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Landen Survey

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To Henry, who provides inspiration, taste and friendship

ABSTRACT. Landen transformations are maps on the coefficients of an integral that preserve its value. We present a brief survey of their appearance in the literature.

1. In the beginning there was Gauss

In the year 1985, one of us had the luxury of attending a graduate course on *Elliptic Functions* given by Henry McKean at the Courant Institute. Among the many beautiful results he described in his unique style, there was a calculation of Gauss: take two positive real numbers a and b, with $a > b$, and form a new pair by replacing a with the arithmetic mean $(a+b)/2$ and b with the geometric mean $\sqrt{ab}$. Then iterate:

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n} \tag{1-1}$$

starting with $a_0 = a$ and $b_0 = b$. Gauss [1799] was interested in the initial conditions $a = 1$ and $b = \sqrt{2}$. The iteration generates a sequence of algebraic numbers which rapidly become impossible to describe explicitly; for instance,

$$a_3 = \frac{1}{2^3}\left((1 + \sqrt[4]{2})^2 + 2\sqrt{2}\,\sqrt[8]{2}\,\sqrt{1+\sqrt{2}}\right) \tag{1-2}$$

is a root of the polynomial

$$G(a) = 16777216a^8 - 16777216a^7 + 5242880a^6 - 10747904a^5 + 942080a^4 - 1896448a^3 + 4436a^2 - 59840a + 1.$$

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The numerical behavior is surprising; a_6 and b_6 agree to 87 digits. It is simple to check that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. \tag{1-3}$$

See (6-1) for details. This common limit is called the *arithmetic-geometric mean* and is denoted by $\text{AGM}(a,b)$. It is the explicit dependence on the initial condition that is hard to discover.

Gauss computed some numerical values and observed that

$$a_{11} \sim b_{11} \sim 1.198140235, \tag{1-4}$$

and then he *recognized* the reciprocal of this number as a numerical approximation to the elliptic integral

$$I = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}. \tag{1-5}$$

It is unclear to the authors how Gauss recognized this number: he simply knew it. (Stirling's tables may have been a help; [Borwein and Bailey 2003] contains a reproduction of the original notes and comments.) He was particularly interested in the evaluation of this definite integral as it provides the length of a lemniscate. In his diary Gauss remarked, *'This will surely open up a whole new field of analysis'* [Cox 1984; Borwein and Borwein 1987].

Gauss' procedure to find an analytic expression for $\text{AGM}(a,b)$ began with the elementary observation

$$\text{AGM}(a,b) = \text{AGM}\left(\frac{a+b}{2}, \sqrt{ab}\right) \tag{1-6}$$

and the homogeneity condition

$$\text{AGM}(\lambda a, \lambda b) = \lambda \text{AGM}(a,b)\ . \tag{1-7}$$

He used (1-6) with $a = (1+\sqrt{k})^2$ and $b = (1-\sqrt{k})^2$, with $0 < k < 1$, to produce

$$\text{AGM}(1+k+2\sqrt{k}, 1+k-2\sqrt{k}) = \text{AGM}(1+k, 1-k). \tag{1-8}$$

He then used the homogeneity of AGM to write

$$\begin{aligned}\text{AGM}(1+k+2\sqrt{k}, 1+k-2\sqrt{k}) &= \text{AGM}\big((1+k)(1+k^*), (1+k)(1-k^*)\big)\\ &= (1+k)\text{AGM}(1+k^*, 1-k^*),\end{aligned}$$

with

$$k^* = \frac{2\sqrt{k}}{1+k}. \tag{1-9}$$

This resulted in the functional equation

$$\mathrm{AGM}(1+k, 1-k) = (1+k)\,\mathrm{AGM}(1+k^*, 1-k^*). \tag{1-10}$$

In his analysis of (1-10), Gauss substituted the power series

$$\frac{1}{\mathrm{AGM}(1+k, 1-k)} = \sum_{n=0}^{\infty} a_n k^{2n} \tag{1-11}$$

into (1-10) and solved an infinite system of nonlinear equations to produce

$$a_n = 2^{-2n} \binom{2n}{n}^2. \tag{1-12}$$

Then he recognized the series as that of an elliptic integral to obtain

$$\frac{1}{\mathrm{AGM}(1+k, 1-k)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}. \tag{1-13}$$

This is a remarkable tour de force.

The function

$$K(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1-k^2 \sin^2 x}} \tag{1-14}$$

is the *elliptic integral of the first kind*. It can also be written in the algebraic form

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{1-15}$$

In this notation, (1-10) becomes

$$K(k^*) = (1+k)K(k). \tag{1-16}$$

This is the *Landen transformation* for the complete elliptic integral. John Landen [1775], the namesake of the transformation, studied related integrals: for example,

$$\kappa := \int_0^1 \frac{dx}{\sqrt{x(1-x^2)}} \,. \tag{1-17}$$

He derived identities such as

$$\kappa = \varepsilon + \sqrt{\varepsilon^2 - \pi}\ , \quad \text{where } \varepsilon := \int_0^{\pi/2} \sqrt{2-\sin^2\theta}\; d\theta\ , \tag{1-18}$$

proven mainly by suitable changes of variables in the integral for ε. In [Watson 1933] the reader will find a historical account of Landen's work, including the above identities.

The reader will find in [Borwein and Borwein 1987] and [McKean and Moll 1997] proofs in a variety of styles. In trigonometric form, the Landen transformation states that

$$G(a,b)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}} \tag{1-19}$$

is invariant under the change of parameters

$$(a,b)\mapsto\left(\frac{a+b}{2},\sqrt{ab}\right).$$

D. J. Newman [1985] presents a very clever proof: the change of variables $x=b\tan\theta$ yields

$$G(a,b)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{(a^2+x^2)(b^2+x^2)}}. \tag{1-20}$$

Now let $x\mapsto x+\sqrt{x^2+ab}$ to complete the proof. Many of the above identities can now be searched for and proven on a computer [Borwein and Bailey 2003].

2. An interlude: the quartic integral

The evaluation of definite integrals of rational functions is one of the standard topics in Integral Calculus. Motivated by the lack of success of symbolic languages, we began a systematic study of these integrals. *A posteriori*, one learns that even rational functions are easier to deal with. Thus we start with one having *a power of a quartic* in its denominator. The evaluation of the identity

$$\int_0^{\infty}\frac{dx}{(x^4+2ax^2+1)^{m+1}}=\frac{\pi}{2^{m+3/2}\,(a+1)^{m+1/2}}P_m(a), \tag{2-1}$$

where

$$P_m(a)=\sum_{l=0}^{m}d_l(m)a^l \tag{2-2}$$

with

$$d_l(m)=2^{-2m}\sum_{k=l}^{m}2^k\binom{2m-2k}{m-k}\binom{m+k}{m}\binom{k}{l}, \tag{2-3}$$

was first established in [Boros and Moll 1999b].

A standard hypergeometric argument yields

$$P_m(a)=P_m^{(\alpha,\beta)}(a), \tag{2-4}$$

where

$$P_m^{(\alpha,\beta)}(a) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m+\beta}{m-k} \binom{m+k+\alpha+\beta}{k} 2^{-k} (a+1)^k \tag{2-5}$$

is the classical Jacobi polynomial; the parameters α and β are given by $\alpha = m+\frac{1}{2}$ and $\beta = -m-\frac{1}{2}$. A general description of these functions and their properties are given in [Abramowitz and Stegun 1972]. The twist here is that they depend on m, which means most of the properties of P_m had to be proven from scratch. For instance, P_m satisfies the recurrence

$$P_m(a) = \frac{(2m-3)(4m-3)a}{4m(m-1)(a-1)} P_{m-2}(a) - \frac{(4m-3)a(a+1)}{2m(m-1)(a-1)} P'_{m-2}(a) + \frac{4m(a^2-1)+1-2a^2}{2m(a-1)} P_m(a).$$

This *cannot* be obtained by replacing $\alpha = m+\frac{1}{2}$ and $\beta = -m-\frac{1}{2}$ in the standard recurrence for the Jacobi polynomials. The reader will find in [Amdeberhan and Moll 2007] several different proofs of (2-1).

The polynomials $P_m(a)$ makes a surprising appearance in the expansion

$$\sqrt{a+\sqrt{1+c}} = \sqrt{a+1}\left(1 - \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{P_{k-1}(a)\, c^k}{2^{k+1}\,(a+1)^k}\right) \tag{2-6}$$

as described in [Boros and Moll 2001a]. The special case $a = 1$ appears in [Bromwich 1926], page 191, exercise 21. Ramanujan had a more general expression, but only for the case $c = a^2$:

$$(a+\sqrt{1+a^2})^n = 1 + na + \sum_{k=2}^{\infty} \frac{b_k(n) a^k}{k!}, \tag{2-7}$$

where, for $k \geq 2$,

$$b_k(n) = \begin{cases} n^2(n^2-2^2)(n^2-4^2)\cdots(n^2-(k-2)^2) & \text{if } k \text{ is even,} \\ n(n^2-1^2)(n^2-3^2)\cdots(n^2-(k-2)^2) & \text{if } k \text{ is odd.} \end{cases} \tag{2-8}$$

This result appears in [Berndt and Bowman 2000] as Corollary 2 to Entry 14 and is machine-checkable, as are many of the identities in this section.

The coefficients $d_l(m)$ in (2-3) have many interesting properties:

• They form a *unimodal sequence*: there exists an index $0 \leq m^* \leq m$ such that $d_j(m)$ increases up to $j = m^*$ and decreases from then on. See [Boros and Moll 1999a] for a proof of the more general statement: *If $P(x)$ is a polynomial* *with nondecreasing, nonnegative coefficients, then the coefficient sequence of* $P(x+1)$ *is unimodal.*

• They form a *log-concave sequence*: define the operator

$$\mathfrak{L}(\{a_k\}) := \{a_k^2 - a_{k-1}a_{k+1}\}$$

acting on sequences of positive real numbers. A sequence $\{a_k\}$ is called log-concave if its image under $\mathfrak{L}$ is again a sequence of positive numbers; i.e. $a_k^2 - a_{k-1}a_{k+1} \geq 0$. Note that this condition is satisfied if and only if the sequence $\{b_k := \log(a_k)\}$ is concave, hence the name. We refer the reader to [Wilf 1990] for a detailed introduction. The log-concavity of $d_l(m)$ was established in [Kauers and Paule 2007] using computer algebra techniques: in particular, cylindrical algebraic decompositions as developed in [Caviness and Johnson 1998] and [Collins 1975].

• They produce interesting polynomials: in [Boros et al. 2001] one finds the representation

$$d_l(m) = \frac{A_{l,m}}{l!\, m!\, 2^{m+l}}, \tag{2-9}$$

with

$$A_{l,m} = \alpha_l(m) \prod_{k=1}^{m} (4k-1) - \beta_l(m) \prod_{k=1}^{m} (4k+1). \tag{2-10}$$

Here α_l and β_l are polynomials in m of degrees l and $l-1$, respectively. For example, $\alpha_1(m) = 2m+1$ and $\beta_1(m) = 1$, so that the coefficient of the linear term of $P_m(a)$ is

$$d_1(m) = \frac{1}{m!\, 2^{m+1}} \left((2m+1) \prod_{k=1}^{m} (4k-1) - \prod_{k=1}^{m} (4k+1) \right). \tag{2-11}$$

J. Little [2005] established the remarkable fact that the polynomials $\alpha_l(m)$ and $\beta_l(m)$ have all their roots on the vertical line $\operatorname{Re} m = -\frac{1}{2}$.

When we showed this to Henry, he simply remarked: *the only thing you have to do now is to let* $l \to \infty$ *and get the Riemann hypothesis*. The proof in [Little 2005] consists in a study of the recurrence

$$y_{l+1}(s) = 2sy_l(s) - \left(s^2 - (2l-1)^2\right) y_{l-1}(s), \tag{2-12}$$

satisfied by $\alpha_l((s-1)/2)$ and $\beta_l((s-1)/2)$. There is no Number Theory in the proof, so it is not likely to connect to the Riemann zeta function $\zeta(s)$, but one never knows.

The arithmetical properties of $A_{l,m}$ are beginning to be elucidated. We have shown that their 2-adic valuation satisfies

$$\nu_2(A_{l,m}) = \nu_2((m+1-l)_{2l}) + l, \tag{2-13}$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the Pochhammer symbol. This expression allows for a combinatorial interpretation of the block structure of these valuations. See [Amdeberhan et al. 2007] for details.

3. The incipient rational Landen transformation

The clean analytic expression in (2-1) is not expected to extend to rational functions of higher order. In our analysis we distinguish according to the domain of integration: the finite interval case, mapped by a bilinear transformation to $[0,\infty)$, and the whole line. In this section we consider the definite integral,

$$U_6(a,b;c,d,e) = \int_0^\infty \frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\,dx, \tag{3-1}$$

as the simplest case on $[0,\infty)$. The case of the real line is considered below. The integrand is chosen to be even by necessity: *none of the techniques in this section work for the odd case*. We normalize two of the coefficients in the denominator in order to reduce the number of parameters. The standard approach for the evaluation of (3-1) is to introduce the change of variables $x = \tan\theta$. This leads to an intractable trigonometric integral.

A different result is obtained if one first symmetrizes the denominator: we say that a polynomial of degree d is *reciprocal* if $Q_d(1/x) = x^{-d}Q_d(x)$, that is, the sequence of its coefficients is a palindrome. Observe that if Q_d is any polynomial of degree d, then

$$T_{2d}(x) = x^d Q_d(x) Q_d(1/x) \tag{3-2}$$

is a reciprocal polynomial of degree $2d$. For example, if

$$Q_6(x) = x^6+ax^4+bx^2+1. \tag{3-3}$$

then

$$T_{12}(x) = x^{12}+(a+b)x^{10}+(a+b+ab)x^8 \\ +(2+a^2+b^2)x^6+(a+b+ab)x^4+(a+b)x^2+1.$$

The numerator and denominator in the integrand of (3-1) are now scaled by $x^6Q_6(1/x)$ to produce a new integrand with reciprocal denominator:

$$U_6 = \int_0^\infty \frac{S_{10}(x)}{T_{12}(x)}\,dx, \tag{3-4}$$

where we write

$$S_{10}(x) = \sum_{j=0}^{5} s_j x^{2j} \text{ and } T_{12}(x) = \sum_{j=0}^{6} t_j x^{2j}. \tag{3-5}$$

The change of variables $x = \tan\theta$ now yields

$$U_6 = \int_0^{\pi/2} \frac{S_{10}(\tan\theta)\, \cos^{10}(\theta)}{T_{12}(\tan\theta)\, \cos^{12}(\theta)}\, d\theta. \tag{3-6}$$

Now let $w = \cos 2\theta$ and use $\sin^2\theta = \frac{1}{2}(1-w)$ and $\cos^2\theta = \frac{1}{2}(1+w)$ to check that the numerator and denominator of the new integrand,

$$S_{10}(\tan\theta)\, \cos^{10}\theta = \sum_{j=0}^{5} s_j \sin^{2j}\theta\, \cos^{10-2j}\theta \tag{3-7}$$

and

$$T_{12}(\tan\theta)\cos^{12}\theta = \sum_{j=0}^{6} t_j \sin^{2j}\theta \cos^{12-2j}\theta = 2^{-6}\sum_{j=0}^{6} t_j (1-w)^j (1+w)^{6-j},$$

are both polynomials in w. The mirror symmetry of T_{12}, reflected in $t_j = t_{6-j}$, shows that the new denominator is an *even* polynomial in w. The symmetry of cosine about $\pi/2$ shows that the terms with odd power of w have a vanishing integral. Thus, with $\psi = 2\theta$, and using the symmetry of the integrand to reduce the integral from $[0, \pi]$ to $[0, \pi/2]$, we obtain

$$U_6 = \int_0^{\pi/2} \frac{r_4 \cos^4\psi + r_2 \cos^2\psi + r_0}{q_6 \cos^6\psi + q_4\cos^4\psi + q_2\cos^2\psi + q_0}\, d\psi. \tag{3-8}$$

The parameters r_j, q_j have explicit formulas in terms of the original parameters of U_6. This even rational function of $\cos\psi$ can now be expressed in terms of $\cos 2\psi$ to produce (letting $\theta \leftarrow 2\psi$)

$$U_6 = \int_0^{\pi} \frac{\alpha_2\cos^2\theta + \alpha_1\cos\theta + \alpha_0}{\beta_3\cos^3\theta + \beta_2\cos^2\theta + \beta_1\cos\theta + \beta_0}\, d\theta. \tag{3-9}$$

The final change of variables $y = \tan\frac{\theta}{2}$ yields a new rational form of the integrand:

$$U_6 = \int_0^{\infty} \frac{c_1 y^4 + d_1 y^2 + e_1}{y^6 + a_1 y^4 + b_1 y^2 + 1}\, dy. \tag{3-10}$$

Keeping track of the parameters, we have established:

THEOREM 3.1. *The integral*

$$U_6 = \int_0^{\infty} \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx \tag{3-11}$$

is invariant under the change of parameters

$$a_1 \leftarrow \frac{ab + 5a + 5b + 9}{(a+b+2)^{4/3}}, \quad b_1 \leftarrow \frac{a+b+6}{(a+b+2)^{2/3}},$$

for the denominator parameters and

$$c_1 \leftarrow \frac{c+d+e}{(a+b+2)^{2/3}}, \quad d_1 \leftarrow \frac{(b+3)c+2d+(a+3)e}{a+b+2}, \quad e_1 \leftarrow \frac{c+e}{(a+b+2)^{1/3}}$$

for those of the numerator.

Theorem 3.1 is the precise analogue of the elliptic Landen transformation (1-1) for the case of a rational integrand. We call (3-12) a *rational Landen transformation*. This construction was first presented in [Boros and Moll 2000].

3.1. Even rational Landen transformations. More generally, there is a similar transformation of coefficients for *any even rational integrand*; details appear in [Boros and Moll 2001b]. We call these *even rational Landen Transformations*. The obstruction in the general case comes from (3-7); one does not get a polynomial in $w = \cos 2\theta$.

The method of proof for even rational integrals can be summarized as follows.

1) Start with an even rational integral:

$$U_{2p} = \int_0^\infty \frac{\text{even polynomial in } x}{\text{even polynomial in } x}\, dx. \tag{3-12}$$

2) Symmetrize the denominator to produce

$$U_{2p} = \int_0^\infty \frac{\text{even polynomial in } x}{\text{even reciprocal polynomial in } x}\, dx. \tag{3-13}$$

The degree of the denominator is doubled.

3) Let $x = \tan\theta$. Then

$$U_{2p} = \int_0^{\pi/2} \frac{\text{polynomial in } \cos 2\theta}{\text{even polynomial in} \cos 2\theta}\, d\theta. \tag{3-14}$$

4) Symmetry produced the vanishing of the integrands with an odd power of $\cos\theta$ in the numerator. We obtain

$$U_{2p} = \int_0^{\pi/2} \frac{\text{even polynomial in } \cos 2\theta}{\text{even polynomial in } \cos 2\theta}\, d\theta. \tag{3-15}$$

5) Let $\psi = 2\theta$ to produce

$$U_{2p} = \int_0^{\pi} \frac{\text{even polynomial in } \cos\psi}{\text{even polynomial in } \cos\psi}\, d\psi. \tag{3-16}$$

Using symmetry this becomes an integral over $[0, \pi/2]$.

6) Let $y = \tan\psi$ and use $\cos\psi = 1/\sqrt{1+y^2}$ to obtain

$$U_{2p} = \int_0^\infty \frac{\text{even polynomial in } y}{\text{even polynomial in } y}\, dy. \tag{3-17}$$

The degree of the denominator is half of what it was in Step 5.

Keeping track of the degrees one checks that the degree of the new rational function is the same as the original one, with new coefficients that appear as functions of the old ones.

4. A geometric interpretation

We now present a geometric foundation of the general even rational Landen transformation (3-12) using the theory of Riemann surfaces. The text [Springer 2002] provides an introduction to this theory, including definitions of objects we will refer to here. The sequence of transformations in Section 3 can be achieved in one step by relating $\tan 2\theta$ to $\tan\theta$. For historical reasons (this is what we did first) we present the details with *cotangent* instead of tangent.

Consider the even rational integral

$$I = \int_0^\infty R(x)\, dx = \frac{1}{2}\int_{-\infty}^\infty R(x)\, dx. \tag{4-1}$$

Introduce the new variable

$$y = R_2(x) = \frac{x^2-1}{2x}, \tag{4-2}$$

motivated by the identity $\cot 2\theta = R_2(\cot\theta)$. The function $R_2 : \mathbb{R} \to \mathbb{R}$ is a two-to-one map. The sections of the inverse are

$$x = \sigma_\pm(y) = y \pm \sqrt{y^2+1}. \tag{4-3}$$

Splitting the original integral as

$$I = \int_{-\infty}^0 R(x)\, dx + \int_0^\infty R(x)\, dx \tag{4-4}$$

and introducing $x = \sigma_+(y)$ in the first and $x = \sigma_-(y)$ in the second integral, yields

$$I = \int_{-\infty}^\infty \left(R_+(y) + R_-(y)\right)\, dy \tag{4-5}$$

where

$$\begin{aligned} R_+(y) &= R(\sigma_+(y)) + R(\sigma_-(y)), \\ R_-(y) &= \frac{y}{\sqrt{y^2+1}}\big(R(\sigma_+(y)) - R(\sigma_-(y))\big). \end{aligned} \tag{4-6}$$

A direct calculation shows that R_+ and R_- are rational functions of degree at most that of R.

The change of variables $y = R_2(x)$ converts the meromorphic differential $\varphi = R(x)\,dx$ into

$$\begin{aligned} R(\sigma_+(y))\frac{d\sigma_+}{dy} + R(\sigma_-(y))\frac{d\sigma_-}{dy} \\ &= \left((R(\sigma_+) + R(\sigma_-)) + \frac{y(R(\sigma_+) - R(\sigma_-))}{\sqrt{y^2+1}} \right) dy \\ &= (R_+(y) + R_-(y))\ dy. \end{aligned}$$

The general situation is this: start with a finite ramified cover $\pi : X \to Y$ of Riemann surfaces and a meromorphic differential φ on X. Let $U \subset Y$ be a simply connected domain that contains no critical values of π, and let $\sigma_1, \dots, \sigma_k : U \to X$ be the distinct sections of π. Define

$$\pi_*\varphi\big|_U = \sum_{j=1}^{k} \sigma_j^* \varphi. \tag{4-7}$$

In [Hubbard and Moll 2003] we show that this construction preserves analytic 1-forms, that is, if φ is an analytic 1-form in X then $\pi_*\varphi$ is an analytic 1-form in Y. Furthermore, for any rectifiable curve γ on Y, we have

$$\int_\gamma \pi_*\varphi = \int_{\pi^{-1}\gamma} \varphi. \tag{4-8}$$

In the case of projective space, this leads to:

LEMMA 4.1. *If $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ is analytic, and $\varphi = R(z)\,dz$ with R a rational function, then $\pi_*\varphi$ can be written as $R_1(z)\,dz$ with R_1 a rational function of degree at most the degree of R.*

This is the generalization of the fact that the integrals in (4-1) and (4-5) are the same.

5. A further generalization

The procedure described in Section 3 can be extended with the rational map R_m, defined by the identity

$$\cot m\theta = R_m(\cot\theta). \tag{5-1}$$

Here $m \in \mathbb{N}$ is arbitrary greater or equal than 2. We present some elementary properties of the rational function R_m.

PROPOSITION 5.1. *The rational function R_m satisfies*:

1) *For* $m \in \mathbb{N}$ *define*

$$P_m(x) := \sum_{j=0}^{\lfloor m/2 \rfloor} (-1)^j \binom{m}{2j} x^{m-2j},$$
$$Q_m(x) := \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} (-1)^j \binom{m}{2j+1} x^{m-(2j+1)}.$$

Then $R_m := P_m/Q_m$.

2) *The function* R_m *is conjugate to* $f_m(x) := x^m$ *via* $M(x) := \dfrac{x+i}{x-i}$; *that is*, $R_m = M^{-1} \circ f_m \circ M$.

3) *The polynomials* P_m *and* Q_m *have simple real zeros given by*

$$p_k := \cot \frac{(2k+1)\pi}{2m} \quad \text{for } 0 \le k \le m-1,$$
$$q_k := \cot \frac{k\pi}{m} \quad \text{for } 1 \le k \le m-1.$$

If we change the domain to the entire real line, we can, using the rational substitutions $R_m(x) \mapsto x$, produce a rational Landen transformation for an arbitrary integrable rational function $R(x) = B(x)/A(x)$ for each integer value of m. The result is a new list of coefficients, from which one produces a second rational function $R^{(1)}(x) = J(x)/H(x)$ with

$$\int_{-\infty}^{\infty} \frac{B(x)}{A(x)}\, dx = \int_{-\infty}^{\infty} \frac{J(x)}{H(x)}\, dx. \tag{5-2}$$

Iteration of this procedure yields a sequence $\boldsymbol{x}_n$, that has a limit $\boldsymbol{x}_\infty$ with convergence of order m, that is,

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}_\infty\| \le C \|\boldsymbol{x}_n - \boldsymbol{x}_\infty\|^m. \tag{5-3}$$

We describe this procedure here in the form of an algorithm; proofs appear in [Manna and Moll 2007a].

Lemma 4.1 applied to the map $\pi(x) = R_m(x)$, viewed as ramified cover of $\mathbb{P}^1$, guarantees the existence of a such new rational function $R^{(1)}$. The question of effective computation of the coefficients of J and H is discussed below. In particular, we show that all these calculations can be done symbolically.

• Algorithm for deriving rational Landen transformations

Step 1. The initial data is a rational function $R(x) := B(x)/A(x)$. We assume that A and B are polynomials with real coefficients and A has no real zeros and write

$$A(x):=\sum_{k=0}^{p} a_k x^{p-k} \text{ and } B(x):=\sum_{k=0}^{p-2} b_k x^{p-2-k}. \tag{5-4}$$

Step 2. Choose a positive integer $m \geq 2$.

Step 3. Introduce the polynomial

$$H(x):=\operatorname{Res}_z(A(z), P_m(z)-xQ_m(z)) \tag{5-5}$$

and write it as

$$H(x):=\sum_{l=0}^{p} e_l x^{p-l}. \tag{5-6}$$

The polynomial H is thus defined as the determinant of the Sylvester matrix which is formed of the polynomial coefficients. As such, the coefficients e_l of $H(x)$ themselves are integer polynomials in the a_i. Explicitly,

$$e_l=(-1)^l a_0^m \prod_{j=1}^{p} Q_m(x_j)\times\sigma_l^{(p)}(R_m(x_1),\ R_m(x_2),\ \dots,\ R_m(x_p)), \tag{5-7}$$

where $\{x_1, x_2, \dots, x_p\}$ are the roots of A, each written according to multiplicity. The functions $\sigma_l^{(p)}$ are the elementary symmetric functions in p variables defined by

$$\prod_{l=1}^{p}(y-y_l)=\sum_{l=0}^{p}(-1)^l\sigma_l^{(p)}(y_1,\dots,y_p)y^{p-l}. \tag{5-8}$$

It is possible to compute the coefficients e_l symbolically from the coefficients of A, without the knowledge of the roots of A.

Also define

$$E(x):=H(R_m(x))\times Q_m(x)^p. \tag{5-9}$$

Step 4. The polynomial A divides E and we denote the quotient by Z. The coefficients of Z are integer polynomials in the a_i.

Step 5. Define the polynomial $C(x):=B(x)Z(x)$.

Step 6. There exists a polynomial $J(x)$, whose coefficients have an explicit formula in terms of the coefficients c_j of $C(x)$, such that

$$\int_{-\infty}^{\infty}\frac{B(x)}{A(x)}\,dx=\int_{-\infty}^{\infty}\frac{J(x)}{H(x)}\,dx. \tag{5-10}$$

This new integrand is the rational function whose existence is guaranteed by Lemma 4.1. The explicit computation of the coefficients of J can be found in [Manna and Moll 2007a]. This is the *rational Landen transformation* of order m.

EXAMPLE 5.1. Completing the algorithm with $m = 3$ and the rational function

$$R(x) = \frac{1}{ax^2 + bx + c}, \tag{5-11}$$

produces the result stated below. Notice that the values of the iterates are ratios of integer polynomials of degree 3, as was stated above. The details of this example appear in [Manna and Moll 2007b].

THEOREM 5.2. *The integral*

$$I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c} \tag{5-12}$$

is invariant under the transformation

$$a \mapsto \frac{a}{\Delta}\big((a+3c)^2 - 3b^2\big), \quad b \mapsto \frac{b}{\Delta}\big(3(a-c)^2 - b^2\big), \quad c \mapsto \frac{c}{\Delta}\big((3a+c)^2 - 3b^2\big), \tag{5-13}$$

where $\Delta := (3a+c)(a+3c) - b^2$. *The condition* $b^2 - 4ac < 0$, *imposed to ensure convergence of the integral*, *is preserved by the iteration*.

EXAMPLE 5.2. In this example we follow the steps described above in order to produce a rational Landen transformation of order 2 for the integral

$$I = \int_{-\infty}^{\infty} \frac{b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4}{a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6}\, dx. \tag{5-14}$$

Recall that the algorithm starts with a rational function $R(x)$ and produces a new function $\mathfrak{L}_2(R(x))$ satisfying

$$\int_{-\infty}^{\infty} R(x)\, dx = \int_{-\infty}^{\infty} \mathfrak{L}_2(R(x))\, dx. \tag{5-15}$$

Step 1. The initial data is $R(x) = B(x)/A(x)$ with

$$A(x) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6, \tag{5-16}$$

and

$$B(x) = b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4. \tag{5-17}$$

The parameter p is the degree of A, so $p = 6$.

Step 2. We choose $m = 2$ to produce a method of order 2. The algorithm employs the polynomials $P_2(z) = z^2 - 1$ and $Q_2(z) = 2z$.

Step 3. The polynomial

$$H(x) := \mathrm{Res}_z(A(z), z^2 - 1 - 2xz) \tag{5-18}$$

is computed with the Mathematica command `Resultant` to obtain

$$H(x) = e_0 x^6 + e_1 x^5 + e_2 x^4 + e_3 x^3 + e_4 x^2 + e_5 x + e_6, \tag{5-19}$$

where

$$\begin{aligned}
e_0 &= 64 a_0 a_6,\\
e_1 &= -32(a_0 a_5 - a_1 a_6), \qquad\qquad (5\text{-}20)\\
e_2 &= 16(a_0 a_4 - a_1 a_5 + 6 a_0 a_6 + a_2 a_6),\\
e_3 &= -8(a_0 a_3 - a_1 a_4 + 5 a_0 a_5 + a_2 a_5 - 5 a_1 a_6 - a_3 a_6),\\
e_4 &= 4(a_0 a_2 - a_1 a_3 + 4 a_0 a_4 + a_2 a_4 - 4 a_1 a_5 - a_3 a_5 + 9 a_0 a_6 + 4 a_2 a_6 + a_4 a_6),\\
e_5 &= -2(a_0 a_1 - a_1 a_2 + 3 a_0 a_3 + a_2 a_3 - 3 a_1 a_4 - a_3 a_4 + 5 a_0 a_5\\
&\qquad + 3 a_2 a_5 + a_4 a_5 - 5 a_1 a_6 - 3 a_3 a_6 - a_5 a_6),\\
e_6 &= (a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6)(a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6).
\end{aligned}$$

The polynomial $H(x)$ is the denominator of the integrand $\mathfrak{L}_2(R(x))$ in (5-15).

In Step 3 we also define

$$E(x) = H(R_2(x)) Q_2^6(x) = H\left(\frac{x^2 - 1}{2x}\right) \cdot (2x)^6. \tag{5-21}$$

The function $E(x)$ is a polynomial of degree 12, written as

$$E(x) = \sum_{k=0}^{12} \alpha_k x^{12-k}. \tag{5-22}$$

Using the expressions for e_j in (5-20) in (5-21) yields

$$\begin{aligned}
\alpha_0 &= \alpha_{12} = 64 a_0 a_6,\\
\alpha_1 &= -\alpha_{11} = -64(a_0 a_5 - a_1 a_6),\\
\alpha_2 &= \alpha_{10} = 64(a_0 a_4 - a_1 a_5 + a_2 a_6),\\
\alpha_3 &= -\alpha_9 = -64(a_0 a_3 - a_1 a_4 + a_2 a_5 - a_3 a_6), \qquad (5\text{-}23)\\
\alpha_4 &= \alpha_8 = 64(a_0 a_2 - a_1 a_3 + a_2 a_4 - a_3 a_5 + a_4 a_6),\\
\alpha_5 &= -\alpha_7 = -64(a_0 a_1 - a_1 a_2 + a_2 a_3 - a_3 a_4 + a_4 a_5 - a_5 a_6),\\
\alpha_6 &= 64(a_0^2 - a_1^2 + a_2^2 - a_3^2 + a_4^2 - a_5^2 + a_6^2).
\end{aligned}$$

Step 4. The polynomial $A(x)$ always divides $E(x)$. The quotient is denoted by $Z(x)$. The values of α_j given in (5-23) produce

$$Z(x) = 64(a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - a_5 x^5 + a_6 x^6). \tag{5-24}$$

Step 5. Define the polynomial $C(x) := B(x)Z(x)$. In this case, C is of degree 10, written as

$$C(x) = \sum_{k=0}^{10} c_k x^{10-k}, \tag{5-25}$$

and the coefficients c_k are given by

$$\begin{aligned}
c_0 &= 64 a_6 b_0, \\
c_1 &= -64(a_5 b_0 - a_6 b_1), \\
c_2 &= 64(a_4 b_0 - a_5 b_1 + a_6 b_2), \\
c_3 &= -64(a_3 b_0 - a_4 b_1 + a_5 b_2 - a_6 b_3), \\
c_4 &= 64(a_2 b_0 - a_3 b_1 + a_4 b_2 - a_5 b_3 + a_6 b_4), \\
c_5 &= -64(a_1 b_0 - a_2 b_1 + a_3 b_2 - a_4 b_3 + a_5 b_4), \\
c_6 &= 64(a_0 b_0 - a_1 b_1 + a_2 b_2 - a_3 b_3 + a_4 b_4), \\
c_7 &= 64(a_0 b_1 - a_1 b_2 + a_2 b_3 - a_3 b_4), \\
c_8 &= 64(a_0 b_2 - a_1 b_3 + a_2 b_4), \\
c_9 &= 64(a_0 b_3 - a_1 b_4), \\
c_{10} &= 64 a_0 b_4.
\end{aligned} \tag{5-26}$$

Step 6 produces the numerator $J(x)$ of the new integrand $\mathfrak{L}_2(R(x))$ from the coefficients c_j given in (5-26). The function $J(x)$ is a polynomial of degree 4, written as

$$J(x) = \sum_{k=0}^{4} j_k x^{4-k}. \tag{5-27}$$

Using the values of (5-26) we obtain

$$\begin{aligned}
j_0 &= 32(a_6 b_0 + a_0 b_4), \qquad (5\text{-}28)\\
j_1 &= -16(a_5 b_0 - a_6 b_1 + a_0 b_3 - a_1 b_4), \\
j_2 &= 8(a_4 b_0 + 3a_6 b_0 - a_5 b_1 + a_0 b_2 + a_6 b_2 - a_1 b_3 + 3a_0 b_4 + a_2 b_4), \\
j_3 &= -4(a_3 b_0 + 2a_5 b_0 + a_0 b_1 - a_4 b_1 - 2a_6 b_1 - a_1 b_2 + a_5 b_2 \\
&\qquad + 2a_0 b_3 + a_3 b_3 - a_6 b_3 - 2a_1 b_4 - a_3 b_4), \\
j_4 &= 2(a_0 b_0 + a_2 b_0 + a_4 b_0 + a_6 b_0 - a_1 b_1 - a_3 b_1 - a_5 b_1 \\
&\qquad + a_0 b_2 + a_2 b_2 + a_4 b_2 + a_6 b_2 - a_1 b_3 \\
&\qquad - a_3 b_3 - a_5 b_3 + a_0 b_4 + a_2 b_4 + a_4 b_4 + a_6 b_4).
\end{aligned}$$

The explicit formula used to compute the coefficients of J can be found in [Manna and Moll 2007a].

The new rational function is

$$\mathfrak{L}_2(R(x)) := \frac{J(x)}{H(x)}, \tag{5-29}$$

with $J(x)$ given in (5-27) and $H(x)$ in (5-19). The transformation is

$$\frac{b_0x^4 + b_1x^3 + b_2x^2 + b_3x + b_4}{a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6} \mapsto \frac{j_0x^4 + j_1x^3 + j_2x^2 + j_3x + j_4}{e_0x^6 + e_1x^5 + e_2x^4 + e_3x^3 + e_4x^2 + e_5x + e_6}.$$

The numerator coefficients are given in (5-20) and the denominator ones in (5-28), explicitly as polynomials in the coefficients of the original rational function. The generation of these polynomials is a completely symbolic procedure.

The first two steps of this algorithm, applied to the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{x^6 + x^3 + 1} = \frac{\pi}{9}\big(2\sqrt{3}\cos(\pi/9) + \sqrt{3}\cos(2\pi/9) + 3\sin(2\pi/9)\big), \tag{5-30}$$

produces the identities

$$\int_{-\infty}^{\infty} \frac{dx}{x^6 + x^3 + 1} = \int_{-\infty}^{\infty} \frac{2(16x^4 + 12x^2 + 2x + 2)}{64x^6 + 96x^4 + 36x^2 + 3}\,dx$$
$$= \int_{-\infty}^{\infty} \frac{4(2816x^4 - 1024x^3 + 8400x^2 - 884x + 5970)}{12288x^6 + 59904x^4 + 87216x^2 + 39601}\,dx.$$

6. The issue of convergence

The convergence of the double sequence (a_n, b_n) appearing in the elliptic Landen transformation (1-1) is easily established. Assume $0 < b_0 \le a_0$, then the arithmetic-geometric inequality yields $b_n \le b_{n+1} \le a_{n+1} \le a_n$. Also

$$0 \le a_{n+1} - b_{n+1} = \frac{1}{2}\frac{(a_n - b_n)^2}{(\sqrt{a_n} + \sqrt{b_n})^2}. \tag{6-1}$$

This shows a_n and b_n have a common limit: $M = \mathrm{AGM}(a, b)$, the arithmetic-geometric of a and b. The convergence is quadratic:

$$|a_{n+1} - M| \le C|a_n - M|^2, \tag{6-2}$$

for some constant $C > 0$ independent of n. Details can be found in [Borwein and Borwein 1987].

The Landen transformations produce maps on the space of coefficients of the integrand. In this section, we discuss the convergence of the rational Landen transformations. This discussion is divided in two cases:

Case 1: the half-line. Let $R(x)$ be an even rational function, written as $R(x) = P(x)/Q(x)$, with

$$P(x) = \sum_{k=0}^{p-1} b_k x^{2(p-1-k)}, \; Q(x) = \sum_{k=0}^{p} a_k x^{2(p-k)}, \tag{6-3}$$

and $a_0 = a_p = 1$. The *parameter space* is

$$\mathfrak{P}_{2p}^{+} = \{(a_1, \dots, a_{p-1}; b_0, \dots, b_{p-1})\} \subset \mathbb{R}^{p-1} \times \mathbb{R}^{p}. \tag{6-4}$$

We write

$$\boldsymbol{a} : = (a_1, \dots, a_{p-1}), \quad \boldsymbol{b} : = (b_0, \dots, b_p). \tag{6-5}$$

Define

$$\Lambda_{2p} = \left\{ (a_1, \dots, a_{p-1}) \in \mathbb{R}^{p-1} : \int_0^\infty R(x)\, dx \text{ is finite} \right\}. \tag{6-6}$$

Observe that the convergence of the integral depends only on the parameters in the denominator.

The Landen transformations provide a map

$$\Phi_{2p} : \mathfrak{P}_{2p}^{+} \to \mathfrak{P}_{2p}^{+} \tag{6-7}$$

that preserves the integral. Introduce the notation

$$\boldsymbol{a}_n = (a_1^{(n)}, \dots, a_{p-1}^{(n)}) \text{ and } \boldsymbol{b}_n = (b_0^{(n)}, \dots, b_p^{(n)}), \tag{6-8}$$

where

$$(\boldsymbol{a}_n, \boldsymbol{b}_n) = \Phi_{2p}(\boldsymbol{a}_{n-1}, \boldsymbol{b}_{n-1}) \tag{6-9}$$

are the iterates of the map Φ_{2p}.

The result that one expects is this:

THEOREM 6.1. *The region Λ_{2p} is invariant under the map Φ_{2p}. Moreover*

$$\boldsymbol{a}_n \to \left(\binom{p}{1}, \binom{p}{2}, \dots, \binom{p}{p-1} \right), \tag{6-10}$$

and there exists a number L, that depends on the initial conditions, such that

$$\boldsymbol{b}_n \to \left(\binom{p-1}{0} L, \binom{p-1}{1} L, \dots, \binom{p-1}{p-1} L \right). \tag{6-11}$$

This is equivalent to saying that the sequence of rational functions formed by the Landen transformations, converge to $L/(x^2+1)$.

This was established in [Hubbard and Moll 2003] using the geometric language of Landen transformations which, while unexpected, is satisfactory.

THEOREM 6.2. *Let φ be a 1-form, holomorphic in a neighborhood of $\mathbb{R} \subset \mathbb{P}^1$. Then*

$$\lim_{n\to\infty} (\pi_*)^n \varphi = \frac{1}{\pi} \left(\int_{-\infty}^{\infty} \varphi \right) \frac{dz}{1+z^2}, \tag{6-12}$$

where the convergence is uniform on compact subsets of U, the neighborhood in the definition of π_*.

The proof is detailed for the map $\pi(z) = \dfrac{z^2-1}{2z} = R_2(z)$, but it extends without difficulty to the generalization R_m.

Theorem 6.2 can be equivalently reformulated as:

THEOREM 6.3. *The iterates of the Landen transformation starting at $(\boldsymbol{a}_0, \boldsymbol{b}_0) \in \mathfrak{P}^+_{2p}$ converge* (*to the limit stated in Theorem* 6.1) *if and only if the integral formed by the initial data is finite.*

It would be desirable to establish this result by purely dynamical techniques. This has been established only for the case $p = 3$. In that case the Landen transformation for

$$U_6 := \int_0^{\infty} \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\, dx \tag{6-13}$$

is

$$a_1 \leftarrow \frac{ab + 5a + 5b + 9}{(a+b+2)^{4/3}}, \quad b_1 \leftarrow \frac{a+b+6}{(a+b+2)^{2/3}}, \tag{6-14}$$

coupled with

$$c_1 \leftarrow \frac{c+d+e}{(a+b+2)^{2/3}}, \quad d_1 \leftarrow \frac{(b+3)c + 2d + (a+3)e}{a+b+2}, \quad e_1 \leftarrow \frac{c+e}{(a+b+2)^{1/3}}.$$

The region

$$\Lambda_6 = \{(a,b) \in \mathbb{R}^2 : U_6 < \infty\} \tag{6-15}$$

is described by the discriminant curve $\mathfrak{R}$, the zero set of the polynomial

$$R(a,b) = 4a^3 + 4b^3 - 18ab - a^2b^2 + 27. \tag{6-16}$$

This zero set, shown in Figure 1, has two connected components: the first one $\mathfrak{R}_+$ contains $(3,3)$ as a cusp and the second one $\mathfrak{R}_-$, given by $R_-(a,b) = 0$, is disjoint from the first quadrant. The branch $\mathfrak{R}_-$ is the boundary of the set Λ_6.

The identity

$$R(a_1, b_1) = \frac{(a-b)^2\, R(a,b)}{(a+b+2)^4}, \tag{6-17}$$

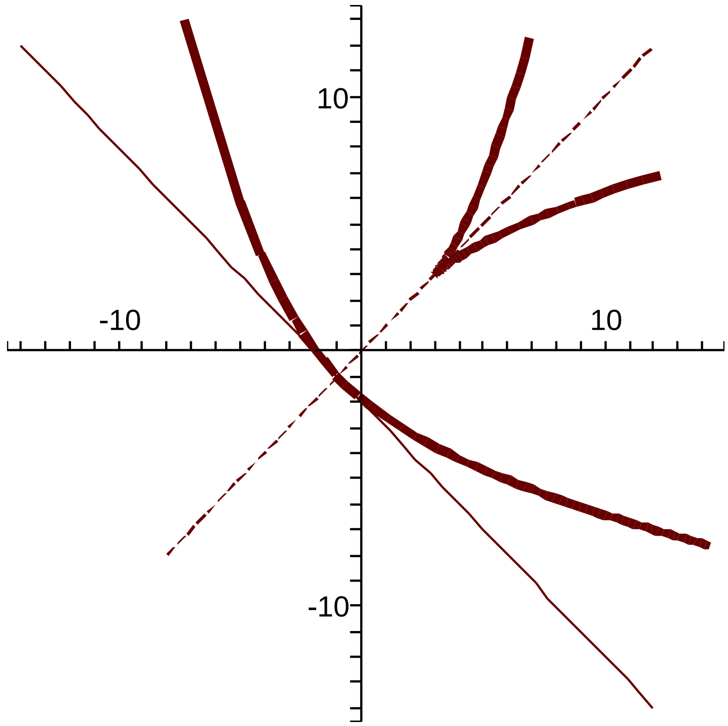

Figure 1. The zero locus of $R(a, b)$.

shows that $\partial\mathfrak{R}$ is invariant under Φ_6. By examining the effect of this map along lines of slope -1, we obtain a direct parametrization of the flow *on* the discriminant curve. Indeed, this curve is parametrized by

$$a(s) = \frac{s^3+4}{s^2} \text{ and } b(s) = \frac{s^3+16}{4s}. \tag{6-18}$$

Then

$$\varphi(s) = \left(\frac{4(s^2+4)^2}{s(s+2)^2} \right)^{1/3} \tag{6-19}$$

gives the image of the Landen transformation Φ_6; that is,

$$\Phi_6(a(s), b(s)) = (a(\varphi(s)), b(\varphi(s))). \tag{6-20}$$

The map Φ_6 has three fixed points: $(3, 3)$, that is superattracting, a saddle point P_2 on the lower branch $\mathfrak{R}_-$ of the discriminant curve, and a third unstable spiral below this lower branch. In [Chamberland and Moll 2006] we prove:

THEOREM 6.4. *The lower branch of the discriminant curve is the curve Λ_6. This curve is also the global unstable manifold of the saddle point P_2. Therefore the iterations of Φ_6 starting at (a, b) converge if and only if the integral*

U_6, formed with the parameters (a, b), is finite. Moreover, $(a_n, b_n) \to (3, 3)$ quadratically and there exists a number L such that $(c_n, d_n, e_n) \to (1, 2, 1)L$.

The next result provides an analogue of the AGM (1-13) for the rational case. The main differences here are that our iterates converge to an *algebraic* number and we achieve *order-m* convergence.

Case 2: The whole line. This works for any choice of positive integer m. Let $R(x)$ be a rational function, written as $R(x) = B(x)/A(x)$. Assume that the coefficients of A and B are real, that A has no real zeros and that $\deg B \le \deg A - 2$. These conditions are imposed to guarantee the existence of

$$I = \int_{-\infty}^{\infty} R(x)\, dx. \tag{6-21}$$

In particular A must be of even degree, and we write

$$A(x) = \sum_{k=0}^{p} a_k x^{p-k} \text{ and } B(x) = \sum_{k=0}^{p-2} b_k x^{p-2-k}. \tag{6-22}$$

We can also require that $\deg(\gcd(A, B)) = 0$.

The class of such rational functions will be denoted by $\mathfrak{R}_p$.

The algorithm presented in Section 5 provides a transformation on the parameters

$$\mathfrak{P}_p\colon = \{a_0, a_1, \dots, a_p; b_0, b_1, \dots, b_{p-2}\} = \mathbb{R}^{p+1} \times \mathbb{R}^{p-1} \tag{6-23}$$

of $R \in \mathfrak{R}_p$ that preserves the integral I. In fact, we produce a family of maps, indexed by $m \in \mathbb{N}$,

$$\mathfrak{L}_{m,p} : \mathfrak{R}_p \to \mathfrak{R}_p,$$

such that

$$\int_{-\infty}^{\infty} R(x)\, dx = \int_{-\infty}^{\infty} \mathfrak{L}_{m,p}(R(x))\, dx. \tag{6-24}$$

The maps $\mathfrak{L}_{m,p}$ induce a *rational Landen transformation*

$$\Phi_{m,p} : \mathfrak{P}_p \to \mathfrak{P}_p \tag{6-25}$$

on the parameter space: we simply list the coefficients of $\mathfrak{L}_{m,p}(R(x))$.

The original integral is written in the form

$$I = \frac{b_0}{a_0} \int_{-\infty}^{\infty} \frac{x^{p-2} + b_0^{-1} b_1 x^{p-3} + b_0^{-1} b_2 x^{p-4} + \cdots + b_0^{-1} b_{p-2}}{x^p + a_0^{-1} a_1 x^{p-1} + a_0^{-1} a_2 x^{p-2} + \cdots + a_0^{-1} a_p}\, dx. \tag{6-26}$$

The Landen transformation generates a sequence of coefficients,

$$\mathfrak{P}_{p,n}\colon = \{a_0^{(n)}, a_1^{(n)}, \dots, a_p^{(n)}; b_0^{(n)}, b_1^{(n)}, \dots, b_{p-2}^{(n)}\}, \tag{6-27}$$

with $\mathfrak{P}_{p,0} = \mathfrak{P}_p$ as in (6-23). We expect that, as $n \to \infty$,

$$\boldsymbol{x}_n := \left(\frac{a_1^{(n)}}{a_0^{(n)}}, \frac{a_2^{(n)}}{a_0^{(n)}}, \ldots, \frac{a_p^{(n)}}{a_0^{(n)}}, \frac{b_1^{(n)}}{b_0^{(n)}}, \frac{b_2^{(n)}}{b_0^{(n)}}, \ldots, \frac{b_{p-2}^{(n)}}{b_0^{(n)}} \right) \tag{6-28}$$

converges to

$$\boldsymbol{x}_\infty := \left(0, \binom{q}{1}, 0, \binom{q}{2}, \ldots, \binom{q}{q}; 0, \binom{q-1}{1}, 0, \binom{q-1}{2}, \ldots, \binom{q-1}{q-1} \right), \tag{6-29}$$

where $q = p/2$. Moreover, we should have

$$\|\boldsymbol{x}_{n+1} - \boldsymbol{x}_\infty\| \le C \|\boldsymbol{x}_n - \boldsymbol{x}_\infty\|^m. \tag{6-30}$$

The invariance of the integral then shows that

$$\frac{b_0^{(n)}}{a_0^{(n)}} \to \frac{1}{\pi} I. \tag{6-31}$$

This produces an iterative method to evaluate the integral of a rational function. The method's convergence is of order m.

The convergence of these iterations, and in particular the bound (6-30), can be established by the argument presented in Section 4. Thus, the transformation $\mathfrak{L}_{m,p}$ leads to a sequence that has order-m convergence. We expect to develop these ideas into an efficient numerical method for integration.

We choose to measure the convergence of a sequence of vectors to 0 in the L_2-norm,

$$\|v\|_2 = \frac{1}{\sqrt{2p-2}} \left(\sum_{k=1}^{2p-2} \|v_k\|^2 \right)^{1/2}, \tag{6-32}$$

and also the L_∞-norm,

$$\|v\|_\infty = \text{Max}\,\{\|v_k\| : 1 \le k \le 2p-2\}. \tag{6-33}$$

The rational functions appearing as integrands have rational coefficients, so, as a measure of their complexity, we take the largest number of digits of these coefficients. This appears in the column marked *size*.

The tables on the next page illustrate the iterates of rational Landen transformations of order 2, 3 and 4, applied to the example

$$I = \int_{-\infty}^{\infty} \frac{3x+5}{x^4 + 14x^3 + 74x^2 + 184x + 208}\, dx = -\frac{7\pi}{12}.$$

The first column gives the L_2-norm of $u_n - u_\infty$, the second its L_∞-norm, the third presents the relative error in (6-31), and in the last column we give the size

n	L_2-norm	L_∞-norm	Error	Size
1	58.7171	69.1000	1.02060	5
2	7.444927	9.64324	1.04473	10
3	4.04691	5.36256	0.945481	18
4	1.81592	2.41858	1.15092	41
5	0.360422	0.411437	0.262511	82
6	0.0298892	0.0249128	0.0189903	164
7	0.000256824	0.000299728	0.0000362352	327
8	1.92454×10^{-8}	2.24568×10^{-8}	1.47053×10^{-8}	659
9	1.0823×10^{-16}	1.2609×10^{-16}	8.2207×10^{-17}	1318

Table 1. Method of order 2.

n	L_2-norm	L_∞-norm	Error	Size
1	15.2207	20.2945	1.03511	8
2	1.97988	1.83067	0.859941	23
3	0.41100	0.338358	0.197044	69
4	0.00842346	0.00815475	0.00597363	208
5	5.05016×10^{-8}	5.75969×10^{-8}	1.64059×10^{-9}	626
6	1.09651×10^{-23}	1.02510×10^{-23}	3.86286×10^{-24}	1878
7	1.12238×10^{-70}	1.22843×10^{-70}	8.59237×10^{-71}	5634

Table 2. Method of order 3.

n	L_2-norm	L_∞-norm	Error	Size
1	7.44927	9.64324	1.04473	10
2	1.81592	2.41858	1.15092	41
3	0.0298892	0.0249128	0.0189903	164
4	1.92454×10^{-8}	2.249128×10^{-8}	1.47053×10^{-8}	659
5	3.40769×10^{-33}	3.96407×10^{-33}	2.56817×10^{-33}	2637

Table 3. Method of order 4.

of the rational integrand. At each step, we verify that the new rational function integrates to $-7\pi/12$.

As expected, for the method of order 2, we observe quadratic convergence in the L_2-norm and also in the L_∞-norm. The size of the coefficients of the integrand is approximately doubled at each iteration.

EXAMPLE 6.1. A method of order 3 for the evaluation of the quadratic integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{ax^2 + bx + c}, \tag{6-34}$$

has been analyzed in [Manna and Moll 2007b]. We refer to Example 5.1 for the explicit formulas of this Landen transformation, and define the iterates accordingly. From there, we prove that the error term,

$$e_n : = \left(a_n - \tfrac{1}{2}\sqrt{4ac - b^2},\, b_n,\, c_n - \tfrac{1}{2}\sqrt{4ac - b^2}\right) \tag{6-35}$$

satisfies $e_n \to 0$ as $n \to \infty$, with cubic rate:

$$\|e_{n+1}\| \le C\|e_n\|^3. \tag{6-36}$$

The proof of convergence is elementary. Therefore

$$(a_n, b_n, c_n) \to \left(\sqrt{ac - b^2/4}, 0, \sqrt{4ac - b^2/4}\right), \tag{6-37}$$

which, in conjunction with (6-34), implies that

$$I = \frac{2}{\sqrt{4ac - b^2}} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}\ , \tag{6-38}$$

exactly as one would have concluded by completing the square. Unlike completing the square, our method extends to a general rational integral over the real line.

7. The appearance of the AGM in diverse contexts

The (elliptic) Landen transformation

$$a_1 \leftarrow \tfrac{1}{2}(a + b), \quad b_1 \leftarrow \sqrt{ab} \tag{7-1}$$

leaving invariant the elliptic integral

$$G(a, b) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} \tag{7-2}$$

appears in many different forms. In this last section we present a partial list of them.

7.1. The elliptic Landen transformation. For the lattice $\mathbb{L} = \mathbb{Z} \oplus \omega\mathbb{Z}$, introduce the *theta-functions*

$$\vartheta_3(x, \omega) : = \sum_{n=-\infty}^{\infty} z^{2n} q^{n^2}, \quad \vartheta_4(x, \omega) : = \sum_{n=-\infty}^{\infty} (-1)^n z^{2n} q^{n^2}, \tag{7-3}$$

where $z = e^{\pi i x}$ and $q = e^{\pi i \omega}$. The condition $\operatorname{Im} \omega > 0$ is imposed to ensure convergence of the series. These functions admit a variety of remarkable identities. In particular, the *null-values* (those with $x = 0$) satisfy

$$\vartheta_4^2(0, 2\omega) = \vartheta_3(0, \omega)\vartheta_4(0, \omega), \quad \vartheta_3^2(0, 2\omega) = \tfrac{1}{2}\big(\vartheta_3^2(0, \omega) + \vartheta_4^2(0, \omega)\big),$$

and completely characterize values of the AGM, leading to the earlier result [Borwein and Borwein 1987]. Grayson [1989] has used the doubling of the period ω to derive the arithmetic-geometric mean from the cubic equations describing the corresponding elliptic curves. See Chapter 3 in [McKean and Moll 1997] for more information. P. Sole et al. [1995; 1998] have proved generalizations of these identities using lattice enumeration methods related to binary and ternary codes.

7.2. A time-one map. We now present a deeper and more modern version of a result known to Gauss: given a sequence of points $\{x_n\}$ on a manifold X, decide whether there is a differential equation

$$\frac{dx}{dt} = V(x), \tag{7-4}$$

starting at x_0 such that $x_n = x(n, x_0)$. Here $x(t, x_0)$ is the unique solution to (7-4) satisfying $x(0, x_0) = x_0$. Denote by

$$\phi_{\text{ellip}}(a, b) = \big(\tfrac{1}{2}(a+b), \sqrt{ab}\big) \tag{7-5}$$

the familiar elliptic Landen transformation. Now take $a, b \in \mathbb{R}$ with $a > b > 0$. Use the null-values of the theta functions to find unique values (τ, ρ) such that

$$a = \rho\vartheta_3^2(0, \tau), \quad b = \rho\vartheta_4^2(0, \tau). \tag{7-6}$$

Finally define

$$x_{\text{ellip}}(t) = (a(t), b(t)) = \rho\big(\vartheta_3^2(0, 2^t\tau), \vartheta_4^2(0, 2^t\tau)\big), \tag{7-7}$$

with $x_{\text{ellip}}(0) = (a, b)$. The remarkable result is [Deift 1992]:

THEOREM 7.1 (DEIFT, LI, PREVIATO, TOMEI). *The map $t \to x_{\text{ellip}}(t)$ is an integrable Hamiltonian flow on X equipped with an appropriate symplectic structure. The Hamiltonian is the complete elliptic integral $G(a, b)$ and the* *angle is (essentially the logarithm of) the second period of the elliptic curve associated with* τ. *Moreover*

$$x_{\text{ellip}}(k) = \phi_{\text{ellip}}^{k}(a,b). \tag{7-8}$$

Thus the arithmetic-geometric algorithm is the time-one map of a completely integrable Hamiltonian flow.

Notice that this theorem shows that the result in question respects some additional structures whose invention postdates Gauss.

A natural question is whether the map (3-12) appears as a time-one map of an interesting flow.

7.3. A quadruple sequence. Several variations of the elliptic Landen appear in the literature. Borchardt [1876] considers the four-term quadratically convergent iteration

$$\begin{aligned} a_{n+1} &= \frac{a_n+b_n+c_n+d_n}{4}, \quad & b_{n+1} &= \frac{\sqrt{a_nb_n}+\sqrt{c_nd_n}}{2}, \\ c_{n+1} &= \frac{\sqrt{a_nc_n}+\sqrt{b_nd_n}}{2}, \quad & d_{n+1} &= \frac{\sqrt{a_nd_n}+\sqrt{b_nc_n}}{2}, \end{aligned} \tag{7-9}$$

starting with $a_0 = a$, $b_0 = b$, $c_0 = c$ and $d_0 = d$. The common limit, denoted by $G(a,b,c,d)$, is given by

$$\frac{1}{G(a,b,c,d)} = \frac{1}{\pi^2}\int_0^{\alpha_3}\int_{\alpha_1}^{\alpha_2} \frac{(x-y)\,dx\,dy}{\sqrt{R(x)R(y)}}, \tag{7-10}$$

where $R(x) = x(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ and the numbers α_j are given by explicit formulas in terms of the parameters a, b, c, d. Details are given in [Mestre 1991].

The initial conditions $(a,b,c,d) \in \mathbb{R}^4$ for which the iteration converges has some interesting invariant subsets. When $a = b$ and $c = d$, we recover the AGM iteration (1-1). In the case that $b = c = d$, we have another invariant subset, linking to an iterative mean described below.

7.4. Variations of AGM with hypergeometric limit. Let $N \in \mathbb{N}$. The analysis of

$$a_{n+1} = \frac{a_n+(N-1)b_n}{N} \text{ and } c_{n+1} = \frac{a_n-b_n}{N}, \tag{7-11}$$

with $b_n = (a_n^N - c_n^N)^{1/N}$, is presented in [Borwein and Borwein 1991]. All the common ingredients appear there: a common limit, fast convergence, theta functions and sophisticated iterations for the evaluation of π. The common limit is denoted by $\mathrm{AG}_N(a,b)$. The convergence is of order N and the limit is identified for small N: we have, for $0<k<1$,

$$\frac{1}{\mathrm{AG}_2(1,k)} = {}_2F_1(1/2,\, 1/2;\, 1;\, 1-k^2), \quad \frac{1}{\mathrm{AG}_3(1,k)} = {}_2F_1(1/3,\, 2/3;\, 1;\, 1-k^2),$$

where

$${}_2F_1(a,\, b;\, c;\, x) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} x^k \tag{7-12}$$

is the classical hypergeometric function. There are integral representations of these as well which parallel (1-13); see [Borwein et al. 2004a], Section 6.1 for details.

Other hypergeometric values appear from similar iterations. For example,

$$a_{n+1} = \frac{a_n + 3b_n}{4} \text{ and } b_{n+1} = \sqrt{b_n(a_n+b_n)/2}, \tag{7-13}$$

have a common limit, denoted by $A_4(a,b)$. It is given by

$$\frac{1}{A_4(1,k)} = {}_2F_1^2(1/4,\, 3/4;\, 1;\, 1-k^2). \tag{7-14}$$

To compute π quartically, start at $a_0 = 1,\ b_0 = (12\sqrt{2}-16)^{1/4}$. Now compute a_n from two steps of AG_2:

$$a_{n+1} = \frac{a_n + b_n}{2}, \text{ and } b_{n+1} = \left(\frac{a_n b_n^3 + b_n a_n^3}{2}\right)^{1/4}. \tag{7-15}$$

Then

$$\pi = \lim_{n\to\infty} 3a_{n+1}^4 \left(1 - \sum_{j=0}^{n} 2^{j+1}(a_j^4 - a_{j+1}^4)\right)^{-1} \tag{7-16}$$

with $|a_{n+1} - \pi| \le C|a_n - \pi|^4$, for some constant $C > 0$. This is much better than the partial sums of

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}. \tag{7-17}$$

The sequences (a_n), (b_n) defined by the iteration

$$a_{n+1} = \frac{a_n + 2b_n}{3}, \quad b_{n+1} = \left(\frac{b_n(a_n^2 + a_n b_n + b_n^2)}{3}\right)^{1/3}, \tag{7-18}$$

starting at $a_0 = 1$, $b_0 = x$ are analyzed in [Borwein and Borwein 1990]. They have a common limit $F(x)$ given by

$$\frac{1}{F(x)} = {}_2F_1\big(1/3, 2/3; 1; 1-x^3\big). \tag{7-19}$$

7.5. Iterations where the limit is harder to find. J. Borwein and P. Borwein [1989] studied the iteration of

$$(a,b) \to \left(\frac{a+3b}{4}, \frac{\sqrt{ab}+b}{2} \right), \tag{7-20}$$

and showed the existence of a common limit $B(a_0,b_0)$. Define $B(x)=B(1,x)$. The study of the iteration (7-20) is based on the functional equation

$$B(x) = \frac{1+3x}{4} B\left(\frac{2(\sqrt{x}+x)}{1+3x} \right). \tag{7-21}$$

and a parametrization of the iterates by theta functions [Borwein and Borwein 1989]. The complete analysis of (7-20) starts with the purely computational observation that

$$B(x) \sim \frac{\pi^2}{3} \log^{-2}(x/4) \quad \text{as } x \to 0. \tag{7-22}$$

H. H. Chan, K. Chua and P. Sole [Heng Huat Chan and Sole 2002] identified the limiting function as

$$B(x) = \left({}_2F_1\left(\frac{1}{3}, \frac{1}{6}; 1; 27\,\frac{x(1-x)^2}{(1+3x)^3} \right) \right)^{-2}, \tag{7-23}$$

valid for $x \in \left(\frac{2}{3}, 1\right)$. A similar hypergeometric expression gives $B(x)$ for $x \in \left(0, \frac{2}{3}\right)$.

7.6. Fast computation of elementary functions. The fast convergence of the elliptic Landen recurrence (1-1) to the arithmetic-geometric mean provides a method for numerical evaluation of the elliptic integral $G(a,b)$. The same idea provides for the fast computation of elementary functions. For example, in [Borwein and Borwein 1984] we find the estimate

$$\left| \log x - \left(G(1,10^{-n}) - G(1,10^{-n}x) \right) \right| < n10^{-2(n-1)}, \tag{7-24}$$

for $0 < x < 1$ and $n \geq 3$.

7.7. A continued fraction. The continued fraction

$$R_\eta(a,b) = \cfrac{a}{\eta + \cfrac{b^2}{\eta + \cfrac{4a^2}{\eta + \cfrac{9b^2}{\eta + \cdots}}}}, \tag{7-25}$$

has an interesting connection to the AGM. In their study of the convergence of $R_\eta(a,b)$, J. Borwein, R. Crandall and G. Fee [Borwein et al. 2004b] established the identity

$$R_\eta\left(\frac{a+b}{2},\sqrt{ab}\right) = \frac{1}{2}\big(R_\eta(a,b)+R_\eta(b,a)\big). \tag{7-26}$$

This identity originates with Ramanujan; the similarity with AGM is now direct.

The continued fraction converges for positive real parameters, but for $a,\ b\in\mathbb{C}$ the convergence question is quite delicate. For example, the even/odd parts of $R_1(1,i)$ converge to distinct limits. See [Borwein et al. 2004b; 2004c] for more details.

7.8. Elliptic Landen with complex initial conditions. The iteration of (1-1) with $a_0,\ b_0\in\mathbb{C}$ requires a choice of square root at each step. Let $a,\ b\in\mathbb{C}$ be nonzero and assume $a\neq\pm b$. A square root c of ab is called the *right choice* if

$$\left|\frac{a+b}{2}-c\right| \leq \left|\frac{a+b}{2}+c\right|. \tag{7-27}$$

It turns out that in order to have a limit for (1-1) one has to make the right choice for all but finitely many indices $n\geq 1$. This is described in detail in [Cox 1984].

7.9. Elliptic Landen with p-adic initial conditions. Let p be a prime and $a,\ b$ be nonzero p-adic numbers. To guarantee that the p-adic series

$$c = a\sum_{i=0}^{\infty}\binom{\frac{1}{2}}{i}\left(\frac{b}{a}-1\right)^i \tag{7-28}$$

converges, and thus defines a p-adic square root of ab, one must assume

$$b/a\equiv 1 \bmod p^\alpha, \tag{7-29}$$

where $\alpha=3$ for $p=2$ and 1 otherwise. The corresponding sequence defined by (1-1) converges for $p\neq 2$ to a common limit: the p-adic AGM. In the case $p=2$ one must assume that the initial conditions satisfy $b/a\equiv 1 \bmod 16$. In the case $b/a\equiv 1 \bmod 8$ but not 1 modulo 16, the corresponding sequence (a_n,b_n) does not converge, but the sequence of so-called *absolute invariants*

$$j_n = \frac{2^8(a_n^4-a_n^2b_n^2+b_n^4)^3}{a_n^4b_n^4(a_n^2-b_n^2)^2} \tag{7-30}$$

converges to a 2-adic integer. Information about these issues can be found in [Henniart and Mestre 1989]. D. Kohel [2003] has proposed a generalization of the AGM for elliptic curves over a field of characteristic $p\in\{2,\ 3,\ 5,\ 7,\ 13\}$. Mestre [2000] has developed an AGM theory for ordinary hyperelliptic curves over a field of characteristic 2. This has been extended to nonhyperelliptic curves of genus 3 curves by Lehavi and Ritzenhaler [2007]. An algorithm for counting points for ordinary elliptic curves over finite fields of characteristic $p > 2$ based on the AGM is presented in [Carls 2004].

7.10. Higher genus AGM. An algorithm analogue to the AGM for abelian integrals of genus 2 was discussed by Richelot [1836; 1837] and Humberdt [1901]. Some details are discussed in [Bost and Mestre 1988]. The case of abelian integrals of genus 3 can be found in [Lehavi and Ritzenhaler 2007].

Gauss was correct: his numerical calculation (1-4) *has grown in many unexpected directions*.

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