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Reality problems in the soliton theory

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Dedicated to Henry McKean

ABSTRACT. This is a survey article dedicated mostly to the theory of real regular finite-gap (algebro-geometrical) periodic and quasiperiodic sine-Gordon solutions. Long period this theory remained unfinished and ineffective, and by that reason practically had no applications. Even for such simple physical quantity as topological charge no formulas existed expressing it through inverse spectral data. A few years ago the present authors solved this problem and made this theory effective. This article contains description of the history and recent achievements. It describes also the reality problems for several other fundamental soliton systems.

1. Introduction

The most powerful method for constructing explicit periodic and quasiperiodic solutions of soliton equations is based on the finite-gap or algebro-geometric approach, developed by Novikov [1974], Dubrovin et al. [1976b], Its and Matveev [1975], Lax [1975], and McKean and van Moerbeke [1975] for $1+1$ systems, and extended by Krichever in 1976 for $2+1$ systems like KP. Already in 1976 new ideas were formulated on how to extend this approach to the $2+1$ systems associated with the spectral theory of the 2D Schrödinger operator restricted to one energy level; see [Manakov 1976; Dubrovin et al. 1976a]. These ideas were developed in 1980s by several people in Moscow's Novikov Seminar, as discussed see below. The "spectral data" characterizing the associated Lax-type operators consist of a Riemann surface (spectral curve)

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equipped with a selected set of points (divisor of poles, infinities). In the finite gap case this Riemann surface has finite genus, and the number of selected point is also finite. The algebro-geometric approach in particular allows one to write down explicit solutions in terms of the Riemann θ functions.

In modern literature very often the problem is assumed to be more or less completely solved if such formulas are derived. However, in some cases this belief is too naive and does not correspond to the needs of real life. It is often necessary to select physically or geometrically relevant classes of solutions corresponding to the source problem: for instance solutions satisfying certain reality conditions, or regular solutions, or bounded solutions. Is this easy or hard?

To reach this goal, the following problems must be solved.

- Problem 1. How to select solutions that are real for real (x, t).
- Problem 2. How to select real nonsingular solutions.
- Problem 3. How to select periodic solutions with a given period (or quasiperiodic solutions with a given group of quasiperiods).

REMARK. We call a solution *nonsingular* if it is nonsingular on the whole real Abel torus. It should remain nonsingular under action of all (real) higher flows from the corresponding integrable hierarchy. The generic x-direction is normally ergodic in the Abel torus, so this definition is equivalent to the standard one. However, for some specific values of constants of motion theoretically we may have solutions, which are regular in the standard sense, but blow-up under the action of the higher symmetries.

For some models like the Korteweg–de Vries equation (KdV), the defocusing noninear Schrödinger equation (NLS) and the Kadomtsev–Petviashvili 2 (KP2) equation, the selection of real and nonsingular solution is straightforward. But for many other models such as KP1, the focusing NLS, the sine-Gordon equation (SG), and the inverse scattering transformation for the Schrödinger operator based at one energy, the problem of selecting real solutions is difficult.

The theory of θ functions is complicated and ineffective. The complexity is hidden behind the simple notations in these formulas.

Our goal is to discuss in more detail the sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u(x,t) = 0. \tag{1-1}$$

In the light-cone coordinates

$$x = 2(\xi + \eta), \quad t = 2(\xi - \eta), \tag{1-2}$$

it has the form

$$u_{\xi\eta} = 4 \sin u, \quad u = u(\xi, \eta). \tag{1-3}$$

According to our definition, the solution $u(x,t)$ is x-periodic with period T if $\exp\{iu\}$ is x-periodic with that period. For the function u we have

$$u(x+T,t) = u(x,t) + 2\pi n, \quad n \in \mathbb{Z}.$$

We call the quantity n the *topological charge* corresponding to the period T.

We call the ratio n/T the *density of topological charge*.

The density of topological charge can be naturally extended to all real generic regular finite-gap (quasiperiodic) solutions. It is the most basic conservation law.

PROBLEM. *Calculate the topological charge of real finite-gap solutions in terms of spectral data.*

We recall that the inverse scattering (spectral) data for the KdV and sine-Gordon systems consist of a Riemann surface (spectral curve) Γ of finite genus g and a collection of points (divisor) $D = \gamma_1 + \cdots + \gamma_g$. (For the NLS and some other systems number of poles may be different from genus.)

In the case of KdV or the finite-gap periodic Schrödinger operator $L = -\partial_x^2 + u(x)$, this surface Γ is hyperelliptic. In the case of the sine-Gordon equation the surface is also hyperelliptic, $\mu^2 = \lambda \prod_{i=1}^{2g}(\lambda - \lambda_i)$, with branching points $(0, \lambda_1, \ldots, \lambda_{2g}, \infty)$. However the classes of admissible Riemann surfaces and divisors for KdV and sine-Gordon are dramatically different, as we shall see below.

The θ-functional formulas for sine-Gordon were obtained in [Kozel and Kotlyarov 1976; Its and Kotlyarov 1976]. The reality problem remained unsolved. Indeed, the class of admissible Riemann surfaces was found in these works; see [Its and Kotlyarov 1976]. The nonzero finite branching points $(\lambda_1, \ldots, \lambda_{2g})$ can be either real negative $(\lambda_1, \ldots, \lambda_{2k}) \in \mathbb{R}$ or complex conjugate with nonzero imaginary part $\lambda_{2k+1} = \bar{\lambda}_{2k+2}, \ldots, \lambda_{2g-1} = \bar{\lambda}_{2g}$. However, no ideas were proposed where the poles are located on the Riemann surface.

In the early 1980s it was realized that this problem is nontrivial; see [McKean 1981; Dybrovin and Novikov 1982; 1982; Ercolani and Forest 1985; Ercolani et al. 1984]. For this reason, periodic finite-gap sine-Gordon theory lacked applications for a long time.

An important idea for how to describe position of poles for the real nonsingular solutions was in fact suggested by Cherednik [1980]. He was the first author who discovered (ineffectively) that for the given admissible real Riemann surface there can be many different real Abel tori generating real nonsingular quasiperiodic solutions. Their number is equal to 2^k where $2k$ is the number of negative real branching points. All real finite-gap solutions are nonsingular for sine-Gordon for the generic Riemann surface. His work was written in the abstract algebro-geometric form, and he never developed these ideas later.

Extending Cherednik's approach on the basis of "algebro-topological" ideas, Dubrovin and Novikov [1982] presented an interesting idea for how to calculate topological charge in terms of the "inverse spectral data". However, as pointed out in [Novikov 1984], there was a mistake in the argument; the proof of the formula proposed in [Dybrovin and Novikov 1982] worked only for a small neighborhood of some very special solutions. The problem remained open till 2001. The complete solution, confirming the Dubrovin–Novikov formula, was obtained in [Grinevich and Novikov 2001] as a development of the algebro-topological approach suggested in [Dybrovin and Novikov 1982]; see also [Grinevich and Novikov 2003a; 2003b]. In [Dubrovin and Natanzon 1982] and [Ercolani and Forest 1985], these components were described as the real subtori in the Jacobian variety $J(\Gamma)$. However this "θ-functional description", which does not involve a specific basis of cycles, did not lead to a formula for the topological charge. As we know now, a good formula for the topological charge can only be written in a very specific basis. We believe that using this basis of cycles one can deduce our formula from the θ-functional expression. It would be good to do that.

2. Physically relevant classes of solutions for the different soliton systems

The Korteweg–de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0, \quad u = u(x,t), \tag{2-1}$$

was originally derived in the theory of water waves. As discovered in early 1960s (see the introduction to [Novikov et al. 1984]), it naturally appears as a first nonvanishing correction for the dispersive nonlinear systems if dissipation can be neglected. In these models only real nonsingular solution are physically relevant.

Integration of the KdV equation is based on the "inverse scattering transform" for the one-dimensional Schrödinger operator

$$L = -\partial_x^2 + u(x,t). \tag{2-2}$$

The selection of real KdV solutions is straightforward.

(1) The spectral curve Γ defined by $\mu^2 = R_{2g+1}(\lambda)$ should be real. This means that $R_{2g+1}(\lambda) = \lambda^{2g+1} + \sum_{i=0}^{2g} p_i \lambda^i$ has real coefficients $p_i \in \mathbb{R}$; equivalently, all roots are either real or form complex conjugate pairs. Therefore we have a holomorphic involution $\tau : (\lambda, \mu) \to (\bar{\lambda}, -\bar{\mu})$ on Γ.

(2) The divisor D should be real with respect to τ: $\tau D = D$, or equivalently, the unordered set of points $\gamma_1, \dots, \gamma_g$ is invariant with respect to τ. Of course, τ may interchange some of them.

Real nonsingular KdV solutions correspond to the following special spectral data:

(1) All branching points of λ_k of Γ are real and distinct. Assume that $\lambda_1 < \lambda_2 < \dots < \lambda_{2g+1}$. Then τ has exactly $g+1$ real ovals over the intervals $a_0 = (-\infty, \lambda_1]$, $a_1 = [\lambda_2, \lambda_3]$, $\dots$, $a_g = [\lambda_{2g}, \lambda_{2g+1}]$.
(2) Each finite oval a_k, $1 \le k \le g$ contains exactly one divisor point $\gamma_k \in a_k$.

REMARK. A real curve of genus g may have at most $g+1$ real oval. Curves with $g+1$ real ovals (the greatest possible number) are called M-curves.

Generic finite-gap solutions are quasiperiodic with g incommensurable periods. How to select x-periodic solutions with prescribed period T? Avoiding any use of algebraic geometry and Riemann surfaces, a nice approach to the characterization of the strictly x-periodic solution in terms of the so-called quasimomentum map was developed by Marchenko and Ostrovskii [1975]. This map was studied in the quantum solid state physics literature in 1959 (see [Kohn 1959]). It is well-defined in the upper half-plane outside of some vertical edges. Its analytical properties were effectively used in [Marchenko and Ostrovskii 1975]. For example the approximation of x-periodic solution (potential) by the finite-gap ones periodic with the same period, was proved. Another approach, based on isoperiodic deformations of finite-gap potentials, was developed by Grinevich and Schmidt in 1995 [Grinevich and Schmidt 1995]. In the KdV case the isoperiodic deformations can be interpreted as the so-called Loewner equations for the corresponding conformal map. We point out that there exists a big literature, dedicated to the KdV solutions with real poles (rational solutions, singular trigonometric and elliptic solutions) — see [Airault et al. 1977], where these ideas were started. These solutions are very important from the mathematical point of view: for example, the dynamics of poles satisfies to the equations of the rational and elliptic Moser–Calogero models respectively. However, they are related neither to nonlinear wave problems nor to the spectral theory of the corresponding Schrödinger operators. So we do not discuss this literature in the present survey article.

The modified Korteweg-de Vries equation has the form:

$$v_t + v_{xxx} - 6v^2 v_x = 0, \quad v = v(x,t). \tag{2-3}$$

It is connected with KdV by the Miura transformation:

$$u(x,t) = v_x(x,t) + v^2(x,t). \tag{2-4}$$

The real nonsingular solutions are physically relevant.

The "complex" nonlinear Schrödinger equation (NLS) is a system of equations for the pair of independent complex functions $q = q(x,t)$, $r = r(x,t)$:

$$\begin{aligned} iq_t + q_{xx} + 2q^2 r &= 0, \\ -ir_t + r_{xx} + 2r^2 q &= 0. \end{aligned} \tag{2-5}$$

This system has two natural real reductions: the defocusing NLS, with $r(x,y) = -\overline{q(x,y)}$, hence

$$iq_t + q_{xx} - 2|q|^2 q = 0, \tag{2-6}$$

and the self-focusing NLS, with $r(x,y) = \overline{q(x,y)}$, hence

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{2-7}$$

These equation describes nonlinear media with dispersion relations depending on the square of the wave amplitude (see [Novikov et al. 1984]). Among the todays applications of NLS is the theory of light propagation in the fiber optics. The sign $+$ or $-$ is determined by the dispersion relation, and the qualitative behavior critically depends on it. From the mathematical point of view, the defocusing NLS system is much simpler because the linear Lax operator is self-adjoint. The focusing NLS is much more complicated. In both cases physical applications requires regular solutions.

The complex NLS spectral data are following: A hyperelliptic Riemann surface Γ with $2g+2$ finite branching points $\lambda_1, \dots, \lambda_{2g+2}$ and $g+1$ divisor points $D = \gamma_1 + \dots + \gamma_{g+1}$. In contrast with the KdV case, there is no branching at ∞.

Solutions of the defocusing NLS correspond to the following spectral data:

(1) Γ is real, i.e. the polynomial $R_{2g+2} = \prod_{k=1}^{2g+2}(\lambda - \lambda_k)$ has real coefficients. Γ is defined by $\mu^2 = R(\lambda)$. The antiholomorphic involution on Γ is defined by the map $\tau : \mathbb{C}^2 \to \mathbb{C}^2$ where $(\lambda, \mu) \to (\bar{\lambda}, -\bar{\mu})$.
(2) The divisor D is real with respect to τ: $\tau D = D$.

Selection of regular solutions is also very similar to the KdV case

(1) All branching points of Γ are real. Therefore Γ has $g+1$ real ovals over the intervals $[\lambda_{2k-1}, \lambda_{2k}]$, $k = 1, \dots, g+1$; that is, Γ is an M-curve.
(2) There is exactly one divisor point at each real oval.

The selection of x-periodic solutions is completely analogous to the KdV case.

We describe the data generating real solutions of the self-focusing NLS equations for regular spectral curves. By Cherednik's theorem [1980], these solutions are automatically nonsingular. The solutions corresponding to singular spectral curves can be obtained as proper degenerations. In contrast with the defocusing case, singular curves may generate regular x-quasiperiodic solutions.

(1) Γ is a real hyperelliptic surface of genus g with $2g+2$ $\lambda_1 < \lambda_2 < \cdots < \lambda_{2g+2}$ finite branching points. There are no branching points on the real line, so they form complex conjugate pairs. The antiholomorphic involution τ acts on the λ-plane as $\tau\lambda = \bar{\lambda}$. The points in Γ lying over the real line are invariant with respect to τ. Equivalently, $\tau : (\lambda, \mu) \to (\bar{\lambda}, \bar{\mu})$.
(2) There exists a meromorphic differential Ω satisfying these conditions:

- $\Omega = (1 + o(1))d\lambda$ at the infinite points of Γ.
- Ω is regular outside infinity. Therefore it has exactly $2g + 2$ zeroes.
- Let $D = \gamma_1 + \cdots + \gamma_{g+1}$. Then the divisor of zeroes of Ω is $D + \tau D$. Therefore, $D + \tau D = 2\infty_1 + 2\infty_2 - K$.

The sine-Gordon equation in the light-cone variables was derived in the end of the nineteenth century. It describes immersions of the negative curvature surfaces into $\mathbb{R}^3$. Assume that an asymptotic coordinate system is chosen (a coordinate system such that coordinate lines have zero normal curvature). The angle between the coordinate lines satisfy (1-3). This means that only real regular solutions such that $u(x, t) \neq 0 \bmod \pi$ are relevant.

The sine-Gordon equation describes also dynamics of the Josephson junctions. In this model $u(x, t)$ is the phase difference between the contacts, therefore the real nonsingular solutions are relevant. However, according to the leading experts in the Superconductivity Theory, the problem always requires boundary problem, so we have to consider either the finite interval or the half-line.

The elliptic sinh-Gordon equation

$$u_{xx} + u_{yy} + 4H \sinh u = 0. \tag{2-8}$$

describes the constant mean curvature surfaces with genus equal to one, outside umbilic points (see the review in [Bobenko 1991]). The constant mean curvature tori have no umbilic points, therefore real nonsingular solutions should be selected. In contrast with soliton equations, all real smooth double-periodic solutions are automatically finite-gap here [Hitchin 1988; Pinkall and Sterling 1989]. This is a consequence of the following observations by Hitchin [1988]: all isospectral flows from the corresponding hierarchy are zero eigenfunctions of the linearized problem. But the linearized system is the two-dimensional (elliptic) Schrödinger operator, and it may have only finite-dimensional space of double-periodic zero eigenfunctions. This means that the hierarchy contains only finitely many linearly independent flows at this point. As a corollary the spectral curve has finite genus. A further development of this idea was used by Novikov and Veselov [1997], who showed that all periodic chains of Laplace transformations consisting of the two-dimensional double-periodic Schrödinger operators with regular coefficients are algebro-geometric (2D analogs of finite-gap operators).

The Boussinesq equation

$$\begin{aligned} u_t &= \eta_x, \\ \eta_t &= -\tfrac{1}{3}u_{xxx} + \tfrac{4}{3}uu_x \end{aligned} \tag{2-9}$$

is used for describing the water waves. For physical applications it is necessary to select real nonsingular solutions. We point out that the problem of selecting such solutions in terms of the finite-gap data remains open.

The Kadomtsev–Petviashvili (KP) equation

$$(u_t + u_{xxx} - 6uu_x)_x + 3\alpha^2 u_{yy} = 0, \quad u = u(x, y, t), \quad \alpha^2 \in \mathbb{R}. \tag{2-10}$$

The auxiliary linear operator for KP has the form

$$L = \alpha\partial_y - \partial_x^2 + u(x, y, t). \tag{2-11}$$

If α is imaginary, we have the so-called KP1 equation, and L is the one-dimensional nonstationary Schrödinger operator. If α is real, L is the parabolic operator. In both cases the real nonsingular solutions are physically relevant only. The necessary and sufficient conditions for the finite-gap spectral data selecting the real nonsingular solutions were found by Dubrovin and Natanzon [1988].

Real nonsingular solutions of the KP-2 equation correspond to the following geometry:

(1) Γ is a algebraic surface of genus g with a marked point and an antiholomorphic involution τ such that the marked point is invariant under the action of τ. The marked point is the essential singularity of the wave function.
(2) τ has exactly $g+1$ fixed oval, i.e. Γ in an M-curve with respect to τ. Denote the oval containing the essential singularity by a_0 and the other ovals by a_n, $n = 1, \dots, g$.
(3) Each oval a_n, $n \neq 0$ contains exactly one divisor point.

In the case of the Kadomtsev–Petviashvili 1 equation the reality constraints on the spectral curve are exactly the same as in the KP 2 case, but the divisor D has a completely different description: There exists a meromorphic differential Ω with exactly one second-order pole located at the marked point such that the divisor of zeroes of Ω is exactly $D + \tau D$. Equivalently, $D + \tau D = 2\infty - K$, where ∞ denotes the marked point. Regular real solutions are generated by the data with the following extra constraint:

The pair (Γ, τ) is of separating type, i.e. after removing all real ovals Γ splits into 2 components.

An important example of "solvable" inverse spectral transform is the one-energy problem for the two-dimensional Schrödinger operator started in [Manakov 1976; Dubrovin et al. 1976a].

$$L = -\partial_x^2 - \partial_y^2 + u(x, y), \tag{2-12}$$

It is well-known that the full set of scattering data for multidimensional Schrödinger operators $n > 1$ is overdetermined. A lot of people have studied this problem; we won't even quote this literature. However, the case $n = 2$ turned out to be very specific. Manakov, Dubrovin, Krichever and Novikov [Manakov 1976; Dubrovin et al. 1976a] started a completely new approach for this specific case, creating inverse scattering theory and the corresponding soliton theory associated with one selected energy level. A lot of work has been done since; see [Novikov and Veselov 1984; Veselov et al. 1985; Grinevich and Novikov 1988] and the review [Grinevich 2000] for additional references. In particular, in the first work [Dubrovin et al. 1976a] they defined the natural analogs of finite-gap potentials for the two-dimensional problem as the potentials, "finite-gap at one energy". Let $u(x, y)$ be double-periodic. Denote the dispersion relation by $\varepsilon_j(k_x, k_y)$. The Fermi-curve at the energy level E_0 is defined by:

$$\varepsilon_j(k_x, k_y) = E_0. \tag{2-13}$$

Denote the complex continuation of the Fermi curve by Γ. The potential $u(x, y)$ is called *finite-gap at one energy* if Γ has finite genus.

For generic spectral data the operators constructed in [Dubrovin et al. 1976a] have generically a nonzero magnetic field, i.e. they have some extra first-order terms:

$$L = -\partial_x^2 - \partial_y^2 + A_1(x, y)\partial_x + A_2(x, y)\partial_y + u(x, y), \tag{2-14}$$

It might happen that $H(x, y) \neq 0$, where $H(x, y) = \partial_x A_2(x, y) - \partial_y A_1(x, y)$. For physical applications it is important to select the case of "potential operators" $A_1(x, y) = A_2(x, y) = 0$ with real potential $u(x, y)$. Sufficient conditions on the spectral data leading to the potential operators were found by Novikov and Veselov [1984]. For double periodic potentials the existence of such forms is necessary; this follows from the direct spectral theory, developed by Krichever [1992]. For the generic regular quasiperiodic potentials "finite-gap for one energy level", this problem remains open. Selection of real potentials here is simple.

How to select the class of regular potentials in terms of algebro-geometrical spectral data? There is no complete solution to this problem. It was shown in [Novikov and Veselov 1984] that if the spectral curve is the so-called M-curve, then the potential $u(x, y)$ is regular, and the operator L is strictly positive (the selected energy level lies below the ground state). An alternative proof of the last statement was obtained by the authors in [Grinevich and Novikov 1988]. The complete characterization of the data generating strictly positive operators (with real regular potentials) was "more or less" clarified but some special features remain unproved rigorously.

If the selected energy level is located above the ground state, the topology of the spectral curve Γ become more complicated. Many classes of spectral data generating real nonsingular solutions were found by Natanzon (see the review in [Natanzon 1995]), but the classification is not complete till now.

3. The sine-Gordon equation

Connections between the sine-Gordon equation and the inverse scattering method were first established by G. Lamb [1971]. The modern approach was started by Ablowitz, Kaup, Newell and Segur [Ablowitz et al. 1973]. It is based on the following zero-curvature representation:[1]

$$\Psi_x = \tfrac{1}{4}(U+V)\Psi, \quad \Psi_t = \tfrac{1}{4}(U-V)\Psi, \tag{3-1}$$

where

$$U = U(\lambda, x, t) = \begin{bmatrix} i(u_x+u_t) & 1 \\ -\lambda & -i(u_x+u_t) \end{bmatrix}, \tag{3-2}$$

$$V = V(\lambda, x, t) = \begin{bmatrix} 0 & -\frac{1}{\lambda}e^{iu} \\ e^{-iu} & 0 \end{bmatrix}. \tag{3-3}$$

As we mentioned above, the finite-gap spectral data consist of

(1) a hyperelliptic Riemann surface Γ defined by $\mu^2 = \lambda \prod_{i=1}^{2g}(\lambda - \lambda_i)$, with $2g+2$ branching points $(0, \lambda_1, \dots \lambda_{2g}, \infty)$; and
(2) the divisor (a collection of points) $D = \gamma_1 + \cdots + \gamma_g$ in Γ.

In our text we always assume that the spectral curve Γ is *generic*, that is, all branching points are distinct.

The construction of complex sine-Gordon solutions is based on the following standard Lemma:

LEMMA 1. *For generic data Γ, D there exists a unique two-component vector-function $\Psi(\gamma, x, t)$ (the Baker–Akhiezer functions) such that*:

(1) *For fixed (x, t) the function $\Psi(\gamma, x, t)$ is meromorphic in the variable $\gamma \in \Gamma$ outside the points* 0, ∞ *and has at most* 1*-st order poles at the divisor points* γ_k, $k = 1, \dots, g$.

[1] The zero-curvature representation for the sine-Gordon equation presented here was, in fact, first written by these authors in subsequent works.

(2) $\Psi(\gamma,x,t)$ *has essential singularities at the points* 0, ∞ *with the following asymptotic*:

$$\Psi(\gamma,x,t)=\begin{pmatrix}1+o(1)\\ i\sqrt{\lambda}+O(1)\end{pmatrix}e^{\frac{i\sqrt{\lambda}}{4}(x+t)}\ \ as\ \ \lambda\to\infty, \tag{3-4}$$

$$\Psi(\gamma,x,t)=\begin{pmatrix}\phi_1(x,t)+o(1)\\ i\sqrt{\lambda}\phi_2(x,t)+O(\lambda)\end{pmatrix}e^{-\frac{i}{4\sqrt{\lambda}}(x-t)}\ \ as\ \ \lambda\to 0, \tag{3-5}$$

with some $\phi_1(x,t)$, $\phi_2(x,t)$.

The sine-Gordon potential $u(x,t)$ *is defined by*

$$u(x,t)=i\ln\frac{\phi_2(x,t)}{\phi_1(x,t)}. \tag{3-6}$$

We denote by $\lambda_k(x,t)$ the projections of the zeroes of the first component of $\Psi(\gamma,x,t)$ to the λ-plane. Then

$$e^{iu(x,t)}=\prod_{j=0}^{g}(-\lambda_j(x,t))\Bigg/\sqrt{\prod_{j=1}^{2g}E_j}. \tag{3-7}$$

REMARK. To be more precise, formulas (3-1)–(3-7) define simultaneously a pair of sine-Gordon solutions $u_1(x,t)$, $u_2(x,t)$, depending on the choice of the branch $1/\sqrt{(\lambda)}$ near the point $\lambda=0$. They are connected by the following relation $u_2(x,t)=u_1(t,x)+\pi$. In the real case it is possible to fix a canonical branch by making the analytical continuation along the real line. This rule is unstable in the following sense: if we add a pair of complex conjugate branching points which are very close to the positive half-line (or, equivalently, open a resonant point), it is a small transformation in terms of the spectral data, but it exchanges u_1 with u_2.

The real sine-Gordon solutions (by Cherednik's lemma they are automatically regular [Čerednik 1980]) correspond to the following data:

(1) Γ is real, i.e. the branching points of Γ are either real, or form complex conjugate pairs. Therefore we have an antiholomorphic involutions $\tau:(\lambda,\mu)\to(\bar{\lambda},\bar{\mu})$. Denote the number of real finite branching points by $2k+1$.
(2) All real branching points lie in the negative half-line $\lambda\le 0$. It is convenient to use following enumeration for the branching points different from 0 and ∞: $0>\lambda_1>\lambda_2>\cdots>\lambda_{2k}$, $\lambda_{2k+1}=\bar{\lambda}_{2k+2},\ldots,\lambda_{2g-1}=\bar{\lambda}_{2g}$.
(3) There exists a meromorphic differential Ω (Cherednik differential) with first order poles at $0,\infty$, holomorphic on $\Gamma\backslash\{0,\infty\}$ with the zeroes at the points $\gamma_1,\ldots,\gamma_g$, $\tau\gamma_1,\ldots,\tau\gamma_g$ (or, equivalently the divisor D satisfy the relation $D+\tau D=0+\infty-K$).

As shown in [Čerednik 1980], the variety of all real potentials corresponding to the given spectral curve Γ consists of 2^k connected components. A characterization of these components in terms of the Abel tori was obtained in [Dubrovin and Natanzon 1982] but this technique did not led to the calculation of topological charge through the inverse spectral data.

Our calculation of the topological charge for the finite-gap sine-Gordon solutions is based on the following effective description of these components, for details of which see [Grinevich and Novikov 2001; 2003a; 2003b]:

Any meromorphic differential with first-order pole at ∞ can be written as

$$\Omega = c\left(1 - \frac{\lambda P_{g-1}(\lambda)}{R(\lambda)^{1/2}}\right)\frac{d\lambda}{2\lambda}, \tag{3-8}$$

where $P_{g-1}(\lambda)$ is a polynomial of degree at most $g-1$. It is also natural to put $c = 1$. In case of the Cherednik differentials the set of zeroes is invariant with respect to τ. Therefore all coefficients of the polynomial $P_{g-1}(\lambda)$ are real.

Take an arbitrary real polynomial $P_{g-1}(\lambda)$. Is it possible to construct a real sine-Gordon solution corresponding to it? The necessary and sufficient condition is this: *the zeroes of Ω can be divided into two groups, $\{\gamma_1, \dots, \gamma_g\}$ and $\{\gamma_{g+1}, \dots, \gamma_{2g}\}$, such that $\tau\gamma_k = \gamma_{k+g}$, $k = 1, \dots, g$.* Equivalently, a polynomial $P_{g-1}(\lambda)$ generates real SG solutions if and only if all real root of Ω have even multiplicity. In generic situation (all roots form distinct complex conjugate pairs) each polynomial $P_{g-1}(\lambda)$ generates 2^g different solutions. To choose one of them one has to say, which point to choose in each complex conjugate pair belonging to D (the second one belongs to τD). In degenerate cases (i.e. if there are real roots) the number of choices is smaller. Al these solutions associated with a given $P_{g-1}(\lambda)$ belong to the same real Abel torus.

DEFINITION. A polynomial $P_{g-1}(\lambda)$ (and the corresponding differential Ω) are called *admissible* if all real roots of Ω have even multiplicity.

Admissible polynomials $P_{g-1}(\lambda)$ can be characterized geometrically. We start by taking the graph of the functions

$$f_{\pm}(\lambda) = \pm\frac{\sqrt{R(\lambda)}}{\lambda}, \tag{3-9}$$

and coloring in black the domains

$$\lambda < 0,\ y^2 < \frac{R(\lambda)}{\lambda^2} \quad \text{and} \quad \lambda > 0,\ y^2 > \frac{R(\lambda)}{\lambda^2}, \tag{3-10}$$

as in Figure 1.

LEMMA 2. *The polynomial $P_{g-1}(\lambda)$ is admissible if and only if the graph of $P_{g-1}(\lambda)$ has no parts inside the black open domains.*

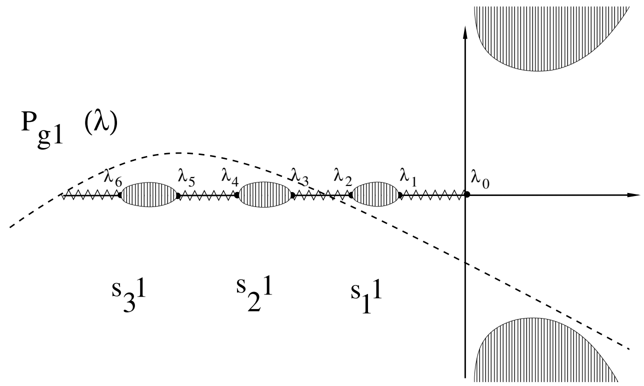

Figure 1.

If the graph does not touch these domains, we have no real divisor points. Real divisor points correspond to the case when the graph touches one of these domains but does not cross the boundary.

Each pair $\lambda_{2j-1}, \lambda_{2j}$ is connected by a black island. The graph of admissible $P_{g-1}(\lambda)$ should go above or below this island, so $P_{g-1}(\lambda) \neq 0$ on at all intervals $[\lambda_{2j}, \lambda_{2j-1}]$, $j \leq k$. We associate with an admissible polynomial $P_{g-1}(\lambda)$ a collection of numbers s_j, $j = 1, \ldots, k$ by the following rule: $s_j = 1$ if the graph of $P_{g-1}(\lambda)$ is positive in the interval $[\lambda_{2j}, \lambda_{2j-1}]$, and $s_j = -1$ otherwise. We call the set s_j the *topological type of the real solution*. There are exactly 2^k possible topological types. Elementary analytic estimates (see [Grinevich and Novikov 2003a]) show that all these components are nonempty. Each connected component is a real Abel torus, on which the x-dynamics defines a straight line. To calculate the density of the topological charge it is sufficient to know the direction of this line and the charges along the basic cycles. This follows from a simple analytic lemma:

LEMMA 3. *Let* $u(\vec{X})$, $X \in \mathbb{R}^n$ *be a smooth function in* $\mathbb{R}^n$ *such that* $\exp(i\,u(\vec{X}))$ *is single-valued on the torus* $\mathbb{R}^n/\mathbb{Z}^n$. *Equivalently, we have* $\exp(i\,u(\vec{X}+\vec{N})) = \exp(i\,u(\vec{X}))$ *for any integer vector* $\vec{N}$, *and*

$$u(X^1, X^2, \ldots, X^k + 1, \ldots, X^n) - u(X^1, X^2, \ldots, X^k, \ldots, X^n) = 2\pi n_k.$$

The numbers n_k *are called the topological charges along the basic cycles* $\mathfrak{A}_k$, $k = 1, \ldots, n$. *Denote by* $u(x)$ *restriction of* $u(\vec{X})$ *to the strait line* $\vec{X} = \vec{X}_0 + x \cdot \vec{v}$, $\vec{v} = (v^1, v^2, \ldots, v^n)$. *Then the density of topological charge*

$$\bar{n} := \lim_{T \to \infty} \frac{u(x+T) - u(x)}{2\pi T}$$

is well-defined; *it does not depend on the point* $\vec{X}_0$, *and*

$$\bar{n} = \sum_{k=1}^{n} n_k v^k. \tag{3-11}$$

The calculation of the direction vector for the x-dynamics is standard; see [Ercolani and Forest 1985], for example. Denote by ω^l the canonical basis of holomorphic differentials on Γ:

$$\omega^l = i\,\frac{\sum\limits_{j=0}^{g-1} D_j^k \lambda^j}{\sqrt{R(\lambda)}}\,d\lambda, \quad D_j^k \in \mathbb{R} \tag{3-12}$$

Then for the components of the x-direction vector we have

$$U_k = \frac{1}{2}\left(D_{g-1}^k + D_0^k \Big/ \sqrt{\prod_{j=1}^{2g} E_j}\right). \tag{3-13}$$

To obtain a simple expression for the basic charges it is critical to use a proper basis of cycles in Γ. In [Grinevich and Novikov 2001; 2003a; 2003b] the authors used the following basis, first suggested in [Dybrovin and Novikov 1982]:

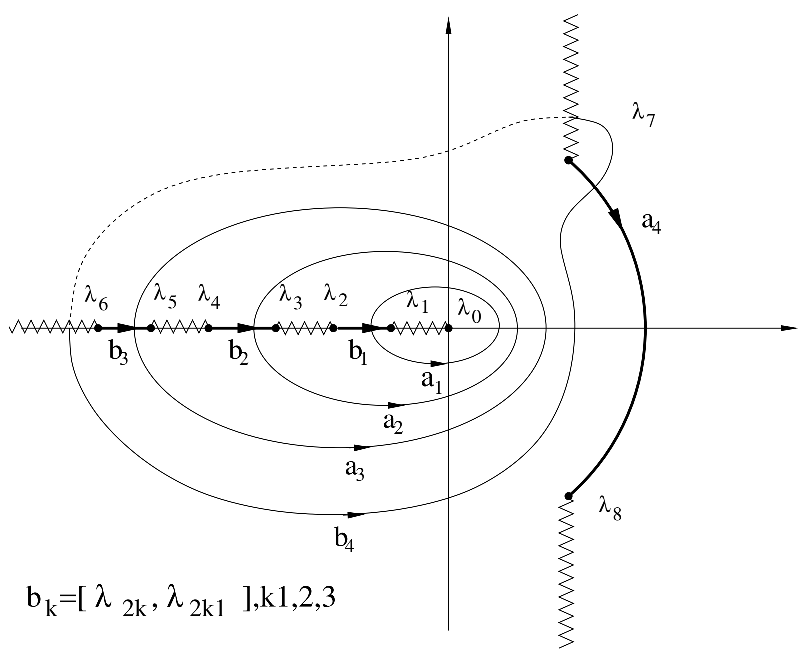

Figure 2.

Here the cycles a_j, $j = 1, \dots, k$ are ovals on the upper sheet of Γ, containing inside the points $\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_{2j-1}$. The cycle b_j, $1 \le j \le k$, lies over the interval $[\lambda_{2j}, \lambda_{2j-1}]$. The cycles a_j, $j = k+1, \dots, g$, lie over paths connecting the pairs λ_{2j-1} and λ_{2j}. We assume that these cycles do not intersect each other, and the cycles a_j, $j = k+1, \dots, g$, do not intersect the negative half-line. The cuts are shown by the zigzag lines. The upper sheet contains the half-line $\lambda > 0$, $\mu > 0$.

Consider a basic cycle $\mathfrak{A}_j$ on the real component of Jacoby torus, represented by the closed curve. The image of this cycle in Γ under the inverse Abel map is a closed oriented curve C_j, formed by the motion of the corresponding divisor points (it may have several connected components). The motion of an individual divisor point does not have to be periodic, after going along the cycle we may obtain a permutation of the divisor points. The curve C_j is homological to the cycle $a_j \in H_1(\Gamma, Z)$. It follows from (3-7) that the topological charge n_j along the cycle $\mathfrak{A}_k$ equals to the winding number of the curve C_j with respect to the point 0. Equivalently

$$n_j = \tilde{C}_j \circ \mathbb{R}_-, \tag{3-14}$$

where $\circ$ denotes the intersection number, $\tilde{C}_j$ denotes the projection of C_j to the λ-plane, $\mathbb{R}_-$ is negative half-line with the standard orientation.

For each point of $\mathfrak{A}_j$ the corresponding divisor $\gamma_1, \dots, \gamma_g$ is admissible. From the characterization of admissible divisors obtained above it is easy to show that the curve C_j does not touch the closed segments on the real line $[-\infty, \lambda_{2m}], \dots,$ $[\lambda_3, \lambda_2]$, $[\lambda_1, 0]$. Therefore any time the curve C_j crosses the negative half-line, it intersects one of the basic cycles b_j, $j = 1, \dots, k$.

This information does not yet suffice to calculate the basic charge, because the orientation of the cycles b_j coincides with the orientation of the negative half-line at one sheet and they are opposite at the other one. For example, in Figure 3 we see two realizations of the cycle a_1, representing different topological types. a_1 is drawn at the upper sheet and a'_1 is drawn at the lower one. We have $a_1 \circ b_1 = a'_1 \circ b_1 = 1$, but $\tilde{a}_1 \circ \mathbb{R}_- = 1$, $\tilde{a}'_1 \circ \mathbb{R}_- = -1$, therefore $n_1 = 1$ and $n_1 = -1$ for these cycles respectively.

Fortunately, the topological type contains information about the sheet where the intersection takes place:

LEMMA 4. *Assume that the cycles C_j intersects the negative half-line at the interval $(\lambda_{2l}, \lambda_{2l-1})$. Then orientations of b_l and $\mathbb{R}_-$ coincide in the intersection point if $(-1)^{l-1} s_l > 0$ and are opposite if $(-1)^{l-1} s_l < 0$.*

Combining all these results we obtain the final formula, expressed in the next theorem:

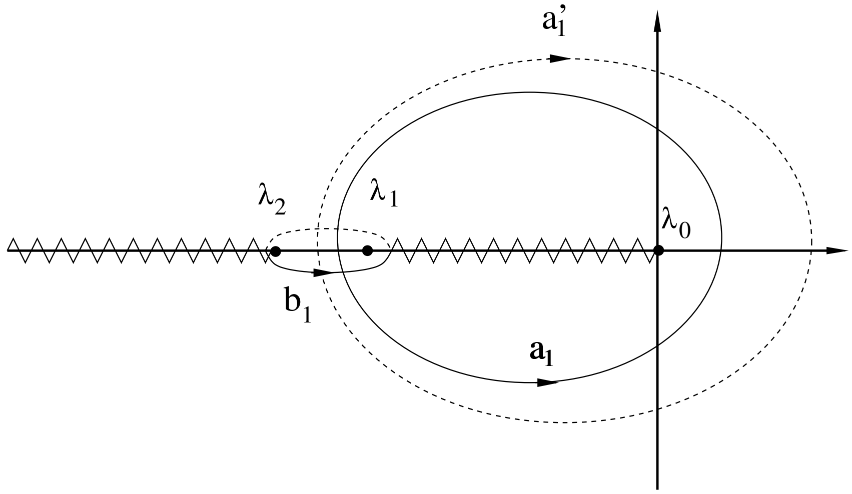

Figure 3.

THEOREM. *The density of topological charge for a real sine-Gordon solution is given by*

$$\bar{n} = \sum_{j=1}^{k} (-1)^{j-1} s_j U_j, \tag{3-15}$$

where the vector U_j is defined by (3-13).

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