (with the closed immersion as the map). X_1 is already smooth, and if one resolves X_0, (that is, normalizes it) there is no compatible way to map X_1 (or even another birational model of X_1) to it, since its preimage by normalization will be two-to-one onto $Z \subset X$! The way this problem is resolved is by creating additional components. So to construct a resolution $Y_\cdot$ we set $Y_1 = Z = X_1$ (since it was already smooth) and set $Y_0 = \overline{X}_0 \coprod Z$ where $\overline{X}_0$ is the normalization of X_0. The map $Y_1 \to Y_0$ just sends Y_1 (isomorphically) to the new component and the map $Y_0 \to X_0$ is the disjoint union of the normalization and inclusion maps.

One should note that although the theorem proving the existence of resolutions of I-schemes is constructive, [GNPP88], it is often easier in practice to construct an ad-hoc resolution.

Now that we have resolutions of I-schemes, we can discuss cubic hyperresolutions of schemes, in fact, even diagrams of schemes have cubic hyperresolutions! First we will discuss a single iterative step in the process of constructing cubic hyperresolutions. This step is called a 2-resolution.

DEFINITION 2.9 [GNPP88, I, 2.7]. Let S be an I-scheme and $Z_\cdot$ a $\square_1^+ \times I$-scheme. We say that $Z_\cdot$ is a 2-*resolution* of S if $Z_\cdot$ is defined by the following Cartesian square (pullback, or fibred product in the category of (reduced) I-schemes) of morphisms of I-schemes:

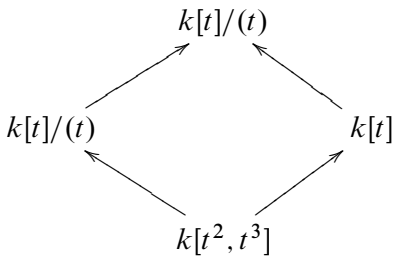
$$\begin{array}{ccc} Z_{11} & \hookrightarrow & Z_{01} \\ \downarrow & & \downarrow f \\ Z_{10} & \hookrightarrow & Z_{00} \end{array}$$

Here

(i) $Z_{00} = S$,
(ii) Z_{01} is a smooth I-scheme,
(iii) The horizontal arrows are closed immersions of I-schemes,
(iv) f is a proper I-morphism, and
(v) Z_{10} contains the discriminant of f; in other words, f induces an isomorphism of $(Z_{01})_i - (Z_{11})_i$ over $(Z_{00})_i - (Z_{10})_i$, for all $i \in \operatorname{Ob} I$.

Clearly 2-resolutions always exist under the same hypotheses that resolutions of I-schemes exist: set Z_{01} to be a resolution, Z_{10} to be discriminant (or any appropriate proper closed sub-I-scheme that contains it), and Z_{11} its (reduced) preimage in Z_{01}.

EXAMPLE 2.10. Let $I = \{0\}$ and let S be the I-scheme $\operatorname{Spec} k[t^2, t^3]$. Let $Z_{01} = \mathbb{A}^1 = \operatorname{Spec} k[t]$ and $Z_{01} \to S = Z_{00}$ be the map defined by $k[t^2, t^3] \to k[t]$. The discriminant of that map is the closed subscheme of $S = Z_{00}$ defined by the map $\phi : k[t^2, t^3] \to k$ that sends t^2 and t^3 to zero. Finally we need to define Z_{11}. The usual fibered product in the category of schemes is $k[t]/(t^2)$, but we work in the category of reduced schemes, so instead the fibered product is simply the associated reduced scheme (in this case $\operatorname{Spec} k[t]/(t)$). Thus our 2-resolution is defined by this diagram of rings:



We need one more definition before defining a cubic hyperresolution,

DEFINITION 2.11 [GNPP88, I, 2.11]. Let r be an integer greater than or equal to 1, and let $X^n_\bullet$ be a $\square^+_n \times I$-scheme, for $1 \le n \le r$. Suppose that for all n, $1 \le n \le r$, the $\square^+_{n-1} \times I$-schemes $X^{n+1}_{00\bullet}$ and $X^n_{1\bullet}$ are equal. Then we define, by induction on r, a $\square^+_r \times I$-scheme

$$Z_\bullet = \operatorname{red}(X^1_\bullet, X^2_\bullet, \ldots, X^r_\bullet)$$

that we call the *reduction* of $(X^1_\bullet, \ldots, X^r_\bullet)$, in the following way: If $r = 1$, one defines $Z_\bullet = X^1_\bullet$, if $r = 2$ one defines $Z_{\bullet\bullet} = \operatorname{red}(X^1_\bullet, X^2_\bullet)$ by

$$Z_{\alpha\beta} = \begin{cases} X^1_{0\beta} & \text{if } \alpha = (0,0), \\ X^2_{\alpha\beta} & \text{if } \alpha \in \square_1, \end{cases}$$

for all $\beta \in \square^+_0$, with the obvious morphisms. If $r > 2$, one defines $Z_\bullet$ recursively as $\operatorname{red}(\operatorname{red}(X^1_\bullet, \ldots, X^{r-1}_\bullet), X^r_\bullet)$.

Finally we are ready to define cubic hyperresolutions.

DEFINITION 2.12 [GNPP88, I, 2.12]. Let S be an I-scheme. A *cubic hyperresolution augmented over* S is a $\square^+_r \times I$-scheme $Z_\bullet$ such that

$$Z_\bullet = \operatorname{red}(X^1_\bullet, \ldots, X^r_\bullet),$$

where $X_\bullet^1$ is a 2-resolution of S, $X_\bullet^{n+1}$ is a 2-resolution of X_1^n for $1 \leq n < r$, and Z_α is smooth for all $\alpha \in \square_r$.

Now that we have defined cubic hyperresolutions, we should note that they exist under reasonable hypotheses:

THEOREM 2.13 [GNPP88, I, 2.15]. *Let S be an I-scheme. Suppose that k is a field of characteristic zero and that I is a finite* (*bounded*) *ordered category. Then there exists $Z_\bullet$, a cubic hyperresolution augmented over S such that*

$$\dim Z_\alpha \leq \dim S - |\alpha| + 1 \quad \textit{for all } \alpha \in \square_r.$$

Below are some examples of cubic hyperresolutions.

EXAMPLE 2.14. Let us begin by computing cubic hyperresolutions of curves so let C be a curve. We begin by taking a resolution $\pi : \overline{C} \to C$ (where $\overline{C}$ is just the normalization). Let P be the set of singular points of C; thus P is the discriminant of π. Finally we let E be the reduced exceptional set of π, therefore we have the Cartesian square

$$\begin{array}{ccc} E & \longrightarrow & \overline{C} \\ \downarrow & & \downarrow \pi \\ P & \longrightarrow & C \end{array}$$

It is clearly already a 2-resolution of C and thus a cubic-hyperresolution of C.

EXAMPLE 2.15. Let us now compute a cubic hyperresolution of a scheme X whose singular locus is itself a smooth scheme, and whose reduced exceptional set of a strong resolution $\pi : \widetilde{X} \to X$ is smooth (for example, any cone over a smooth variety). As in the previous example, let Σ be the singular locus of X and E the reduced exceptional set of π, Then the Cartesian square of reduced schemes

$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \pi \\ \Sigma & \longrightarrow & X \end{array}$$

is in fact a 2-resolution of X, just as in the case of curves above.

The obvious algorithm used to construct cubic hyperresolutions does not construct hyperresolutions in the most efficient or convenient way possible. For example, applying the obvious algorithm to the intersection of three coordinate planes gives us the following.

EXAMPLE 2.16. Let $X \cup Y \cup Z$ be the three coordinate planes in $\mathbb{A}^3$. In this example we construct a cubic hyperresolution using the obvious algorithm.

What makes this construction different, is that the dimension is forced to drop when forming the discriminant of a resolution of a diagram of schemes.

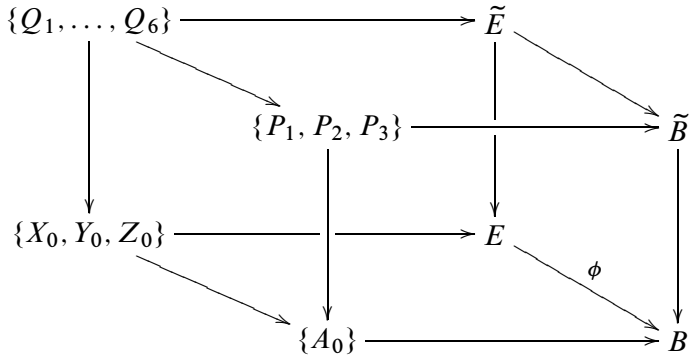
Again we begin the algorithm by taking a resolution and the obvious one is $\pi : (X \sqcup Y \sqcup Z) \to (X \cup Y \cup Z)$. The discriminant is $B = (X \cap Y) \cup (X \cap Z) \cup (Y \cap Z)$, the three coordinate axes. The fiber product making the square below Cartesian is simply the exceptional set E shown:

$$\begin{array}{ccc} E = ((X\cap Y)\cup(X\cap Z)) \sqcup ((Y\cap X)\cup(Y\cap Z)) \sqcup ((Z\cap X)\cup(Z\cap Y)) & \longrightarrow & X \sqcup Y \sqcup Z \\ \downarrow \phi & & \downarrow \pi \\ B = (X\cap Y) \cup (X\cap Z) \cup (Y\cap Z) & \longrightarrow & X \cup Y \cup Z \end{array}$$

We now need to take a 2-resolution of the $\underline{2}$-scheme $\phi : E \to B$. We take the obvious resolution that simply separates irreducible components. This gives us $\widetilde{E} \to \widetilde{B}$ mapping to $\phi : E \to B$. The discriminant of $\widetilde{E} \to E$ is a set of three points X_0, Y_0 and Z_0 corresponding to the origins in X, Y and Z respectively. The discriminant of the map $\widetilde{B} \to B$ is simply identified as the origin A_0 of our initial scheme $X \cup Y \cup Z$ (recall B is the union of the three axes). The union of that with the images of X_0, Y_0 and Z_0 is again just A_0. The fiber product of the diagram

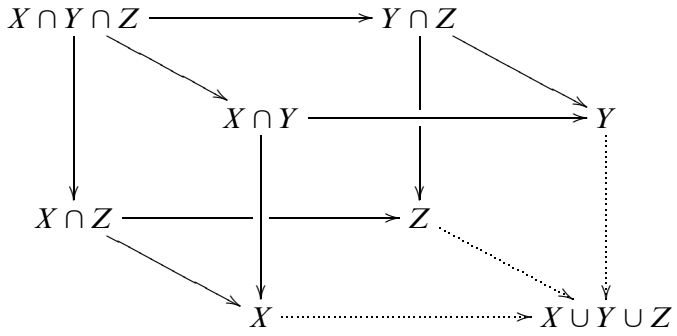
$$\begin{array}{ccc} & & (\widetilde{E} \to \widetilde{B}) \\ & & \downarrow \\ (\{X_0, Y_0, Z_0\} \to \{A_0\}) & \longrightarrow & (\phi : E \to B) \end{array}$$

can be viewed as $\{Q_1, \ldots, Q_6\} \to \{P_1, P_2, P_3\}$ where Q_1 and Q_2 are mapped to P_1 and so on (remember E was the disjoint union of the coordinate axes of X, of Y, and of respectively Z, so $\widetilde{E}$ has six components and thus six origins). Thus we have the diagram



which we can combine with previous diagrams to construct a cubic hyperresolution.

REMARK 2.17. It is possible to find a cubic hyperresolution for the three coordinate planes in $\mathbb{A}^3$ in a different way. Suppose that S is the union of the three coordinate planes (X, Y, and Z) of $\mathbb{A}^3$. Consider the $\square_2$ or $\square_2^+$ scheme defined by the diagram below, where the dotted arrows are those in $\square_2^+$ but not in $\square_2$.



One can verify that this is also a cubic hyperresolution of $X \cup Y \cup Z$.

Now we discuss sheaves on diagrams of schemes, as well as the related notions of push-forward and its right derived functors.

DEFINITION 2.18 [GNPP88, I, 5.3–5.4]. Let $X_\cdot$ be an I-scheme (or even an I-topological space). We define a *sheaf (or pre-sheaf) of abelian groups $F^\cdot$ on $X_\cdot$* to be the following data:

(i) A sheaf (pre-sheaf) F^i of abelian groups over X_i, for all $i \in \operatorname{Ob} I$, and
(ii) An X_ϕ-morphism of sheaves $F^\phi : F^i \to (X_\phi)_* F^j$ for all morphisms $\phi : i \to j$ of I, required to be compatible in the obvious way.

Given a morphism of diagrams of schemes $f_\cdot : X_\cdot \to Y_\cdot$ one can construct a push-forward functor for sheaves on $X_\cdot$.

DEFINITION 2.19 [GNPP88, I, 5.5]. Let $X_\cdot$ be an I-scheme, $Y_\cdot$ a J-scheme, $F^\cdot$ a sheaf on $X_\cdot$, and $f_\cdot : X_\cdot \to Y_\cdot$ a morphism of diagrams of schemes. We define $(f_\cdot)_* F^\cdot$ in the following way. For each $j \in \operatorname{Ob} J$ we define

$$((f_\cdot)_* F^\cdot)^j = \varprojlim (Y_\phi)_* (f_{i*} F^i)$$

where the inverse limit traverses all pairs (i, ϕ) where $\phi : f(i) \to j$ is a morphism in J^{op}.

REMARK 2.20. In many applications, J will simply be the category $\{0\}$ with one object and one morphism (for example, cubic hyperresolutions of schemes). In that case one can merely think of the limit as traversing I.

REMARK 2.21. One can also define a functor f^*, show that it has a right adjoint and that that adjoint is f_* as defined above [GNPP88, I, 5.5].

DEFINITION 2.22 [GNPP88, I, Section 5]. Let $X_\cdot$ and $Y_\cdot$ be diagrams of topological spaces over I and J respectively, $\Phi : I \to J$ a functor, $f_\cdot : X_\cdot \to Y_\cdot$ a Φ-morphism of topological spaces. If $G^\cdot$ is a sheaf over $Y_\cdot$ with values in a complete category C, one denotes by $f_\cdot^* G^\cdot$ the sheaf over $X_\cdot$ defined by

$$(f_\cdot^* G^\cdot)^i = f_i^*(G^{\Phi(i)}),$$

for all $i \in \mathrm{Ob}\, I$. One obtains in this way a functor

$$f_\cdot^* : \mathsf{Sheaves}(Y_\cdot, \mathsf{C}) \to \mathsf{Sheaves}(X_\cdot, \mathsf{C})$$

Given an I-scheme $X_\cdot$, one can define the category of sheaves of abelian groups $\mathsf{Ab}(X_\cdot)$ on $X_\cdot$ and show that it has enough injectives. Next, one can even define the derived category $D^+(X_\cdot, \mathsf{Ab}(X_\cdot))$ by localizing bounded below complexes of sheaves of abelian groups on $X_\cdot$ by the quasi-isomorphisms (those that are quasi-isomorphisms on each $i \in I$). One can also show that $(f_\cdot)_*$ as defined above is left exact so that it has a right derived functor $R(f_\cdot)_*$ [GNPP88, I, 5.8-5.9]. In the case of a cubic hyperresolution of a scheme $f : X_\cdot \to X$,

$$R((f_\cdot)_* F^\cdot) = R \varprojlim (R f_{i*} F^i)$$

where the limit traverses the category I of $X_\cdot$.

Final remark. We end our excursion into the world of hyperresolutions here. There are many other things to work out, but we will leave them to the interested reader. Many "obvious" statements need to be proved, but most are relatively straightforward once one gets comfortable using the appropriate language. For those and many more statements, including the full details of the construction of the Du Bois complex and many applications, the reader is encouraged to read [GNPP88].

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