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Partizan Splittles

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ABSTRACT. Splittles is a nim variant in which a move consists of removing tokens from one heap, and (optionally) splitting the remaining heap into two. The possible numbers of tokens that can legally be removed are fixed, but the two players might have different *subtraction sets.* The nature of the game, and the analysis techniques employed, vary dramatically depending on the subtraction sets.

1. Introduction

Partizan Splittles is a game played by two players, conventionally called Left and Right. A position in the game consists of a number of heaps of tokens and a move requires a player to choose a heap, remove some positive number, s, of tokens from the heap and optionally to split the remaining heap (if there are two or more tokens remaining) into two heaps. Two sets of positive integers S_L and S_R are fixed in advance, and there is an additional restriction that when Left moves she must choose $s \in S_L$, while Right must choose $s \in S_R$ at his turn. The sets S_L and S_R are called the *subtraction sets* of Left and Right respectively.

It is sometimes convenient to represent a position pictorially by one-dimensional blocks of boxes rather than heaps of tokens. A move is to remove a contiguous block of boxes; moves in the middle of a block are tantamount to splitting a heap.

In this paper, we address several possible restrictions on S_L and S_R, each of which yields a game whose analysis requires different techniques from combinatorial game theory. For some choices of S_L and S_R, canonical forms are

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readily available, while for others, canonical forms are complex and uninformative, while temperature theory and the relatively new techniques of reduced canonical forms reveal a great deal of information.

While impartial *octal games* [BCG01] are well-studied, there has been surprisingly little work on partizan variants. In a partizan octal game, the two players have different octal codes indicating their legal moves. Each code is a sequence of octal digits, $\mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\ldots$, where each $\mathbf{d}_i$ includes a **1**, **2**, and/or **4**, indicating whether it is legal to remove i tokens and leave 0, 1, and/or 2 heaps, respectively. In Partizan Splittles, the octal codes consist entirely of **0**s and **7**s. For instance, if $S_L = \{1,4\}$ and $S_R = \{1,5\}$, then the game is **0.7007** versus **0.70007**.

Thane Plambeck [Pla95], as well as Calistrate and Wolfe, have unpublished results in the game where players cannot split into two heaps; the octal codes for these games consist entirely of **0**s and **3**s.

2. $\{1, \textbf{odds}\}$ versus $\{1, \textbf{odds}\}$

Our first example is simple.

THEOREM 1. *If* 1 *is an element of both subtraction sets and all the elements of both subtraction sets are odd numbers*, *then*

$$G_n = \begin{cases} 0 & \text{if } n \text{ is even}, \\ * & \text{if } n \text{ is odd}. \end{cases}$$

PROOF. Each move changes the parity of the total number of tokens in all the heaps, and in the final position, which has zero tokens, this total is even. Thus, the game is she-loves-me she-loves-me-not. □

3. $\{1\}$ versus $\{k\}$

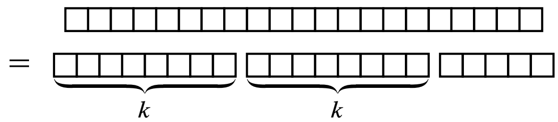
THEOREM 2. *If $S_L = \{1\}$ and $S_R = \{k\}$, then G_n is arithmetic-periodic with period k and saltus $\{k-1 \mid 0\}$. In particular*,

$$G_n = \begin{cases} n & \text{if } n < k, \\ \{k-1 \mid 0\} + G_{n-k} & \text{if } n \geq k. \end{cases}$$

We can write G_n more naturally with period $2k$ and saltus $k-1$ as

$$G_n = \begin{cases} n & \text{if } n < k, \\ \{n-1 \mid n-k\} & \text{if } k \leq n < 2k, \\ k-1+G_{n-2k} & \text{if } n \geq 2k. \end{cases}$$

PROOF. The proof follows in part from the fact that the conjectured saltus is exactly G_k. So the theorem asserts that one can treat a single heap as a collection of heaps of size k and (possibly) a single remaining heap of size less than k without changing its value. For instance:



Clearly, $G_a = a$ for $0 \le a < k$, since only Left can move from such a position. Likewise, $G_k = \{k-1 \mid 0\}$.

In a general position, it suffices to show that any move that straddles a period boundary is dominated by one that does not, for then the game reduces to its "decomposed" form. Left's moves never straddle a boundary. As for Right's moves, assume inductively that shorter positions achieve their conjectured values and decompose at period boundaries. Observe that if $a+b=k+c$ for $0 \le a, b, c < k$, then $G_a + G_b = a + b = k + c$ exceeds $G_k + G_c = \{k-1 \mid 0\} + c$. Hence, Right prefers the latter move, avoiding a boundary. $\square$

4. $\{1, 2k\}$ versus $\{1, 2k+1\}$

In this case, too, we can find exact values for all G_n. The sequence is arithmetic-periodic with period $4k$ and saltus $\uparrow^{\rightarrow 2}$. (Note that $\uparrow^{\rightarrow 2} = \uparrow + \uparrow^2 = \{\uparrow \mid *\}$, and that $\uparrow^2 = \{0 \mid \downarrow *\}$ is positive and infinitesimal with respect to $\uparrow$.)

THEOREM 3. *If $S_L = \{1, 2k\}$ and $S_R = \{1, 2k+1\}$ then*

$$G_{4jk+i} = \begin{cases} 0 + j.\uparrow^{\rightarrow 2} & \text{if } i \text{ is even and } 0 \le i < 2k, \\ * + j.\uparrow^{\rightarrow 2} & \text{if } i \text{ is odd and } 0 \le i < 2k, \\ \uparrow + j.\uparrow^{\rightarrow 2} & \text{if } i \text{ is even and } 2k \le i < 4k, \\ \uparrow * + j.\uparrow^{\rightarrow 2} & \text{if } i \text{ is odd and } 2k \le i < 4k. \end{cases}$$

That is, the values are given by

$$\overbrace{\hspace{6em}}^{2k}$$

0	$*$	0	$*$	$\dots$	0	$*$
$\uparrow$	$\uparrow *$	$\uparrow$	$\uparrow *$	$\dots$	$\uparrow$	$\uparrow *$
Period $4k$, *saltus* $\uparrow^{\rightarrow 2}$						

PROOF. When $n < 4k$, it is easy to confirm that the proposed values of G_n are correct. It thus suffices to show that $G_{n+4k} - G_n = \uparrow^{\rightarrow 2}$. Assume, inductively, that the conjectured values are correct for heap sizes less than $n + 4k$. First, moves by either player that split $-G_n$ into $-G_a - G_b$ can be countered by splitting G_{n+4k} into $G_{a+4k} + G_b$, leaving zero by induction.

$$
\begin{tikzcd}[column sep=small]
 & & G_{n+4k} - G_n - \uparrow^{\rightarrow 2} \arrow[dll] \arrow[dl] \arrow[d] \\
G_n+G_{2k}-G_n-\uparrow^{\rightarrow 2} \arrow[d] & G_{n+2k-1}+G_{2k}-G_n-\uparrow^{\rightarrow 2} \arrow[d] & G_{n+4k}-G_n-* \arrow[d] \\
G_n+G_{2k}-G_n-\uparrow & \begin{array}{c} G_n+G_{2k-2}+G_{2k}-G_n-\uparrow^{\rightarrow 2} \\ = G_n+0+\uparrow-G_n-\uparrow^{\rightarrow 2} < 0 \end{array} & G_n+G_{2k-1}-G_n-*
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
G_{n+4k} - G_n - \uparrow^{\rightarrow 2} \arrow[d] \arrow[dr] & \\
G_n+G_{2k-1}-G_n-\uparrow^{\rightarrow 2} \arrow[d] & G_{n+4k}-G_n-\uparrow \arrow[d] \\
G_n+G_{2k-1}-G_n-* & G_n+G_{2k}-G_n-\uparrow
\end{tikzcd}
$$

Figure 1. Diagrams showing $G_{n+4k} = G_n + \uparrow^{\rightarrow 2}$. The first tree shows Right's winning responses to Left's moves, while second shows how Left wins when Right moves first. Except in one case, the response leaves a game equal to 0.

Next, observe that Left's moves from G_{n+4k} that leave a heap of size $2k$ are at least as good as her other moves. Similarly, Right's dominant splitting moves from G_{n+4k} remove $2k+1$ tokens, leaving a heap of size $2k-1$. (When convincing yourself of these last assertions, it helps to keep in mind that all $G_a + G_b$ for fixed $a+b$ have the same $*$-parity, and alternate rows add $\uparrow$ and $\uparrow^2$.) Figure 1 summarizes how the second player wins in response to Left's (respectively, Right's) options not yet dispensed with. $\square$

COROLLARY 4. *The values in the last theorem remain unchanged when*

- *Left has additional odd options*, *and/or*
- *Left has additional options exceeding* $2k$, *and/or*
- *Right has additional odd options between* 1 *and* $2k+1$.

PROOF. In all cases these new options are dominated. $\square$

5. $\{1, \text{others}\}$ versus $\{1, 3, 5, \ldots, 2k+1\}$

While the actual values might be quite complex and depend on the specific choice for S_L, we can describe a few properties of the sequence G_n.

THEOREM 5. *Suppose* $1 \in S_L$, *and either* $S_R = \{1, 3, 5, \ldots, 2k+1\}$ *for some* k *or* $S_R = \{1, 3, 5, \ldots\}$. *The following relations hold*:

$$G_n \le G_{n+2} \tag{5-1}$$
$$G_{2n+1} = G_{2n} + * \tag{5-2}$$
$$G_n \le G_i + G_j \quad (\textit{for } S_R \textit{ finite and } n - i - j \ge 2k \textit{ even}) \tag{5-3}$$

PROOF. For (5-1), in most sequences of play, Left wins $G_{n+2} - G_n$ by matching options naturally, removing the same number of tokens as Right did from the opposite heap. Left can then leave a position of the form $G_{a+2} + G_b - G_a - G_b$, for $a, b \ge 0$, which is ≥ 0 by induction. The only exception is if Right removes $n+1$ from the first heap. In this case, Left removes $n-1$ from the second, leaving $G_1 - G_1 = 0$.

We show (5-1) and (5-1) in tandem by induction. In particular, we assume that (5-1) holds for $n' \le n$ when proving (5-1), but that (5-1) holds for $n' < n$ when proving (5-1).

For (5-1), we wish to show the second player wins on the difference game

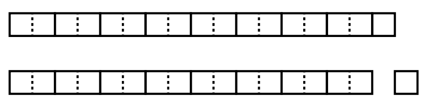
$$G_{2n+1} - G_{2n} - * = G_{2n+1} - G_{2n} - G_1.$$

We can depict this game as

Left's moves **Right's moves**

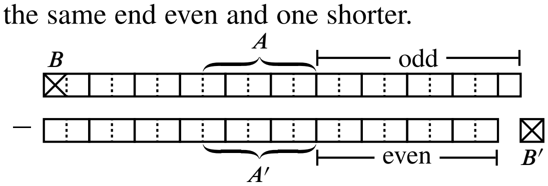
$\{1, \text{others}\}$ $\{1, 3, 5, \ldots, 2k+1\}$

$\{1, 3, 5, \ldots, 2k+1\}$ $\{1, \text{others}\}$



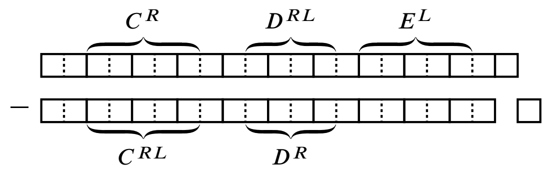
The roles of the players are reversed in the second row, it being the negative of the game $G_{2n} + G_1$. So in the second row, Left removes elements from S_R while Right removes elements from S_L.

If either player removes r boxes from the top row, leaving a block of length i odd (and, perhaps a second block of either parity), the other player can counter symmetrically by removing r boxes from the bottom, leaving a block of length $i-1$. The resulting position is 0 by induction. The reverse is also true; the second player can respond to moves on the bottom row that leave an even-length block. Pictorially, moves A on top leaving one end odd match up with moves A' on bottom leaving the same end even and one shorter.



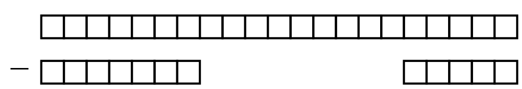
Also shown are moves B which take a single box from one end of the top row, which match up with the move B' taking the lone box on the bottom row.

So we are left with cases that split the top row into two even-length heaps or that split the bottom row into two odd-length heaps. Only Right can do the latter, for it requires removing an even number. Left's responses parallel Right's moves as below. If Right removes C^R on the top, Left's reply of C^{RL} wins by application of (5-1) then multiple applications of (5-1). Similarly Left wins after Right's D^R and Left's D^{RL}.

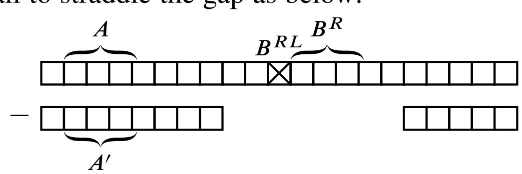


We are now left with the single case when Left removes an odd number from the top row, splitting it into two even-sized heaps, as in E^L above. Right responds by removing as large an (odd) number as possible from one of these two even-sized heaps. If one of the heaps is of size $\leq 2k+2$ (or S_R is infinite), Right leaves that heap a singleton, canceling the single box on the second row, and wins by (5-1). Otherwise, he has taken away $2k+1$, and wins by (5-1) and (5-1).

Lastly, to prove (5-1), we show that Left wins moving second on $G_n - G_i - G_j$:

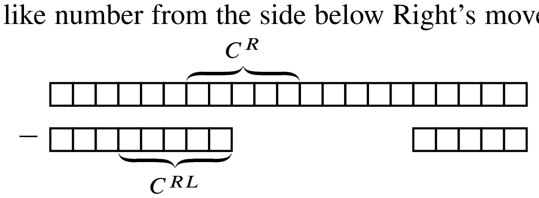


The gap in the bottom row is of even length and at least $2k$. Left can respond to moves that fail to straddle the gap as below:



In particular moves outside the gap match up with moves in the other row, winning by induction. Left responds to moves inside the gap by responding on the *odd side:* Since the gap was of even length, and Right can remove only odd numbers, the gap is split into an even length and an odd length. Left then wins by application of (5-1) to both sides.

A Right move that straddles the gap can only straddle one side. Left responds by removing a like number from the side below Right's move:



Since the parity of the number of boxes in each row is preserved, each segment can be shortened to an even length by an even number of applications of (5-1), which, since $* + * = 0$ and $* = -*$, leave the game value unchanged. Left then proceeds to win by (5-1) applied to both sides. □

6. Odd versus Even

In this section we consider the partizan splitting game where S_L is the set of all (positive) odd integers and S_R is the set of all (positive) even integers. A salient feature is that the endgame is overwhelmingly favorable to Left: $G_1 = 1$, but there are *no* positions with negative right stop, since Left has a move from every nonterminal position. One would expect, therefore, that Left should prefer to split each position into as many components as possible, preferably odd in size, while Right should aim to annihilate each component as quickly as possible. Since Right will naturally give preference to destroying the largest heaps, one might also expect that Left would prefer to split each heap as evenly as possible.

As is so often the case, the canonical forms of *Odd versus Even* are a mess, but the *orthodox* moves — as defined by Berlekamp [Ber96], Definition 10 — realize these intuitions precisely. Left's orthodox strategy is to split as evenly as possible; Right's is to consume the largest available heap. Furthermore, we will see that from positions of the form G_{2^k-1} — where it is most crucial that Left split evenly — these are the *unique* orthodox options (Theorem 11).

The game also exhibits a fascinating logarithmic behavior: if Left and Right play orthodox strategies, with Left splitting evenly and Right consuming what he can, then the game will last for $O(\log n)$ moves. Furthermore, from positions of the form G_{2^k-1} — where it is most crucial that Left split evenly — Left's *only* orthodox move is the even split. By contrast, we note that, G_{31} has seven canonical Left options.

The main result is the following theorem, which gives the mean, $m(G_n)$, and temperature, $t(G_n)$, of every single-heap Odd versus Even position.

THEOREM 6. *Fix $n \geq 1$ and let k be such that $2^k \leq n < 2^{k+1}$. Then $m(G_n) = \mu_n$ and $t(G_n) = \tau_n$, where*

$$\tau_n = \frac{\lfloor n/2 \rfloor + 1}{2^k} + \frac{k}{2} - 1; \qquad \mu_n = \begin{cases} \tau_n & \text{if } n \text{ is even}, \\ \tau_n + 1 & \text{if } n \text{ is odd}. \end{cases}$$

To prove Theorem 6, we will define H_n to be the auxiliary game where $S_L = \{1\}$ and S_R is the set of even integers. We will first show that $m(H_n) = \mu_n$ and $t(H_n) = \tau_n$, and then argue that the means and temperatures do not change when Left's subtraction set includes other odd integers.

We will need several lemmas. The first shows that if n is odd, then $H_n = H_{n-1} + 1$ (canonically). This reduces Theorem 6 to the case where n is even.

LEMMA 7. *Let n be odd. Then*

$$H_{n+1} - H_n < 1, \tag{6-1}$$
$$H_n - H_{n-1} = 1. \tag{6-2}$$

PROOF. We proceed by induction on n.

To prove (6-1), consider

$$1 + H_n - H_{n+1}.$$

Left can win by moving immediately to $1 + H_n$; this value is positive since $R_0(H_n) \geq 0$. If Right moves to $1 + H_a + H_b - H_{n+1}$, Left counters to $1 + H_a + H_b - H_{a+1} - H_b$. This is a winning move by induction on (6-1) or (6-1), depending on whether a is odd or even, respectively. If Right moves to $1 + H_n - H_a - H_b$, then since $n+1$ is even, $a+b$ is odd and hence one of a, b (say a) must be odd. Left counters to $1 + H_{a-1} + H_b - H_a - H_b$, which is 0 by induction.

To prove (6-1) we show that

$$H_n - H_{n-1} - 1$$

is a second-player win. If Right moves to $H_a + H_b - H_{n-1} - 1$, then since n is odd, $a+b$ is also odd and hence one of a, b (say a) must be odd. Left counters to $H_a + H_b - H_{a-1} - H_b - 1$, which by induction is equal to 0. Likewise, if Right moves to $H_n - H_a - H_b - 1$, then since $n-1$ is even, $a+b$ is odd and hence one of a, b (say a) must be even. Left counters to $H_{a+1} + H_b - H_a - H_b - 1$. Finally, if Right moves to $H_n - H_{n-1}$, Left simply responds with $H_{n-1} - H_{n-1}$.

Conversely, if Left moves to $H_a + H_b - H_{n-1} - 1$, then since $n > 1$ we can assume without loss of generality that $a > 0$. Right counters to $H_a + H_b - H_{a-1} - H_b - 1$. By induction, this is 0 if a is odd, and negative if a is even. If instead Left moves to $H_n - H_a - H_b - 1$, then since $n-1$ is even, $a+b$ is odd and hence one of a, b (say a) must be odd. Right counters to $H_{a+1} + H_b - H_a - H_b - 1$, which is negative by induction on (6-1). □

The rather dry arithmetic of the τ_n and μ_n is described in the next two lemmas.

LEMMA 8. *Fix $n > 2$ and let k be such that $2^k \leq n < 2^{k+1}$. Then*

$$\tau_n - \tau_{n-2} = \frac{1}{2^k}.$$

PROOF. We may assume that n is even, for if $n' = n+1$ is odd, we have $2^k \leq n, n' < 2^{k+1} \leq$ and

$$\tau_{n'} - \tau_{n'-2} = \tau_n - \tau_{n-2} = \frac{1}{2^k}.$$

We separate cases into $n = 2^k$ and $n \neq 2^k$.

If $n = 2^k$,

$$\tau_{2^k} - \tau_{2^k-2} = \Big(\frac{2^{k-1}+1}{2^k} + \frac{k}{2} - 1\Big) - \Big(\frac{(2^{k-1}-1)+1}{2^{k-1}} + \frac{k-1}{2} - 1\Big)$$
$$= \Big(\frac{1}{2} + \frac{1}{2^k} + \frac{k}{2} - 1\Big) - \Big(1 + \frac{k}{2} - \frac{1}{2} - 1\Big) = \frac{1}{2^k}.$$

If $n \neq 2^k$,

$$\tau_n - \tau_{n-2} = \Big(\frac{n/2+1}{2^k} + \frac{k}{2} - 1\Big) - \Big(\frac{(n-2)/2+1}{2^k} + \frac{k}{2} - 1\Big)$$
$$= \Big(\frac{n}{2^{k+1}} + \frac{1}{2^k} + \frac{k}{2} - 1\Big) - \Big(\frac{n}{2^{k+1}} + \frac{k}{2} - 1\Big) = \frac{1}{2^k}. \qquad \square$$

Lemma 8 shows that the τ_n are (nonstrictly) increasing; and therefore, up to parity, so are the μ_n. Furthermore, up to parity, the rate of increase is decreasing. This fact will be critical in the proof of Theorem 6, since it quantifies the intuition that Left prefers to split as evenly as possible.

LEMMA 9. *Fix $n > 2$ and let k be such that $2^k \le n < 2^{k+1}$. Then*

$$\mu_n + \mu_{n-1} = \frac{n+1}{2^k} + k - 1.$$

PROOF. Again, we separate into the same two cases. If $n = 2^k$,

$$\mu_{2^k} + \mu_{2^k-1} = \Big(\frac{2^{k-1}+1}{2^k} + \frac{k}{2} - 1\Big) + \Big(\frac{(2^{k-1}-1)+1}{2^{k-1}} + \frac{k-1}{2}\Big)$$
$$= \Big(\frac{1}{2} + \frac{1}{2^k} + \frac{k}{2} - 1\Big) + \Big(1 + \frac{k}{2} - \frac{1}{2}\Big) = \frac{1}{2^k} + k.$$

Since $n/2^k = 1$, this yields the desired equality.

When $n \neq 2^k$, notice that exactly one of n, $n-1$ is odd, and in either case $\lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor = n - 1$. So,

$$\tau_n + \tau_{n-1} = \Big(\frac{\lfloor n/2 \rfloor + 1}{2^k} + \frac{k}{2} - 1\Big) + \Big(\frac{\lfloor (n-1)/2 \rfloor + 1}{2^k} + \frac{k}{2} - 1\Big)$$
$$= \frac{(n-1)+2}{2^k} + k - 2 = \frac{n+1}{2^k} + k - 2.$$

Since exactly one of n, $n-1$ is odd, we have $\mu_n + \mu_{n-1} = \tau_n + \tau_{n-1} + 1$, as needed. $\square$

PROOF. (of Theorem 6) As noted in the exposition, we first show that $m(H_n) = \mu_n$ and $t(H_n) = \tau_n$, and then generalize to the G_n. The proof is by induction on n. The base cases $H_1 = 1$ and $H_2 = \{1 \mid 0\}$ are easily verified. At odd stages of the induction, the result is an immediate corollary of Lemma 7, so fix an even $n > 2$.

Left has a move to $H_n^L = H_{n/2} + H_{n/2-1}$. By induction, we know that

$$m(H_n^L) = m(H_{n/2}) + m(H_{n/2-1}) = \mu_{n/2} + \mu_{n/2-1}$$

and since $2^{k-1} \le n/2 < 2^k$, Lemma 9 implies that

$$\mu_{n/2} + \mu_{n/2-1} = \frac{n/2+1}{2^{k-1}} + (k-1) - 1 = \frac{\lfloor n/2 \rfloor + 1}{2^{k-1}} + k - 2.$$

Furthermore,

$$t(H_n^L) \le \max\{\tau_{n/2}, \tau_{n/2-1}\} = \tau_{n/2}.$$

Now Right can remove the entire heap, moving to $H_n^R = 0$. Therefore

$$\frac{m(H_n^L) - m(H_n^R)}{2} = \frac{m(H_n^L)}{2} = \frac{\lfloor n/2 \rfloor + 1}{2^k} + \frac{k}{2} - 1 = \tau_n.$$

Now certainly $\tau_n > \tau_{n/2}$. Therefore

$$\frac{m(H_n^L) - m(H_n^R)}{2} > \max\{t(H_n^L), t(H_n^R)\}.$$

If H_n^L and H_n^R were the *only* options of H_n, then by an elementary thermographic argument, we would have

$$t(H_n) = \tau_n \quad \text{and} \quad m(H_n) = \frac{m(H_n^L) + m(H_n^R)}{2} = \frac{m(H_n^L)}{2} = \tau_n = \mu_n.$$

Certainly both players have other options available, so we conclude the proof by showing that H_n^L and H_n^R are thermally optimal at all temperatures $t \ge \tau_{n/2}$. Since $\tau_{n/2}$ is an upper bound for $t(H_n^L)$, it suffices to show that, for any other options $H_n^{L'}$, $H_n^{R'}$, we have $m(H_n^{L'}) \le m(H_n^L)$ and $m(H_n^{R'}) \ge m(H_n^R)$.

This is trivial in the case of Right options, since no *Odd versus Even* position can have negative mean.

Therefore, consider some arbitrary Left option $H_a + H_b$, with $a > b$ and $a + b = n - 1$. We necessarily have $a \ge n/2$ and $n/2 - 1 \ge b$, with $a - n/2 = (n/2-1) - b$. Now since exactly one of $n/2, n/2-1$ is odd, repeated applications of Lemma 8 imply that

$$\tau_a - \tau_{n/2} \le \tau_{n/2-1} - \tau_b.$$

It follows that

$$\tau_a + \tau_b < \tau_{n/2} + \tau_{n/2-1}$$

and hence, since $a + b$ and $n/2 + (n/2 - 1)$ are both odd,

$$\mu_a + \mu_b < \mu_{n/2} + \mu_{n/2-1}.$$

This completes the proof for the H_n. To conclude, we show (again by induction on n) that Left's additional options in G_n convey no thermographic advantage. For suppose Left has a move from G_n to $G_a + G_b$, where $a+b = n-2c-1$ for some $c \geq 0$. By induction we may assume that $m(G_i) = \mu_i$ and $t(G_i) = \tau_i$ for all $i < n$. But just as before, we have

$$\mu_a + \mu_b \leq \mu_{a+2c} + \mu_b \leq \mu_{n/2} + \mu_{n/2-1}$$

so that $G_a + G_b$ is thermally dominated at temperatures $t \geq \tau_{n/2}$. □

We conclude with a neat little theorem on orthodox moves.

DEFINITION 10. Let G be a game and fix $t \geq 0$. A Left option G^L is said to be *orthodox at temperature t* if $R_t(G^L) = L_t(G^L) + t$. Likewise, a Right option G^R is orthodox at temperature t if $L_t(G^R) = R_t(G) - t$. We say that an option is *orthodox* if it is orthodox at temperature $t(G)$.

That is, an orthodox move is one that achieves the best possible score at temperature $t(G)$.

THEOREM 11. *Let $k > 2$ and $n = 2^k - 1$. Left's only orthodox move from G_n is to $G_n^L = G_{2^{k-1}-1} + G_{2^{k-1}-1}$.*

PROOF. Since n is odd, Left must split G_n into two heaps that are either both even or both odd. It is easily seen that those options with both heaps even are badly dominated, so it suffices to show that G_n^L is strictly optimal among those options with both heaps odd.

By Lemma 8,

$$\mu_{2^{k-1}-1} - \mu_{2^{k-1}-3} = \frac{1}{2^{k-2}}, \quad \text{but} \quad \mu_{2^{k-1}+1} - \mu_{2^{k-1}-1} = \frac{1}{2^{k-1}}.$$

Therefore,

$$\mu_{2^{k-1}+1} + \mu_{2^{k-1}-3} < \mu_{2^{k-1}-1} + \mu_{2^{k-1}-1}.$$

Repeated application of Lemma 8 also shows that

$$\mu_a + \mu_b \leq \mu_{2^{k-1}+1} + \mu_{2^{k-1}-3}$$

for every other choice of a, b both odd with $a + b < n$. Therefore G_n^L has the *strictly* highest mean among the Left options of G_n. But we also know that

$$t(G_{2^{k-1}-1} + G_{2^{k-1}-1}) \leq \tau_{2^{k-1}-1} < \tau_n$$

so G_n^L is the unique optimal move at temperature τ_n. □

7. $\{1, \mathbf{odds}\}$ versus $\{2, 4\}$

Suppose S_L is any set of odd numbers containing 1, and $S_R = \{2, 4\}$. The values G_n for these games can be quite complex. For example, when $S_L = \{1, 3\}$ the canonical form of G_{14} contains 611 stops! Furthermore, the exact value of G_n depends strongly on the specific set S_L (if $S_L = \{1\}$ then the canonical form of G_{14} has only 6 stops). Although it is not practical to solve for G_n exactly, we can find a very good approximation for G_n. In particular, let $f(n)$ be the arithmetic-periodic sequence with period 4 and saltus 3/4 defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ 1/2 & \text{if } n = 2, \\ 3/2 & \text{if } n = 3, \\ f(n-4) + 3/4 & \text{if } n \geq 4. \end{cases}$$

The main theorem of this section is that G_n is infinitesimally close to $f(n) \cdot 1*$ for any choice of S_L that contains 1 and zero or more other odd numbers. We begin by briefly reviewing some definitions and results that will be required.

Infinitesimals. Write $L_0(G)$ and $R_0(G)$ for the Left and Right stops of G, respectively. A game G is *infinitesimal* if $L_0(G) = R_0(G) = 0$. Write $G \equiv_{\text{Inf}} H$ when G and H differ by an infinitesimal; we also say that G is H-ish. If $G \equiv_{\text{Inf}} H$, then $L_0(G) = L_0(H)$ and $R_0(G) = R_0(H)$. The converse is in general not true, but if x is a number and $L_0(G) = R_0(G) = x$, then it is true that $G \equiv_{\text{Inf}} x$, in which case we say that G is numberish.

Write $G \geq_{\text{Inf}} H$ if there is some infinitesimal ε such that $G \geq H + \varepsilon$, and similarly for $G \leq_{\text{Inf}} H$. A Left option G^L of G is *Inf-dominated* if $G^{L'} \geq_{\text{Inf}} G^L$ for some other Left option $G^{L'}$, and similarly for Right options.

In [GS07] it is shown that if $G \geq_{\text{Inf}} H$, then $L_0(G) \geq L_0(H)$ and $R_0(G) \geq R_0(H)$. More importantly, they show:

PROPOSITION 12. If G is not a number and G' is obtained from G by repeatedly

(i) eliminating Inf-dominated options, and
(ii) replacing any option H with $H' \equiv_{\text{Inf}} H$, then $G' \equiv_{\text{Inf}} G$.

Norton multiplication. Fix a game $U > 0$. The Norton product $G \cdot U$ is defined by

$$G \cdot U = \begin{cases} 0 \ \text{ or } \ \overbrace{U + U + \cdots + U}^{G \text{ times}} \ \text{ or } \ \overbrace{-U - U - \cdots - U}^{-G \text{ times}} & \text{if } G \text{ is an integer}, \\ \left\{ G^L \cdot U + (U + I) \;\middle|\; G^R \cdot U - (U + I) \right\} & \text{otherwise.} \end{cases}$$

where I ranges over all Left and Right incentives of G. We will use the following properties of Norton multiplication, which are proved in [BCG01].

PROPOSITION 13. Let U be any positive game. Then:

(i) If $G = H$, then $G \cdot U = H \cdot U$ (independence of form).
(ii) $G \geq H$ if and only if $G \cdot U \geq H \cdot U$ (monotonicity).
(iii) $(G + H) \cdot U = G \cdot U + H \cdot U$ (distributivity).

For our purposes, we take $U = 1*$. Since the only Left or Right incentive of $1*$ is $*$, we have

$$G \cdot 1* = \{G^L \cdot 1* + 1 \mid G^R \cdot 1* - 1\}$$

when G is not an integer. We note that $G \cdot 1*$ is equal to G *overheated* from $1*$ to 1, an operation defined in [BCG01]. It is easy to verify by induction that if x is a number then $L_0(x \cdot 1*) = \lceil x \rceil$ and $R_0(x \cdot 1*) = \lfloor x \rfloor$.

LEMMA 14. *If $x = a/4$ for some integer a, then*

$$\{(x - 1/4) \cdot 1* + 1 \mid (x + 1/4) \cdot 1* - 1\} = x \cdot 1*$$

PROOF. If x is not an integer, then $x = \{x - 1/4 \mid x + 1/4\}$, so the result follows from the definition of Norton multiplication and Proposition 13(i). Otherwise, by symmetry, it suffices to show that Right has no winning move from

$$\{(x - 1/4) \cdot 1* + 1 \mid (x + 1/4) \cdot 1* - 1\} - x \cdot 1*.$$

If Right moves in the first component, then the resulting game is

$$(x + 1/4) \cdot 1* - 1 - x \cdot 1* = (1/4) \cdot 1* - 1$$

which we can verify is $\rhd 0$, so Left wins. If Right moves in the second component, which has the effect of subtracting $*$, then Left responds in the first, and the resulting game is

$$(x - 1/4) \cdot 1* + 1 - x \cdot 1* - * = (-1/4) \cdot 1* + 1* = (3/4) \cdot 1*$$

which we can verify is ≥ 0, so again Left wins. □

PROOF OF MAIN RESULT. We will now show that $G_n \equiv_{\mathrm{Inf}} f(n) \cdot 1*$. Our proof is by induction. Suppose the result holds for all $m < n$. It is convenient to assume that $n \geq 4$; for $n < 4$ we can easily validate the result by hand. We begin by showing that in the game G_n, there are only one Left and one Right option that need to be considered.

LEMMA 15. *The Left options of G_n are Inf-dominated by*

$$G_{n-4} + G_3 \equiv_{\mathrm{Inf}} (f(n) + 3/4) \cdot 1*.$$

The Right options are Inf-dominated by

$$G_{n-4} \equiv_{\mathrm{Inf}} (f(n) - 3/4)\cdot 1*.$$

PROOF. The Left options of G_n are $G_{n-k-a} + G_k$ with $a \in S_L$. Since $f(m) < f(m+2)$ for all m and $G_m \equiv_{\mathrm{Inf}} f(m)\cdot 1*$ for $m < n$, $G_{n-k-a} + G_k \leq_{\mathrm{Inf}} G_{n-k-1} + G_k$ so we may assume that $a = 1$. Next, since f is arithmetic-periodic with period 4, $G_{n-k-1} + G_k \equiv_{\mathrm{Inf}} G_{n-k+3} + G_{k-4}$ for $k \geq 4$, so we may assume that $k < 4$. This leaves us with four options to consider, which are infinitesimally close to:

$$f(n-1)\cdot 1*,\ (f(n-2)+1)\cdot 1*,\ (f(n-3)+1/2)\cdot 1*,\ (f(n-4)+3/2)\cdot 1*$$

It is easy to verify that for all m we have

$$f(m) + 3/2 \geq f(m+3), \quad f(m) + 1/2 \geq f(m+2), \quad f(m) + 1 \geq f(m+1),$$

from which it follows that $(f(n-4)+3/2)\cdot 1* = (f(n)+3/4)\cdot 1*$ Inf-dominates the others.

The proof for the Right options is similar. Since $f(m) < f(m+2)$ and f is arithmetic-periodic with period 4, we need only consider the four options $G_{n-k-4} + G_k$ with $k < 4$, which are infinitesimally close to

$$f(n-4)\cdot 1*,\ (f(n-5)+1)\cdot 1*,\ (f(n-6)+1/2)\cdot 1*,\ (f(n-7)+3/2)\cdot 1*$$

The same three inequalities as before show that $f(n-4)\cdot 1* = (f(n)-3/4)\cdot 1*$ Inf-dominates the others (note that for $n = 4, 5, 6$, not all the other options exist, but this does not affect the result). □

Next we show that G_n has the same Left and Right stops as $f(n)\cdot 1*$.

LEMMA 16. $L_0(G_n) = \lceil f(n) \rceil$ *and* $R_0(G_n) = \lfloor f(n) \rfloor$.

PROOF. First we compute $\max\{R_0(G_n^L)\}$ and $\min\{L_0(G_n^R)\}$. By Lemma 15,

$$\max\{R_0(G_n^L)\} = R_0((f(n)+3/4)\cdot 1*) = \lfloor f(n) + 3/4 \rfloor = \lceil f(n) \rceil.$$

The last equality follows from the fact that $f(n)$ is of the form $a/4$ for some integer a. Similarly,

$$\min\{L_0(G_n^R)\} = L_0((f(n)-3/4)\cdot 1*) = \lceil f(n) - 3/4 \rceil = \lfloor f(n) \rfloor.$$

If G_n is not a number then we are done, as then $L_0(G_n) = \max\{R_0(G_n^L)\}$ and $R_0(G_n) = \min\{L_0(G_n^R)\}$. If G_n is a number then $G_n = L_0(G_n) = R_0(G_n)$, but

$$L_0(G_n) \geq \max\{R_0(G_n^L)\} = \lceil f(n) \rceil \geq \lfloor f(n) \rfloor = \min\{L_0(G_n^R)\} \geq R_0(G_n)$$

so in fact we must have equality throughout, which means that $f(n)$ is also an integer, and again we are done. □

From Lemma 16 it follows immediately that when $f(n)$ is an integer, $G_n \equiv_{\text{Inf}} f(n) \equiv_{\text{Inf}} f(n)\cdot 1*$. Finally, if $f(n)$ is not an integer, then by Lemma 16, G_n is not numberish. So by Lemma 15, Proposition 12 and Lemma 14,

$$\begin{aligned} G_n &\equiv_{\text{Inf}} \{(f(n)+3/4)\cdot 1* \mid (f(n)-3/4)\cdot 1*\} \\ &\equiv_{\text{Inf}} \{(f(n)-1/4)\cdot 1* + 1 \mid (f(n)+1/4)\cdot 1* - 1\} = f(n)\cdot 1*. \end{aligned} \qquad \square$$

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