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Basic zeta functions and some applications in physics

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1. Introduction

It is the aim of these lectures to introduce some basic zeta functions and their uses in the areas of the Casimir effect and Bose–Einstein condensation. A brief introduction into these areas is given in the respective sections; for recent monographs on these topics see [8; 22; 33; 34; 57; 67; 68; 71; 72]. We will consider exclusively spectral zeta functions, that is, zeta functions arising from the eigenvalue spectrum of suitable differential operators. Applications like those in number theory [3; 4; 23; 79] will not be considered in this contribution.

There is a set of technical tools that are at the very heart of understanding analytical properties of essentially every spectral zeta function. Those tools are introduced in Section 2 using the well-studied examples of the Hurwitz [54], Epstein [38; 39] and Barnes zeta function [5; 6]. In Section 3 it is explained how these different examples can all be thought of as being generated by the same mechanism, namely they all result from eigenvalues of suitable (partial) differential operators. It is this relation with partial differential operators that provides the motivation for analyzing the zeta functions considered in these lectures. Motivations come for example from the questions "Can one hear the shape of a drum?", "What does the Casimir effect know about a boundary?", and "What does a Bose gas know about its container?" The first two questions are considered in detail in Section 4. The last question is examined in Section 5, where we will see how zeta functions can be used to analyze the phenomenon of Bose–Einstein condensation. Section 6 will point towards recent developments for the analysis of spectral zeta functions and their applications.

2. Some basic zeta functions

In this section we will construct analytical continuations of basic zeta functions. From these we will determine the meromorphic structure, residues at singular points and special function values.

2.1. Hurwitz zeta function. We start by considering a generalization of the Riemann zeta function

$$\zeta_R(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2-1}$$

DEFINITION 2.1. Let $s \in \mathbb{C}$ and $0 < a < 1$. Then for $\mathrm{Re}\, s > 1$ the Hurwitz zeta function is defined by

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Clearly, $\zeta_H(s,1) = \zeta_R(s)$. Results for $a = 1 + b > 1$ follow by observing

$$\zeta_H(s,1+b) = \sum_{n=0}^{\infty} \frac{1}{(n+1+b)^s} = \zeta_H(s,b) - \frac{1}{b^s}.$$

In order to determine properties of the Hurwitz zeta function, one strategy is to express it in term of 'known' zeta functions like the Riemann zeta function.

THEOREM 2.2. *For $0 < a < 1$ we have*

$$\zeta_H(s,a) = \frac{1}{a^s} + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} a^k \zeta_R(s+k).$$

PROOF. Note that for $|z| < 1$ we have the binomial expansion

$$(1-z)^{-s} = \sum_{k=0}^{\infty} \frac{\Gamma(s+k)}{\Gamma(s)k!} z^k.$$

So for $\mathrm{Re}\, s > 1$ we compute

$$\begin{aligned}
\zeta_H(s,a) &= \frac{1}{a^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} \frac{1}{\left(1+\frac{a}{n}\right)^s} = \frac{1}{a^s} + \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} \left(\frac{a}{n}\right)^k \\
&= \frac{1}{a^s} + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} a^k \sum_{n=1}^{\infty} \frac{1}{n^{s+k}} \\
&= \frac{1}{a^s} + \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(s)k!} a^k \zeta_R(s+k). \qquad \square
\end{aligned}$$

From here it is seen that $s = 1$ is the only pole of $\zeta_H(s, a)$ with $\operatorname{Res} \zeta_H(1, a) = 1$.

In determining certain function values of $\zeta_H(s, a)$ the following polynomials will turn out to be useful.

DEFINITION 2.3. For $x \in \mathbb{C}$ we define the Bernoulli polynomials $B_n(x)$ by the equation

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n, \qquad \text{where } |z| < 2\pi. \tag{2-2}$$

Examples are $B_0(x) = 1$ and $B_1(x) = x - \frac{1}{2}$. The numbers $B_n(0)$ are called Bernoulli numbers and are denoted by B_n. Thus

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \qquad \text{where } |z| < 2\pi. \tag{2-3}$$

LEMMA 2.4. *The Bernoulli polynomials satisfy*

(1) $$B_n(x) = \textstyle\sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

(2) $$B_n(x+1) - B_n(x) = nx^{n-1} \qquad \text{if } n \geq 1,$$

(3) $$(-1)^n B_n(-x) = B_n(x) + nx^{n-1},$$

(4) $$B_n(1-x) = (-1)^n B_n(x).$$

EXERCISE 1. Use relations (2-2) and (2-3) to show assertions (1)–(4).

We now establish elementary properties of $\zeta_H(s, a)$.

THEOREM 2.5. *For* $\operatorname{Re} s > 1$ *we have*

$$\zeta_H(s, a) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-at}}{1 - e^{-t}} dt. \tag{2-4}$$

Furthermore, for $k \in \mathbb{N}_0$ *we have*

$$\zeta_H(-k, a) = -\frac{B_{k+1}(a)}{k+1}.$$

PROOF. We use the definition of the gamma function and have

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u} du = \lambda^s \int_0^{\infty} t^{s-1} e^{-t\lambda} dt. \tag{2-5}$$

This shows the first part of the theorem,

$$\begin{aligned} \zeta_H(s, a) &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-t(n+a)} dt = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{n=0}^{\infty} e^{-t(n+a)} dt \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-at}}{1 - e^{-t}} dt. \end{aligned}$$

Furthermore we have

$$\begin{aligned}\zeta_H(s,a) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}\frac{te^{-ta}}{1-e^{-t}}dt \\ &= \frac{1}{\Gamma(s)}\int_0^1 t^{s-2}\frac{(-t)e^{-ta}}{e^{-t}-1}dt+\frac{1}{\Gamma(s)}\int_1^\infty t^{s-2}\frac{(-t)e^{-ta}}{e^{-t}-1}dt.\end{aligned}$$

The integral in the second term is an entire function of s. Given that the gamma function has singularities at $s=-k$, $k\in\mathbb{N}_0$, only the first term can possibly contribute to the properties $\zeta_H(-k,a)$ considered. We continue and write

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^1 t^{s-2}\frac{(-t)e^{-ta}}{e^{-t}-1}dt &= \frac{1}{\Gamma(s)}\int_0^1 t^{s-2}\sum_{n=0}^\infty\frac{B_n(a)}{n!}(-t)^n dt \\ &= \frac{1}{\Gamma(s)}\sum_{n=0}^\infty\frac{B_n(a)}{n!}\frac{(-1)^n}{s+n-1},\end{aligned}$$

which provides the analytical continuation of the integral to the complex plane. From here we observe again

$$\operatorname{Res}\zeta_H(1,a) = B_0(a)=1.$$

Furthermore the second part of the theorem follows:

$$\begin{aligned}\zeta_H(-k,a) &= \lim_{\varepsilon\to 0}\frac{1}{\Gamma(-k+\varepsilon)}\frac{B_{k+1}(a)}{(k+1)!}\frac{(-1)^{k+1}}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}(-1)^k k!\varepsilon\frac{B_{k+1}(a)}{(k+1)!}\frac{(-1)^{k+1}}{\varepsilon} = -\frac{B_{k+1}(a)}{k+1}.\end{aligned}$$

□

The disadvantage of the representation (2-4) is that it is valid only for $\operatorname{Re}s>1$. This can be improved by using a complex contour integral representation. The starting point is the following representation for the gamma function [46].

LEMMA 2.6. *For $z\notin\mathbb{Z}$ we have*

$$\Gamma(z)=-\frac{1}{2i\sin(\pi z)}\int_{\mathcal{C}}(-t)^{z-1}e^{-t}dt,$$

where the anticlockwise contour $\mathcal{C}$ consists of a circle $\mathcal{C}_3$ of radius $\varepsilon<2\pi$ and straight lines $\mathcal{C}_1$, respectively $\mathcal{C}_2$, just above, respectively just below, the x-axis; see Figure 1.

PROOF. Assume $\operatorname{Re}z>1$. As the integrand remains bounded along $\mathcal{C}_3$, no contributions will result as $\varepsilon\to 0$. Along $\mathcal{C}_1$ and $\mathcal{C}_2$ we parametrize as given in

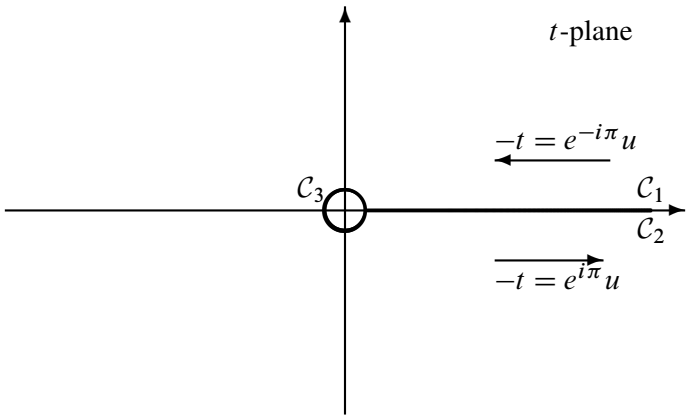

Figure 1. Contour in Lemma 2.6.

Figure 1 and thus for $\operatorname{Re} z > 1$

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\mathcal{C}}(-t)^{z-1}e^{-t}dt &= \int_{\infty}^{0}e^{-i\pi(z-1)}u^{z-1}e^{-u}du+\int_{0}^{\infty}e^{i\pi(z-1)}u^{z-1}e^{-u}du\\ &= -\int_{0}^{\infty}u^{z-1}e^{-u}\left(e^{i\pi z}-e^{-i\pi z}\right)du\\ &= -2i\sin(\pi z)\int_{0}^{\infty}u^{z-1}e^{-u}du,\end{aligned}$$

which implies the assertion by analytical continuation. □

This representation for the gamma function can be used to show the following result for the Hurwitz zeta function.

THEOREM 2.7. *For $s \in \mathbb{C}$, $s \notin \mathbb{N}$, we have*

$$\zeta_H(s,a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}}\frac{(-t)^{s-1}e^{-ta}}{1-e^{-t}}dt,$$

with the contour $\mathcal{C}$ given in Figure 1.

PROOF. We follow the previous calculation to note

$$\int_{\mathcal{C}}\frac{(-t)^{s-1}e^{-ta}}{1-e^{-t}}dt = -2i\sin(\pi s)\int_{0}^{\infty}t^{s-1}\frac{e^{-ta}}{1-e^{-t}}dt,$$

and we use [46]

$$\sin(\pi s)\Gamma(s) = \frac{\pi}{\Gamma(1-s)}$$

to conclude the assertion. □

From here, properties previously given can be easily derived. For $s \in \mathbb{Z}$ the integrand does not have a branch cut and the integral can easily be evaluated using the residue theorem. The only possible singularity enclosed is at $t = 0$ and to read off the residue we use the expansion

$$-(-t)^{s-2}\frac{(-t)e^{-ta}}{e^{-t}-1} = -(-t)^{s-2}\sum_{n=0}^{\infty}\frac{B_n(a)}{n!}(-t)^n.$$

2.2. Barnes zeta function. The Barnes zeta function is a multidimensional generalization of the Hurwitz zeta function.

DEFINITION 2.8. Let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > d$ and $c \in \mathbb{R}_+$, $\vec{r} \in \mathbb{R}_+^d$. The Barnes zeta function is defined as

$$\zeta_{\mathcal{B}}(s, c|\vec{r}) = \sum_{\vec{m}\in\mathbb{N}_0^d}\frac{1}{(c+\vec{m}\cdot\vec{r})^s}. \tag{2-6}$$

If $c = 0$ it is understood that the summation ranges over $\vec{m} \neq \vec{0}$.

For $\vec{r} = \vec{1}_d := (1, 1, \ldots, 1, 1)$, the Barnes zeta function can be expanded in terms of the Hurwitz zeta function.

EXAMPLE 2.9. Consider $d = 2$ and $\vec{r} = (1, 1)$. Then

$$\begin{aligned}\zeta_{\mathcal{B}}(s, c|\vec{1}_2) &= \sum_{\vec{m}\in\mathbb{N}_0^2}\frac{1}{(c+m_1+m_2)^s} = \sum_{k=0}^{\infty}\frac{k+1}{(c+k)^s} = \sum_{k=0}^{\infty}\frac{k+c+1-c}{(c+k)^s}\\ &= \zeta_H(s-1, c) + (1-c)\zeta_H(s, c).\end{aligned}$$

EXAMPLE 2.10. Let $e_k^{(d)}$ be the number of possibilities to write an integer k as a sum over d non-negative integers. We then can write

$$\zeta_{\mathcal{B}}(s, c|\vec{1}_d) = \sum_{\vec{m}\in\mathbb{N}_0^d}\frac{1}{(c+m_1+\cdots+m_d)^s} = \sum_{k=0}^{\infty}e_k^{(d)}\frac{1}{(c+k)^s}.$$

The coefficient $e_k^{(d)}$ can be determined for example as follows. Consider

$$\begin{aligned}\frac{1}{(1-x)^d} &= \frac{1}{1-x}\cdots\frac{1}{1-x} = \Big(\sum_{l_1=0}^{\infty}x^{l_1}\Big)\cdots\Big(\sum_{l_d=0}^{\infty}x^{l_d}\Big)\\ &= \sum_{l_1=0}^{\infty}\cdots\sum_{l_d=0}^{\infty}x^{l_1+\cdots+l_d} = \sum_{k=0}^{\infty}e_k^{(d)}x^k.\end{aligned}$$

On the other side, using the binomial expansion

$$\frac{1}{(1-x)^d} = \sum_{k=0}^{\infty} \frac{\Gamma(d+k)}{\Gamma(d)k!} x^k = \sum_{k=0}^{\infty} \frac{(d+k-1)!}{(d-1)!k!} x^k = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} x^k.$$

This shows

$$\zeta_{\mathcal{B}}(s, c|\vec{1}_d) = \sum_{k=0}^{\infty} \binom{d+k-1}{d-1} \frac{1}{(c+k)^s},$$

which, once the dimension d is specified, allows to write the Barnes zeta function as a sum of Hurwitz zeta functions along the lines in Example 2.9.

It is possible to obtain similar formulas for r_i rational numbers [27; 28].

For some properties of the Barnes zeta function the use of complex contour integral representations turns out to be the best strategy.

THEOREM 2.11. *We have the following representations*:

$$\begin{aligned} \zeta_{\mathcal{B}}(s, c|\vec{r}) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-ct}}{\prod_{j=1}^d (1-e^{-r_j t})} dt \\ &= -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} (-t)^{s-1} \frac{e^{-ct}}{\prod_{j=1}^d (1-e^{-r_j t})} dt, \end{aligned}$$

with the contour $\mathcal{C}$ given in Figure 1.

EXERCISE 2. Use equation (2-5), and again Lemma 2.6, to prove Theorem 2.11.

The residues of the Barnes zeta function and its values at non-positive integers are best described using generalized Bernoulli polynomials [70].

DEFINITION 2.12. We define the generalized Bernoulli polynomials $B_n^{(d)}(x|\vec{r})$ by the equation

$$\frac{e^{-xt}}{\prod_{j=1}^d (1-e^{-r_j t})} = \frac{(-1)^d}{\prod_{j=1}^d r_j} \sum_{n=0}^{\infty} \frac{(-t)^{n-d}}{n!} B_n^{(d)}(x|\vec{r}).$$

Using Definition 2.12 in Theorem 2.11 one immediately obtains the following properties of the Barnes zeta function.

THEOREM 2.13.

(1) $$\operatorname{Res} \zeta_{\mathcal{B}}(z, c|\vec{r}) = \frac{(-1)^{d+z}}{(z-1)!(d-z)! \prod_{j=1}^d r_j} B_{d-z}^{(d)}(c|\vec{r}), \qquad z = 1, 2, \ldots, d,$$

(2) $$\zeta_{\mathcal{B}}(-n, c|\vec{r}) = \frac{(-1)^d n!}{(d+n)! \prod_{j=1}^d r_j} B_{d+n}^{(d)}(c|\vec{r}).$$

EXERCISE 3. Use the first representation of $\zeta_{\mathcal{B}}(s,c|\vec{r})$ in Theorem 2.11 together with Definition 2.12 to show Theorem 2.13. Follow the steps of the proof in Theorem 2.5.

EXERCISE 4. Use the second representation of $\zeta_{\mathcal{B}}(s,c|\vec{r})$ in Theorem 2.11 together with Definition 2.12 and the residue theorem to show Theorem 2.13.

2.3. Epstein zeta function. We now consider zeta functions associated with sums of squares of integers [38; 39].

DEFINITION 2.14. Let $s \in \mathbb{C}$ with $\mathrm{Re}\, s > d/2$ and $c \in \mathbb{R}_+$, $\vec{r} \in \mathbb{R}_+^d$. The Epstein zeta function is defined as

$$\zeta_{\mathcal{E}}(s,c|\vec{r}) = \sum_{\vec{m}\in\mathbb{Z}^d} \frac{1}{(c+r_1m_1^2+r_2m_2^2+\cdots+r_dm_d^2)^s}.$$

If $c=0$ it is understood that the summation ranges over $\vec{m} \neq \vec{0}$.

LEMMA 2.15. *For* $\mathrm{Re}\, s > d/2$, *we have*

$$\zeta_{\mathcal{E}}(s,c|\vec{r}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{\vec{m}\in\mathbb{Z}^d} e^{-t(r_1m_1^2+\cdots+r_dm_d^2+c)} dt.$$

PROOF. This follows as before from property (2-5) of the gamma function. □

As we have noted in the proof of Theorem 2.5, it is the small-t behavior of the integrand that determines residues of the zeta function and special function values. The way the integrand is written in Lemma 2.15 this $t \to 0$ behavior is not easily read off. A suitable representation is obtained by using the Poisson resummation [53].

LEMMA 2.16. *Let* $r \in \mathbb{C}$ *with* $\mathrm{Re}\, r > 0$ *and* $t \in \mathbb{R}_+$, *then*

$$\sum_{l=-\infty}^{\infty} e^{-trl^2} = \sqrt{\frac{\pi}{tr}} \sum_{l=-\infty}^{\infty} e^{-\frac{\pi^2}{rt}l^2}.$$

EXERCISE 5. If $F(x)$ is continuous such that

$$\int_{-\infty}^{\infty} |F(x)|dx < \infty,$$

then we define its Fourier transform by

$$\hat{F}(u) = \int_{-\infty}^{\infty} F(x)e^{-2\pi i x u}\, dx.$$

If

$$\int_{-\infty}^{\infty} |\hat{F}(u)|\, du < \infty,$$

then we have the Fourier inversion formula

$$F(x) = \int_{-\infty}^{\infty} \hat{F}(u)\ e^{2\pi i x u}\ du.$$

Show the following Theorem: Let $F \in L^1(\mathbb{R})$. Suppose that the series

$$\sum_{n\in\mathbb{Z}} F(n+v)$$

converges absolutely and uniformly in v, and that

$$\sum_{m\in\mathbb{Z}} |\hat{F}(m)| < \infty.$$

Then

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \hat{F}(n) e^{2\pi i n v}.$$

Hint: Note that

$$G(v) = \sum_{n\in\mathbb{Z}} F(n+v)$$

is a function of v of period 1.

EXERCISE 6. Apply Exercise 5 with a suitable function $F(x)$ to show the Poisson resummation formula Lemma 2.16.

In Lemma 2.16 it is clearly seen that the only term on the right hand side that is not exponentially damped as $t \to 0$ comes from the $l = 0$ term. Using the resummation formula for all d sums in Lemma 2.15, after resumming the $\vec{m} = \vec{0}$ term contributes

$$\begin{aligned}\zeta_{\mathcal{E}}^{\vec{0}}(s, c|\vec{r}) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{\pi^{d/2}}{t^{d/2}\sqrt{r_1\cdots r_d}} e^{-ct} dt \\ &= \frac{\pi^{d/2}}{\sqrt{r_1\cdots r_d}\ \Gamma(s)} \int_0^\infty t^{s-d/2-1} e^{-ct} dt = \frac{\pi^{d/2}}{\sqrt{r_1\cdots r_d}} \frac{\Gamma\left(s-\frac{d}{2}\right)}{\Gamma(s) c^{s-d/2}}.\end{aligned}$$

All other contributions after resummation are exponentially damped as $t \to 0$ and can be given in terms of modified Bessel functions [46].

DEFINITION 2.17. Let $\operatorname{Re} z^2 > 0$. We define the modified Bessel function $K_\nu(z)$ by

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}}\ t^{-\nu-1} dt.$$

Performing the resummation in Lemma 2.15 according to Lemma 2.16, with Definition 2.17 one obtains the following representation of the Epstein zeta function valid in the whole complex plane [34; 78].

THEOREM 2.18. *We have*

$$\zeta_{\mathcal{E}}(s,c|\vec{r}) = \frac{\pi^{d/2}}{\sqrt{r_1\cdots r_d}}\frac{\Gamma\left(s-\frac{d}{2}\right)}{\Gamma(s)}c^{\frac{d}{2}-s} + \frac{2\pi^s c^{\frac{d-2s}{4}}}{\Gamma(s)\sqrt{r_1\cdots r_d}} \times \sum_{\vec{n}\in\mathbb{Z}^d/\{\vec{0}\}} \left(\frac{n_1^2}{r_1}+\cdots+\frac{n_d^2}{r_d}\right)^{\frac{1}{2}\left(s-\frac{d}{2}\right)} K_{\frac{d}{2}-s}\left(2\pi\sqrt{c}\left(\frac{n_1^2}{r_1}+\cdots+\frac{n_d^2}{r_d}\right)^{\frac{1}{2}}\right).$$

EXERCISE 7. Show Theorem 2.18 along the lines indicated.

From Definition 2.17 it is clear that the Bessel function is exponentially damped for large $\mathrm{Re}\, z^2$. As a result the representation above is numerically very effective as long as the argument of $K_{d/2-s}$ is large. The terms involving the Bessel functions are analytic for all values of s, the first term contains poles. As an immediate consequence of the properties of the gamma function one can show the following properties of the Epstein zeta function.

THEOREM 2.19. *For d even, $\zeta_{\mathcal{E}}(s,c|\vec{r})$ has poles at $s=\frac{d}{2},\frac{d}{2}-1,\ldots,1$, whereas for d odd they are located at $s=\frac{d}{2},\frac{d}{2}-1,\ldots,\frac{1}{2},-\frac{2l+1}{2}$, $l\in\mathbb{N}_0$. Furthermore,*

$$\mathrm{Res}\,\zeta_{\mathcal{E}}(j,c|\vec{r}) = \frac{(-1)^{\frac{d}{2}+j}\pi^{\frac{j}{2}}c^{\frac{d}{2}-j}}{\sqrt{r_1\cdots r_d}\;\Gamma(j)\Gamma\left(\frac{d}{2}-j+1\right)},$$

$$\zeta_{\mathcal{E}}(-p,c|\vec{r}) = \begin{cases} 0 & \text{for } d \text{ odd}, \\ \dfrac{(-1)^{\frac{d}{2}}p!\pi^{\frac{d}{2}}c^{\frac{d}{2}+p}}{\sqrt{r_1\cdots r_d}\;\Gamma\left(\frac{d}{2}+p+1\right)} & \text{for } d \text{ even}. \end{cases}$$

EXERCISE 8. Use Theorem 2.18 and properties of the gamma function to show Theorem 2.19.

This concludes the list of examples for zeta functions to be considered in what follows. A natural question is what the motivations are to consider these zeta functions. Before we describe a few aspects relating to this question let us mention how all these zeta functions, and many others, result from a common principle.

3. Boundary value problems and associated zeta functions

In this section we explain how the considered zeta functions, and others, are all associated with eigenvalue problems of (partial) differential operators.

EXAMPLE 3.1. Let $M = [0, L]$ be some interval and consider the *Dirichlet* boundary value problem.

$$P\phi_n(x) := -\frac{\partial^2}{\partial x^2}\phi_n(x) = \lambda_n\phi_n(x), \qquad \phi_n(0) = \phi_n(L) = 0.$$

The solutions to the boundary value problem have the general form

$$\phi_n(x) = A\sin(\sqrt{\lambda_n}x) + B\cos(\sqrt{\lambda_n}x).$$

Imposing the Dirichlet boundary condition shows we need

$$\phi_n(0) = B = 0, \qquad \phi_n(L) = A\sin(L\sqrt{\lambda_n}) = 0,$$

which implies

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad n \in \mathbb{N}.$$

We only need to consider $n \in \mathbb{N}$ because non-positive integers lead to linearly dependent eigenfunctions. The zeta function $\zeta_P(s)$ associated with this boundary value problem is defined to be the sum over all eigenvalues raised to the power $(-s)$, namely

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}, \qquad \operatorname{Re} s > \tfrac{1}{2}.$$

So here the associated zeta function is a multiple of the zeta function of Riemann,

$$\zeta_P(s) = \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^{-2s} = \left(\frac{L}{\pi}\right)^{2s}\zeta_R(2s).$$

EXAMPLE 3.2. The previous example can be easily generalized to higher dimensions. We consider explicitly two dimensions; for the higher dimensional situation see [1]. Let $M = \{(x, y) | x \in [0, L_1], y \in [0, L_2]\}$. We consider the boundary value problem with Dirichlet boundary conditions on M, that is

$$\begin{aligned} P\phi_{n,m}(x, y) &= \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + c\right)\phi_{n,m}(x, y) = \lambda_{n,m}\phi_{n,m}(x, y), \\ \phi_{n,m}(0, y) &= \phi_{n,m}(L_1, y) = \phi_{n,m}(x, 0) = \phi_{n,m}(x, L_2) = 0. \end{aligned}$$

Using the process of separation of variables, eigenfunctions are seen to be

$$\phi_{n,m}(x, y) = A\sin\left(\frac{n\pi x}{L_1}\right)\sin\left(\frac{m\pi y}{L_2}\right),$$

with the eigenvalues

$$\lambda_{n,m} = \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2 + c, \qquad n, m \in \mathbb{N}.$$

The associated zeta function therefore is

$$\zeta_P(s) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \left[\left(\frac{n\pi}{L_1} \right)^2 + \left(\frac{m\pi}{L_2} \right)^2 + c \right]^{-s},$$

which can be expressed in terms of the Epstein zeta function given in Definition 2.14 as follows:

$$\zeta_P(s) = \tfrac{1}{4} \zeta_{\mathcal{E}} \left(s, c \,\Big|\, \left(\left(\frac{\pi}{L_1} \right)^2, \left(\frac{\pi}{L_2} \right)^2 \right) \right) - \tfrac{1}{4} \zeta_{\mathcal{E}} \left(s, c \,\Big|\, \left(\frac{\pi}{L_1} \right)^2 \right) - \tfrac{1}{4} \zeta_{\mathcal{E}} \left(s, c \,\Big|\, \left(\frac{\pi}{L_2} \right)^2 \right) + \tfrac{1}{4} c^{-s}. \quad (3\text{-}1)$$

EXAMPLE 3.3. Similarly one can consider periodic boundary conditions instead of Dirichlet boundary conditions, this means the manifold M is given by $M = S^1 \times S^1$. In this case the eigenfunctions have to satisfy

$$\begin{aligned} \phi_{n,m}(0,y) &= \phi_{n,m}(L_1,y), & \frac{\partial}{\partial x}\phi_{n,m}(0,y) &= \frac{\partial}{\partial x}\phi_{n,m}(L_1,y), \\ \phi_{n,m}(x,0) &= \phi_{n,m}(x,L_2), & \frac{\partial}{\partial y}\phi_{n,m}(x,0) &= \frac{\partial}{\partial y}\phi_{n,m}(x,L_2). \end{aligned}$$

This shows that

$$\phi_{n,m}(x,y) = A e^{i2\pi nx/L_1} \; e^{i2\pi my/L_2},$$

which implies for the eigenvalues

$$\lambda_{n,m} = \left(\frac{2\pi n}{L_1} \right)^2 + \left(\frac{2\pi m}{L_2} \right)^2 + c, \qquad (n,m) \in \mathbb{Z}^2.$$

The associated zeta function therefore is

$$\zeta_P(s) = \zeta_{\mathcal{E}}\left(s, c|\vec{r} \right), \qquad \vec{r} = \left(\left(\frac{2\pi}{L_1} \right)^2, \left(\frac{2\pi}{L_2} \right)^2 \right).$$

Clearly, in d dimensions one finds

$$\zeta_P(s) = \zeta_{\mathcal{E}}\left(s, c|\vec{r} \right), \qquad \vec{r} = \left(\left(\frac{2\pi}{L_1} \right)^2, \ldots, \left(\frac{2\pi}{L_d} \right)^2 \right).$$

EXAMPLE 3.4. As a final example we consider the Schrödinger equation of atoms in a harmonic oscillator potential. In this case $M = \mathbb{R}^3$, and the eigenvalue equation reads

$$\left(-\frac{\hbar^2}{2m}\Delta + \frac{m}{2}\left(\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2\right)\right)\phi_{n_1,n_2,n_3}(x,y,z) = \lambda_{n_1,n_2,n_3}\phi_{n_1,n_2,n_3}(x,y,z).$$

This differential equation is augmented by the condition that eigenfunctions must be square integrable, $\phi_{n_1,n_2,n_3}(x,y,z) \in \mathcal{L}^2(\mathbb{R}^3)$. As is well known, this gives the eigenvalues

$$\lambda_{n_1,n_2,n_3} = \hbar\omega_1\left(n_1 + \tfrac{1}{2}\right) + \hbar\omega_2\left(n_2 + \tfrac{1}{2}\right) + \hbar\omega_3\left(n_3 + \tfrac{1}{2}\right),$$

for $(n_1, n_2, n_3) \in \mathbb{N}_0^3$. This clearly leads to the Barnes zeta function

$$\zeta_P(s) = \zeta_{\mathcal{B}}(s, c|\vec{r}),$$

where

$$c = \tfrac{1}{2}\hbar(\omega_1 + \omega_2 + \omega_3), \qquad \vec{r} = \hbar\,(\omega_1, \omega_2, \omega_3)\,.$$

If $M = \mathbb{R}$ is chosen the Hurwitz zeta function results.

The examples above illustrate how the zeta functions considered in Section 2 are all related in a natural way to eigenvalues of specific boundary value problems. In fact, zeta functions in a much more general context are studied in great detail. For our purposes the relevant setting is the setting of Laplace-type operators on a Riemannian manifold M, possibly with a boundary ∂M. Laplace-type means the operator P can be written as

$$P = -g^{jk}\nabla_j^V\nabla_k^V - E,$$

where g^{jk} is the metric of M, ∇^V is the connection on M acting on a smooth vector bundle V over M, and where E is an endomorphism of V. Imposing suitable boundary conditions, eigenvalues λ_n and eigenfunctions ϕ_n do exist,

$$P\phi_n(x) = \lambda_n\phi_n(x),$$

and assuming $\lambda_n > 0$ the zeta function is defined to be

$$\zeta_P(s) = \sum_{n=1}^{\infty}\lambda_n^{-s}$$

for $\operatorname{Re} s$ sufficiently large. If there are modes with $\lambda_n = 0$ those have to be excluded from the sum. Also, if finitely many eigenvalues are negative the zeta function can be defined by choosing nonstandard definitions of the principal value for the argument of complex numbers, but we will not need to consider those cases.

4. Some motivations to consider zeta functions

There are many situations where properties of zeta functions in the above context of Laplace-type operators are needed. In the following we present a few of them, but many more can be found for example in the context of number theory [3; 4; 23; 79] and quantum field theory [8; 14; 15; 16; 26; 30; 31; 33; 41; 42; 57; 74].

4.1. Can one hear the shape of a drum? Let M be a two-dimensional membrane representing a drum with boundary ∂M. The drum is fixed along its boundary. Then possible vibrations of the drum and its fundamental tones are described by the eigenvalue problem

$$-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi_n(x,y)=\lambda_n\phi_n(x,y), \qquad \phi_n(x,y)|_{(x,y)\in\partial M}=0.$$

Here, (x, y) denotes the variables in the plane, the eigenfunctions $\phi_n(x, y)$ describe the amplitude of the vibrations and λ_n its fundamental tones. In 1966 Kac [56] asked if just by listening with a perfect ear, so by knowing all the fundamental tones λ_n, it is possible to hear the shape of the drum. One problem in answering this question is, of course, that in general it will be impossible to write down the eigenvalues λ_n in a closed form and to read off relations with the shape of the drum directly. Instead one has to organize the spectrum intelligently in form of a spectral function to reveal relationships between the eigenvalues and the shape of the drum. In this context a particularly fruitful spectral function is the heat kernel

$$K(t)=\sum_{n=1}^{\infty}e^{-\lambda_n t},$$

which as t tends to zero clearly diverges. Given that some relations between the fundamental tones and properties of the drum are hidden in the $t\to 0$ behavior let us consider this asymptotic behavior very closely. Before we come back to the setting of the drum, let us use a few examples to get an idea what the structure of the $t\to 0$ behavior of the heat kernel is expected to be.

EXAMPLE 4.1. Let $M=S^1$ be the circle with circumference L and let $P=-\partial^2/\partial x^2$. Imposing periodic boundary conditions eigenvalues are

$$\lambda_k=\left(\frac{2\pi k}{L}\right)^2, \qquad k\in\mathbb{Z},$$

and the heat kernel reads

$$K_{S^1}(t)=\sum_{k=-\infty}^{\infty}e^{-(2\pi k/L)^2 t}.$$

From Lemma 2.16 we find the $t \to 0$ behavior

$$K_{S^1}(t) = \frac{1}{\sqrt{4\pi t}} L + (\text{exponentially damped terms}) .$$

With the obvious notation this could be written as

$$K_{S^1}(t) = \frac{1}{\sqrt{4\pi t}} \operatorname{vol} M + (\text{exponentially damped terms}) .$$

EXAMPLE 4.2. The heat kernel for the d-dimensional torus $M = S^1 \times \cdots \times S^1$ with $P = -\Delta$ clearly gives a product of the above and thus

$$K_M(t) = K_{S^1}(t) \times \cdots \times K_{S^1}(t) = \frac{1}{(4\pi t)^{d/2}} \operatorname{vol} M + \text{e.d.t.}$$

EXAMPLE 4.3. To avoid the impression that there is always just one term that is not exponentially damped consider M as above but $P = -\Delta + m^2$. Then

$$\begin{aligned} K(t) &= e^{-m^2 t} K_M(t) = e^{-m^2 t} \left(\frac{1}{(4\pi t)^{d/2}} \operatorname{vol} M + \text{e.d.t.} \right) \\ &= \frac{1}{(4\pi)^{d/2}} \operatorname{vol} M \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} m^{2\ell} t^{\ell - \frac{d}{2}} + \text{e.d.t.} \end{aligned}$$

In fact, the structure of the heat kernel observed in this last example is the structure observed for the general class of Laplace-type operators.

THEOREM 4.4. *Let M be a d-dimensional smooth compact Riemannian manifold without boundary and let*

$$P = -g^{jk} \nabla_j^V \nabla_k^V - E,$$

where g^{jk} is the metric of M, ∇^V is the connection on M acting on a smooth vector bundle V over M, and where E is an endomorphism of V. Then as $t \to 0$,

$$K(t) \sim \sum_{k=0}^{\infty} a_k \; t^{k - d/2}$$

with the so-called heat kernel coefficients a_k.

PROOF. See, e.g., [44]. □

In Example 4.3 one sees that

$$a_k = \frac{1}{(4\pi)^{d/2}} \frac{(-1)^k}{k!} m^{2k} \operatorname{vol} M.$$

In general, the heat kernel coefficients are significantly more complicated and they depend upon the geometry of the manifold M and the endomorphism E [44].

Up to this point we have only considered manifolds without boundary. In order to consider in more detail questions relating to the drum, let us now see what relevant changes in the structure of the small-t heat kernel expansion occur if boundaries are present.

EXAMPLE 4.5. Let $M = [0, L]$ and $P = -\partial^2/\partial x^2$ with Dirichlet boundary conditions imposed. Normalized eigenfunctions are then given by

$$\varphi_\ell(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi \ell x}{L}$$

and the associated eigenvalues are

$$\lambda_\ell = \left(\frac{\pi \ell}{L}\right)^2, \qquad \ell \in \mathbb{N}.$$

Using Lemma 2.16 this time we obtain

$$K(t) = \frac{1}{\sqrt{4\pi t}} \operatorname{vol} M - \tfrac{1}{2} + (\text{exponentially damped terms}). \tag{4-1}$$

Notice that in contrast to previous results we have integer and half-integer powers in t occurring.

EXERCISE 9. There is a more general version of the Poisson resummation formula than the one given in Lemma 2.16, namely

$$\sum_{\ell=-\infty}^{\infty} e^{-t(\ell+c)^2} = \sqrt{\frac{\pi}{t}} \sum_{\ell=-\infty}^{\infty} e^{-\frac{\pi^2}{t}\ell^2 - 2\pi i \ell c}. \tag{4-2}$$

Apply Exercise 5 with a suitable function $F(x)$ to show equation (4-2).

EXERCISE 10. Consider the setting described in Example 4.5. The local heat kernel is defined as the solution of the equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) K(t, x, y) = 0$$

with the initial condition

$$\lim_{t \to 0} K(t, x, y) = \delta(x, y).$$

In terms of the quantities introduced in Example 4.5 it can be written as

$$K(t, x, y) = \sum_{\ell=1}^{\infty} \varphi_\ell(x) \varphi_\ell(y) e^{-\lambda_\ell t}.$$

Use the resummation (4-2) for $K(t, x, y)$ and the fact that

$$K(t) = \int_0^L K(t, x, x)dx$$

to rediscover the result (4-1).

EXERCISE 11. Let $M = [0, L]$ and

$$P = -\frac{\partial^2}{\partial x^2} + m^2$$

with Dirichlet boundary conditions imposed. Find the small-t asymptotics of the heat kernel.

EXERCISE 12. Let $M = [0, L] \times S^1 \times \cdots \times S^1$ be a d-dimensional manifold and

$$P = -\frac{\partial^2}{\partial x^2} + m^2.$$

Impose Dirichlet boundary conditions on $[0, L]$ and periodic boundary conditions on the circle factors. Find the small-t asymptotics of the heat kernel.

As the examples and exercises above suggest, one has the following result.

THEOREM 4.6. *Let M be a d-dimensional smooth compact Riemannian manifold with smooth boundary and let*

$$P = -g^{jk}\nabla_j^V \nabla_k^V - E,$$

where g^{jk} is the metric of M, ∇^V is the connection on M acting on a smooth vector bundle V over M, and where E is an endomorphism of V. We impose Dirichlet boundary conditions. Then as $t \to 0$,

$$K(t) \sim \sum_{k=0,\frac{1}{2},1,\ldots}^{\infty} a_k \, t^{k-d/2}$$

with the heat kernel coefficients a_k.

PROOF. See, e.g., [44]. □

As for the manifold without boundary case, Theorem 4.4, the heat kernel coefficients depend upon the geometry of the manifold M and the endomorphism E, and in addition on the geometry of the boundary. Note, however, that in contrast to Theorem 4.4 the small-t expansion contains integer and half-integer powers in t.

The same structure of the small-t asymptotics is found for other boundary conditions like Neumann or Robin, see [44], and the coefficients then also depend on the boundary condition chosen. In particular, for Dirichlet boundary conditions one can show the identities

$$a_0 = (4\pi)^{-d/2} \operatorname{vol} M, \qquad a_{1/2} = (4\pi)^{-(d-1)/2} \left(-\tfrac{1}{4}\right) \operatorname{vol} \partial M, \tag{4-3}$$

a result going back to McKean and Singer [66]. In the context of the drum, what the formula shows is that by listening with a perfect ear one can indeed hear certain properties like the area of the drum and the circumference of its boundary. But as has been shown by Gordon, Webb and Wolpert [45], one cannot hear all details of the shape.

EXERCISE 13. Use Exercise 12 to verify the general formulas (4-3) for the heat kernel coefficients.

Instead of using the heat kernel coefficients to make the preceding statements, one could equally well have used zeta function properties for equivalent statements. Consider the setting of Theorem 4.6. The associated zeta function is

$$\zeta_P(s) = \sum_{n=1}^{\infty} \lambda_n^{-s},$$

where it follows from Weyl's law [80; 81] that this series is convergent for $\operatorname{Re} s > d/2$. The zeta function is related with the heat kernel by

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} K(t)\, dt, \tag{4-4}$$

where equation (2-5) has been used. This equation allows us to relate residues and function values at certain points with the small-t behavior of the heat kernel. In detail,

$$\operatorname{Res} \zeta_P(z) = \frac{a_{(d/2)-z}}{\Gamma(z)}, \qquad z = \frac{d}{2}, \frac{d-1}{2}, \ldots, \frac{1}{2}, -\frac{2n+1}{2},\ n \in \mathbb{N}_0, \tag{4-5}$$

$$\zeta_P(-q) = (-1)^q q! a_{\frac{d}{2}+q}, \qquad q \in \mathbb{N}_0. \tag{4-6}$$

Keeping in mind the vanishing of the heat kernel coefficients a_k with half-integer index for $\partial M = \varnothing$, see Theorem 4.4, this means for d even the poles are actually located only at $z = d/2, d/2-1, \ldots, 1$. In addition, for d odd we get $\zeta_P(-q) = 0$ for $q \in \mathbb{N}_0$.

EXERCISE 14. Use Theorem 4.6 and proceed along the lines indicated in the proof of Theorem 2.5 to show equations (4-5) and (4-6).

Going back to the setting of the drum properties of the zeta function relate with the geometry of the surface. In particular, from (4-3) and (4-5) one can show the identities

$$\operatorname{Res}\zeta_P(1)=\frac{\operatorname{vol} M}{4\pi},\qquad \operatorname{Res}\zeta_P\left(\tfrac{1}{2}\right)=-\frac{\operatorname{vol}\partial M}{2\pi},$$

and the remarks below equation (4-3) could be repeated.

4.2. What does the Casimir effect know about a boundary? We next consider an application in the context of quantum field theory in finite systems. The importance of this topic lies in the fact that in recent years, progress in many fields has been triggered by the continuing miniaturization of all kinds of technical devices. As the separation between components of various systems tends towards the nanometer range, there is a growing need to understand every possible detail of quantum effects due to the small sizes involved.

Very generally speaking, effects resulting from the finite extension of systems and from their precise form are known as the Casimir effect. In modern technical devices this effect is responsible for up to 10% of the forces encountered in microelectromechanical systems [19; 20]. Casimir forces are of direct practical relevance in nanotechnology where, e.g., sticking of mobile components in micromachines might be caused by them [76]. Instead of fighting the occurrence of the effect in technological devices, the tendency is now to try and take technological advantage of the effect.

Experimental progress in recent years has been impressive and for some configurations allows for a detailed comparison with theoretical predictions. The best tested situations are those of parallel plates [12] and of a plate and a sphere [20; 21; 62; 63; 69]; recently also a plate and a cylinder has been considered [13; 37]. Experimental data and theoretical predictions are in excellent agreement, see, e.g., [8; 25; 61; 64]. This interplay between theory and experiments, and the intriguing technological applications possible, are the main reasons for the heightened interest in this effect in recent years.

In its original form, the effect refers to the situation of two uncharged, parallel, perfectly conducting plates. As predicted by Casimir [17], the plates should attract with a force per unit area, $F(a)\sim 1/a^4$, where a is the distance between the plates. Two decades later Boyer [10] found a repulsive pressure of magnitude $F(R)\sim 1/R^4$ for a perfectly conducting spherical shell of radius R. Up to this day an intuitive understanding of the opposite signs found is lacking. One of the main questions in the context of the Casimir effect therefore is how the occurring forces depend on the geometrical properties of the system considered. Said differently, the question is "What does the Casimir effect know about a boundary?" In the absence of general answers one approach consists in accu-

mulating further knowledge by adding bits of understanding based on specific calculations for specific configurations. Several examples will be provided in this section and we will see the dominant role the zeta functions introduced play. However, before we come to specific settings let us briefly introduce the zeta function regularization of the Casimir energy and force that we will use later.

We will consider the Casimir effect in a quantum field theory of a non-interacting scalar field under *external* conditions. The action in this case is [55]

$$S[\Phi] = -\frac{1}{2}\int_M \Phi(x)\,(\Delta - V(x))\,\Phi(x)\;dx, \tag{4-7}$$

describing a scalar field $\Phi(x)$ in the background potential $V(x)$. We assume the Riemannian manifold M to be of the form $M = S^1 \times M_s$, where the circle S^1 of radius β is used to describe finite temperature $T = 1/\beta$ and M_s, in general, is a d-dimensional Riemannian manifold with boundary. For the action (4-7) the corresponding field equations are

$$(\Delta - V(x))\Phi(x) = 0. \tag{4-8}$$

If M_s has a boundary ∂M_s, these equations of motion have to be supplemented by boundary conditions on ∂M_s. Along the circle, for a scalar field, periodic boundary conditions are imposed.

Physical properties like the Casimir energy of the system are conveniently described by means of the path-integral functionals

$$Z[V] = \int e^{-S[\Phi]}\;D\Phi, \tag{4-9}$$

where we have neglected an infinite normalization constant, and the functional integral is to be taken over all fields satisfying the boundary conditions. Formally, equation (4-9) is easily evaluated to be

$$\Gamma[V] = -\ln Z[V] = \tfrac{1}{2}\ln\det\big((-\Delta + V(x))/\mu^2\big), \tag{4-10}$$

where μ is an arbitrary parameter with dimension of a mass to adjust the dimension of the arguments of the logarithm.

EXERCISE 15. In order to motivate equation (4-10) show that for P a positive definite Hermitian $(N \times N)$-matrix one has

$$\int_{\mathbb{R}^n} e^{-(x,Px)/2}(dx) = (\det P)^{-1/2},$$

where

$$(dx) = d^n x (2\pi)^{-n/2}.$$

For $P = -\Delta + V(x)$ and interpreting the scalar product (x, Px) as an $L^2(M)$-product, one is led to (4-10) by identifying $D\Phi$ with (dx).

Equation (4-10) is purely formal, because the eigenvalues λ_n of $-\Delta+V(x)$ grow without bound for $n\to\infty$ and thus expression (4-10) needs further explanations.

In order to motivate the basic definition let P be a Hermitian $(N\times N)$-matrix with positive eigenvalues λ_n. Clearly

$$\ln\det P=\sum_{n=1}^{N}\ln\lambda_n=-\frac{d}{ds}\sum_{n=1}^{N}\lambda_n^{-s}\bigg|_{s=0}=-\frac{d}{ds}\zeta_P(s)\bigg|_{s=0},$$

and the determinant of P can be expressed in terms of the zeta function associated with P. This very same definition, namely

$$\ln\det P=-\zeta_P'(0) \tag{4-11}$$

with

$$\zeta_P(s)=\sum_{n=1}^{\infty}\lambda_n^{-s} \tag{4-12}$$

is now applied to differential operators as in (4-10). Here, the series representation is valid for $\operatorname{Re}s$ large enough, and in (4-11) the unique analytical continuation of the series to a neighborhood about $s=0$ is used.

This definition was first used by the mathematicians Ray and Singer [73] to give a definition of the Reidemeister–Franz torsion. In physics, this regularization scheme took its origin in ambiguities of dimensional regularization when applied to quantum field theory in curved spacetime [29; 51]. For applications beyond the ones presented here see, e.g., [14; 15; 26; 30; 31; 41; 42; 74].

The quantity $\Gamma[V]$ is called the effective action and the argument V indicates the dependence of the effective action on the external fields. The Casimir energy is obtained from the effective action via

$$E=\frac{\partial}{\partial\beta}\Gamma[V]=-\frac{1}{2}\frac{\partial}{\partial\beta}\zeta'_{P/\mu^2}(0). \tag{4-13}$$

Here, we will only consider the zero temperature Casimir energy

$$E_{\mathrm{Cas}}=\lim_{\beta\to\infty}E \tag{4-14}$$

and we will next derive a suitable representation for E_{Cas}. We want to concentrate on the influence of boundary conditions and therefore we set $V(x)=0$. The relevant operator to be considered therefore is

$$P=-\frac{\partial^2}{\partial\tau^2}-\Delta_s,$$

where $\tau \in S^1$ is the imaginary time and Δ_s is the Laplace operator on M_s. In order to analyze the zeta function associated with P we note that eigenfunctions, respectively eigenvalues, are of the form

$$\begin{aligned} \phi_{n,j}(\tau, y) &= \frac{1}{\beta} e^{2\pi i n \tau/\beta} \varphi_j(y), \\ \lambda_{n,j} &= \left(\frac{2\pi n}{\beta}\right)^2 + E_j^2, \qquad n \in \mathbb{Z}, \end{aligned}$$

with

$$-\Delta_s \varphi_j(y) = E_j^2 \varphi_j(y),$$

where $y \in M_s$. For the non-self-interacting case considered here, E_j are the one-particle energy eigenvalues of the system. The relevant zeta function therefore has the structure

$$\zeta_P(s) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} \left(\left(\frac{2\pi n}{\beta}\right)^2 + E_j^2 \right)^{-s}. \tag{4-15}$$

We repeat the analysis outlined previously, namely we use equation (2-5) and we apply Lemma 2.16 to the n-summation. In this process the zeta function

$$\zeta_{P_s}(s) = \sum_{j=1}^{\infty} E_j^{-2s}$$

and the heat kernel

$$K_{P_s}(t) = \sum_{j=1}^{\infty} e^{-E_j^2 t} \sim \sum_{k=0,\frac{1}{2},1,\ldots}^{\infty} a_k \; t^{k-(d/2)}$$

of the spatial section are the most natural quantities to represent the answer,

$$\begin{aligned} \zeta_P(s) &= \frac{1}{\Gamma(s)} \sum_{n=-\infty}^{\infty} \int_0^{\infty} t^{s-1} e^{-(2\pi n/\beta)^2 t} K_{P_s}(t)\, dt \\ &= \frac{\beta}{\sqrt{4\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \zeta_{P_s}\left(s-\tfrac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}\,\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-\frac{3}{2}} e^{-\frac{n^2\beta^2}{4t}} K_{P_s}(t)\, dt. \end{aligned}$$

For the Casimir energy we need ($D = d + 1$)

$$\begin{aligned}\zeta'_{P/\mu^2}(0) &= \zeta'_P(0) + \zeta_P(0)\ln\mu^2 \\ &= -\beta\Big(FP\ \zeta_{P_s}\big(-\tfrac{1}{2}\big) + 2(1-\ln 2)\operatorname{Res}\zeta_{P_s}\big(-\tfrac{1}{2}\big) - \frac{1}{\beta}\zeta_P(0)\ln\mu^2\Big) \\ &\quad + \frac{\beta}{\sqrt{\pi}}\sum_{n=1}^{\infty}\int_0^{\infty} t^{-3/2} e^{-\left(\frac{n^2\beta^2}{4t}\right)} K_{P_s}(t)\,dt \\ &= -\beta\left(FP\ \zeta_{P_s}\big(-\tfrac{1}{2}\big) - \frac{1}{\sqrt{4\pi}} a_{D/2}\big((\ln\mu^2) + 2(1-\ln 2)\big)\right) \\ &\quad + \frac{\beta}{\sqrt{\pi}}\sum_{n=1}^{\infty}\int_0^{\infty} t^{-3/2} e^{-\frac{n^2\beta^2}{4t}} K_{P_s}(t)\,dt, \end{aligned} \tag{4-16}$$

with the finite part FP of the zeta function and where equations (4-5) and (4-6) together with the fact that

$$K_M(t) = K_{S^1}(t)\ K_{P_s}(t)$$

have been used, in particular

$$\operatorname{Res}\zeta_{P_s}\big(-\tfrac{1}{2}\big) = -\frac{a_{D/2}}{2\sqrt{\pi}}, \qquad \zeta_P(0) = \frac{\beta}{\sqrt{4\pi}} a_{D/2}. \tag{4-17}$$

At $T = 0$ we obtain for the Casimir energy, see equations (4-13) and (4-14),

$$E_{\text{Cas}} = \lim_{\beta\to\infty} E = \frac{1}{2} FP\ \zeta_{P_s}\big(-\tfrac{1}{2}\big) - \frac{1}{2\sqrt{4\pi}} a_{D/2}\ln\tilde{\mu}^2, \tag{4-18}$$

with the scale $\tilde{\mu} = (\mu e/2)$. Equation (4-18) implies that as long as $a_{D/2} \neq 0$ the Casimir energy contains a finite ambiguity and renormalization issues need to be discussed. Note from (4-17) that whenever $\zeta_{P_s}\big(-\frac{1}{2}\big)$ is finite no ambiguity exists because $a_{D/2} = 0$. In the specific examples chosen later we will make sure that these ambiguities are absent and therefore a discussion of renormalization will be unnecessary.

In a purely formal calculation one essentially is also led to equation (4-18). As mentioned, in the quantum field theory of a free scalar field the eigenvalues of a Laplacian are the square of the energies of the quantum fluctuations. Writing the Casimir energy as (one-half) the sum over the energy of all quantum fluctuations one has

$$E_{\text{Cas}} = \frac{1}{2}\sum_{k=0}^{\infty} \lambda_k^{1/2}, \tag{4-19}$$

and a formal identification "shows" that

$$E_{\text{Cas}} = \tfrac{1}{2}\zeta_{P_s}\big(-\tfrac{1}{2}\big). \tag{4-20}$$

Clearly, the expression (4-19) is purely formal as the series diverges. However, when $\zeta_{P_s}\left(-\frac{1}{2}\right)$ turns out to be finite this formal identification yields the correct result. Otherwise, the ambiguities given in (4-18) remain as discussed above.

An alternative discussion leading to definition (4-18) can be found in [7].

As a first example let us consider the configuration of two parallel plates a distance a apart analyzed originally by Casimir [17]. For simplicity we concentrate on a scalar field instead of the electromagnetic field and we impose Dirichlet boundary conditions on the plates. The boundary value problem to be solved therefore is

$$-\Delta u_k(x,y,z) = \lambda_k u_k(x,y,z),$$

with $u_k(0,y,z) = u_k(a,y,z) = 0$.

For the time being, we compactify the (y,z)-directions to a torus with perimeter length R and impose periodic boundary conditions in these directions. Later on, the limit $R \to \infty$ is performed to recover the parallel plate configuration. Using separation of variables one obtains normalized eigenfunctions in the form

$$u_{\ell_1\ell_2\ell}(x,y,z) = \sqrt{\frac{2}{aR^2}}\,\sin\frac{\pi\ell x}{a}\,e^{i2\pi\ell_1 y/R}e^{i2\pi\ell_2 z/R}$$

with eigenvalues

$$\lambda_{\ell_1\ell_2\ell} = \left(\frac{2\pi\ell_1}{R}\right)^2 + \left(\frac{2\pi\ell_2}{R}\right)^2 + \left(\frac{\pi\ell}{a}\right)^2, \quad (\ell_1,\ell_2)\in\mathbb{Z}^2, \quad \ell\in\mathbb{N}.$$

This means we have to study the zeta function

$$\zeta(s) = \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2}\sum_{\ell=1}^{\infty}\left(\left(\frac{2\pi\ell_1}{R}\right)^2 + \left(\frac{2\pi\ell_2}{R}\right)^2 + \left(\frac{\pi\ell}{a}\right)^2\right)^{-s}. \qquad (4\text{-}21)$$

As $R\to\infty$ the Riemann sum turns into an integral and we compute using polar coordinates in the (y,z)-plane

$$\begin{aligned}
\zeta(s) &= \left(\frac{R}{2\pi}\right)^2\sum_{\ell=1}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(k_1^2+k_2^2+\left(\frac{\pi\ell}{a}\right)^2\right)^{-s}dk_2\,dk_1\\
&= \left(\frac{R}{2\pi}\right)^2\sum_{\ell=1}^{\infty}2\pi\int_0^{\infty}k\left(k^2+\left(\frac{\pi\ell}{a}\right)^2\right)^{-s}dk\\
&= \frac{R^2}{2\pi}\frac{1}{2(1-s)}\sum_{\ell=1}^{\infty}\left(k^2+\left(\frac{\pi\ell}{a}\right)^2\right)^{-s+1}\Big|_0^{\infty}\\
&= -\frac{R^2}{4\pi(1-s)}\sum_{\ell=1}^{\infty}\left(\frac{\pi\ell}{a}\right)^{2(-s+1)} = -\frac{R^2}{4\pi(1-s)}\left(\frac{\pi}{a}\right)^{2-2s}\zeta_R(2s-2).
\end{aligned}$$

Setting $s = -\frac{1}{2}$ as needed for the Casimir energy we obtain

$$\zeta\left(-\tfrac{1}{2}\right) = -\frac{R^2}{4\pi}\ \frac{2}{3}\ \left(\frac{\pi}{a}\right)^3 \zeta_R(-3) = -\frac{R^2\pi^2}{720a^3}. \tag{4-22}$$

The resulting Casimir force *per area* is

$$F_{\text{Cas}} = -\frac{\partial}{\partial a}\frac{E_{\text{Cas}}}{R^2} = -\frac{\pi^2}{480a^4}. \tag{4-23}$$

Note that this computation takes into account only those quantum fluctuations from between the plates. But in order to find the force acting on the, say, right plate the contribution from the right to this plate also has to be counted. To find this part we place another plate at the position $x = L$ where at the end we take $L \to \infty$. Following the preceding calculation, we simply have to replace a by $L - a$ to see that the associated zeta function produces

$$\zeta\left(-\tfrac{1}{2}\right) = -\frac{R^2\pi^2}{720(L-a)^3}$$

and the contribution to the force on the plate at $x = a$ reads

$$F_{\text{Cas}} = \frac{\pi^2}{480(L-a)^4}.$$

This shows the plate at $x = a$ is always attracted to the closer plate. As $L \to \infty$ it is seen that equation (4-23) also describes the total force on the plate at $x = a$ for the parallel plate configuration.

EXERCISE 16. Consider the Casimir energy that results in the previous discussion when the compactification length R is kept finite. Use Lemma 2.18 to give closed answers for the energy and the resulting force. Can the force change sign depending on a and R?

More realistically plates will have a finite extension. An interesting setting that we are able to analyze with the tools provided are pistons. These have received an increasing amount of interest because they allow the unambiguous prediction of forces [18; 52; 58; 65; 77].

Instead of having parallel plates let us consider a box with side lengths L_1, L_2 and L_3. Although it is possible to find the Casimir force acting on the plate at $x = L_1$ resulting from the interior of the box, the exterior problem has remained unsolved until today. No analytical procedure is known that allows to obtain the Casimir energy or force for the outside of the box. This problem is avoided by adding on another box with side lengths $L - L_1, L_2$ and L_3 such that the wall at $x = L_1$ subdivides the bigger box into two chambers. The wall at $x = L_1$ is assumed to be movable and is called the piston. Each chamber can be dealt

with separately and total energies and forces are obtained by adding up the two contributions. Assuming again Dirichlet boundary conditions and starting with the left chamber, the relevant spectrum reads

$$\lambda_{\ell_1\ell_2\ell_3} = \left(\frac{\pi\ell_1}{L_1}\right)^2 + \left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2, \quad \ell_1,\ell_2,\ell_3 \in \mathbb{N}, \tag{4-24}$$

and the associated zeta function is

$$\zeta(s) = \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}} \left(\left(\frac{\pi\ell_1}{L_1}\right)^2 + \left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2\right)^{-s}. \tag{4-25}$$

One way to proceed is to rewrite (4-25) in terms of the Epstein zeta function in Definition 2.14.

EXERCISE 17. Use Lemma 2.18 in order to find the Casimir energy for the inside of the box with side lengths L_1, L_2 and L_3 and with Dirichlet boundary conditions imposed.

Instead of using Lemma 2.18 we proceed as follows. We write first

$$\zeta(s) = \frac{1}{2}\sum_{\ell_1=-\infty}^{\infty}\sum_{\ell_2,\ell_3=1}^{\infty}\left(\left(\frac{\pi\ell_1}{L_1}\right)^2 + \left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2\right)^{-s} - \frac{1}{2}\sum_{\ell_2,\ell_3=1}^{\infty}\left(\left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2\right)^{-s}. \tag{4-26}$$

This shows that it is convenient to introduce

$$\zeta_{\mathcal{C}}(s) = \sum_{\ell_2,\ell_3=1}^{\infty}\left(\left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2\right)^{-s}. \tag{4-27}$$

We note that this could be expressed in terms of the Epstein zeta function given in Definition 2.14. However, it will turn out that this is unnecessary.

Also, to simplify the notation let us introduce

$$\mu^2_{\ell_2\ell_3} = \left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2.$$

Using equation (2-5) for the first line in (4-26) we continue

$$\zeta(s) = \frac{1}{2\Gamma(s)}\sum_{\ell_1=-\infty}^{\infty}\sum_{\ell_2,\ell_3=1}^{\infty}\int_0^\infty t^{s-1}\exp\left(-t\left(\left(\frac{\pi\ell_1}{L_1}\right)^2 + \mu^2_{\ell_2\ell_3}\right)\right)dt - \frac{1}{2}\zeta_{\mathcal{C}}(s).$$

We now apply the Poisson resummation in Lemma 2.16 to the ℓ_1-summation and therefore we get

$$\zeta(s) = \frac{L_1}{2\sqrt{\pi}\ \Gamma(s)} \sum_{\ell_1=-\infty}^{\infty} \sum_{\ell_2,\ell_3=1}^{\infty} \int_0^\infty t^{s-3/2} \exp\Big(-\frac{L_1^2\ell_1^2}{t} - t\mu_{\ell_2\ell_3}^2\Big)\, dt \\ -\tfrac{1}{2}\zeta_{\mathcal{C}}(s). \quad (4\text{-}28)$$

The $\ell_1 = 0$ term gives a $\zeta_{\mathcal{C}}$-term, the $\ell_1 \neq 0$ terms are rewritten using (2.17). The outcome reads

$$\zeta(s) = \frac{L_1\Gamma\big(s-\frac{1}{2}\big)}{2\sqrt{\pi}\ \Gamma(s)}\zeta_{\mathcal{C}}\big(s-\tfrac{1}{2}\big) - \tfrac{1}{2}\zeta_{\mathcal{C}}(s) \\ + \frac{2L_1^{s+\frac{1}{2}}}{\sqrt{\pi}\ \Gamma(s)} \sum_{\ell_1,\ell_2,\ell_3=1}^{\infty} \Big(\frac{\ell_1^2}{\mu_{\ell_2\ell_3}^2}\Big)^{\frac{1}{2}(s-\frac{1}{2})} K_{\frac{1}{2}-s}(2L_1\ell_1\mu_{\ell_2\ell_3}). \quad (4\text{-}29)$$

We need the zeta function about $s = -\frac{1}{2}$ in order to find the Casimir energy and Casimir force.

Let $s = -\frac{1}{2} + \varepsilon$. In order to expand equation (4-29) about $\varepsilon = 0$ we need to know the pole structure of $\zeta_{\mathcal{C}}(s)$. From equation (2.18) it is expected that $\zeta_{\mathcal{C}}(s)$ has at most a first order pole at $s = -\frac{1}{2}$ and that it is analytic about $s = -1$. So for now let us simply assume the structure

$$\zeta_{\mathcal{C}}\big(-\tfrac{1}{2}+\varepsilon\big) = \frac{1}{\varepsilon}\operatorname{Res}\zeta_{\mathcal{C}}\big(-\tfrac{1}{2}\big) + \operatorname{FP}\zeta_{\mathcal{C}}\big(-\tfrac{1}{2}\big) + \mathcal{O}(\varepsilon),$$
$$\zeta_{\mathcal{C}}(-1+\varepsilon) = \zeta_{\mathcal{C}}(-1) + \varepsilon\zeta_{\mathcal{C}}'(-1) + \mathcal{O}(\varepsilon^2),$$

where $\operatorname{Res}\zeta_{\mathcal{C}}\big(-\frac{1}{2}\big)$ and $\operatorname{FP}\zeta_{\mathcal{C}}\big(-\frac{1}{2}\big)$ will be determined later. With this structure assumed, we find

$$\zeta\big(-\tfrac{1}{2}+\varepsilon\big) = \frac{1}{\varepsilon}\Big(\frac{L_1}{4\pi}\zeta_{\mathcal{C}}(-1) - \tfrac{1}{2}\operatorname{Res}\zeta_{\mathcal{C}}\big(-\tfrac{1}{2}\big)\Big) \\ + \frac{L_1}{4\pi}\big(\zeta_{\mathcal{C}}'(-1) + \zeta_{\mathcal{C}}(-1)(\ln 4 - 1)\big) - \tfrac{1}{2}\operatorname{FP}\zeta_{\mathcal{C}}\big(-\tfrac{1}{2}\big) \\ - \frac{1}{\pi}\sum_{\ell_1,\ell_2,\ell_3=1}^{\infty}\left|\frac{\mu_{\ell_2\ell_3}}{\ell_1}\right| K_1\left(2L_1\ell_1\mu_{\ell_2\ell_3}\right). \quad (4\text{-}30)$$

This shows that the Casimir energy for this setting is unambiguously defined only if $\zeta_{\mathcal{C}}(-1) = 0$ and $\operatorname{Res}\zeta_{\mathcal{C}}\big(-\frac{1}{2}\big) = 0$.

EXERCISE 18. Show the following analytical continuation for $\zeta_{\mathcal{C}}(s)$:

$$\zeta_{\mathcal{C}}(s) = -\frac{1}{2}\left(\frac{L_3}{\pi}\right)^{2s}\zeta_R(2s) + \frac{L_2\Gamma\left(s-\frac{1}{2}\right)}{2\sqrt{\pi}\ \Gamma(s)}\left(\frac{L_3}{\pi}\right)^{2s-1}\zeta_R(2s-1) \qquad (4\text{-}31)$$
$$+\frac{2L_2^{s+1/2}}{\sqrt{\pi}\ \Gamma(s)}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=1}^{\infty}\left(\frac{\ell_2 L_3}{\pi\ell_3}\right)^{s-1/2}K_{\frac{1}{2}-s}\left(\frac{2\pi L_2\ell_2\ell_3}{L_3}\right).$$

Read off that $\zeta_{\mathcal{C}}(-1) = \operatorname{Res}\zeta_{\mathcal{C}}\left(-\frac{1}{2}\right) = 0$.

Using the results from Exercise 18 the Casimir energy, from equation (4-30), can be expressed as

$$E_{\text{Cas}} = \frac{L_1}{8\pi}\zeta'_{\mathcal{C}}(-1) - \tfrac{1}{4}\text{FP}\ \zeta_{\mathcal{C}}\left(-\tfrac{1}{2}\right) - \frac{1}{2\pi}\sum_{\ell_1,\ell_2,\ell_3=1}^{\infty}\left|\frac{\mu_{\ell_2\ell_3}}{\ell_1}\right|K_1(2L_1\ell_1\mu_{\ell_2\ell_3}). \qquad (4\text{-}32)$$

EXERCISE 19. Use representation (4-31) to give an explicit representation of the Casimir energy (4-32).

For the force this shows

$$F_{\text{Cas}} = -\frac{1}{8\pi}\zeta'_{\mathcal{C}}(-1) + \frac{1}{2\pi}\sum_{\ell_1,\ell_2,\ell_3=1}^{\infty}\left|\frac{\mu_{\ell_2\ell_3}}{\ell_1}\right|\frac{\partial}{\partial L_1}K_1(2L_1\ell_1\mu_{\ell_2\ell_3}). \qquad (4\text{-}33)$$

EXERCISE 20. Use Definition 2.17 to show that $K_\nu(x)$ is a monotonically decreasing function for $x \in \mathbb{R}_+$.

EXERCISE 21. Determine the sign of $\zeta'_{\mathcal{C}}(-1)$. What is the sign of the Casimir force as $L_1 \to \infty$? What about $L_1 \to 0$?

Remember that the results given describe the contributions from the interior of the box only. The contributions from the right chamber are obtained by replacing L_1 with $L - L_1$. This shows for the right chamber

$$E_{\text{Cas}} = \frac{L-L_1}{8\pi}\zeta'_{\mathcal{C}}(-1) - \tfrac{1}{4}\text{FP}\ \zeta_{\mathcal{C}}\left(-\tfrac{1}{2}\right)$$
$$-\frac{1}{2\pi}\sum_{\ell_1,\ell_2,\ell_3=1}^{\infty}\left|\frac{\mu_{\ell_2\ell_3}}{\ell_1}\right|K_1(2(L-L_1)\ell_1\mu_{\ell_2\ell_3}),$$
$$F_{\text{Cas}} = \frac{1}{8\pi}\zeta'_{\mathcal{C}}(-1) + \frac{1}{2\pi}\sum_{\ell_1,\ell_2,\ell_3=1}^{\infty}\left|\frac{\mu_{\ell_2\ell_3}}{\ell_1}\right|\frac{\partial}{\partial L_1}K_1(2(L-L_1)\ell_1\mu_{\ell_2\ell_3}).$$

Adding up, the total force on the piston is

$$F_{\text{Cas}}^{\text{tot}} = \frac{1}{2\pi} \sum_{\ell_1,\ell_2,\ell_3=1}^{\infty} \left| \frac{\mu_{\ell_2\ell_3}}{\ell_1} \right| \frac{\partial}{\partial L_1} K_1(2L_1\ell_1\mu_{\ell_2\ell_3}) + \frac{1}{2\pi} \sum_{\ell_1,\ell_2,\ell_3=1}^{\infty} \left| \frac{\mu_{\ell_2\ell_3}}{\ell_1} \right| \frac{\partial}{\partial L_1} K_1(2(L-L_1)\ell_1\mu_{\ell_2\ell_3}). \quad (4\text{-}34)$$

This shows, using the results of Exercise 20, that the piston is always attracted to the closer wall.

Although we have presented the analysis for a piston with rectangular cross-section, our result in fact holds in much greater generality. The fact that we analyzed a rectangular cross-section manifests itself in the spectrum (4-24), namely the part

$$\left(\frac{\pi\ell_2}{L_2}\right)^2 + \left(\frac{\pi\ell_3}{L_3}\right)^2$$

is a direct consequence of it. If instead we had considered an arbitrary cross-section $\mathcal{C}$, the relevant spectrum had the form

$$\lambda_{\ell_1 i} = \left(\frac{\pi\ell_1}{L_1}\right)^2 + \mu_i^2,$$

where, assuming still Dirichlet boundary conditions on the boundary of the cross-section $\mathcal{C}$, μ_i^2 is determined from

$$-\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi_i(y,z) = \mu_i^2\phi_i(y,z), \qquad \phi_i(y,z)\Big|_{(y,z)\in\partial\mathcal{C}} = 0.$$

Proceeding in the same way as before, replacing $\mu_{\ell_2\ell_3}$ with μ_i and introducing $\zeta_{\mathcal{C}}(s)$ as the zeta function for the cross-section,

$$\zeta_{\mathcal{C}}(s) = \sum_{i=1}^{\infty} \mu_i^{-2s},$$

equation (4-28) remains valid, as well as equations (4-29) and (4-30). So also for an arbitrary cross-section the total force on the piston is described by equation (4-34) with the replacements given and the piston is attracted to the closest wall.

EXERCISE 22. In going from equation (4-28) to (4-29) we used the fact that $\mu_{\ell_2\ell_3}^2 > 0$. Above we used $\mu_i^2 > 0$ which is true because we imposed Dirichlet boundary conditions. Modify the calculation if boundary conditions are chosen (like Neumann boundary conditions) that allow for d_0 zero modes $\mu_i^2 = 0$ [58].

We have presented the piston set-up for three spatial dimensions, but a similar analysis can be performed in the presence of extra dimensions [58]. Once this kind of calculation is fully understood for the electromagnetic field it is hoped that future high-precision measurements of Casimir forces for simple configurations such as parallel plates can serve as a window into properties of the dimensions of the universe that are somewhat hidden from direct observations.

As we have seen for the example of the piston, there are cases where an unambiguous prediction of Casimir forces is possible. Of course the set-up we have chosen was relatively simple and for many other configurations even the sign of Casimir forces is unknown. This is a very active field of research; some references are [8; 36; 43; 67; 68; 75]. Further discussion is provided in the Conclusions.

5. Bose–Einstein condensation of Bose gases in traps

We now turn to applications in statistical mechanics. We have chosen to apply the techniques in a quantum mechanical system described by the Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\Delta + V(x,y,z)\right)\phi_k(x,y,z) = \lambda_k \phi_k(x,y,z), \tag{5-1}$$

that is we consider a gas of quantum particles of mass m under the influence of the potential $V(x,y,z)$. Specifically, later we will consider in detail the harmonic oscillator potential

$$V(x,y,z) = \frac{m}{2}(\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2)$$

briefly mentioned in Example 3.4, as well as a gas confined in a finite cavity.

Thermodynamic properties of a *Bose gas*, which is what we shall consider in the following, are described by the (grand canonical) partition sum

$$q = -\sum_{k=0}^{\infty} \ln\bigl(1 - e^{-\beta(\lambda_k - \mu)}\bigr), \tag{5-2}$$

where β is the inverse temperature and μ is the chemical potential. We assume the index $k = 0$ labels the unique ground state, that is, the state with smallest energy eigenvalue λ_0. From this partition sum all thermodynamical properties are obtained. For example the particle number is

$$N = \frac{1}{\beta}\frac{\partial q}{\partial \mu}\bigg|_{T,V} = \sum_{k=0}^{\infty} \frac{1}{e^{\beta(\lambda_k - \mu)} - 1}, \tag{5-3}$$

where the notation $(\partial q/\partial \mu|_{T,V})$ indicates that the derivative has to be taken with temperature T and volume V kept fixed. The particle number is the most important quantity for the phenomenon of Bose–Einstein condensation. Although this phenomenon was predicted more than 80 years ago [9; 32] it was only relatively recently experimentally verified [2; 11; 24]. Bose–Einstein condensation is one of the most interesting properties of a system of bosons. Namely, under certain conditions it is possible to have a phase transition at a critical value of the temperature in which all of the bosons can condense into the ground state. In order to understand at which temperature the phenomenon occurs a detailed study of N, or alternatively q, is warranted. This is the subject of this section.

We first note that from the fact that the particle number in each state has to be non-negative it is clear that $\mu < \lambda_0$ has to be imposed. It is seen in (5-2) that as $\beta \to 0$ (high temperature limit) the behavior of q cannot be easily understood. But contour integral techniques together with the zeta function information provided makes the analysis feasible and it will allow for the determination of the critical temperature of the Bose gas.

Let us start by noting that from

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad \text{for } |x| < 1,$$

the partition sum can be rewritten as

$$q = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n} e^{-\beta(\lambda_k - \mu)n}. \tag{5-4}$$

The $\beta \to 0$ behavior is best found using the following representation of the exponential.

EXERCISE 23. Given that

$$\lim_{|y|\to\infty} |\Gamma(x+iy)|\ e^{\frac{\pi}{2}|y|}\ |y|^{\frac{1}{2}-x} = \sqrt{2\pi}, \quad x, y \in \mathbb{R},$$

and

$$\Gamma(z) = \sqrt{2\pi} e^{\left(z-\frac{1}{2}\right)\log z - z} \left(1 + o(1)\right),$$

as $|z| \to \infty$, show that

$$e^{-a} = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} a^{-t}\ \Gamma(t)\ dt, \tag{5-5}$$

valid for $\sigma > 0$, $|\arg\ a| < \dfrac{\pi}{2} - \delta$, $0 < \delta \leq \pi/2$.

Before we apply this result to the partition sum (5-4) let us use a simple example to show how this formula allows us to determine asymptotic behavior of certain series in a relatively straightforward fashion. From Lemma 2.16 we know that

$$\begin{aligned}\sum_{\ell=1}^{\infty} e^{-\beta\ell^2} &= \frac{1}{2}\sum_{\ell=-\infty}^{\infty} e^{-\beta\ell^2} - \frac{1}{2} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\beta}} - \frac{1}{2} + \sqrt{\frac{\pi}{\beta}}\sum_{\ell=1}^{\infty} e^{-\frac{\pi^2}{\beta}\ell^2}. \end{aligned} \tag{5-6}$$

As $\beta \to 0$ it is clear that the series on the left diverges and Lemma 2.16 shows that the leading behavior is described by a $1/\sqrt{\beta}$ term, followed by a constant term, followed by exponentially damped corrections. Let us see how we can easily find the polynomial behavior as $\beta \to 0$ from (5-5). We first write

$$\sum_{\ell=1}^{\infty} e^{-\beta\ell^2} = \sum_{\ell=1}^{\infty} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} (\beta\ell^2)^{-t}\,\Gamma(t)\,dt.$$

Here, $\sigma > 0$ is assumed by Exercise 23. However, in order to be allowed to interchange summation and integration we need to impose $\sigma > \frac{1}{2}$ and find

$$\begin{aligned}\sum_{\ell=1}^{\infty} e^{-\beta\ell^2} &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \beta^{-t}\,\Gamma(t)\sum_{\ell=1}^{\infty}\ell^{-2t}\,dt \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \beta^{-t}\,\Gamma(t)\zeta_R(2t)\,dt.\end{aligned}$$

In order to find the small-β behavior, the strategy now is to shift the contour to the left. In doing so we cross over poles of the integrand generating polynomial contributions in β. For this example, the right most pole is at $t = \frac{1}{2}$ (pole of the zeta function of Riemann) and the next pole is at $t = 0$ (from the gamma function). Those are all singularities present as $\zeta_R(-2n) = 0$ for $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\sum_{\ell=1}^{\infty} e^{-\beta\ell^2} &= \beta^{-1/2}\ \Gamma\left(\tfrac{1}{2}\right)\tfrac{1}{2} + \beta^0\cdot 1\cdot\zeta_R(0) + \frac{1}{2\pi i}\int_{\tilde{\sigma}-i\infty}^{\tilde{\sigma}+i\infty}\beta^{-t}\,\Gamma(t)\zeta_R(2t)\,dt \\ &= \frac{1}{2}\sqrt{\frac{\pi}{\beta}} - \frac{1}{2} + \frac{1}{2\pi i}\int_{\tilde{\sigma}-i\infty}^{\tilde{\sigma}+i\infty}\beta^{-t}\,\Gamma(t)\zeta_R(2t)\,dt,\end{aligned}$$

where $\tilde{\sigma} < 0$ and where contributions from the horizontal lines between $\tilde{\sigma} \pm i\infty$ and $\sigma \pm i\infty$ are neglected. For the remaining contour integral plus the neglected horizontal lines one can actually show that they will produce the exponentially

damped terms as given in (5-6). How exactly this actually happens has been described in detail in [35].

EXERCISE 24. Argue how $\sum_{n=1}^{\infty} e^{-\beta n^{\alpha}}$, $\beta > 0, \alpha > 0$, behaves as $\beta \to 0$ by using the procedure above. Determine the leading three terms in the expansion assuming that the contributions from the contour at infinity can be neglected.

EXERCISE 25. Find the leading three terms of the small-β behavior of

$$\sum_{n=1}^{\infty} \log(1 - e^{-\beta n})$$

assuming that the contributions from the contour at infinity can be neglected.

We next apply these ideas to the partition sum (5-4). As a further warmup, for simplicity, let us first set $\mu = 0$. Not specifying λ_k for now and using

$$\zeta(s) = \sum_{k=0}^{\infty} \lambda_k^{-s}$$

for $\operatorname{Re} s > M$ large enough to make this series convergent, we write

$$\begin{aligned} q &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n} e^{-\beta \lambda_k n} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} (\beta \lambda_k n)^{-t} \Gamma(t)\, dt \\ &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \beta^{-t} \Gamma(t) \Big(\sum_{n=1}^{\infty} n^{-t-1} \Big) \Big(\sum_{k=0}^{\infty} \lambda_k^{-t} \Big)\, dt \\ &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \beta^{-t} \Gamma(t) \zeta_R(t+1) \zeta(t)\, dt. \end{aligned}$$

Here $\sigma > M$ is needed for the algebraic manipulations to be allowed. It is clearly seen that the integrand has a double pole at $t = 0$. The right most pole (at M) therefore comes from $\zeta(t)$, and the location of this pole determines the leading $\beta \to 0$ behavior of the partition sum.

For the harmonic oscillator potential, in the notation of Example 3.4, the Barnes zeta function occurs and we have

$$q = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \beta^{-t} \Gamma(t) \zeta_R(t+1) \zeta_{\mathcal{B}}(t, c|\vec{r})\, dt. \tag{5-7}$$

The location of the poles and its residues are known for the Barnes zeta function, see Definition 2.12 and Theorem 2.13, in particular one has

$$\operatorname{Res} \zeta_{\mathcal{B}}(3, c|\vec{r}) = \frac{1}{2\hbar^3 \Omega^3},$$

where, as is common, the geometric mean of the oscillator frequencies

$$\Omega = (\omega_1\omega_2\omega_3)^{1/3}$$

has been used. The leading order of the partition sum therefore is

$$q = \frac{\pi^4}{90}\frac{1}{(\beta\hbar\Omega)^3} + \mathcal{O}(\beta^{-2}).$$

EXERCISE 26. Use Definition 2.12 and Theorem 2.13 to find the subleading order of the small-β expansion of the partition sum q.

EXERCISE 27. Consider the harmonic oscillator potential in d dimensions and find the leading and subleading order of the small-β expansion of the partition sum q.

If instead of considering a Bose gas in a trap we consider the gas in a finite three-dimensional cavity M with boundary ∂M we have to augment the Schrödinger equation (5-1) by boundary conditions. We choose Dirichlet boundary conditions and thus the results for the heat kernel coefficients (4-3) are valid.

From equation (4-5) we also conclude that the rightmost pole of $\zeta(s)$ is located at $s = 3/2$ and that

$$\operatorname{Res}\zeta\left(\tfrac{3}{2}\right) = \frac{a_0}{\Gamma\left(\frac{3}{2}\right)} = \frac{\operatorname{vol} M}{4\pi^2};$$

furthermore the next pole is located at $s = 1$. For this case, the leading order of the partition sum therefore is

$$q = \frac{1}{(4\pi\beta)^{3/2}}\zeta_R\left(\tfrac{5}{2}\right)\operatorname{vol} M + \mathcal{O}(\beta^{-1}).$$

One way to read this result is that the Bose gas does know the volume of its container because it can be found from the partition sum. This is completely analogous to the statement for the drum where we used the heat kernel instead of the partition sum.

Subleading orders of the partition sum reveal more information about the cavity, see the following exercise. But as for the drums, the gas does not know all the details of the shape of the cavity because there are different cavities leading to the same eigenvalue spectrum [45]. Those cavities cannot be distinguished by the above analysis.

EXERCISE 28. Consider a Bose gas in a d-dimensional cavity M with boundary ∂M. Use (4-3) and (4-5) to find the leading and subleading order of the small-β expansion of the partition sum q. What does the Bose gas know about its container, meaning what information about the container can be read of from the high-temperature behavior of the partition sum?

In order to examine the phenomenon of Bose–Einstein condensation we have to consider non-vanishing chemical potential. Close to the phase transition, as we will see, more and more particles have to reside in the ground state and the value of the chemical potential will be close to the smallest eigenvalue, which is the 'critical' value for the chemical potential, $\mu_c = \lambda_0$. Near the phase transition, for the expansion to be established, it will turn out advantageous to rewrite $\lambda_k - \mu$ such that the small quantity $\mu_c - \mu$ appears,

$$\lambda_k - \mu = \lambda_k - \mu_c + \mu_c - \mu = \lambda_k - \lambda_0 + \mu_c - \mu.$$

Given the special role of the ground state, we separate off its contribution and write

$$q = q_0 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n} e^{-\beta n(\lambda_k - \lambda_0)} \; e^{-\beta n(\mu_c - \mu)}.$$

Note that the k-sum starts with $k = 1$, which means that the ground state is not included in this summation. Employing the representation (5-5) only to the first exponential factor and proceeding as before we obtain

$$q = q_0 + \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \beta^{-t} \Gamma(t) \mathrm{Li}_{1+t}\big(e^{-\beta(\mu_c - \mu)}\big) \zeta_0(t)\, dt, \tag{5-8}$$

with the polylogarithm

$$\mathrm{Li}_n(x) = \sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell^n}, \tag{5-9}$$

and the spectral zeta function

$$\zeta_0(s) = \sum_{k=1}^{\infty} (\lambda_k - \lambda_0)^{-s}.$$

In order to determine the small-β behavior of expression (5-8) let us discuss the pole structure of the integrand. Given $\mu_c - \mu > 0$, the polylogarithm $\mathrm{Li}_{1+t}(e^{-\beta(\mu_c - \mu)})$ does not generate any poles. Concentrating on the harmonic oscillator, we find

$$\mathrm{Res}\, \zeta_0(3) = \frac{1}{2(\hbar\Omega)^3}, \quad \mathrm{Res}\, \zeta_0(2) = \frac{1}{2\hbar^2} \left(\frac{1}{\omega_1\omega_2} + \frac{1}{\omega_1\omega_3} + \frac{1}{\omega_2\omega_3} \right).$$

Note that $\zeta_0(s)$ is the Barnes zeta function as given in Definition 2.8 with $c = 0$ where we have to exclude $\vec{m} = \vec{0}$ from the summation. However, clearly the residues at $s = 3$ and $s = 2$ can still be obtained from Theorem 2.13 with $c \to 0$ taken.

Shifting the contour to the left we now find

$$q = q_0 + \frac{1}{(\beta\hbar\Omega)^3}\mathrm{Li}_4\left(e^{-\beta(\mu_c-\mu)}\right) + \frac{1}{2(\beta\hbar)^2}\mathrm{Li}_3\left(e^{-\beta(\mu_c-\mu)}\right)\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_2\omega_3}\right)+\cdots$$

In order to find the particle number N we need the relation for the polylogarithm

$$\frac{\partial \mathrm{Li}_n(x)}{\partial x} = \frac{1}{x}\mathrm{Li}_{n-1}(x),$$

which follows from (5-9). So

$$N = N_0 + \frac{1}{(\beta\hbar\Omega)^3}\mathrm{Li}_3\left(e^{-\beta(\mu_c-\mu)}\right) + \frac{1}{2(\beta\hbar)^2}\mathrm{Li}_2\left(e^{-\beta(\mu_c-\mu)}\right)\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_2\omega_3}\right)+\cdots$$

EXERCISE 29. Use (5-5) and (5-9) to show

$$\mathrm{Li}_n\left(e^{-x}\right) = \zeta_R(n) - x\zeta_R(n-1)+\cdots$$

valid for $n > 2$. What does the subleading term look like for $n = 2$?

As the critical temperature is approached $\mu \to \mu_c$ and with Exercise 29 the particle number close to the transition temperature becomes

$$N = N_0 + \frac{\zeta_R(3)}{(\beta\hbar\Omega)^3} + \frac{\zeta_R(2)}{2(\beta\hbar)^2}\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_2\omega_3}\right)+\cdots \quad (5\text{-}10)$$

The second and third terms give the number of particles in the *excited* levels (at high temperature close to the phase transition).

The critical temperature is defined as the temperature where all excited levels are completely filled such that lowering the temperature the ground state population will start to build up. This means the defining equation for the critical temperature $T_c = 1/\beta_c$ in the approximation considered is

$$N = \frac{1}{(\beta_c\hbar\Omega)^3}\zeta_R(3) + \frac{1}{2(\beta_c\hbar)^2}\zeta_R(2)\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_2\omega_3}\right). \quad (5\text{-}11)$$

Solving for β_c one finds

$$T_c = T_0\left(1 - \frac{\zeta_R(2)}{3\zeta_R(3)^{2/3}}\,\delta\, N^{-1/3}\right).$$

Here, T_0 is the critical temperature in the bulk limit ($N \to \infty$)

$$T_0 = \hbar\Omega\left(\frac{N}{\zeta_R(3)}\right)^{1/3}$$

and

$$\delta = \tfrac{1}{2}\Omega^{2/3}\left(\frac{1}{\omega_1\omega_2}+\frac{1}{\omega_1\omega_3}+\frac{1}{\omega_2\omega_3}\right).$$

Different approaches can be used to obtain the same answers [47; 48; 49; 50].

If only a few thousand particles are used in the experiment the finite-N correction is actually quite important. For example the first successful experiments on Bose–Einstein condensates were done with rubidium [2] at frequencies $\omega_1 = \omega_2 = 240\pi/\sqrt{8}\ \mathrm{s}^{-1}$ and $\omega_3 = 240\pi\,\mathrm{s}^{-1}$. With $N = 2000$ one finds $T_c \sim 31.9\mathrm{nK} = 0.93\ T_0$ [59], a significant correction compared to the thermodynamic limit.

EXERCISE 30. Consider the Bose gas in a d-dimensional cavity. Find the particle number and the critical temperature along the lines described for the harmonic oscillator. What is the correction to the critical temperature caused by the finite size of the cavity? (For a solution to this problem see [60].)

6. Conclusions

In these lectures some basic zeta functions are introduced and used to analyze the Casimir effect and Bose–Einstein condensation for particular situations. The basic zeta functions considered are the Hurwitz, the Barnes and the Epstein zeta function. Although these zeta functions differ from each other they have one property in common: they are based upon a sequence of numbers that is explicitly known and given in closed form. The analysis of these zeta functions and of the indicated applications in physics is heavily based on this explicit knowledge in that well-known summation formulas are used.

In most cases, however, an explicit knowledge of the eigenvalues of, say, a Laplacian will not be available and an analysis of the associated zeta functions will be more complicated. In recent years a new class of examples where eigenvalues are defined implicitly as solutions to transcendental equations has become accessible. In some detail let us assume that eigenvalues are determined by equations of the form

$$F_\ell(\lambda_{\ell,n}) = 0 \tag{6-1}$$

with ℓ, n suitable indices. For example when trying to find eigenvalues and eigenfunctions of the Laplacian whenever possible one resorts to separation of variables and ℓ and n would be suitable 'quantum numbers' labeling eigenfunctions. To be specific consider a scalar field in a three dimensional ball of radius R with Dirichlet boundary conditions. The eigenvalues λ_k for this situation,

with k as a multiindex, are thus determined through

$$-\Delta\phi_k(x) = \lambda_k \phi_k(x), \qquad \phi_k(x)|_{|x|=R} = 0.$$

In terms of spherical coordinates (r, Ω), a complete set of eigenfunctions may be given in the form

$$\phi_{l,m,n}(r, \Omega) = r^{-1/2} J_{l+1/2}\big(\sqrt{\lambda_{l,n}} r\big) Y_{l,m}(\Omega),$$

where $Y_{l,m}(\Omega)$ are spherical surface harmonics [40], and J_ν are Bessel functions of the first kind [46]. Eigenvalues of the Laplacian are determined as zeroes of Bessel functions. In particular, for a given angular momentum quantum number l, imposing Dirichlet boundary conditions, eigenvalues $\lambda_{l,n}$ are determined by

$$J_{l+1/2}\big(\sqrt{\lambda_{l,n}} R\big) = 0. \tag{6-2}$$

Although some properties of the zeroes of Bessel functions are well understood [46], there is no closed form for them available and we encounter the situation described by (6-1). In order to find properties of the zeta function associated with this kind of boundary value problems the idea is to use the argument principle or Cauchy's residue theorem. For the situation of the ball one writes the zeta function in the form

$$\zeta(s) = \sum_{l=0}^{\infty} (2l+1) \frac{1}{2\pi i} \int_\gamma k^{-2s} \frac{\partial}{\partial k} \ln J_{l+1/2}(kR)\, dk, \tag{6-3}$$

where the contour γ runs counterclockwise and must enclose all solutions of (6-2). The factor $(2l+1)$ represents the degeneracy for each angular momentum l and the summation is over all angular momenta. The integrand has singularities exactly at the eigenvalues and one can show that the residues are one such that the definition of the zeta function is recovered. More generally, in other coordinate systems, one would have, somewhat symbolically,

$$\zeta(s) = \sum_j d_j \frac{1}{2\pi i} \int_\gamma k^{-2s} \frac{\partial}{\partial k} \ln F_j(k)\, dk, \tag{6-4}$$

the task being to construct the analytical continuation of this object. The details of the procedure will depend very much on the properties of the special function F_j that enters, but often all the information needed can be found [57]. Nevertheless, for many separable coordinate systems this program has not been performed but efforts are being made in order to obtain yet unknown precise values for the Casimir energy for various geometries.

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