

Zeta functions and chaos

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1. Introduction

This paper is an expanded version of lectures given at MSRI in June of 2008. It provides an introduction to various zeta functions emphasizing zeta functions of a finite graph and connections with random matrix theory and quantum chaos.

For the number theorist, most zeta functions are multiplicative generating functions for something like primes (or prime ideals). The Riemann zeta is the chief example. There are analogous functions arising in other fields such as Selberg's zeta function of a Riemann surface, Ihara's zeta function of a finite connected graph. All of these are introduced in Section 2. We will consider the Riemann hypothesis for the Ihara zeta function and its connection with expander graphs.

Chapter 3 starts with the Ruelle zeta function of a dynamical system, which will be shown to be a generalization of the Ihara zeta. A determinant formula is proved for the Ihara zeta function. Then we prove the graph prime number theorem.

In Section 4 we define two more zeta functions associated to a finite graph: the edge and path zetas. Both are functions of several complex variables. Both are reciprocals of polynomials in several variables, thanks to determinant formulas. We show how to specialize the path zeta to the edge zeta and then the edge zeta to the original Ihara zeta. The Bass proof of Ihara's determinant formula for the Ihara zeta function is given. The edge zeta allows one to consider graphs with weights on the edges. This is of interest for work on quantum graphs. See [Smilansky 2007] or [Horton et al. 2006b].

Lastly we consider what the poles of the Ihara zeta have to do with the eigenvalues of a random matrix. That is the sort of question considered in quantum chaos theory. Physicists have long studied spectra of Schrödinger operators and random matrices thanks to the implications for quantum mechanics where eigenvalues are viewed as energy levels of a system. Number theorists such as A.

Odlyzko have found experimentally that (assuming the Riemann hypothesis) the high zeros of the Riemann zeta function on the line $\text{Re}(s) = 1/2$ have spacings that behave like the eigenvalues of a random Hermitian matrix. Thanks to our two determinant formulas we will see that the Ihara zeta function, for example, has connections with spectra of more than one sort of matrix.

References [Terras 2007] and [Terras 2010] may be helpful for more details on some of these matters. The first is some introductory lectures on quantum chaos given at Park City, Utah in 2002. The second is a draft of a book on zeta functions of graphs.

2. Three zeta functions

2.1. The Riemann zeta function Riemann's zeta function for $s \in \mathbb{C}$ with $\text{Re}(s) > 1$ is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

In 1859 Riemann extended the definition of zeta to an analytic function in the whole complex plane except for a simple pole at $s = 1$. He also showed that there is a *functional equation*

$$\Lambda(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \Lambda(1-s). \quad (2-1)$$

The *Riemann hypothesis*, or *RH*, says that the nonreal zeros of $\zeta(s)$ (equivalently those with $0 < \text{Re } s < 1$) are on the line $\text{Re } s = \frac{1}{2}$. It is equivalent to giving an explicit error term in the prime number theorem stated below. The Riemann hypothesis has been checked to the 10^{13} -th zero as of 12 October 2004, by Xavier Gourdon with the help of Patrick Demichel. See Ed Pegg Jr.'s website for an article called the Ten Trillion Zeta Zeros: <http://www.maa.org/editorial/mathgames>. Proving (or disproving) the Riemann hypothesis is one of the million-dollar problems on the Clay Mathematics Institute website.

There is a duality between the primes and the zeros of zeta, given analytically through the Hadamard product formula as various sorts of explicit formulas. See [Davenport 1980] and [Murty 2001]. Such results lead to the *prime number theorem* which says

$$\#\{p = \text{prime} \mid p \leq x\} \sim \frac{x}{\log x}, \quad \text{as } x \rightarrow \infty.$$

The spacings of high zeros of zeta have been studied by A. Odlyzko; see the page www.dtc.umn.edu/~odlyzko/doc/zeta.htm. He has found that experimentally they look like the spacings of the eigenvalues of random Hermitian matrices (GUE). We will say more about this in the last section. See also [Conrey 2003].

EXERCISE 1. Use Mathematica to make a plot of the Riemann zeta function. *Hint.* The function `Zeta[s]` in Mathematica can be used to compute the Riemann zeta function.

There are many other kinds of zeta function. One is the Dedekind zeta of an algebraic number field F such as $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$, where primes are replaced by prime ideals \mathfrak{p} in the ring of integers O_F (which is $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$, if $F = \mathbb{Q}(\sqrt{2})$). Define the *norm of an ideal* of O_F to be $N\mathfrak{a} = |O_F/\mathfrak{a}|$. Then the *Dedekind zeta function* is defined for $\text{Re } s > 1$ by

$$\zeta(s, F) = \prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1},$$

where the product is over all prime ideals of O_F . The Riemann zeta function is $\zeta(s, \mathbb{Q})$.

Hecke gave the analytic continuation of the Dedekind zeta to all complex s except for a simple pole at $s = 1$. And he found the functional equation relating $\zeta(s, F)$ and $\zeta(1-s, F)$. The value at 0 involves the interesting number h_F —the class number of O_F which measures how far O_F is from having unique factorization into prime numbers rather than prime ideals ($h_{\mathbb{Q}(\sqrt{2})} = 1$). Also appearing in $\zeta(0, F)$ is the regulator which is a determinant of logarithms of units (i.e., elements $u \in O_F$ such that $u^{-1} \in O_F$). For $F = \mathbb{Q}(\sqrt{2})$, the regulator is $\log(1 + \sqrt{2})$. The formula is

$$\zeta(0, F) = -\frac{hR}{w}, \quad (2-2)$$

where w is the number of roots of unity in F ($w = 2$ for $F = \mathbb{Q}(\sqrt{2})$). One has $\zeta(0, \mathbb{Q}) = -\frac{1}{2}$. See [Stark 1992] for an introduction to this subject meant for physicists.

2.2. The Selberg zeta function. This zeta function is associated to a compact (or finite volume) Riemannian manifold. Assuming M has constant curvature -1 , it can be realized as a quotient of the *Poincaré upper half-plane*

$$H = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}.$$

The *Poincaré arc length* element is

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

which can be shown invariant under fractional linear transformation

$$z \mapsto \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{R}, ad - bc > 0.$$

It is not hard to see that *geodesics*—curves minimizing the Poincaré arc length—are half-lines and semicircles in H orthogonal to the real axis. Calling these geodesics straight lines creates a model for non-Euclidean geometry since Euclid's fifth postulate fails. There are infinitely many geodesics through a fixed point not meeting a given geodesic.

The *fundamental group* Γ of M acts as a discrete group of distance-preserving transformations. The favorite group of number theorists is the *modular group* $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ of 2×2 matrices of determinant one and integer entries or the quotient $\bar{\Gamma} = \Gamma / \{\pm I\}$. However the Riemann surface $M = \mathrm{SL}(2, \mathbb{Z}) \backslash H$ is not compact, although it does have finite volume.

Selberg defined primes in the compact Riemannian manifold $M = \Gamma \backslash H$ to be primitive closed geodesics C in M . Here *primitive* means you only go around the curve once.

Define the *Selberg zeta function*, for $\mathrm{Re}(s)$ sufficiently large, as

$$Z(s) = \prod_{[C]} \prod_{j \geq 1} (1 - e^{-(s+j)v(C)}).$$

The product is over all primitive closed geodesics C in $M = \Gamma \backslash H$ of Poincaré length $v(C)$. By the Selberg trace formula (which we do not discuss here), there is a duality between the lengths of the primes and the spectrum of the Laplace operator on M . Here

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Moreover one can show that the Riemann hypothesis (suitably modified to fit the situation) can be proved for Selberg zeta functions of compact Riemann surfaces.

EXERCISE 2. Show that $Z(s+1)/Z(s)$ has a product formula which is more like that for the Riemann zeta function.

The closed geodesics in $M = \Gamma \backslash H$ correspond to geodesics in H itself. One can show that the endpoints of such geodesics in \mathbb{R} (the real line = the boundary of H) are fixed by hyperbolic elements of Γ ; i.e., the matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with trace $a+d > 2$. Primitive closed geodesics correspond to hyperbolic elements that generate their own centralizer in Γ .

Some references for this subject are [Selberg 1989] and [Terras 1985].

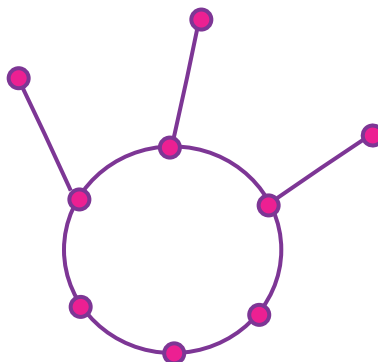


Figure 1. An example of a bad graph for zeta functions.

2.3. The Ihara zeta function. We will see that the Ihara zeta function of a graph has similar properties to the preceding zetas. A good reference for graph theory is [Biggs 1974].

First we must figure out what primes in graphs are. Recalling what they are for manifolds, we expect that we need to look at closed paths that minimize distance. What is distance? It is the number of oriented edges in a path.

First suppose that X is a finite connected unoriented graph. Thus it is a collection of vertices and edges. Usually we assume the graph is not a cycle or a cycle with hair (i.e., degree 1 vertices). Thus Figure 1 is a bad graph. We do allow our graphs to have loops and multiple edges however.

Let E be the set of unoriented (or undirected) edges of X and V the set of vertices. We orient (or direct) the edges arbitrarily and label them $e_1, e_2, \dots, e_{|E|}$. An example is shown in Figure 2. Then we label the inverse edges (meaning

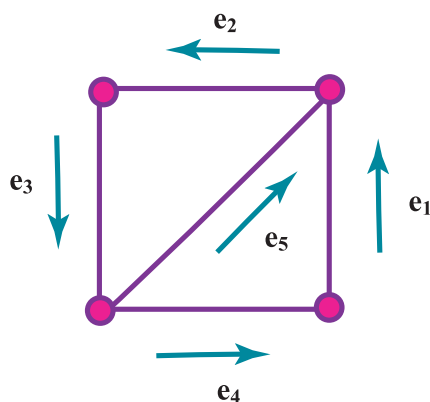


Figure 2. We choose an arbitrary orientation of the edges of a graph. Then we label the inverse edges via $e_{j+|E|} = e_j^{-1}$, for $j = 1, \dots, 5$.

the edge with the opposite orientation) $e_{j+|E|} = e_j^{-1}$, for $j = 1, \dots, |E|$. The oriented edges give an alphabet which we use to form words representing the paths in our graph.

Now we can define primes in the graph X . They correspond to closed geodesics in compact manifolds. They are equivalence classes $[C]$ of tailless primitive closed paths C . We define these last adjectives in the next paragraph.

A path or walk $C = a_1 \cdots a_s$, where a_j is an oriented or directed edge of X , is said to have a *backtrack* if $a_{j+1} = a_j^{-1}$, for some $j = 1, \dots, s-1$. A path $C = a_1 \cdots a_s$ is said to have a *tail* if $a_s = a_1^{-1}$. The *length* of $C = a_1 \cdots a_s$ is $s = \nu(C)$. A *closed path* means the starting vertex is the same as the terminal vertex. The closed path $C = a_1 \cdots a_s$ is called a *primitive or prime path* if it has no backtrack or tail and $C \neq D^f$, for $f > 1$. For the path $C = a_1 \cdots a_s$, the *equivalence class* $[C]$ means

$$[C] = \{a_1 \cdots a_s, a_2 \cdots a_s a_1, \dots, a_s a_1 \cdots a_{s-1}\}.$$

That is, we call two prime paths *equivalent* if we get one from the other by changing the starting point. A *prime* in the graph X is an equivalence class $[C]$ of prime paths.

Examples of primes in a graph. For the graph in Figure 2, we have primes $[C] = [e_2 e_3 e_5]$, $[D] = [e_1 e_2 e_3 e_4]$, $E = [e_1 e_2 e_3 e_4 e_1 e_{10} e_4]$. Here $e_{10} = e_5^{-1}$ and the lengths of these primes are: $\nu(C) = 3$, $\nu(D) = 4$, $\nu(E) = 7$. We have infinitely many primes since $E_n = [(e_1 e_2 e_3 e_4)^n e_1 e_{10} e_4]$ is prime for all $n \geq 1$. But we don't have unique factorization into primes. The only nonprimes are powers of primes.

DEFINITION 3. The *Ihara zeta function* is defined for $u \in \mathbb{C}$, with $|u|$ sufficiently small by

$$\zeta(u, X) = \prod_{[P]} (1 - u^{\nu(P)})^{-1}, \quad (2-3)$$

where the product is over all primes $[P]$ in X . Recall that $\nu(P)$ denotes the length of P .

EXERCISE 4. How small should $|u|$ be for convergence of $\zeta(u, X)$?

Hint. See formula (3-5) below for $\log \zeta(u, X)$.

There are two determinant formulas for the Ihara zeta function (see formulas (2-4) and (3-1) below). The first was proved in general by Bass [1992] and Hashimoto [1989], as Ihara considered the special case of *regular* graphs (those all of whose vertices have the same *degree*; i.e., the same number of oriented edges coming out of the vertex) and in fact was considering p -adic groups and not graphs. Moreover the degree had to be $1 + p^e$, where p is a prime number.

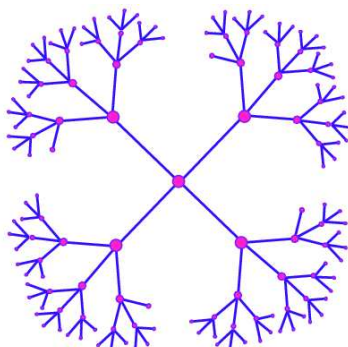


Figure 3. Part of the 4-regular tree. The tree itself is infinite.

The (vertex) *adjacency matrix* A of X is a $|V| \times |V|$ matrix whose i, j entry is the number of directed edges from vertex i to vertex j . The matrix Q is defined to be a diagonal matrix whose j -th diagonal entry is 1 less than the degree of the j -th vertex. If there is a loop at a vertex, it contributes 2 to the degree.

Then we have the *Ihara determinant formula*

$$\zeta(u, X)^{-1} = (1 - u^2)^{r-1} \det(I - Au + Qu^2). \quad (2-4)$$

Here r is the rank of the fundamental group of the graph. This is $r = |E| - |V| + 1$. In Section 4 we will give a version of Bass's proof of this formula.

In the case of regular graphs, one can prove the formula using the Selberg trace formula for the graph realized as a quotient $\Gamma \backslash T$, where T is the universal covering tree of the graph and Γ is the fundamental group of the graph. A graph T is a *tree* if it is a connected graph without any closed backtrackless paths. For a tree to be regular, it must be infinite. We will discuss covering graphs in the last section of this paper. For a discussion of the Selberg trace formula on $\Gamma \backslash T$, see the last chapter of [Terras 1999].

Figure 3 shows part of the 4-regular tree T_4 . As the tree is infinite, we cannot put the whole thing on a page. It can be identified with the 3-adic quotient $\mathrm{SL}(2, \mathbb{Q}_3) / \mathrm{SL}(2, \mathbb{Z}_3)$. A finite 4-regular graph X is a quotient of T_4 modulo the fundamental group of X .

EXAMPLE 5. The *tetrahedron graph* K_4 is the complete graph on 4 vertices and its zeta function is given by

$$\zeta(u, K_4)^{-1} = (1 - u^2)^2 (1 - u)(1 - 2u)(1 + u + 2u^2)^3.$$

EXAMPLE 6. Let $X = K_4 - e$ be the graph obtained from K_4 by deleting an edge e . See Figure 2. Then

$$\zeta(u, X)^{-1} = (1 - u^2)(1 - u)(1 + u^2)(1 + u + 2u^2)(1 - u^2 - 2u^3).$$

EXERCISE 7. Compute the Ihara zeta functions of your favorite graphs; e.g., the cube, the icosahedron, the buckyball or soccer ball graph.

EXERCISE 8. Obtain a functional equation for the Ihara zeta function of a $(q + 1)$ -regular graph. It will relate $\zeta(u, X)$ and $\zeta(1/qu, X)$.

Hint. Use the Ihara determinant formula (2-4).

There are various possible answers to this question. One answer is:

$$\Lambda_X(u) = (1 - u^2)^{r-1+n/2} (1 - q^2 u^2)^{n/2} \zeta_X(u) = (-1)^n \Lambda_X\left(\frac{1}{qu}\right).$$

In the special case of a $(q + 1)$ -regular graph the substitution $u = q^{-s}$ makes the Ihara zeta more like Riemann zeta. That is we set $f(s) = \zeta(q^{-s}, X)$ when X is $(q + 1)$ -regular. Then the functional equation relates $f(s)$ and $f(1 - s)$. See Exercise 8.

The *Riemann hypothesis* for Ihara's zeta function of a $(q + 1)$ -regular graph says that

$$\zeta(q^{-s}, X) \text{ has no poles with } 0 < \operatorname{Re} s < 1 \text{ unless } \operatorname{Re} s = \frac{1}{2}. \quad (2-5)$$

It turns out (using the Ihara determinant formula again) that the Riemann hypothesis means that the graph is *Ramanujan*; i.e., the nontrivial spectrum of the adjacency matrix of the graph is contained in the spectrum of the adjacency operator on the universal covering tree which is the interval $[-2\sqrt{q}, 2\sqrt{q}]$. This definition was introduced by Lubotzky, Phillips and Sarnak [Lubotzky et al. 1988], who showed that for each fixed degree of the form $p^e + 1$, $p = \text{prime}$, there is a family of Ramanujan graphs X_n with $|V(X_n)| \rightarrow \infty$. Ramanujan graphs are of interest to computer scientists because they provide efficient communication networks. The graph is a good expander.

EXERCISE 9. Show that for a $(q + 1)$ -regular graph the Riemann hypothesis is equivalent to saying that the graph is Ramanujan; i.e. if λ is an eigenvalue of the adjacency matrix A of the graph such that $|\lambda| \neq q + 1$, then $|\lambda| \leq 2\sqrt{q}$.

Hint. Use the Ihara determinant formula (2-4).

What is an expander graph? There are 4 ideas.

(1) There is a spectral property of some matrix associated to our finite graph X . Choose one of three matrices:

- (a) the (vertex) adjacency matrix A ,
- (b) the Laplacian $D - A$ or $I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$, where D is the diagonal matrix of degrees of vertices, or
- (c) the edge adjacency matrix W_1 to be defined in the next section.

Following [Lubotzky 1995], a graph is Ramanujan if the spectrum of the adjacency matrix for X is inside the spectrum of the analogous operator on the universal covering tree of X . One could ask for the analogous property of the other operators such as the Laplacian or the edge adjacency matrix.

- (2) X behaves like a random graph in some sense.
- (3) Information is passed quickly in the gossip network based on X . The graph has a large expansion constant. This is defined by formula (2-6) below.
- (4) The random walker on the graph gets lost FAST.

DEFINITION 10. For sets of vertices S, T of X , define

$$E(S, T) = \{e \mid e \text{ is edge of } X \text{ with one vertex in } S \text{ and the other vertex in } T\}.$$

DEFINITION 11. If S is a set of vertices of X , we say the *boundary* is $\partial S = E(S, X - S)$.

DEFINITION 12. A graph X with vertex set V and $n = |V|$ has *expansion constant*

$$h(X) = \min_{\substack{S \subset V \\ |S| \leq n/2}} \frac{|\partial S|}{|S|}. \tag{2-6}$$

The expansion constant is an analog of the Cheeger constant for differentiable manifolds. References for these things include [Chung 2007; Hoory et al. 2006; Terras 1999; 2010]. The first of these references gives relations between the expansion constant and the *spectral gap* $\lambda_X = \min \{\lambda_1, 2 - \lambda_{n-1}\}$ if $0 = \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n$ are the eigenvalues of $I - D^{-1/2}AD^{-1/2}$. Fan Chung proves that $2h_X \geq \lambda_X \geq h_X^2/2$. This is an analog of the Cheeger inequality in differential geometry. She also connects these inequalities with webpage search algorithms of the sort used by Google.

The possible locations of poles u of $\zeta(u, X)$ of a $(q + 1)$ -regular graph can be found in Figure 4. The poles satisfying the Riemann hypothesis are those on the circle of radius $1/\sqrt{q}$. Any nontrivial pole; i.e., $u \neq \pm 1, \pm 1/q$, which is not on that circle is a non-RH pole. In the $(q + 1)$ -regular graph case, $1/q$ is always the closest pole of the Ihara zeta to the origin.

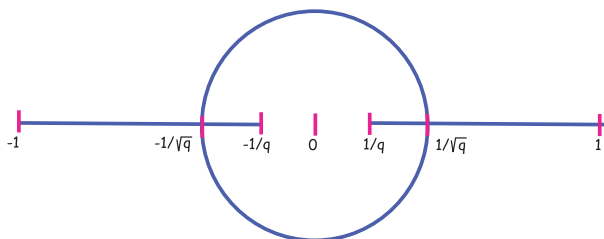


Figure 4. Possible locations of poles of zeta for a $(q + 1)$ -regular graph.

EXERCISE 13. Show that Figure 4 correctly locates the position of possible poles of the Ihara zeta function of a $(q + 1)$ -regular graph.

Hint. Use the Ihara determinant formula (2-4).

The *Alon conjecture* for regular graphs says that the RH is *approximately* true for “most” regular graphs See [Friedman 2008] for a proof. See [Miller and Novikoff 2008] for experiments leading to the conjecture that the percent of regular graphs exactly satisfying the RH approaches 27% as the number of vertices approaches infinity. The argument involves the Tracy–Widom distribution from random matrix theory.

Newland [2005] performed graph analogs of Odlyzko’s experiments on the spacings of imaginary parts of zeros of Riemann zeta. See Figure 5 below and Figure 8 on page 173.

An obvious question is: What is the meaning of the RH for irregular graphs? To understand this we need a definition.

DEFINITION 14. $R_X = R$ is the *radius of the largest circle of convergence* of the Ihara zeta function.

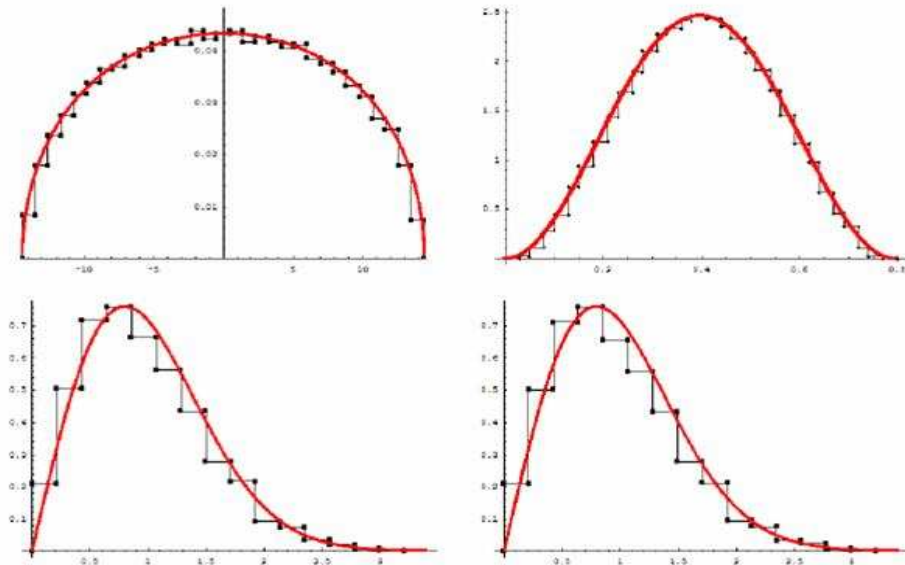


Figure 5. For a pseudo-random regular graph with degree 53 and 2000 vertices, generated by Mathematica, the top row shows the distributions of the eigenvalues of the adjacency matrix on the left and imaginary parts of the Ihara zeta poles on the right. The bottom row contains their respective level spacings. The red line on the bottom left is the Wigner surmise for the GOE $y = ((\pi x/2))e^{-\pi x^2/4}$. From [Newland 2005].

As a power series in the complex variable u , the Ihara zeta function has non-negative coefficients. Thus, by a classic theorem of Landau, both the series and the product defining $\zeta_X(u)$ will converge absolutely in a circle $|u| < R_X$ with a singularity (pole of order 1 for connected X) at $u = R_X$. See [Apostol 1976, p. 237] for Landau's theorem.

Define the *spectral radius* of a matrix M to be the maximum absolute value of all eigenvalues of M . We will see in the next section that by the Perron–Frobenius theorem in linear algebra (see [Horn and Johnson 1990], for example), $1/R_X$ is the spectral radius of the edge adjacency matrix W_1 which will be defined at the beginning of the next section. To apply the theorem, one must show that the edge adjacency matrix of a graph (under our usual assumptions) satisfies the necessary hypotheses. See [Terras and Stark 2007]. It is interesting to see that the quantity R_X can be viewed from two points of view; complex analysis and linear algebra.

For a $(q+1)$ -regular graph $R_X = 1/q$. If the graph is not regular, one sees by experiment that generally there is no functional equation. Thus when we make the change of variables $u = R^s$ in our zeta, the critical strip $0 \leq \operatorname{Re} s \leq 1$ is too large. We should only look at half of it and our Riemann hypothesis becomes:

The graph theory RH for irregular graphs:

$$\zeta(u, X) \text{ is pole free in } R < |u| < \sqrt{R}. \quad (2-7)$$

If the graph is $(q+1)$ -regular (by the functional equations), this is equivalent to the Riemann hypothesis stated earlier in formula (2-5).

To investigate this we need to define the edge adjacency matrix W_1 , which is found in the next section. We will consider examples in the last section.

EXERCISE 15. Consider the graph $X = K_4 - e$ from Exercise 6. Show that the poles of $\zeta(u, X)$ are not invariant under the map $u \rightarrow R/u$. This means there is no functional equation of the sort that occurs for regular graphs in Exercise 8. Do the poles satisfy the Riemann hypothesis? Do they satisfy a weak RH meaning that $\zeta(u, X)$ is pole free in $R < |u| < 1/\sqrt{q}$?

3. Ruelle's zeta function of a dynamical system, a determinant formula, and the graph prime number theorem

3.1. The edge adjacency matrix of a graph and another determinant formula for the Ihara zeta. In this section we consider some zeta functions that arise from those in algebraic geometry. For the Ihara zeta function, we prove a simple determinant formula. We also consider a proof of the graph theory prime number theorem.

DEFINITION 16. The *edge adjacency matrix* W_1 is defined to be the $2|E| \times 2|E|$ matrix with i, j entry 1 if edge i feeds into edge j (meaning that the terminal vertex of edge i is the initial vertex of edge j) provided that edge i is not the inverse of edge j .

We will soon prove a second determinant formula for the Ihara zeta function:

$$\zeta(u, X)^{-1} = \det(I - uW_1). \quad (3-1)$$

COROLLARY 17. *The poles of the Ihara zeta are the reciprocals of the eigenvalues of W_1 .*

Recall that R is the radius of convergence of the product defining the Ihara zeta function. By the corollary it is the reciprocal of the spectral radius of W_1 as well as the closest pole of zeta to the origin. It is necessarily positive. See the last chapter of [Horn and Johnson 1990] or [Apostol 1976, p. 237] for Landau's theorem, which implies the same thing.

There are many proofs of formula (3-1). We give the dynamical systems version here. There is another related proof in Section 4.1 and in [Terras 2010].

3.2. Ruelle zeta (aka dynamical systems zeta or Smale zeta). For the material here see [Ruelle 1994], [Bedford et al. 1991] or [Lagarias 1999]. Ruelle's motivation for his definition came partially from [Artin and Mazur 1965], whose authors were in turn inspired by the zeta function of a projective nonsingular algebraic variety V of dimension n over a finite field k with q elements. If N_m denotes the number of points of V with coordinates in the degree m extension field of k , the *zeta function of a variety V over a finite field* is

$$Z_V(u) = \exp\left(\sum_{m \geq 1} \frac{N_m u^m}{m}\right). \quad (3-2)$$

Example of varieties are given by taking solutions of polynomial equations over finite fields; e.g., $x^2 + y^2 = 1$ and $y^2 = x^3 + ax + b$. You actually have to look at the homogeneous version of the equations in projective space. For more information on these zeta functions, see [Lorenzini 1996, p. 280] or [Rosen 2002].

Let F be the *Frobenius map* taking a point on the variety with coordinates x_i to the point with coordinates x_i^q . Here q is the number of elements in the finite field k . Define $\text{Fix}(F^m) = \{x \in M \mid F^m(x) = x\}$; then $N_m = |\text{Fix}(F^m)|$.

Weil conjectured that zeta satisfies a functional equation relating the values $Z_V(u)$ and $Z_V(\frac{1}{q^n u})$. He also conjectured that

$$Z_V(u) = \prod_{j=0}^{2n} P_j(u)^{(-1)^{j+1}},$$

where the P_j are polynomials with zeros of absolute value $q^{-j/2}$. Weil proved the conjectures for the case of curves ($n = 1$). The proof was later simplified. See Rosen [Rosen 2002]. For general n , the Weil conjectures were proved by Deligne. Further, the P_j have a cohomological meaning as $\det(1 - uF^* |_{H^j(V)})$. Here the Frobenius has induced an action on the ℓ -adic étale cohomology. The case that $n = 1$ is very similar to that of the Ihara zeta function for a $(q + 1)$ -regular graph.

Artin and Mazur replace the Frobenius of V with a diffeomorphism f of a smooth compact manifold M such that its iterates f^k all have isolated fixed points. The *Artin–Mazur zeta function* is defined by

$$\zeta(u) = \exp\left(\sum_{m \geq 1} \frac{u^m}{m} |\text{Fix}(f^m)|\right). \tag{3-3}$$

The Ruelle zeta function involves a function $f : M \rightarrow M$ on a compact manifold M . Assume $\text{Fix}(f^m)$ is finite for all $m \geq 1$. The (first type of) *Ruelle zeta* is defined for a matrix valued function $\varphi : M \rightarrow \mathbb{C}^{d \times d}$ by

$$\zeta(u) = \exp\left(\sum_{m \geq 1} \frac{u^m}{m} \sum_{x \in \text{Fix}(f^m)} \text{Tr} \prod_{k=0}^{m-1} \varphi(f^k(x))\right). \tag{3-4}$$

Here we consider only the special case that $d = 1$ and φ is identically 1, when formula (3-4) looks exactly like formula (3-3).

Let I be a finite nonempty set (*our alphabet*). For a graph X , I is the set of directed edges. The transition matrix t is a matrix of zeros and ones with indices in I . In the case of a graph X , t is the 0,1 edge adjacency W_1 from Definition 16.

Since $I^{\mathbb{Z}}$ is compact, the following subset is closed:

$$\Lambda = \{(\xi_k)_{k \in \mathbb{Z}} \mid t_{\xi_k \xi_{k+1}} = 1 \text{ for all } k\}.$$

In the graph case, $t = W_1$ and $\xi \in \Lambda$ corresponds to a path without backtracking.

A continuous function $\tau : \Lambda \rightarrow \Lambda$ such that $\tau(\xi)_k = \xi_{k+1}$ is called a *subshift of finite type*. In the graph case, this shifts the path right, assuming the paths go from left to right.

Then we can find a new formula for the Ihara zeta function which shows that it is a Ruelle zeta. To understand this formula, we need a definition.

DEFINITION 18. $N_m = N_m(X)$ is the number of closed paths of length m without backtracking and tails in the graph X .

From Definition 2-3 of the Ihara zeta, we prove in the next paragraph that

$$\log \zeta(u, X) = \sum_{m \geq 1} \frac{N_m}{m} u^m. \tag{3-5}$$

Compare this formula with formula (3-2) defining the zeta function of a projective variety over a finite field.

To prove formula (3-5), take the logarithm of Definition 2-3 where the product is over primes $[P]$ in the graph X :

$$\begin{aligned} \log \zeta(u, X) &= \log \left(\prod_{[P] \text{ prime}} (1 - u^{\nu(P)})^{-1} \right) = - \sum_{[P]} \log(1 - u^{\nu(P)}) \\ &= \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} u^{j\nu(P)} = \sum_P \sum_{j \geq 1} \frac{1}{j\nu(P)} u^{j\nu(P)} \\ &= \sum_P \sum_{j \geq 1} \frac{1}{\nu(P^j)} u^{\nu(P^j)} \\ &= \sum_{\substack{C \text{ closed, backtrackless,} \\ \text{tailless path}}} \frac{1}{\nu(C)} u^{\nu(C)} = \sum_{m \geq 1} \frac{N_m}{m} u^m. \end{aligned}$$

Here we have used the power series for $\log(1 - x)$ to see the third equality. Then the fourth equality comes from the fact that there are $\nu(P)$ elements in the equivalence class $[P]$, for any prime $[P]$. The sixth equality is proved using the fact that any closed backtrackless tailless path C in the graph is a power of some prime path P . The last equality comes from Definition 18 of N_m .

If the subshift of finite type τ is as defined above for the graph X , we have

$$|\text{Fix}(\tau^m)| = N_m. \quad (3-6)$$

It follows from this result and formula (3-5) that the Ihara zeta is a special case of the Ruelle zeta.

Next we claim that

$$N_m = \text{Tr}(W_1^m). \quad (3-7)$$

To see this, set $B = W_1$, with entries b_{ef} , for oriented edges e, f . Then

$$\text{Tr}(W_1^m) = \text{Tr}(B^m) = \sum_{e_1, \dots, e_m} b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1},$$

where the sum is over all oriented edges of the graph. The b_{ef} are 0 unless edge e feeds into edge f without backtracking; i.e., the terminal vertex of e is the initial vertex of f and $f \neq e^{-1}$. Thus $b_{e_1 e_2} b_{e_2 e_3} \cdots b_{e_m e_1} = 1$ means that the path $C = e_1 e_2 \cdots e_m$ is closed, backtrackless, tailless of length m .

It follows that:

$$\begin{aligned} \log \zeta(u, X) &= \sum_{m \geq 1} \frac{u^m}{m} \text{Tr}(W_1^m) = \text{Tr} \left(\sum_{m \geq 1} \frac{u^m}{m} W_1^m \right) \\ &= \text{Tr}(\log(I - uW_1)^{-1}) = \log \det(I - uW_1)^{-1}. \end{aligned}$$

Here we have used formula (3-7) and the continuous linear property of trace. Then we need the power series for the matrix logarithm and the following exercise.

EXERCISE 19. Show that $\exp \operatorname{Tr}(A) = \det(\exp A)$, for any matrix A . To prove this, you need to know that there is a nonsingular matrix B such that $BAB^{-1} = T$ is upper triangular. See your favorite linear algebra book.

This proves formula (3-1) for the Ihara zeta function which says $\zeta(u, X) = \det(I - uW_1)^{-1}$. This is known as the Bowen-Lanford theorem for subshifts of finite type in the context of Ruelle zeta functions.

3.3. Graph prime number theorem. Next we prove the graph prime number theorem. This requires two definitions and a theorem.

DEFINITION 20. The *prime counting function* is

$$\pi(n) = \#\{\text{primes } [P] \mid n = \nu(P) = \text{length of } P\}.$$

DEFINITION 21. The *greatest common divisor of the prime path lengths* is

$$\Delta_X = \gcd\{\nu(P) \mid [P] \text{ prime of } X\}.$$

Kotani and Sunada prove the following theorem. Their proof makes heavy use of the Perron–Frobenius theorem from linear algebra. A proof can also be found in [Terras 2010].

THEOREM 22 [Kotani and Sunada 2000]. *Assume, as usual, that the graph X is connected, has fundamental group of rank $r > 1$, and has no degree 1 vertices.*

(1) *Every pole u of $\zeta_X(u)$ satisfies $R_X \leq |u| \leq 1$, with R_X from Definition 14, and*

$$q^{-1} \leq R_X \leq p^{-1}. \quad (3-8)$$

(2) *For a graph X , if $q+1$ is the maximum degree of X and $p+1$ is the minimum degree of X , then every nonreal pole u of $\zeta_X(u)$ satisfies the inequality*

$$q^{-1/2} \leq |u| \leq p^{-1/2}. \quad (3-9)$$

(3) *The poles of ζ_X on the circle $|u| = R_X$ have the form $R_X e^{2\pi ia/\Delta_X}$, where $a = 1, \dots, \Delta_X$. Here Δ_X is from Definition 21.*

EXERCISE 23. Look up [Kotani and Sunada 2000] and figure out their proof of the result that we needed in proving the prime number theorem. Another version of this proof can be found in [Terras 2010].

THEOREM 24 (GRAPH PRIME NUMBER THEOREM). *Assume X satisfies the hypotheses of the preceding theorem. Suppose that R_X is as in Definition 14. If $\pi(m)$ and Δ_X are as in Definitions 20 and 21, then $\pi(m) = 0$ unless Δ_X divides m . If Δ_X divides m , we have*

$$\pi(m) \sim \Delta_X \frac{R_X^{-m}}{m} \quad \text{as } m \rightarrow \infty.$$

PROOF. If N_m is as in Definition 18, then formula (3-5) implies we have

$$u \frac{d}{du} \log \zeta_X(u) = \sum_{m \geq 1} N_m u^m. \quad (3-10)$$

Now observe that the defining formula for the Ihara zeta function can be written as

$$\zeta_X(u) = \prod_{n \geq 1} (1 - u^n)^{-\pi(n)}.$$

Then

$$u \frac{d}{du} \log \zeta_X(u) = \sum_{n \geq 1} \frac{n\pi(n)u^n}{1 - u^n} = \sum_{m \geq 1} \sum_{d|m} d\pi(d)u^m.$$

Here the inner sum is over all positive divisors of m . Thus we obtain the *relation between N_m and $\pi(n)$*

$$N_m = \sum_{d|m} d\pi(d).$$

This sort of relation occurs frequently in number theory and combinatorics. It is inverted using the *Möbius function* $\mu(n)$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ for distinct primes } p_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the *Möbius inversion formula* (which can be found in any elementary number theory book),

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d. \quad (3-11)$$

Next use formula (3-1) to see that

$$u \frac{d}{du} \log \zeta_X(u) = -u \frac{d}{du} \sum_{\lambda \in \text{Spec } \mathcal{W}_1} \log(1 - \lambda u) = \sum_{\lambda \in \text{Spec } \mathcal{W}_1} \sum_{n \geq 1} (\lambda u)^n.$$

From this, we get the *formula relating N_m and the spectrum of the edge adjacency \mathcal{W}_1* :

$$N_m = \sum_{\lambda \in \text{Spec } \mathcal{W}_1} \lambda^m. \quad (3-12)$$

The dominant terms in this last sum are those coming from $\lambda \in \text{Spec } W_1$ such that $|\lambda| = R^{-1}$, with $R = R_X$ from Definition 14.

By Theorem 22, the largest absolute value of an eigenvalue λ occurs Δ_X times with these eigenvalues having the form $e^{2\pi ia/\Delta_X} R^{-1}$, where $a = 1, \dots, \Delta_X$. we see that

$$\pi(n) \sim \frac{1}{n} \sum_{|\lambda| \text{ maximal}} \lambda^n = \frac{R^{-n}}{n} \sum_{a=1}^{\Delta_X} e^{2\pi ian/\Delta_X}.$$

The orthogonality relations for exponential sums (see [Terras 1999]) which are basic to the theory of the finite Fourier transform say that

$$\sum_{a=1}^{\Delta_X} e^{2\pi ian/\Delta_X} = \begin{cases} 0 & \text{if } \Delta_X \text{ does not divide } n, \\ \Delta_X & \text{if } \Delta_X \text{ divides } n. \end{cases} \quad (3-13)$$

The graph prime number theorem follows. □

Note that the Riemann hypothesis gives information on the size of the error term in the prime number theorem.

EXERCISE 25. Fill in the details in the proof of the graph theory prime number theorem. In particular, prove the orthogonality relations for exponential sums which implies formula (3-13) above.

As is the case for the Riemann zeta function, it is clear that the graph theory Riemann hypothesis gives information on the size of the error term in the prime number theorem.

EXAMPLE 26 (TETRAHEDRON OR K_4). We saw that

$$\zeta_{K_4}(u)^{-1} = (1 - u^2)^2 (1 - u) (1 - 2u) (1 + u + 2u^2)^3.$$

From this we find that

$$\begin{aligned} u \frac{d}{du} \log \zeta_X(u) &= \sum_{m \geq 1} N_m u^m \\ &= 24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + \dots \end{aligned}$$

Then the question becomes what are the corresponding $\pi(n)$? We see that $\pi(3) = N_3/3 = 8$, $\pi(4) = N_4/4 = 6$; $\pi(5) = N_5 = 0$. Then, because $\pi(1) = \pi(2) = 0$, we have

$$N_6 = \sum_{d|5} d\pi(d) = 3\pi(3) + 6\pi(6),$$

which implies $\pi(6) = 12$.

It follows from the fact that there are paths of lengths 3 and 4 that the greatest common divisor of the lengths of the prime paths $\Delta = 1$.

The poles of zeta for K_4 are $\{1, 1, 1, -1, \frac{1}{2}, a, a, b, b, b\}$, where

$$a = \frac{1 + \sqrt{-7}}{4}, \quad b = \frac{1 - \sqrt{-7}}{4}.$$

Then $|a| = |b| = 1/\sqrt{2}$. The closest pole of zeta to the origin is $\frac{1}{2}$.

The prime number theorem for K_4 says that

$$\pi(m) \sim \frac{2^m}{m}, \quad \text{as } m \rightarrow \infty.$$

EXAMPLE 27 (TETRAHEDRON MINUS AN EDGE, $X = K_4 - e$). We saw that

$$\zeta_X(u)^{-1} = (1 - u^2)(1 - u)(1 + u^2)(1 + u + 2u^2)(1 - u^2 - 2u^3).$$

From this, we have

$$\begin{aligned} u \frac{d}{du} \log \zeta_X(u) &= \sum_{m \geq 1} N_m u^m \\ &= 12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + \dots \end{aligned}$$

It follows that $\pi(3) = 4$, $\pi(4) = 2$, $\pi(5) = 0$, $\pi(6) = 2$. Again Δ , the gcd of lengths of primes, is equal to 1.

The poles of zeta for $K_4 - e$ are $\{1, 1, -1, i, -i, a, b, \alpha, \beta, \bar{\beta}\}$. Here

$$a = \frac{1 + \sqrt{-7}}{4}, \quad b = \frac{1 - \sqrt{-7}}{4},$$

and $\alpha = R$ is the real root of the cubic, while $\beta, \bar{\beta}$ are the remaining (nonreal) roots of the cubic.

The prime number theorem for $K_4 - e$ becomes, for $1/\alpha \cong 1.5$

$$\pi(m) \sim \frac{\alpha^{-m}}{m}, \quad \text{as } m \rightarrow \infty.$$

EXERCISE 28. Compute the Ihara zeta function of your favorite graph and then use formula (3-10) to compute the first 5 nonzero N_m . State the prime number theorem explicitly for this graph. Next do the same computations for the graph with one edge removed.

EXERCISE 29. (a) Show that the radius of convergence of the Ihara zeta function of a $(q + 1)$ -regular graph is $R = 1/q$.

(b) A graph is a *bipartite graph* if and only if the set of vertices can be partitioned into 2 disjoint sets S, T such that no vertex in S is adjacent to any other vertex in S and no vertex in T is adjacent to any other vertex in T . Assume your graph is non bipartite and prove the prime number theorem using the Ihara determinant formula (2-4).

(c) What happens if the graph is $(q + 1)$ -regular graph and bipartite?

EXERCISE 30. List all the zeta functions you can and what they are good for. The website www.maths.ex.ac.uk/~mwatkins lists lots of them.

Now that we have the prime number theorem, we can also produce analogs of the explicit formulas of analytic number theory. That is, we seek an analog of Weil’s explicit formula for the Riemann zeta function. In Weil’s original work he used the result to formulate an equivalent statement to the Riemann hypothesis. See [Weil 1952].

Our analog of the von Mangoldt function from elementary number theory is N_m . Using formula (3-1), we have

$$\begin{aligned} u \frac{d}{du} \log \zeta(u, X) &= -u \frac{d}{du} \sum_{\lambda \in \text{Spec } W_1} \log(1 - \lambda u) \\ &= \sum_{\lambda \in \text{Spec } W_1} \frac{\lambda u}{1 - \lambda u} = - \sum_{\rho \text{ pole of } \zeta} \frac{u}{u - \rho}. \end{aligned} \tag{3-14}$$

Then it is not hard to prove the following result following the method of [Murty 2001, p. 109].

PROPOSITION 31 (AN EXPLICIT FORMULA). *Let $0 < a < R$, where R is the radius of convergence of $\zeta(u, X)$. Assume $h(u)$ is meromorphic in the plane and holomorphic outside the circle of center 0 and radius $a - \varepsilon$, for small $\varepsilon > 0$. Assume also that $h(u) = O(|u|^p)$ as $|u| \rightarrow \infty$ for some $p < -1$. Also assume that its transform $\widehat{h}_a(n)$ decays rapidly enough for the right hand side of the formula to converge absolutely. Then if N_m is as in Definition 3-10, we have*

$$\sum_{\rho} \rho h(\rho) = \sum_{n \geq 1} N_n \widehat{h}_a(n),$$

where the sum on the left is over the poles of $\zeta(u, X)$ and

$$\widehat{h}_a(n) = \frac{1}{2\pi i} \oint_{|u|=a} u^n h(u) du.$$

PROOF. We follow the method of [Murty 2001, p. 109]. Look at

$$\frac{1}{2\pi i} \oint_{|u|=a} \left(u \frac{d}{du} (\log \zeta(u, X)) \right) h(u) du.$$

Use Cauchy's integral formula to move the contour over to the circle $|u| = b > 1$. Then let $b \rightarrow \infty$. Also use formulas (3-14) and (3-10). Note that $N_n \sim \Delta_X / R_X^m$, as $m \rightarrow \infty$. \square

Such explicit formulas are basic to work on the pair correlation of complex zeros of zeta (see [Montgomery 1973]). They can also be viewed as an analog of Selberg's trace formula. See [Horton 2007] or [Terras and Wallace 2003] for discussion of Selberg's trace formula for a $q + 1$ regular graph. In these papers various kernels (e.g., Green's, characteristic functions of intervals, heat) were plugged in to the trace formula deducing various things such as McKay's theorem on the distribution of eigenvalues of the adjacency matrix and the Ihara determinant formula for the Ihara zeta. It would be an interesting research project to do the same sort of thing for irregular graphs.

4. Edge and path zeta functions and their determinant formulas; connections with quantum chaos

4.1. Proof of Ihara's determinant formula for Ihara zeta. Before we give our version of the Bass proof of formulas (2-4) and (3-1) from [Stark and Terras 2000], we define a new graph zeta function with many complex variables. We orient and label the edges of our undirected graph as usual.

DEFINITION 32. The *edge matrix* W for graph X is a $2m \times 2m$ matrix with a, b entry corresponding to the oriented edges a and b . This a, b entry is the complex variable w_{ab} if edge a feeds into edge b (i.e., the terminal vertex of a is the starting vertex of b) and $b \neq a^{-1}$ and the a, b entry is 0 otherwise.

DEFINITION 33. Given a path C in X , which is written as a product of oriented edges $C = a_1 a_2 \cdots a_s$, the *edge norm* of C is

$$N_E(C) = w_{a_1 a_2} w_{a_2 a_3} \cdots w_{a_{s-1} a_s} w_{a_s a_1}.$$

The *edge Ihara zeta function* is

$$\zeta_E(W, X) = \prod_{[P]} (1 - N_E(P))^{-1},$$

where the product is over primes in X . Here assume that all $|w_{ab}|$ are sufficiently small for convergence.

Properties and applications of edge zeta

- (1) By the definitions, if you set all nonzero variables in W equal to u , the *edge zeta function specializes to the Ihara zeta function*; i.e.,

$$\zeta_E(W, X) \Big|_{\substack{0 \neq w_{ab} = u \\ \text{for all } a,b}} = \zeta(u, X). \tag{4-1}$$

- (2) If you cut or delete an edge of a graph, you can compute the edge zeta for the new graph with one less edge by setting all variables equal to 0 if the cut or deleted edge or its inverse appear in a subscript.
- (3) The edge zeta allows one to define a zeta function for a weighted or quantum graph. See [Smilansky 2007] or [Horton et al. 2006b].
- (4) There is an application of the edge zeta to error correcting codes. See [Koetter et al. 2005].

The following result is a generalization of formula (3-1).

THEOREM 34 (DETERMINANT FORMULA FOR THE EDGE ZETA).

$$\zeta_E(W, X) = \det(I - W)^{-1}.$$

We prove the theorem after giving an example.

EXAMPLE 35 (DUMBBELL GRAPH). Figure 6 shows the labeled picture of the dumbbell graph X . For this graph we find that

$$\zeta_E(W, X)^{-1} = \det \begin{pmatrix} w_{11}-1 & w_{12} & 0 & 0 & 0 & 0 \\ 0 & -1 & w_{23} & 0 & 0 & w_{26} \\ 0 & 0 & w_{33}-1 & 0 & w_{35} & 0 \\ 0 & w_{42} & 0 & w_{44}-1 & 0 & 0 \\ w_{51} & 0 & 0 & w_{54} & -1 & 0 \\ 0 & 0 & 0 & 0 & w_{65} & w_{66}-1 \end{pmatrix}.$$

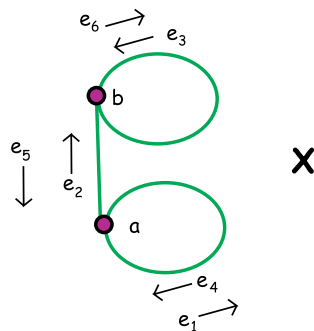


Figure 6. The dumbbell graph.

If we cut or delete the vertical edges which are edges e_2 and e_5 , we should specialize all the variables with 2 or 5 in them to be 0. This yields the edge zeta function of the subgraph with the vertical edge removed, and incidentally diagonalizes the matrix W . This also diagonalizes the edge matrix W . Of course the resulting graph consists of 2 disconnected loops. So zeta is the product of two loop zetas.

EXERCISE 36. Do another example computing the edge zeta function of your favorite graph. Then see what happens if you delete an edge.

PROOF OF THEOREM 34. First note that, from the Euler product for the edge zeta function, we have

$$-\log \zeta_E(W, X) = \sum_{[P]} \sum_{j \geq 1} \frac{1}{j} N_E(P)^j.$$

Then, since there are $\nu(P)$ elements in $[P]$, we have

$$-\log \zeta_E(W, X) = \sum_{\substack{m \geq 1 \\ j \geq 1}} \frac{1}{jm} \sum_{\substack{P \\ \nu(P)=m}} N_E(P)^j.$$

It follows that

$$-\log \zeta_E(W, X) = \sum_C \frac{1}{\nu(C)} N_E(C).$$

This comes from the fact that any closed path C without backtracking or tail has the form P^j for a prime path P . Then by the Exercise below, we see that

$$-\log \zeta_E(W, X) = \sum_{m \geq 1} \frac{1}{m} \text{Tr}(W^m).$$

Finally, again using the Exercise below, we see that the right hand side of the preceding formula is $\log \det(I - W)^{-1}$. This proves the theorem. \square

EXERCISE 37. Prove that

$$\sum_C \frac{1}{\nu(C)} N_E(C) = \sum_{m \geq 1} \frac{1}{m} \text{Tr}(W^m) = \log \det(I - W)^{-1}.$$

Hints. (1) For the first equality, you need to think about $\text{Tr}(W^m)$ as an m -fold sum of products of w_{ij} in terms of closed paths C of length m just as we did in proving formula (3-7) above.

(2) Exercise 19 says that $\det(\exp(B)) = e^{\text{Tr}(B)}$. Then write $\log((I - W)^{-1}) = B$, using the matrix logarithm (which converges for small w_{ij}), and see that

$$\log \det((I - W)^{-1}) = \text{Tr}(\log(I - W)^{-1}).$$

Theorem 34 gives another proof of formula (3-1) for the Ihara zeta by specializing all the nonzero w_{ij} to be u .

Next we give a version of Bass's proof of the Ihara determinant formula (2-4) using formula (3-1). In what follows, n is the number of vertices of X and m is the number of unoriented edges of X .

First define some matrices. Set $J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$. Then define the $n \times 2m$ start matrix S and the $n \times 2m$ terminal matrix T by setting

$$s_{ve} = \begin{cases} 1 & \text{if } v \text{ is the starting vertex of the oriented edge } e, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$t_{ve} = \begin{cases} 1 & \text{if } v \text{ is the terminal vertex of the oriented edge } e, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 38 (SOME MATRIX IDENTITIES). *Using these definitions, the following formulas hold. We write tM for the transpose of the matrix M .*

- (1) $SJ = T$ and $TJ = S$.
- (2) If A is the adjacency matrix of X and $Q + I_n$ is the diagonal matrix whose j th diagonal entry is the degree of the j th vertex of X , then $A = S{}^tT$ and $Q + I_n = S{}^tS = T{}^tT$.
- (3) The 0,1 edge adjacency W_1 from Definition 16 satisfies $W_1 + J = {}^tTS$.

PROOF. (1) This comes from the fact that the starting (terminal) vertex of edge e_j is the terminal (starting) vertex of edge $e_{j+|E|}$, according to our edge numbering system.

(2) Consider

$$(S{}^tT)_{ab} = \sum_e s_{ae}t_{be}.$$

The right hand side is the number of oriented edges e such that a is the initial vertex and b is the terminal vertex of e , which is the a, b entry of A . Note that $A_{a,a} = 2 \cdot \text{number of loops at vertex } a$. Similar arguments prove the second formula.

(3) We have

$$({}^tTS)_{ef} = \sum_v t_{ve}s_{vf}.$$

The sum is 1 if and only if edge e feeds into edge f , even if $f = e^{-1}$. □

BASS'S PROOF OF THE GENERALIZED IHARA DETERMINANT FORMULA (2-4). In the following identity all matrices are $(n + 2m) \times (n + 2m)$, where the

first block is $n \times n$, if n is the number of vertices of X and m is the number of unoriented edges of X . Use the preceding proposition to see that

$$\begin{pmatrix} I_n & 0 \\ {}^tT & I_{2m} \end{pmatrix} \begin{pmatrix} I_n(1-u^2) & Su \\ 0 & I_{2m}-W_1u \end{pmatrix} = \begin{pmatrix} I_n-Au+Qu^2 & Su \\ 0 & I_{2m}+Ju \end{pmatrix} \begin{pmatrix} I_n & 0 \\ {}^tT-{}^tSu & I_{2m} \end{pmatrix}.$$

EXERCISE 39. Check this equality.

Take determinants to obtain

$$(1-u^2)^n \det(I-W_1u) = \det(I_n-Au+Qu^2) \det(I_{2m}+Ju).$$

To finish the proof of formula (2-4), observe that

$$I+Ju = \begin{pmatrix} I & Iu \\ Iu & I \end{pmatrix}$$

implies

$$\begin{pmatrix} I & 0 \\ -Iu & I \end{pmatrix} (I+Ju) = \begin{pmatrix} I & Iu \\ 0 & I(1-u^2) \end{pmatrix}.$$

Thus $\det(I+Ju) = (1-u^2)^m$. Since $r-1 = m-n$ for a connected graph, formula (2-4) follows. \square

EXERCISE 40. Read about quantum graphs and consider the properties of their zeta functions. See [Horton et al. 2006a; 2006b; 2008], as well as the other papers in those volumes. Another reference is [Smilansky 2007].

4.2. The path zeta function of a graph. First we need a few definitions. A *spanning tree* T for graph X means a tree which is a subgraph of X containing all the vertices of X .

The *fundamental group* of a topological space such as our graph X has elements which are closed directed paths starting and ending at a fixed basepoint $v \in X$. Two paths are equivalent if and only if one can be continuously deformed into the other (i.e., one is homotopic to the other within X , while still starting and ending at v). The product of 2 paths a, b means the path obtained by first going around a then b .

It turns out (by the Seifert-von Kampen theorem, for example) that the fundamental group of graph X is a free group on r generators, where r is the number of edges left out of a spanning tree for X . Let us try to explain this a bit. More information can be found in [Hatcher 2002], [Massey 1967, p. 198], or [Gross and Tucker 2001].

From the graph X construct a new graph $X^\#$ by shrinking a spanning tree T of X to a point. The new graph will be a bouquet of r loops as in Figure 7. The fundamental group of X is the same as that of $X^\#$. Why? The quotient map $X \rightarrow X/T$ is what algebraic topologists call a homotopy equivalence. This

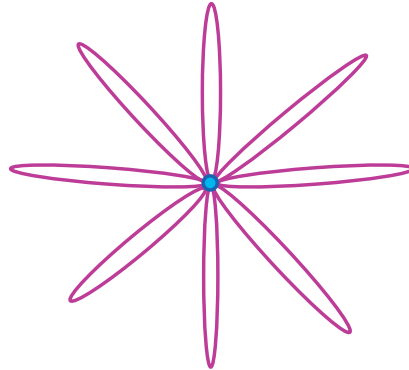


Figure 7. A bouquet of loops.

means that intuitively you can continuously deform one graph into the other without changing the topology.

The fundamental group of the bouquet of r loops in Figure 7 is the free group on r generators. The generators are the directed loops! The elements are the words in these loops.

EXERCISE 41. Show that $r - 1 = |E| - |V|$.

EXERCISE 42. The *complexity* κ_X of a graph is defined to be the number of spanning trees in X . Use the matrix-tree theorem (see [Biggs 1974]) to prove that

$$\left[\frac{d^r}{du^r} \zeta_X^{-1}(u) \right] \Big|_{u=1} = r!(-1)^{r+1} 2^r (r-1) \kappa_X.$$

This is an analog of formula (2-2) for the Dedekind zeta function of a number field at 0. The complexity is considered to be an analog of the class number of a number field.

Here we look at a zeta function invented by Stark. It has several advantages over the edge zeta. It can be used to compute the edge zeta with smaller determinants. It gives the edge zeta for a graph in which an edge has been fused; i.e., shrunk to one vertex.

Choose a spanning tree T of X . Then T has $|V| - 1 = n - 1$ edges. We denote the oriented versions of these *edges left out of the spanning tree* T (or “deleted” edges of T) and their inverses by

$$e_1, \dots, e_r, e_1^{-1}, \dots, e_r^{-1}.$$

Denote the remaining (oriented) *edges in the spanning tree* by T

$$t_1, \dots, t_{n-1}, t_1^{-1}, \dots, t_{n-1}^{-1}.$$

Any backtrackless, tailless cycle on X is uniquely (up to starting point on the tree between last and first e_k) determined by the ordered sequence of e_k 's it passes through. The free group of rank r generated by the e_k 's puts a group structure on backtrackless tailless cycles which is equivalent to the fundamental group of X .

There are 2 elementary reduction operations for paths written down in terms of directed edges just as there are elementary reduction operations for words in the fundamental group of X . This means that if a_1, \dots, a_s and e are taken from the e_k 's and their inverses, the two *elementary reduction operations* are:

- (i) $a_1 \cdots a_{i-1} e e^{-1} a_{i+2} \cdots a_s \cong a_1 \cdots a_{i-1} a_{i+2} \cdots a_s$
- (ii) $a_1 \cdots a_s \cong a_2 \cdots a_s a_1$

Using the first elementary reduction operation, each equivalence class of words corresponds to a group element and a word of minimum length in an equivalence class is *reduced* word in group theory language. Since the second operation is equivalent to conjugating by a_1 , an equivalence class using both elementary reductions corresponds to a conjugacy class in the fundamental group. A word of minimum length using both elementary operations corresponds to finding words of minimum length in a conjugacy class in the fundamental group. If a_1, \dots, a_s are taken from e_1, \dots, e_{2r} , a word $C = a_1 \cdots a_s$ is of minimum length in its conjugacy class if and only if $a_{i+1} \neq a_i^{-1}$, for $1 \leq i \leq s-1$ and $a_1 \neq a_s^{-1}$.

This is equivalent to saying that C corresponds to a *backtrackless, tailless* cycle under the correspondence above. Equivalent cycles correspond to conjugate elements of the fundamental group. A conjugacy class $[C]$ is *primitive* if a word of minimal length in $[C]$ is not a power of another word. We will say that a word of minimal length in its conjugacy class is *reduced in its conjugacy class*. From now on, we assume a representative element of $[C]$ is chosen which is reduced in $[C]$.

DEFINITION 43. The $2r \times 2r$ *path matrix* Z has ij entry given by the complex variable z_{ij} if $e_i \neq e_j^{-1}$ and by 0 if $e_i = e_j^{-1}$.

The path matrix Z has only one zero entry in each row unlike the edge matrix W from Definition 32 which is rather sparse. Next we imitate the definition of the edge zeta function.

DEFINITION 44. Define the *path norm* for a primitive path $C = a_1 \cdots a_s$ reduced in its conjugacy class $[C]$, where $a_i \in \{e_1^{\pm 1}, \dots, e_r^{\pm 1}\}$ as

$$N_P(C) = z_{a_1 a_2} \cdots z_{a_{s-1} a_s} z_{a_s a_1}.$$

Then the *path zeta* is defined for small $|z_{ij}|$ to be

$$\zeta_P(Z, X) = \prod_{[C]} (1 - N_P(C))^{-1},$$

where the product is over primitive reduced conjugacy classes $[C]$ other than the identity class.

We have similar results to those for the edge zeta.

THEOREM 45.

$$\zeta_P(Z, X)^{-1} = \det(I - Z).$$

PROOF. Imitate the proof of Theorem 34 for the edge zeta. □

The path zeta function is the same for all graphs with the same fundamental group. Next we define a procedure called *specializing the path matrix to the edge matrix* which will allow us to specialize the path zeta function to the edge zeta function. Use the notation above for the edges e_i left out of the spanning tree T and denote the edges of T by t_j . A prime cycle C is first written as a product of generators of the fundamental group and then as a product of actual edges e_i and t_k . Do this by inserting $t_{k_1} \cdots t_{k_s}$ which is the unique non backtracking path on T joining the terminal vertex of e_i and the starting vertex of e_j if e_i and e_j are successive deleted or non-tree edges in C . Now *specialize the path matrix Z to $Z(W)$ with entries*

$$z_{ij} = w_{e_i t_{k_1}} w_{t_{k_1} t_{k_2}} \cdots w_{t_{k_{s-1}} t_{k_s}} w_{t_{k_s} e_j}. \tag{4-2}$$

THEOREM 46. *Using the specialization procedure just given, we have*

$$\zeta_P(Z(W), X) = \zeta_E(W, X).$$

EXAMPLE 47 (THE DUMBBELL AGAIN). Recall that the edge zeta of the dumbbell graph of Figure 6 was evaluated by a 6×6 determinant. The path zeta requires a 4×4 determinant. Take the spanning tree to be the vertical edge. One finds, using the determinant formula for the path zeta and the specialization of the path to edge zeta:

$$\zeta_E(W, X)^{-1} = \det \begin{pmatrix} w_{11}-1 & w_{12}w_{23} & 0 & w_{12}w_{26} \\ w_{35}w_{51} & w_{33}-1 & w_{35}w_{54} & 0 \\ 0 & w_{42}w_{23} & w_{44}-1 & w_{42}w_{26} \\ w_{65}w_{51} & 0 & w_{65}w_{54} & w_{66}-1 \end{pmatrix}. \tag{4-3}$$

If we shrink the vertical edge to a point (which we call *fusion* or *contraction*), the edge zeta of the new graph is obtained by replacing any $w_{x_2}w_{2_y}$ (for $x, y = 1, 3, 4, 6$) which appear in formula (4-3) by w_{x_y} and any $w_{x_5}w_{5_y}$ (for $x, y = 1, 3, 4, 6$) by w_{x_y} . This gives the zeta function of the new graph obtained from the dumbbell, by fusing the vertical edge.

EXERCISE 48. Compute the path zeta function for $K_4 - e$ (the tetrahedron minus one edge) and then specialize it to the edge zeta function of the graph.

EXERCISE 49. Write a Mathematica program to specialize the path matrix Z to the matrix $Z(W)$ so that $\zeta_P(Z(W), X) = \zeta_E(W, X)$.

4.3. Connections with quantum chaos. A reference with some background on random matrix theory and quantum chaos is [Terras 2007]. In Figure 5 (page 154) we saw the experimental connections between the statistics of spectra of random real symmetric matrices and the statistics of the imaginary parts of s at the poles of the Ihara zeta function $\zeta(q^{-s}, X)$ for a $(q+1)$ -regular graph X . This is analogous to the connection between the statistics of the imaginary parts of zeros of the Riemann zeta function and the statistics of the spectra of random Hermitian matrices. At this point one should look at the figure produced by Bohigas and Giannoni comparing spacings of spectral lines from nuclear physics with those from number theory and billiards. Sarnak added lines from the spectrum of the Poincaré Laplacian on the fundamental domain of the modular group and I added eigenvalues of finite upper half-plane graphs. See [Terras 2007, p. 337].

Suppose you must arrange the eigenvalues E_i of a random symmetric matrix in decreasing order: $E_1 \geq E_2 \geq \cdots E_n$ and then normalize the eigenvalues so that the mean of the level spacings $E_i - E_{i+1}$ is 1. Wigner's surmise from 1957 says that the normalized level (eigenvalue) spacing histogram is approximated by the function $\frac{1}{2}\pi x \exp(-\pi x^2/4)$. In 1960 Gaudin and Mehta found the correct distribution function which is close to Wigner's. The correct distribution function is called the GOE distribution. A reference is [Mehta 1967]. The main property of this distribution is its vanishing at the origin (often called *level repulsion* in the physics literature). This differs in a big way from the spacing density of a Poisson random variable which is e^{-x} .

Many experiments have been performed with spacings of eigenvalues of the Laplace operator for a manifold such as the fundamental domain for a discrete group Γ acting on the upper half-plane H or the unit disc. The experiments of Schmit give the spacings of the eigenvalues of the Laplacian on $\Gamma \backslash H$ for an arithmetic Γ . To define "arithmetic" we must first define *commensurable subgroups* A, B of a group C . This means that $A \cap B$ has finite index both in A and B . Then suppose that Γ is an algebraic group over \mathbb{Q} as in [Borel and Mostow 1966, p. 4]. One says that Γ is *arithmetic* if there is a faithful rational representation ρ into the general linear group of $n \times n$ nonsingular matrices such that ρ is defined over the rationals and $\rho(\Gamma)$ is commensurable with $\rho(\Gamma) \cap \text{GL}(n, \mathbb{Z})$. Roughly we are saying that the integers are hiding somewhere in the definition of Γ . See [Borel and Mostow 1966] for more information. Arithmetic and nonarithmetic subgroups of $\text{SL}(2, \mathbb{C})$ are discussed in [Elstrodt et al. 1998].

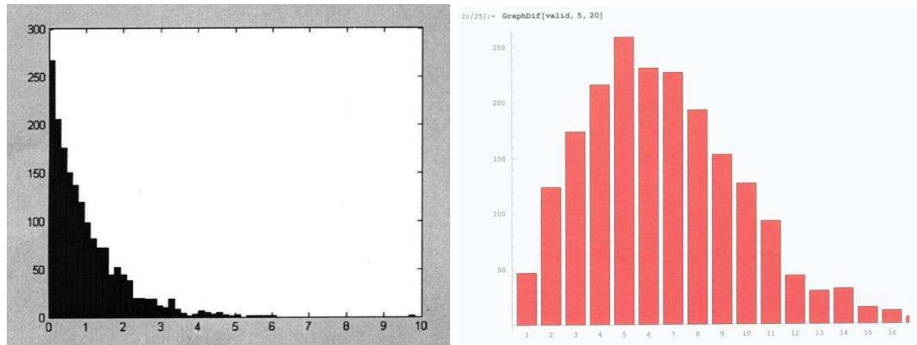


Figure 8. From [Newland 2005]. Spacings of the poles of the Ihara zeta for (left) a finite euclidean graph $\text{Euc}_{1999}(2,1)$ as defined in [Terras 1999], and (right) a random regular graph from Mathematica with 2000 vertices and degree 71.

Experiments of Schmit [1991] compared spacings of eigenvalues of the Laplacian for arithmetic and nonarithmetic groups acting on the unit disc. Schmit found that the arithmetic group had spacings that were close to Poisson while the nonarithmetic group spacings looked GOE.

Newland [2005] did experiments on spacings of poles of the Ihara zeta for regular graphs. When the graph was a certain Cayley graph for an abelian group which we called a Euclidean graph in [Newland 2005], he found Poisson spacings. When the graph was random, he found GOE spacings (actually a transform of GOE coming from the relationship between the eigenvalues of the adjacency matrix of the graph and the zeta poles). Figure 8, left, shows the spacing histogram for the poles of the Ihara zeta for a finite Euclidean graph $\text{Euc}_{1999}(2,1)$ as in Chapter 5 of [Terras 1999]. It is a Cayley graph for a finite abelian group. The right part of the same figure shows the spacing histogram for the poles of the Ihara zeta of a random regular graph as given by Mathematica with 2000 vertices and degree 71. The moral is that the spacings for the poles of the Ihara zeta of a Cayley graph of an abelian group look Poisson while, for a random graph, the spacings look GOE. The difference between the two parts of Figures 8 is similar to that between the spacings of the Laplacian for arithmetic and nonarithmetic groups.

Our plan for the rest of this section is to investigate the spacings of the poles of the Ihara zeta function of a random graph and compare the result with spacings for covering graphs both random and with abelian Galois group. By formula (3-1), this is essentially the same as investigating the spacings of the eigenvalues of the edge adjacency matrix W_1 from Definition 16. Here, although W_1 is not symmetric, the nearest neighbor spacing can be studied. If the eigenvalues of the matrix are $\lambda_i, i = 1, \dots, 2m$, we want to look at $v_i = \min\{|\lambda_i - \lambda_j| \mid j \neq i\}$.

The question becomes: what function best approximates the histogram of the v_i , assuming they are normalized to have mean 1?

References for the study of spacings of eigenvalues of nonsymmetric matrices include [Ginibre 1965; LeBoeuf 1999; Mehta 1967]. The *Wigner surmise for nonsymmetric matrices* is

$$4\Gamma\left(\frac{5}{4}\right)x^3\exp\left(-\Gamma\left(\frac{5}{4}\right)x^4\right). \quad (4-4)$$

Since our matrix W_1 is real and has certain special properties, this may not be the correct Wigner surmise. In what follows some experiments are performed. The following proposition gives some of the properties of W_1 .

PROPOSITION 50 (PROPERTIES OF W_1).

- (i) $W_1 = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$, where B and C are symmetric real, A is real with transpose tA . The diagonal entries of B and C are 0.
- (ii) The sum of the entries of the i -th row of W_1 is the degree of the vertex which is the start of edge i .

PROOF. See [Horton 2007] or [Terras 2010]. □

Our first experiment involves the eigenvalues of a random matrix with block form $\begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix}$, where B and C are symmetric and 0 on the diagonal. We used Matlab's `randn(N)` command to get matrices A, B, C with normally distributed entries. There is a result known as the Girko circle law which says that the eigenvalues of a set of random $n \times n$ real matrices with independent entries with a standard normal distribution should be approximately uniformly distributed in a circle of radius \sqrt{n} for large n . References are [Bai 1997; Girko 1984; Tao and Vu 2009]. A plot of the eigenvalues of a random matrix with the properties of W_1 appears in Figure 9, left. Note the symmetry with respect to the real axis, since our matrix is real. Another interesting fact is that the circle radius is not exactly that which Girko predicts. The spacing distribution for this random matrix is compared with the nonsymmetric Wigner surmise in formula (4-4) in Figure 9, right.

Our next experiments concern the spectra of actual W_1 matrices for graphs. First recall that the eigenvalues of W_1 are the reciprocals of the poles of the Ihara zeta function. You should also recall the graph theory Riemann hypothesis given in formula (2-7) as well as Theorem 22 of Kotani and Sunada. Figure 11 shows Ihara zeta poles for three graphs. The left half of Figures 12, 13, and 14 plot the eigenvalues of the W_1 matrix as well as circles of radius $\sqrt{p} \leq 1/\sqrt{R} \leq \sqrt{q}$, where $p + 1$ is the minimum degree of vertices of our graph and $q + 1$ is the maximum degree. Then R is from Definition 14 and $1/R$ is the spectral radius

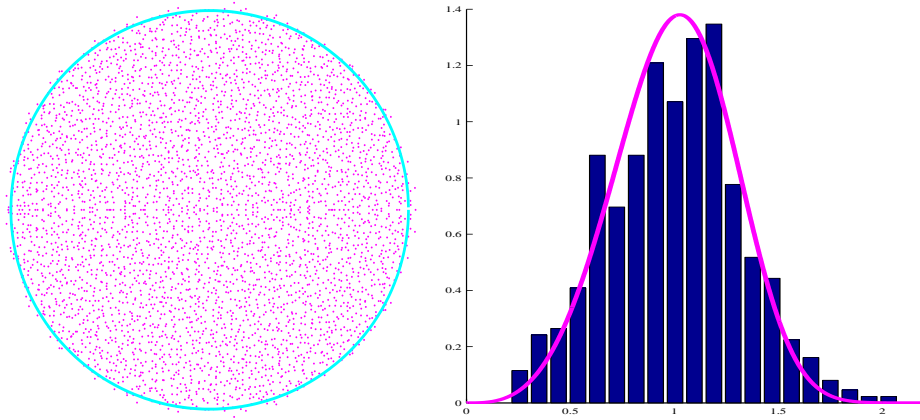


Figure 9. Left: Matlab-obtained spectrum of a random 2000×2000 matrix with the properties of W_1 except that the entries are not 0 and 1. The circle, centered at the origin, has radius $\frac{1}{2}(1 + \sqrt{2})\sqrt{2000} \approx 54.0$ rather than $\sqrt{2000}$ as in Girko's circle law. Right: The normalized nearest neighbor spacing for the spectrum of the same matrix. The curve is the Wigner surmise from formula (4-4).

of W_1 . Theorem 22 of Kotani and Sunada says the spectra cannot ever fill up a circle. They must lie in an annulus.

Figures 10 and 11 give the results of some Mathematica experiments on the distribution of the poles of zeta for various graphs constructed using the RealizeDegreeSequence command to create a few graphs with various degree sequences and then contracting vertices to join these graphs together. The first

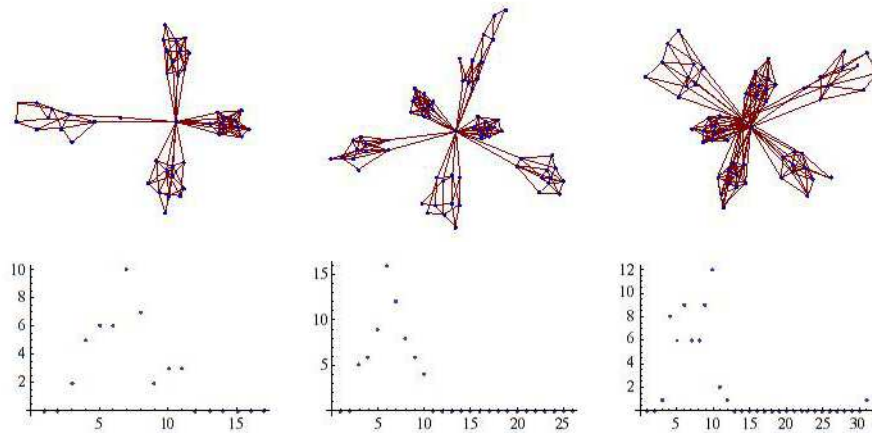


Figure 10. Mathematica-generated graphs (top row) and histograms of degrees (bottom row). The graphs were constructed using RealizeDegreeSequence and vertex contraction.

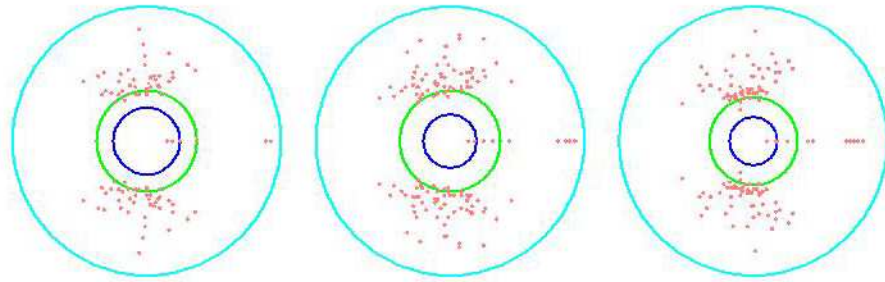


Figure 11. Mathematica-determined poles (pink points) of the Ihara zetas of the graphs in Figure 10. The middle green circle is the Riemann hypothesis circle with radius \sqrt{R} , for R the closest pole to 0. The inner circle has radius $1/\sqrt{q}$, where $q + 1$ is the maximum degree and the outer circle has radius $1/\sqrt{p}$, where $p + 1$ is the minimum degree. Many poles are inside the green (middle) circle and thus violate the Riemann hypothesis.

Figure 11 shows the poles of the Ihara zetas of the graphs in Figure 10. Many poles appear inside the green circle rather than outside as the RH would say.

Figure 12, left, shows a Matlab experiment giving the spectrum of the edge adjacency matrix W_1 for a “random graph”. The inner circle has radius \sqrt{p} , the green circle has radius $1/\sqrt{R}$. The outer circle has radius \sqrt{q} . The middle green circle is the Riemann hypothesis circle. Because the eigenvalues of W_1 are reciprocals of the poles of zeta, now the RH says the spectrum should be inside the middle circle. The RH looks approximately true.

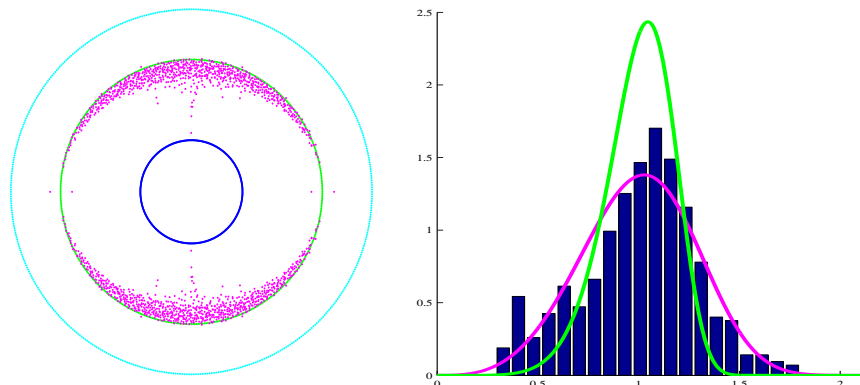


Figure 12. Left: Matlab-obtained eigenvalues of the edge adjacency matrix W_1 for a random graph (pink points). The inner circle has radius \sqrt{p} , the middle (green) circle has radius $1/\sqrt{R}$, and the outer one radius \sqrt{q} . Because the eigenvalues of W_1 are reciprocals of the poles of zeta, the Riemann hypothesis says the spectrum should be inside the green circle, which seems approximately true. The graph has 800 vertices, mean degree $\cong 13.125$, edge probability $\cong 0.0164$. Right: The histogram of the nearest neighbor spacings of the spectrum of W_1 for the same random graph, versus the modified Wigner surmise from formula (4-5) with $\omega = 3$ and 6.

Figure 12, right, shows the histogram of the nearest neighbor spacings of the spectrum of the same random graph versus a *modified Wigner surmise* (with $\omega = 3, 6$)

$$(\omega + 1)\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^\omega \exp\left(-\Gamma\left(\frac{\omega+2}{\omega+1}\right)^{\omega+1} x^{\omega+1}\right). \quad (4-5)$$

When $\omega = 3$, this is the original Wigner surmise.

Next we consider some experiments involving covering graphs. An example of a graph covering is the cube covering the tetrahedron. The theory mimics the theory of extensions of algebraic number fields. In particular, there is an analog of Galois theory and Artin L-functions attached to representations of the Galois group. This helps to explain the factorizations of the zeta functions in our earlier examples. See [Stark and Terras 2000] and [Terras 2010].

DEFINITION 51. If the graph has no multiple edges and loops we can say that the graph Y is an *unramified covering* of the graph X if we have a covering map $\pi : Y \rightarrow X$ which is an onto graph mapping (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the collection of points adjacent to $y \in Y$ is mapped 1-1 onto the collection of points adjacent to $x \in X$.

The definition in the case of loops and multiple edges can be found in [Stark and Terras 2000] or [Terras 2010]. It requires directing edges and requiring the covering map to preserve local directed neighborhoods of a vertex.

DEFINITION 52. If Y/X is a d -sheeted covering with projection map $\pi : Y \rightarrow X$, we say that it is a *normal covering* when there are d graph automorphisms $\sigma : Y \rightarrow Y$ such that $\pi \circ \sigma = \pi$. The Galois group $G(Y/X)$ is the set of these maps σ .

For covering graphs one can say more about the expected shape of the spectrum of the edge adjacency matrix or equivalently describe the region bounding the poles of the Ihara zeta. Angel, Friedman and Hoory give in [Angel et al. 2007] a method to compute the region encompassing the spectrum of the analogous operator to the edge adjacency matrix W_1 on the universal cover of a graph X . In Section 2 we mentioned the Alon conjecture for regular graphs. Angel, Friedman and Hoory give an *analog of the Alon conjecture for irregular graphs*. Roughly their conjecture says that the new edge adjacency spectrum of a large random covering graph is near the edge adjacency spectrum of the universal covering. Here “new” means not occurring in the spectrum of W_1 for the base graph. This conjecture can be shown to imply the approximate Riemann hypothesis for the new poles of a large random cover.

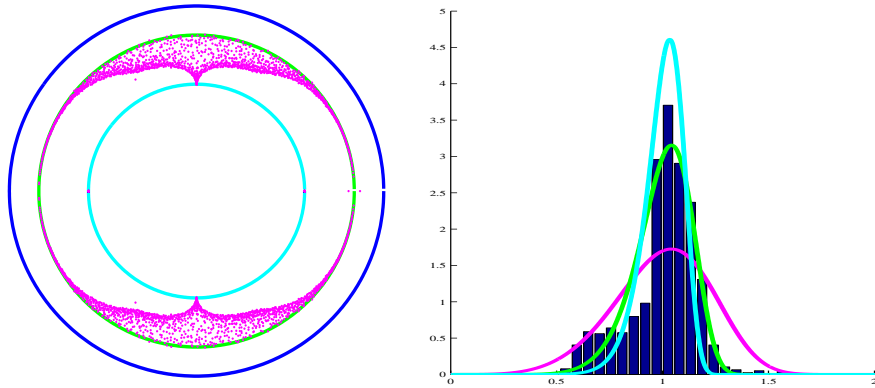


Figure 13. Left: Matlab-obtained eigenvalues (pink points) of the edge adjacency matrix of a random cover of the base graph made of two loops with an extra vertex on one loop. Thus we plot the reciprocals of the poles of zeta. The inner circle has radius 1, the middle circle has radius $1/\sqrt{R}$, and the outer one radius $\sqrt{3}$. The Riemann hypothesis is approximately true for this graph zeta. The cover has 801 sheets (copies of a spanning tree). Right: The nearest neighbor spacings for the spectrum of the edge adjacency matrix of the previous graph compared with three versions of the modified Wigner surmise from formula (4-5), with $\omega = 3, 6, 9$.

We show some examples related to this conjecture. Figure 13, left, shows the spectrum of the edge adjacency matrix of a random cover of the base graph consisting of two loops with an extra vertex on one loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is approximately true for this graph zeta.

Figure 13, right, shows the nearest neighbor spacings for the points in this same spectrum, compared with the modified Wigner surmise in formula (4-5), for various small values of ω .

Figure 14, left, shows the spectrum of the edge adjacency matrix for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the base graph consisting of two loops with an extra vertex on one loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$, with R as in Definition 14. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis looks very false.

Figure 14, right, shows the histogram of the nearest neighbor spacings for the spectrum of the edge adjacency matrix of the graph in the preceding figure compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (4-4).

EXERCISE 53. Compute more examples of poles of zeta functions of graphs. In particular, it would be interesting to look at graphs with degrees satisfying a power law d^{-e} , where $2 \leq e \leq 3$, say.

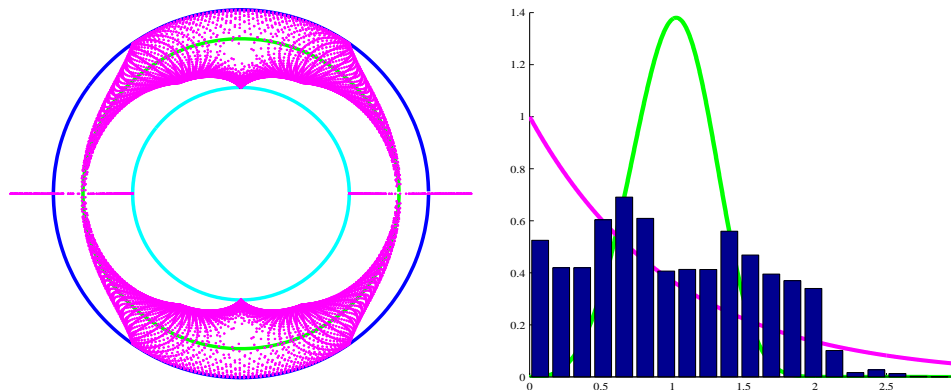


Figure 14. Left: Matlab-obtained eigenvalues (pink points) of the edge adjacency matrix W_1 for a Galois $\mathbb{Z}_{163} \times \mathbb{Z}_{45}$ covering of the base graph consisting of two loops with an extra vertex on one loop. The inner circle has radius 1. The middle circle has radius $1/\sqrt{R}$, with R as in Definition 14. The outer circle has radius $\sqrt{3}$. The Riemann hypothesis is very false. Left: The histogram of the nearest neighbor spacings for the spectrum of the same edge adjacency matrix W_1 , compared with spacings of a Poisson random variable (e^{-x}) and the Wigner surmise from formula (4-4).

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