Probability, Geometry and Integrable Systems
MSRI Publications
Volume **55**, 2007

The Riccati map in random Schrödinger and random matrix theory

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For H. P. McKean, who taught us this trick.

ABSTRACT. We discuss the relevance of the classical Riccati substitution to the spectral edge statistics in some fundamental models of one-dimensional random Schrödinger and random matrix theory.

1. Introduction

The Riccati map amounts to the observation that the Schrödinger eigenvalue problem $Q\psi = \lambda\psi$ for $Q = -d^2/dx^2 + q(x)$ is transformed into the first order relation

$$q(x) = \lambda + p'(x) + p^2(x) \tag{1-1}$$

upon setting $p(x) = \psi'(x)/\psi(x)$. That this simple fact has deep consequences for the problem of characterizing the spectrum of Q with a random potential q has been known for some time. It also turns out to be important for related efforts in random matrix theory (RMT). We will describe some of the recent progress on both fronts.

Random operators of type Q arise in the description of disordered systems. Their use goes back to Schmidt [1957], Lax and Phillips [1958], and Frisch and Lloyd [1960] in connection with disordered crystals, represented by potentials in the form of trains of signed random masses, randomly placed on the line. Consider instead the case of white noise potential, $q(x) = b'(x)$ with a standard brownian motion $x \mapsto b(x)$, which may be viewed as a simplifying caricature

Rider was supported in part by NSF grant DMS-0505680.

of the above. The problem $Q\psi = \lambda\psi$ then reads $d\psi'(x) = \psi(x)\,db(x) + \lambda\psi(x)\,dx$ and is solvable for $\psi \in C^{3/2}$.

A first order statistic of interest is the integrated density of states $N(\lambda) = \lim_{L\to\infty} L^{-1} \times \{\text{the number of eigenvalues} \le \lambda\}$, in which we take Q on the interval $[0, L]$ with say Dirichlet boundary conditions. Build the sine-like solution $\psi_0(x,\lambda)$ of $Q\psi_0 = \lambda\psi_0$ with $\psi_0(0) = 0$ and $\psi_0'(0) = 1$. The pair $x \mapsto (\psi_0(x), \psi_0'(x))$ is clearly Markovian, as is the ratio $x \mapsto p(x) := \psi_0'(x)/\psi_0(x)$. Further, the latter solves a version of (1-1) which can only be interpreted as to say that p performs the diffusion with infinitesimal generator

$$\mathfrak{G} = (1/2)\partial^2/\partial p^2 - (\lambda + p^2)\partial/\partial p. \tag{1-2}$$

This motion begins at $p(0) = +\infty$, which is an entrance barrier, hits the exit barrier $-\infty$ at the first root $\mathfrak{m}_1$ of $\psi_0(x,\lambda) = 0$, then reappears at $+\infty$ whereupon everything starts afresh.

Now, to count the eigenvalues below a level λ is to count the number of roots of $\psi_0(x,\lambda)$ before $x = L$, and so the number of (independent) passages from $+\infty$ to $-\infty$ of the p motion. If this number is n, then L approximates $\mathfrak{s}_n = \mathfrak{m}_1 + \cdots + \mathfrak{m}_n$, the sum of the first n passage times, so that, by the law of large numbers

$$\frac{1}{N(\lambda)} = \lim_{n\to\infty} \mathfrak{s}_n/n = E[\mathfrak{m}_1] = \sqrt{2\pi}\int_0^\infty e^{-(p^3/6+\lambda p)}\,\frac{dp}{\sqrt{p}},$$

as may be worked out from the speed and scale associated with (1-2). This computation is due to Halperin [1965]; see also [Fukushima and Nakao 1976/77].

As for the fluctuations, McKean [1994] proved, via Riccati, that

$$\lim_{L\to\infty} P\Big(\frac{L}{\pi}(-\Lambda_0(L))^{1/2}\exp\big[-\tfrac{8}{3}(-\Lambda_0(L))^{3/2}\big] > x\Big) = \begin{cases} 1 & \text{for } x < 0, \\ e^{-x} & \text{for } x \ge 0, \end{cases} \tag{1-3}$$

where $\Lambda_0(L)$ pertains to the operator $-\frac{d^2}{dx^2} + b'(x)$ acting on $[0, L]$ with Dirichlet, Neumann, or periodic conditions. While a step forward, (1-3) is still thermodynamic in nature. More desirable is to use the Riccati trick to capture local spectral statistics in a fixed volume, and this is where the main part of our story begins.

Cambronero and McKean [1999] took the point of view that the Riccati map (1-1) represents a change of measure from potential, or q-path, space to the space of p-paths, resulting in an explicit functional integral formula for the probability density of Λ_0 under periodic conditions (Hill's equation). The method extends from white noise q, to any periodic diffusion potential of brownian motion type plus restoring drift. Section 2 describes all this. Given such integral expressions, the next natural task is to describe the shape of the ground state eigenvalue density. A summary of the results thus far makes up Section 3, with an emphasis on the differences between the white noise case, and the roughly universal nature of the shape for nice Gaussian potentials. Section 4 is devoted to the surprising recent discovery that a 1-d random Schrödinger operator and thus, via Riccati, the explosion probability of a certain diffusion figure into the celebrated Tracy–Widom laws of RMT along with their generalizations. We finish up with a collection of open questions.

Further background. As indicated, the Riccati substitution is a basic tool in the study of 1-d random Schrödinger, as may be gleaned from the comprehensive book [Carmona and Lacroix 1990]. Indeed, (1-3) is only one instance of a ground state limit theorem. For a large class of Markovian potentials it is understood that the spectrum is Poissonian and that the large volume limit of the edge eigenvalues follow standard i.i.d. extremal laws; see [Molčanov 1980/81] or [Grenkova et al. 1983]. The second reference also shows that the limit can be joint Gaussian (and so exhibit repulsion) when the Lyapunov exponent is degenerate at the spectral edge. In all these results the normalization depends on the smoothness of the potential, and this is one reason that (1-3) deserves to be set apart. Additionally, our shape results for the ground state density (Section 3) should be compared with the large body of work on the Lifschitz tails dating back to the 70's. Ideas connected to that work can in fact be used to obtain tail estimates on the *distribution function* in the case of continuous Gaussian potentials in a finite volume, including even multiple dimensions (exactly such bounds turn up in recent work on the parabolic Anderson model [Gärtner et al. 2000]). Finally, there is an extensive literature on the almost sure behavior of Λ_0 in the more physical $d > 1$ setting with Poisson-bump or Gibbsian type potentials; see [Merkl 2003; Sznitman 1998] and the many references therein. Our point here though is to focus on the ground state *density* and the approach inspired by McKean.

2. The Riccati map as a change of measure

Let $Q\psi = -\psi'' + q\psi = \lambda\psi$ be Hill's equation with standard white noise potential $q(x)$ on the circle $0 \le x < 1 = S^1$. Bring in the sine and cosine-like solutions $\psi_0(x, \lambda)$ and $\psi_1(x, \lambda)$ satisfying $\psi_1(0) = 0, \psi_0'(0) = 1, \psi_1(0) = 1, \psi_1'(0) = 0$, and also the discriminant $\Delta(\lambda) = \frac{1}{2}[\psi_0(1, \lambda) + \psi_1'(1, \lambda)]$. The latter is an entire function of order $1/2$ and encodes the spectrum: $\Delta = \pm 1$ at the periodic/antiperiodic eigenvalues. In particular, if $\Lambda_0 = \Lambda_0(q)$ is the ground state eigenvalue for Q, $\Delta(\lambda)$ decreases from the left to its value $\Delta = 1$ at $\lambda = \Lambda_0$. Moreover, $Q\psi = \lambda\psi$ has a solution with multiplier m (a solution for which $\psi(x+1) = m\psi(x)$), if and only if $m = \Delta(\lambda) \pm \sqrt{\Delta^2(\lambda) - 1}$. There is a positive solution of this type with $0 < m < \infty$ only when $\lambda \le \Lambda_0$, in which case there are actually two such solutions with multipliers m_+ and $m_- = 1/m_+$; these fall together ($m_+ = m_- = 1$) at the periodic ground state when $\lambda = \Lambda_0$.

The corresponding Riccati equation,

$$q(x) = \lambda + p'(x) + p^2, \tag{2-1}$$

determines p as a diffusion on S^1 solving the stochastic differential equation

$$dp(x) = db(x) - (\lambda + p^2(x))\, dx,$$

provided that $\Lambda_0(q) \ge \lambda$. In fact, if such a solution p exists and ϕ is a smooth periodic function with $\int_0^1 \phi^2(x)\, dx = 1$, then

$$\int_0^1 \big((\phi'(x))^2 + q(x)\phi^2(x)\big)\, dx \ge \lambda,$$

and therefore $\Lambda_0 \ge \lambda$. Conversely, if $\Lambda_0 \ge \lambda$ we have just explained that there is a positive solution $\psi(x)$ of $Q\psi = \lambda\psi$ with multiplier: $\psi(x+1) = m\psi(x)$ and $m \ge 1$. It follows that $p = \psi'/\psi$ solves (2-1) and satisfies the side condition $\int_0^1 p(x)\, dx = \log m \ge 0$.

This defines the Riccati map. In the $p \to q$ direction, it is one-to-one on $H = \big[\int_0^1 p = 0\big]$, and also on $H^+ = \big[\int_0^1 p \ge 0\big]$. The set H^+ is mapped onto $[\Lambda_0(q) \ge \lambda]$, while the mean-zero condition in p-space H coincides with $m = 1$ and so the event $[\Lambda_0(q) = \lambda]$.

Distribution of the ground state eigenvalue. Cambronero and McKean [1999] used the map above between $[\Lambda_0 \ge \lambda]$ and $\big[\int_0^1 p \ge 0\big]$ to express the white noise measure of the former in terms of a circular brownian motion (CBM) integral over the latter. The CBM is formed by the standard brownian motion loop space with $p(0) = p(1)$, which is then distributed according to $P(p(0) \in da) = (1/\sqrt{2\pi})\, da$. The result is,

$$Q_*[\Lambda_0(q) \ge \lambda] = \sqrt{\frac{2}{\pi}} \int_{H^+} e^{-\frac{1}{2}\int_0^1 (\lambda + p^2(x))^2\, dx} \sinh\bigg(\int_0^1 p\bigg)\, dP_*(p), \tag{2-2}$$

where Q_* and P_* henceforth denote the white noise and CBM measures. By a more elaborate computation, considering the Riccati map on the product space of the potential and logarithmic multiplier $\log m$, [Cambronero and McKean 1999] also establishes a formula for the probability density $f(\lambda) = \frac{d}{d\lambda} Q_*[\Lambda_0 \le \lambda]$. In particular,

$$f(\lambda) = \frac{1}{\sqrt{2\pi}} \int_H e^{-\frac{1}{2}\int_0^1 (\lambda + p^2(x))^2\, dx} A(p)\, dP_0(p), \tag{2-3}$$

where $A(p) = \int_0^1 e^{2\int_0^x p} \times \int_0^1 e^{-2\int_0^x p}$ and P_0 is the CBM conditioned so that $\int_0^1 p = 0$. Unlike CBM which has infinite total mass, P_0 is a proper Gaussian probability measure on paths.

REMARK. The distribution (2-2) may be differentiated to produce the density in the form

$$f(\lambda) = \sqrt{\frac{2}{\pi}} \int_{H^+} \left(\lambda + \int_0^1 p\right) e^{-\frac{1}{2}\int_0^1 (\lambda + p^2(x))^2\, dx} \sinh\left(\int_0^1 p\right) dP_*(p),$$

equating an integral over the half-space H^+ to an integral over its boundary H. One might suppose that the present is related to (2-3) by the appropriate function-space divergence theorem, and this in fact is verified in [Cambronero and McKean 1999].

Formally, the Riccati map relates the white noise measure to CBM via

$$dQ_* = \exp\left(-\frac{1}{2}\int_0^1 q^2\right) \frac{d^\infty q}{(2\pi/0+)^{\infty/2}} = \exp\left(-\frac{1}{2}\int_0^1 (\lambda + p^2)^2\right) |J|\, dP_*,$$

where

$$dP_* = \exp\left(-\frac{1}{2}\int_0^1 |p'|^2\right) \frac{d^\infty p}{(2\pi 0+)^{\infty/2}}$$

is the CBM in symbols, and the Jacobian J is to be determined. One may be tempted to employ the Cameron–Martin formula and claim that

$$dQ_* = \exp\left(-\frac{1}{2}\int_0^1 (\lambda + p^2)^2\right) \exp\left(\int_0^1 p\right) dP_*,$$

that is, $|J| = \exp\left(\int_0^1 p\right)$. But this does not apply here, the equation (2-1) being understood with periodic, and not initial, conditions.

The next section contains a sketch of the proper Jacobian calculation and so the verification of (2-2). This is followed by (the outline of) two proofs of the density formula (2-3). Last, it is explained how both types of expressions may be extended to a class of periodic diffusion potentials.

Jacobian of the Riccati map and distribution of Λ_0. The needed Jacobian is obtained by passing through the finite–dimensional distributions of Q_* and P_*. These spaces are furnished with a discrete version of the transformation (2-1) for which we can compute $|J|$ by hand. Afterward, limits may be performed to pin down the "infinite dimensional" Jacobian.

The appropriate discrete version of Riccati's transformation reads

$$q_i = \lambda + n^2(e^{hp_{i+1}} - 2 + e^{-hp_i}), \quad i = 0, \dots, n-1, \tag{2-4}$$

carrying $\mathbb{R}^n$ to $\mathbb{R}^n$, where $h = \frac{1}{n}$, and $q_n = q_0$ and $p_n = p_0$. Notice that, for hp_i small,

$$q_i \simeq \lambda + n(p_{i+1} - p_i) + \tfrac{1}{2}(p_{i+1}^2 + p_i^2)$$

provides an approximation to (2-1). Also, one easily computes that

$$|J| = \frac{2}{h^n}\left|\sinh\left(\sum_{i=0}^{n-1} p_i h\right)\right|$$

for the map (2-4). This expression vanishes only when $\left[\sum p_i h = 0\right]$, and this discrete form of Riccati is actually one-to-one on the region $\left[\sum p_i h > 0\right]$ onto $[\lambda^{(n)} \geq \lambda]$, $\lambda^{(n)}$ being the ground state of the discrete version of Hill's equation with potential vector $(q_0, \ldots, q_{n-1})$.

Next, bring in the discrete white noise

$$\overline{q}_i = n \int_{\frac{i}{n}}^{\frac{i+1}{n}} q = n(b_{i+1} - b_i), \quad \text{with } b_i = b\left(\frac{i}{n}\right) \text{ and a standard brownian motion } b(\cdot). \tag{2-5}$$

Assuming that $\Lambda_0(q) > \lambda$, it holds that $\lambda^{(n)}(q) > \lambda$ for all large values of n. Also, denoting by $\overline{p_0 \cdots p_{n-1}}$ the polygonal path determine by the points $p_0, \ldots, p_n$, and similarly for q, it may be checked that:

LEMMA 2.1. *For almost every white noise path q, with $\Lambda_0(q) > \lambda$, $\overline{p_0 \cdots p_{n-1}}$ converges uniformly to the solution $p(x) = \psi'(x,\lambda)/\psi(x,\lambda)$of* (2-1).

As a consequence, if H_N denotes the set of white noise paths q for which $\lambda^{(n)}(q) > \lambda$ for all $n \geq N$, and $\max|p_i| \leq N$ for all $n \geq N$, then $Q_*(H_N) \to 1$, as $N \to \infty$. This allows one to further restrict the discrete transform to

$$D_N = H_N \cap \{q : \max_{i=0,\ldots,n-1} |b_{i+1} - b_i| \leq 2\sqrt{h \log n} \text{ for all } n \geq N\},$$

where the convergence may be controlled. (By Levy's modulus of continuity $Q_*(D_N)$ tends to 1, so this is enough.) Now, on D_N and taking $\lambda = 0$ for convenience, one has

$$-\frac{1}{2}\sum_{i=0}^{n-1} q_i^2 h = -\frac{1}{2h}\sum_{i=0}^{n-1}(p_{i+1} - p_i)^2 - \frac{1}{8}\sum_{i=0}^{n-1}(p_{i+1}^2 + p_i^2)^2 h + R_n,$$

with a remainder $R_n \to 0$ boundedly. The discrete white noise measure

$$\exp\left(-\frac{1}{2}\sum q_i^2 h\right)\frac{dq_0 \ldots dq_{n-1}}{(2\pi/h)^{n/2}}$$

may then be written as

$$\sqrt{\frac{2}{\pi}}\exp\biggl(-\frac{1}{8}\sum_{i=0}^{n-1}(p_{i+1}^2+p_i^2)^2h+R_n\biggr)\biggl|\sinh\sum_{i=0}^{n-1}p_ih\biggr|\,d\mu_n,$$

where

$$d\mu_n=\sqrt{2\pi}\exp\biggl(-\frac{1}{2h}\sum_{i=0}^{n-1}(p_{i+1}-p_i)^2\biggr)\frac{dp_0\dots dp_{n-1}}{(2\pi h)^{n/2}}.$$

Thus, for a bounded continuous function ϕ of the path q vanishing off D_N, it holds that

$$\begin{aligned}\int_{[\lambda_0\ge 0]}&\phi(q)\,dQ_*\\ &=\lim_{n\to\infty}\int_{\mathbb{R}^n}\phi_n(q_0,\dots,q_{n-1})\exp\biggl(-\frac{1}{2}\sum_{i=0}^{n-1}q_i^2h\biggr)\frac{dq_0\dots dq_{n-1}}{(2\pi/h)^{n/2}}\\ &=\lim_{n\to\infty}\sqrt{\frac{2}{\pi}}\int_{\mathbb{R}^n}\hat{\phi}_n(p_0,\dots,p_{n-1})\,d\nu_n,\end{aligned}$$

in which

$$d\nu_n=\exp\biggl(-\frac{1}{8}\sum_{i=0}^{n-1}(p_{i+1}^2+p_i^2)^2h+R_n\biggr)\sinh\biggl(\sum_{i=0}^{n-1}p_ih\biggr)\,d\mu_n,$$

ϕ_n denotes ϕ evaluated on the discrete q-path, and $\hat{\phi}_n(p):=\phi_n(q)$. Then, by dominated convergence we have the identity

$$\int_{[\Lambda_0\ge 0]}\phi(q)\,dQ_*=\sqrt{\frac{2}{\pi}}\int_{H^+}\hat{\phi}(p)\,\exp\biggl(-\frac{1}{2}\int_0^1p^4\biggr)\sinh\biggl(\int_0^1p\biggr)\,dP_*,$$

where $\hat{\phi}(p)$ is defined through the Riccati correspondence; it is sensible along with $\phi(q)$. A standard argument will extend the picture to any bounded continuous ϕ and also to $\lambda\neq 0$. To summarize:

THEOREM 2.2. *If Q_* is the restriction of the white noise measure to the region* $[\lambda_0(q)\ge\lambda]$, *and if P_* is the restriction of circular brownian motion measure to H^+, then*

$$dQ_*=\sqrt{\frac{2}{\pi}}\exp\biggl(-\frac{1}{2}\int_0^1(\lambda+p^2)^2\biggr)\sinh\biggl(\int_0^1p\biggr)\,dP_*.$$

The formula (2-2) *for the distribution of $\Lambda_0(q)$ follows immediately.*

REMARK. As an entertaining aside one learns that

$$\lim_{\lambda\to-\infty}\int_{H^+}\exp\Big(-\frac{1}{2}\int_0^1(\lambda+p^2)^2\Big)\sinh\Big(\int_0^1 p\Big)\,dP_*=\sqrt{\frac{\pi}{2}},$$

which is not at all obvious.

The measure induced by Q_* on $[\Lambda_0=\lambda]$ and the density formula. Here is a way to understand (2-3) not reported in [Cambronero and McKean 1999]. To start, define Q_λ by

$$\int_{[\Lambda_0=\lambda]}\phi(q)\,dQ_\lambda=\lim_{h\to 0}\frac{1}{h}\int\phi(q)\chi_{[\lambda\le\Lambda_0\le\lambda+h]}\,dQ_*, \tag{2-6}$$

for any bounded continuous ϕ.

Next, being analytic, $\Delta(\lambda)$ is locally bounded in both λ and $|b|$, and the same is true of $\dot\Delta(\lambda)=(d/d\lambda)\,\Delta(\lambda)$ and $\ddot\Delta(\lambda)$. So, $\Delta(\lambda)=1+(\Lambda_0-\lambda)\,|\dot\Delta(\Lambda_0)|+O(h^2)$ with $\lambda\le\Lambda\le\lambda+h$. It follows that

$$m=\Delta+\sqrt{\Delta^2-1}=1+\sqrt{2\,(\Lambda_0-\lambda)\,|\dot\Delta(\Lambda_0)|}+O(h),$$

and for $q=\lambda+p'+p^2$, we also conclude

$$\int_0^1 p=\log m=\sqrt{2\,(\Lambda_0-\lambda)\,|\dot\Delta(\lambda_0)|}+O(h).$$

Coupled with the classical fact that

$$-2\dot\Delta(\Lambda_0)=\int_0^1\psi^2(t)\,dt\int_0^1\frac{dt}{\psi^2(t)}.$$

for ψ the periodic ground state, $2|\dot\Delta(\lambda_0)|=A(p_0)(1+O(h))$ where $p_0=p-\int_0^1 p$ and

$$A(p_0)=\int_0^1 e^{-2\int_0^x p_0}dx\int_0^1 e^{2\int_0^x p_0}dx.$$

Now introduce the identity

$$\int_H\phi(p)B^2(p)\,dP_0(p)\\=\lim_{\varepsilon\downarrow 0}\frac{2}{\varepsilon^2}\int\phi(p)\sinh\Big(\int_0^1 p\Big)\,1_{[0\le\int_0^1 p\le B(p-\int_0^1 p)\varepsilon]}\,dP_*(p),$$

which is proved directly from the definition of P_0 as the conditional P_*; it holds for bounded continuous ϕ and a large class of $B:H\to\mathbb{R}^+$ including $B(\cdot)=\sqrt{A(\cdot)}$. With that choice, the previous estimates can be used to effectively replace $\big\{0\le\int_0^1 p\le\big(A(p-\int p)\big)^{1/2}\varepsilon\big\}$ with $\{0\le\Lambda_0\le\varepsilon^2\}$. If that substitution is made, we understand at once that the measure Q_λ induced by Q_* on $[\Lambda_0 = \lambda]$ satisfies

$$dQ_\lambda = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\int_0^1 (\lambda + p^2)^2\right) A(p)\, dP_0$$

under the Riccati transformation, and this is equivalent to (2-3).

Joint distribution of $(q, \log m)$ and a second proof. Perhaps a more formulaic route to the density formula is available by way of the joint transformation

$$(q, \log m) \leftrightarrow (p, \lambda).$$

Given (p, λ) with p in the CBM space, we set $\log m = \int_0^1 p$ and $q = \lambda + p' + p^2$. Mapping back, given $(q, \log m)$ with q in the white noise space, we take $\lambda \le \Lambda_0(q)$ so that $\Delta(\lambda, q) = \frac{1}{2}(m + \frac{1}{m})$. This λ is unique since $\dot{\Delta}(\lambda, q) < 0$ for $\lambda < \Lambda_0(q)$. One may then choose ψ to be the positive Hill's solution with multiplier m and set $p = \psi'/\psi$. This (p, λ) pair is thus unique and will reproduce the original $(q, \log m)$, showing that the augmented Riccati map is one to one and onto.

To compute the joint distribution of q and $\log m$ in terms of p and λ, [Cambronero and McKean 1999] again considers the approximating discrete (one-to-one and onto) transformation

$$(p_0, \ldots, p_{n-1}, \lambda) \longrightarrow (q_0, \ldots, q_{n-1}, \log m),$$

from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$, defined by

$$q_i = \lambda + n^2(e^{hp_{i+1}} - 2 + e^{-hp_i}), \quad \log m = \sum_{i=0}^{n-1} p_i h, \tag{2-7}$$

where $h = \dfrac{1}{n}$ and $p_n = p_0$. The corresponding Jacobian is now

$$h^n\,|J_n| = \sum_{i=0}^{n-1} \frac{h}{m\varphi_i^2} \sum_{k=i+1}^{i+n} \varphi_k^2 h + O(h) \ \text{ for } \ \varphi_i = \exp\left(\sum_{j=1}^{i} p_j h\right).$$

As before, the discrete white noise $\times d \log m$ measure may then be reexpressed as in

$$\begin{aligned}
&\exp\left(-\frac{1}{2}\sum q_i^2 h\right) \frac{dq_0 \ldots dq_{n-1}}{(2\pi/h)^{n/2}} \times d\log m \\
&\quad = \exp\Biggl(-\frac{1}{2h}\sum_{i=0}^{n-1}(p_{i+1} - p_i)^2 - \frac{1}{8}\sum_{i=0}^{n-1}(p_{i+1}^2 + p_i^2)^2 h \\
&\qquad\qquad - \frac{\lambda}{2}\sum_{i=0}^{n-1}(p_i^2 + p_{i+1}^2)h - \frac{\lambda^2}{2} + R_n\Biggr) \frac{h^n|J_n|}{(2\pi h)^{n/2}}\, dp_0 \ldots dp_{n-1}\, d\lambda,
\end{aligned}$$

where again $R_n \to 0$ boundedly on certain sets of large measure. Thus, on any such set, we have

$$\lim_{n\to\infty} h^n |J_n| = A(p) = \int_0^1 \frac{dx}{m\varphi^2(x)} \int_x^{x+1} \varphi^2(y)\,dy,$$

for $\varphi(x) = \exp(\int_0^x p)$, and it is only a bit more effort to arrive at the following.

THEOREM 2.3. *For any bounded ϕ, compactly supported with respect to $a = \log m$, we have*

$$\int \phi(Q,a)\,dQ_*\,da = \int \phi\big(\lambda + p' + p^2,\ \textstyle\int p\big)\,\Theta(p,\lambda)\,dP_*\,d\lambda,$$

where

$$\Theta(p,\lambda) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\int_0^1 (\lambda + p^2)^2 - \int_0^1 p\right) A(p).$$

In brief, $dQ_*\,da = \Theta(p,\lambda)\,dP_*\,d\lambda$.

Now employ the relation between $dQ_*\,da$ and $dP_*\,d\lambda$ as follows. First,

$$\begin{aligned} Q_*[\lambda \le \Lambda_0(q) \le \lambda+\varepsilon] &= \frac{1}{\delta}\int_0^\delta \int_{[\lambda\le\Lambda_0(q)\le\lambda+\varepsilon]} dQ_*\,da \\ &= \frac{1}{\delta}\int_{[0\le\int p\le\delta]} \Theta(p,\lambda)\chi_{[\lambda\le\Lambda_0(\lambda+p'+p^2)\le\lambda+\varepsilon]}\,dP_*\,d\lambda. \end{aligned}$$

The left-hand side is independent of δ, so for $\delta\to 0$ we find

$$Q_*[\lambda \le \Lambda_0(q) \le \lambda + \varepsilon] = \int_H \Theta(p,\lambda)\chi_{[\lambda\le\Lambda_0(\lambda+p'+p^2)\le\lambda+\varepsilon]}\,dP_0\,d\lambda.$$

Now $\int_0^1 p = 0$ implies $\Lambda_0(\lambda + p' + p^2) = \lambda$, and therefore

$$\varepsilon^{-1} Q_*[\lambda \le \Lambda_0(q) \le \lambda + \varepsilon] = \int_H \left(\varepsilon^{-1}\int_\lambda^{\lambda+\varepsilon} \Theta(p,\lambda)\,d\lambda\right) dP_0.$$

As $\varepsilon \to 0$, the left-hand side converges to $f(\lambda) = (d/d\lambda)\,Q_*[\Lambda_0 \le \lambda]$, and the integrand on the right-hand side converges to $\Theta(p,\lambda)$. Moreover, there is the needed domination to prove that

$$f(\lambda) = \int_H \Theta(p,\lambda)\,dP_0 = \frac{1}{\sqrt{2\pi}} \int_H e^{-\frac{1}{2}\int(\lambda+p^2)^2} A(p)\,dP_0,$$

as advertised.

Ornstein–Uhlenbeck type potentials. The methods above extend from white noise potentials to a whole class of q's which perform a periodic diffusion. For example, let $\hat{Q}$ denote periodic Ornstein–Uhlenbeck (OU) measure (of mass m). This is the rotation invariant Gaussian process on S^1 arrived at by conditioning the OU paths so that $q(0) = q(1)$ and then distributing that common point according to the stationary measure for the full-line OU.

Similarly to white noise one gets:

THEOREM 2.4 [Cambronero and McKean 1999]. *Under the transformation* $q = \lambda + p' + p^2$, *the periodic OU measure* $\hat{Q}$, *restricted to* $[\Lambda_0 \geq \lambda]$, *is transformed into the measure* $dP_0\, d\alpha$ *according to*

$$\int_{[\Lambda_0 \geq \lambda]} \phi\, d\hat{Q} = C \int_H \int_{\mathcal{I}(p')}^{\infty} \phi(\lambda + p' + p^2) e^{-\frac{1}{2}m^2 \int_0^1 (\lambda + p' + p^2)^2} G(\alpha, p')\, d\alpha\, dP_0(p'), \tag{2-8}$$

with $C = (4/\sqrt{2\pi}) \sinh(m/2)$, $p = \alpha + \int_0^t p'$, $\mathcal{I}(p') = -\int_0^1 \int_0^t p'$, *and*

$$G(\alpha, p') = \exp\left(\int_0^1 (p'^3 - 2p^2 p' + p^2)\, dt \right) \sinh\left(\int_0^1 p \right).$$

In particular, the distribution is read off upon setting $\phi \equiv 1$ in (2-8), providing the analogue of (2-2). Further, one can move on to other potentials of type brownian motion plus drift,

$$dq(x) = db(x) - m(q)\, dx,$$

where it is assumed that m is an odd function with $m(q) > 0$ for $q > 0$ to avoid explosion. The periodic versions of these processes are built in the same way as for OU; the added condition

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}(m'(q) - m^2(q))}\, dq < \infty \tag{2-9}$$

being required to ensure the periodic measure has finite total mass.

THEOREM 2.5 [Cambronero 1996]. *Let* Q_* *be a periodic diffusion with odd drift* $m(q)$ *subject to* $m(q) > 0$ *for* $q > 0$ *and* (2-9). *Then*

$$Q_*[\Lambda_0 \geq \lambda] = 2C_0 \int_H \int_{\mathcal{I}(p')}^{\infty} \exp\left(-\frac{1}{2} \int_0^1 F(\lambda + p'(x) + p^2(x))\, dx \right) G(\alpha, p')\, d\alpha\, dP_0,$$

where $F = -m' + m^2$, *and* $C_0^{-1} = \int \exp\bigl(-\frac{1}{2}\int_0^1 F(q)\bigr) dP_*$ *is a normalizing constant.*

And, again by considering joint distributions of q and the multiplier, there is also a formula for the density.

THEOREM 2.6 [Cambronero 1996]. *The density of* Λ_0 *under* Q_* *is given by*

$$f(\lambda) = C_0 \int_H e^{-\frac{1}{2}\int F(\lambda+p'+p^2)} \mathscr{E}(p')\, dP_0(p')$$

where $p = \mathscr{I}(p') + \int_0^t p'$ *and* $\mathscr{E}(p') = \exp\bigl(\int_0^1 (p'^3 - 2p^2 p' + p^2)\bigr) A(p)$.

After this parade of formulae, it is probably helpful to write out the linear (OU or $m(q) = mq$) case in full:

$$f_{OU}(\lambda) = \sqrt{\frac{2}{\pi}} \sinh \frac{m}{2} \int_H e^{-\frac{1}{2}m^2 \int_0^1 (\lambda+p'+p^2)^2} e^{\int_0^1 (p'^3 - 2p^2 p' + p^2)} A(p)\, dP_0(p'). \tag{2-10}$$

It is now p' that is locally brownian. Starting with white noise, p is CBM under the Riccati map. Starting with an additional derivative in potential space results in an additional derivative in p-space. The added dependence in the field makes integrals like (2-10) harder to analyze than their white noise counterparts. This is the subject of the next section.

3. Ground state energy asymptotics

As an application of the above integral expressions we consider the shape of the ground state energy density for various random potentials. We begin again in the white noise case, for which detailed asymptotics are available:

THEOREM 3.1 [Cambronero et al. 2006]. *Let* $f_{WN}(\lambda)$ *denote the density function for* $\Lambda_0(q)$, *the minimal eigenvalue for Hill's operator on the circle of perimeter one with white noise potential. Then*

$$f_{WN}(\lambda) = \sqrt{\frac{\lambda}{\pi}} \exp\Bigl(-\frac{1}{2}\lambda^2 - \frac{1}{\sqrt{2}}\lambda^{1/2}\Bigr)(1 + o(1)),$$

as $\lambda \to +\infty$ *and*,

$$f_{WN}(\lambda) = \frac{4}{3\pi}|\lambda| \exp\Bigl(-\frac{8}{3}|\lambda|^{3/2} - \frac{1}{2}|\lambda|^{1/2}\Bigr)(1 + o(1)),$$

as $\lambda \to -\infty$.

The overall asymmetry has an intuitive explanation: level-repulsion holds down the right tail, while a large negative deviation can be affected by a single excursion of the potential. The $3/2$-exponent in $\lambda \to -\infty$ direction is shared by the allied tail in the Tracy–Widom laws of RMT, but more on this later.

The above result stems from the second version of the density:

$$f_{WN}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_H e^{-\frac{1}{2}\int_0^1 (\lambda + p^2)^2} A(p)\, dP_0(p),$$

where P_0 is the CBM conditioned to be mean-zero. In either the $\lambda \to +\infty$ or $\lambda \to -\infty$ direction, the leading order, or logarithmic scale, asymptotics of f_{WN} are governed by those of the infimum of

$$I_\lambda(p) := \frac{1}{2}\int_0^1 \big(\lambda + p^2(x)\big)^2 dx + \frac{1}{2}\int_0^1 \big(p'(x)\big)^2 dx, \tag{3-1}$$

over $p \in H$. When $\lambda \to +\infty$ it is plain that it is most advantageous for the path p to sit in a vicinity of the origin, which already accounts for the appraisal $f_{WN}(\lambda) \sim e^{-\lambda^2/2}$. For a more complete picture, $\int_0^1 (\lambda + p^2)^2$ may be expanded, and both $A(p)$ and $e^{-1/2\int_0^1 p^4}$ are seen to be unimportant in comparison with $e^{-\lambda \int_0^1 p^2}$. That is, $E_0[e^{-\lambda \int_0^1 p^2 - 1/2\int_0^1 p^4} A(p)] \simeq E_0[e^{-\lambda \int_0^1 p^2}]$, and the computation is finished with aid of the explicit formula

$$\int_H e^{-\lambda \int_0^1 p^2} dP_0(p) = \frac{\sqrt{\lambda/2}}{\sinh \sqrt{\lambda/2}}.$$

All this had already been noticed in [Cambronero and McKean 1999].

The behavior as $\lambda \to -\infty$ is far less transparent. Now there is the possibility of cancellation in the first part of the variational formula $\int_0^1 (|\lambda| - p^2)^2$, compelling the path to live near $\pm\sqrt{-\lambda}$. However, the mean-zero condition ($p \in H$) dictates that p must its time between these two levels, while sharp transitions from $-\sqrt{-\lambda}$ to $+\sqrt{-\lambda}$ or back are penalized by the energy $\int_0^1 p'^2$. The heavier left tail is the outcome of this competition.

Getting started, the Euler–Lagrange equation for any $\lambda < 0$ minimizer p_λ of (3-1) may be computed,

$$p_\lambda'' = 2p_\lambda^3 - 2p_\lambda^2, \tag{3-2}$$

and solved explicitly in terms of the Jacobi elliptic function sin-amp,

$$p_\lambda(x) = k\sqrt{|\lambda|} \times \mathrm{sn}(\sqrt{|\lambda|}x, k), \tag{3-3}$$

with modulus satisfying $k^2 \simeq 1 - 16e^{-\sqrt{|\lambda|}/2}$ to fix the period at one.[1] Substituting back yields $I_\lambda(p_\lambda) \sim \frac{8}{3}|\lambda|^{3/2}$, and there follows the first-order large-deviation type estimate

$$f_{WN}(\lambda) \simeq \exp\Big(-\frac{8}{3}|\lambda|^{3/2}\Big) \quad \text{for } \lambda \to -\infty.$$

[1] Technical aside: the equation (3-2) reported in [Cambronero et al. 2006] includes an additive constant, but this was later understood to vanish in [Ramírez and Rider 2006].

Toward more exact asymptotics, there are various degeneracy problems that need to be addressed. First is the obvious lack of uniqueness: any translation $p_\lambda^a(x) = p_\lambda(x+a)$ of (3-3) also minimizes I_λ. Second, and more obscure, is an asymptotic degeneracy in the direction of the low lying eigenfunctions of the Hessian of I_λ.

The translational issue is dealt with by conditioning: the minimizing path is pinned at zero at some predetermined point. Then, by a change of measure computation, we arrive at the following Rice-type formula. With $\{p_\lambda^a\}$ the one-parameter family of minimizers, $d(\,\cdot\,,\{p_\lambda^a\})$ the sup-norm distance to that family, and any $\varepsilon > 0$, we have

$$E_0\Big[e^{-\frac{1}{2}\int_0^1(|\lambda|-p^2)^2}A(p),\, d(p,\{p_\lambda^a\}) \le \varepsilon\sqrt{|\lambda|}\Big]$$
$$= E_0^0\Big[e^{-\frac{1}{2}\int_0^1(|\lambda|-p^2)^2}A(p)R(p),\, d(p,\{p_\lambda^a\}) \le \varepsilon\sqrt{|\lambda|}\Big]\, P_0\bigg(\int_0^1 \phi_1^\lambda p = 0\bigg).$$

Here, ϕ_1^λ is the $L^2(S^1)$-normalized derivative of p_λ (the derivative generating all translations), E_0^0 is now the CBM conditioned so that both $\int_0^1 p = 0$ and $\int_0^1 \phi_1^\lambda p = 0$, and $R(p)$ is a Radon–Nikodym factor which we will not make explicit. On the left-hand side, note that the integral is localized about the full family of minimizers. On the right-hand side, it is easy to see that the intersection of a small tube about $\{p_\lambda^a\}$ and the plane $\Big[p : \int_0^1 p\phi_1^\lambda = 0\Big]$ may be replaced with a similarly small neighborhood about $p_\lambda^0 = p_\lambda$. In this way the expectation has in fact been localized about a fixed path.

Next, the obvious shift $p \to p + p_\lambda$ results in

$$f_{WN}(\lambda) \simeq$$
$$e^{-I_\lambda(p_\lambda)} E_0^0\Big[e^{-\frac{1}{2}\int_0^1(q_\lambda+2\lambda)p^2} S(p,p_\lambda),\, \|p\|_\infty \le \varepsilon\sqrt{|\lambda|}\Big]\, P_0\bigg(\int_0^1 \phi_1^\lambda p = 0\bigg),$$

where

$$S(p,p_\lambda) = e^{-2\int_0^1 p_\lambda p^3 - \frac{1}{2}\int_0^1 p^4} A(p+p_\lambda)R(p+p_\lambda),$$
$$q_\lambda(x) = 6|\lambda|k^2 \mathrm{sn}^2(\sqrt{|\lambda|}x, k).$$

One expects the Gaussian measure tied to the quadratic form

$$Q_\lambda = -\frac{d^2}{dx^2} + q_\lambda(x) + 2\lambda \tag{3-4}$$

to dominate the higher order nonlinearities in $S(\,\cdot\,, p_\lambda)$ and focus the path at $p = 0$. This deterministic Hill's operator Q_λ is of course the Hessian of I_λ, and it is no small piece of good fortune that it coincides with one of Lamé's finite-gap operators for which simple spectrum and corresponding eigenfunctions are explicitly computable [Ince 1940]. Information about the rest of the spectrum is obtained from a beautiful formula of Hochstadt [1961] for the discriminant.

HOCHSTADT'S FORMULA. *Let Q be finite-gap with $2g+1$ simple eigenvalues. Then $\Delta(\lambda) = 2\cos\psi(\lambda)$ with*

$$\psi(\lambda) = \frac{\sqrt{-1}}{2}\int_{\lambda_0}^{\lambda} \frac{(s-\lambda_1')\cdots(s-\lambda_g')}{\sqrt{-(s-\lambda_0)\cdots(s-\lambda_{2g})}}\,ds, \tag{3-5}$$

in which $\lambda_1' < \cdots < \lambda_g'$ are the points $\lambda_{2\ell-1} < \lambda_\ell' < \lambda_{2\ell}$, where $\Delta'(\lambda) = 0$. They are determined from the simple spectrum through the requirement $\psi(\lambda_{2\ell}) - \psi(\lambda_{2\ell-1}) = 0$ for $\ell = 1, 2, \ldots g$.

In the case of Q_λ for example we have $g = 2$.

Moving on, as alluded to just above we claim that, for $\lambda \to -\infty$ and all $\varepsilon > 0$ sufficiently small:

$$E_0^0\Big[e^{2\int_0^1 p_\lambda p^3 - \frac{1}{2}\int_0^1 p^4} A(p+p_\lambda)R(p+p_\lambda)e^{-\frac{1}{2}\int_0^1 (q_\lambda(x)+2\lambda)p^2}, \|p\|_\infty \le \varepsilon\sqrt{|\lambda|}\Big] \\ = A(p_\lambda)R(p_\lambda)Z(\lambda)(1+o(1)), \tag{3-6}$$

where

$$Z(\lambda) = E_0^0\big[e^{-\frac{1}{2}\int_0^1 (q_\lambda(x)+2\lambda)p^2(x)\,dx}\big]P_0\bigg(\int_0^1 \phi_1^\lambda(x)p(x)\,dx = 0\bigg).$$

This rests on the coercive properties of the measure $e^{-\frac{1}{2}\int_0^1 (q_\lambda+2\lambda)p^2}\,d\,\mathrm{CBM}(p)$ restricted to $\int_0^1 p = 0$ and $\int_0^1 p\phi_1^\lambda = 0$, which is to say, on the spectral gap of Q_λ restricted to the same space. Here lies the second degeneracy in the problem. This gap actually goes to zero as $\lambda \to -\infty$, making the estimate (3-6) rather laborious and hard to imagine without having the Q_λ spectrum explicitly at hand.

Taking the last appraisal for granted, it remains to find a closed expression for $Z(\lambda)$. This plays the role of the usual (though now infinite-dimensional) Gaussian correction in any Laplace-type analysis, and the fact is

$$Z(\lambda) \simeq C(\lambda_0, \ldots \lambda_4; c_0, \ldots, c_4) \times \frac{1}{\sqrt{\Delta^2(2|\lambda|) - 4}}. \tag{3-7}$$

The prefactor $C(\cdot)$ is a rational function of the (explicitly known) simple spectrum of Q_λ (eigenvalues λ_k and corresponding norming-constants c_k, $k = 0, \ldots, 4$). Hochstadt's formula now comes to the rescue, expressing the discriminant Δ back in terms of the same $\lambda_0, \ldots, \lambda_4$. Putting together the asymptotics of $A(p_\lambda)$, $R(\lambda)$, and those for $Z(\lambda)$ via the above expression will complete the proof for the left tail.

REMARK. It is enlightening to run the Riccati correspondence in reverse, the concentration of p about p_λ resulting in an optimal potential of the form

$$q(x;\lambda) = \lambda + p'_\lambda(x) + p^2_\lambda(x) \simeq -2|\lambda| \operatorname{sech}^2\bigl(\sqrt{|\lambda|}\,(x-1/2)\bigr).$$

That is, the white noise "path" must perform a single excursion of depth $O(\lambda)$ in an $O(|\lambda|^{-1/2})$ span to produce a large negative eigenvalue.

Nice gaussian potentials. Ay this point is natural to ask: To what extent is the white noise result universal for some class of potentials? The general case remains a question for the future; we describe here what is known for a class of *nice* stationary Gaussian potentials q.

The periodic diffusion setting is not the appropriate theater to explore questions of universality; certainly the details of the force F in Theorem 2.6 will play out in the shape of the density. Instead we consider the case that q is a stationary Gaussian process of periodicity one, continuous (and so, nice) such that

$$E[q(x)] = 0, \quad E[q(x)q(y)] = K(x-y), \tag{3-8}$$

with K satisfying the technical condition $\bar{K} = \int_0^1 K(x)\,dx > 0$. There is of course a point in common with the previously discussed potentials, namely periodic OU with mass m, in which case $K(z) = \frac{1}{2m}\left(\frac{e^{mz}}{e^m-1} - \frac{e^{-mz}}{e^{-m}-1}\right)$. Generally, however, the Cambronero–McKean formulas do not carry over to this Gaussian potential framework. Because the Riccati map is nonlinear, it is not always the case that q, under the $(\,\cdot\,, K^{-1}, \cdot\,)$ Gaussian measure, and p, under the $(\,\cdot\,, DK^{-1}D\,\cdot\,)$ Gaussian measure, are absolutely continuous. Take for example the situation when only a finite number of modes in the spectral expansion of q are charged.

For these reasons we rely on a yet another formula for the density, the idea behind which is to carry out the Riccati map on only part of the space. Denote by P the measure of q and let $\hat{P}$ be the measure induced on $\hat{q} = q - \int_0^1 q$. Then, this new formula for the density, established in [Ramírez and Rider 2006], is

$$f_K(\lambda) = \frac{1}{\sqrt{2\pi}} \int_H \exp\left(-\frac{1}{2\bar{K}}(\lambda + \Phi(\hat{q}))^2\right) d\hat{P}(\hat{q}). \tag{3-9}$$

Here, Φ is some implicitly defined nonlinear functional of the path, expressible through the Riccati map. When available, the Cambronero–McKean formula is certainly more powerful, being so explicit. On the other hand, (3-9) suffices to uncover the asymptotic shape of the density.

THEOREM 3.2 [Ramírez and Rider 2006]. *The probability density function f_K for $\Lambda_0(q)$ corresponding to any Gaussian random potential as above is C^∞ and* *satisfies*

$$\lim_{\lambda\to+\infty}\frac{1}{\lambda^2}\log f_K(\lambda) = -\frac{1}{2\bar{K}}, \quad \lim_{\lambda\to-\infty}\frac{1}{\lambda^2}\log f_K(\lambda) = -\frac{1}{2K(0)}.$$

As in the white noise case, the computation for the right tail is relatively simple, stemming from an optimal potential $q \simeq \lambda$ (or $p \simeq 0$). Further, the estimate for the left tail is connected with q-paths concentrating around $q(x) \simeq \lambda K(x)/K(0)$. That is, the covariance structure provides just enough freedom for the path to oscillate in accordance with K itself. Very loosely speaking, this falls in line with the white noise result where, after rescaling, the minimizing potential approaches a Dirac delta, which is the right kernel K for that process.

Lastly, we should reiterate the connection between Theorem 3.2 and well known Lifschitz tail results. For example, Pastur [1972] proved that, with $Q_L = -\Delta + q(x)$ on the cube of side-length L in $\mathbb{R}^d$ and q stationary Gaussian with covariance K satisfying a Hölder estimate: $\lim_{\lambda\to-\infty}\lambda^{-2}\log N(\lambda) = -1/(2K(0))$ where $N(\lambda)$ equals the $L\uparrow\infty$ density of states. Moreover, the basic method employed will provide tail bounds on the distribution function of the ground state eigenvalue for a large class of continuous q and $L < \infty$. From here, our own result could very well be anticipated. On the other hand, we know of no way to access the density function directly other than through the Riccati-as-a-change-of-measure idea.

4. General Tracy–Widom laws

The study of detailed limit theorems at the spectral edge is far more highly developed in RMT than in random Schrödinger. This is easiest to describe for the Gaussian Unitary Ensemble (GUE). GUE is an $n\times n$ Hermitian matrix ensemble M comprised of independent complex Gaussians: $M_{ij} = \overline{M_{ji}} \sim \mathcal{N}_{\mathbb{C}}(0, 1/4)$, while $M_{ii} \sim \mathcal{N}(0, 1/2)$. Equivalently, it is drawn from the distribution with increment $dP(M) = \frac{1}{Z}e^{-\mathrm{tr}M^2}\,dM$; dM denoting Lebesgue measure on the space of n-dimensional Hermitian matrices and $Z < \infty$ a normalizing factor.

Regarding spectral properties, GUE is integrable in so far as the full joint density of eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ is known:

$$\begin{aligned} P_{GUE}(\lambda_1,\lambda_2,\dots,\lambda_n) &= \frac{1}{Z_n}e^{-\sum_{k=1}^n \lambda_k^2}\prod_{k<j}|\lambda_j-\lambda_k|^2 \\ &= \frac{1}{n!}\det\bigl(K_n(\lambda_j,\lambda_k)\bigr)_{1\le j,k\le n}. \end{aligned} \tag{4-1}$$

On the second line, $K_n(\lambda,\mu)$ is the kernel for the projection onto the span of the first n Hermite polynomials in $L^2(\mathbb{R}, e^{-\mu^2})$; it follows from line one by simple row operations in the square Vandermonde component of the density. In fact, all finite dimensional correlations are expressed in terms of determinants of the same kernel. As a consequence there is the explicit formula at the spectral edge,[2]

$$P(\lambda_{max} \le \lambda) = \det\bigl(I - K_n \mathbf{1}_{(\lambda,\infty)}\bigr),$$

the right-hand side denoting the Fredholm determinant of the integral operator associated with K_n restricted to (λ, ∞). The classical Plancherel–Rotach asymptotics for Hermite polynomials and a marvelous identity from [Tracy and Widom 1994] now provide the distributional limit as $n \to \infty$:

$$P_{GUE}\Bigl(\frac{1}{\sqrt{2}} n^{1/6}(\lambda_{max} - \sqrt{2n}) \le \lambda\Bigr) \to \exp\biggl(-\int_\lambda^\infty (s-\lambda) u^2(s)\, ds\biggr) =: F_{GUE}(\lambda). \quad (4\text{-}2)$$

Here $u(s)$ is the solution of $u'' = su + 2u^3$ (Painlevé II) subject to $u(s) \sim Ai(s)$ (the standard Airy function) as $s \to +\infty$.

Associated with GUE are the Gaussian Orthogonal and Symplectic Ensembles (GOE and GSE) of real symmetric or self-dual quaternion Gaussian matrices. These are again integrable, with joint eigenvalue densities of a similar shape to line one of (4-1), the power two on the absolute Vandermonde interaction term being replaced by a 1 or 4. While not determinantal in the same way, there are again closed expressions for the largest eigenvalue distribution, and, at the same basic scalings, limit laws due to Tracy and Widom [1996]:

$$F_{G(O/S)E}(\lambda) = \begin{cases} \exp\bigl(-\frac{1}{2}\int_\lambda^\infty (s-\lambda) u^2(s)\, ds\bigr) \exp\bigl(-\frac{1}{2}\int_\lambda^\infty u(s)\, ds\bigr), \\ \exp\bigl(-\frac{1}{2}\int_{\lambda'}^\infty (s-\lambda') u^2(s)\, ds\bigr) \cosh\bigl(\int_{\lambda'}^\infty u(s)\, ds\bigr). \end{cases} \quad (4\text{-}3)$$

with $\lambda' = 2^{2/3}\lambda$ and u is the same solution of Painlevé II. For each of these three special ensembles there are also Painlevé expressions for the limiting distribution of the scaled second and higher largest eigenvalues, see again [Tracy and Widom 1994] and [Dieng 2005].

While striking in and of themselves, these results of Tracy–Widom have surprising importance in physics, combinatorics, multivariate statistics, engineering, and applied probability. A few highlights include [Johansson 2000; Baik et al. 1999; Johnstone 2001; Baryshnikov 2001]. From a probabilist's perspective, the laws (4-2) and (4-3) should be regarded as important new points in the space of distributions. In particular, one would like to understand $F_{G(O/U/S)E}$ in the same way that we do the Normal or Poisson distribution, being able to set down a few characterizing conditions. As it stands, the Tracy–Widom laws

[2] In RMT it is customary here to look at largest, rather than smallest, eigenvalues as is the case in random Schrödinger.

seem to live in the realm of integrable systems: we know of many interesting examples in which they arise, but that is about all.

One avenue to a deeper understanding of the Tracy–Widom laws would be some explanation of how to interpolate between them. For any $\beta > 0$, consider the following measure on n real points

$$P_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} e^{-\beta \sum_{k=1}^{n} \lambda_k^2} \prod_{k<j} |\lambda_j - \lambda_k|^\beta. \tag{4-4}$$

G(O/U/S)E occur for $\beta = 1, 2, 4$; on physical grounds β plays the role of inverse temperature in a 1-d "coulomb" gas. The limiting distribution of the largest P_β-point would give us a general Tracy–Widom law. While there appears to be no hope of integrating out a correlation for general beta, Dumitriu and Edelman [2002] have discovered a matrix model for all $\beta > 0$. The fact is: with $g_1, g_2, \ldots$ i.i.d. unit Gaussians and each χ_r an independent chi random variables of parameter r, the symmetric tridiagonal ensemble

$$H_n^\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} g_1 & \chi_{\beta(n-1)} & & & \\ \chi_{\beta(n-1)} & \sqrt{2} g_2 & \chi_{\beta(n-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{\beta 2} & \sqrt{2} g_{n-1} & \chi_\beta \\ & & & \chi_\beta & \sqrt{2} g_n \end{bmatrix} \tag{4-5}$$

has joint eigenvalue law with density (4-4).

The simplicity of (4-5) opens up the possibility of scaling the operator itself rather than dealing with the eigenvalue law. Formally invoking the central limit theorem in the form $\chi_{(n-k)\beta} \approx \sqrt{n\beta} - (\sqrt{\beta} k/2n) + (1/\sqrt{2})\mathcal{N}(0,1)$ in the off-diagonal entries, one can readily understand the conjecture of Sutton and Edelman [Edelman and Sutton 2007] that the rescaled matrices

$$-\tilde{H}_n^\beta = -\frac{\sqrt{2}}{\sqrt{\beta}} n^{1/6} (H_n^\beta - \sqrt{2\beta n} I),$$

should go over into

$$\mathcal{H}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}} b'(x) \tag{4-6}$$

in the $n \to \infty$, or continuum, limit. As before, b' indicates a white noise, and the scaling in (4) corresponds to that for the spectral edge in the known $\beta = 1, 2, 4$ cases. Thus, were it to hold, the above correspondence would entail that the low-lying eigenvalues of $\mathcal{H}_\beta$ agree in distribution with the limiting largest eigenvalues of H_n^β. Recently, the second two authors and B. Virág [Ramírez et al. 2006] have proved this conjecture.

THEOREM 4.1. *([Ramírez et al. 2006]) Let $\lambda_{\beta,1} \geq \lambda_{\beta,2} \geq \cdots$ be the ordered eigenvalues of the β-ensemble H_n^β, and $\Lambda_0(\beta) \leq \Lambda_1(\beta) \leq \cdots$ the spectral points of $\mathcal{H}_\beta$ in $L^2(\mathbb{R}_+)$ with Dirichlet conditions at $x = 0$. Then, for any finite k, the family*

$$\left\{ \sqrt{\frac{2}{\beta}} n^{1/6} (\lambda_{\beta,\ell} - \sqrt{2\beta n}) \right\}_{\ell=1,\ldots,k}$$

converges in distribution as $n \to \infty$ to $\{-\Lambda_0(\beta), -\Lambda_1(\beta), \cdots, -\Lambda_{k-1}(\beta)\}$.

Part of this result is the fact that the Schrödinger operator $\mathcal{H}_\beta$, referred to as the Stochastic Airy operator for obvious reasons, has an almost surely finite ground state eigenvalue Λ_0, as well as well defined higher eigenvalues Λ_1, and so on. Though no longer on a finite volume, the compactifying linear restoring force proves enough to tame the white noise at infinity. It is also remarked that the proof of Theorem 4.1 is actually made almost surely — eigenvalue by eigenvalue — after coupling the noise in the matrix model H_n^β to the brownian motion $b(x)$ in the limiting $\mathcal{H}_\beta$.

Next recall that the densities f_β of the $\beta = 1, 2, 4$ Tracy–Widom laws satisfy

$$f_\beta(\lambda) \sim e^{-\frac{1}{24}\beta|\lambda|^3}$$

for $\lambda \to -\infty$ and

$$f_\beta(\lambda) \sim e^{-\frac{2}{3}\beta\lambda^{3/2}}$$

for $\lambda \to +\infty$. Coupled with Theorem 4.1 this sheds new light on the results just discussed for the shape of the ground state eigenvalue density of the simple Hill operator $-d^2/dx^2 + b'(x)$. Moving into the spectrum, white noise on S^1 and white noise plus linear force on $\mathbb{R}^+$ certainly should give rise to different phenomena. On the other hand, when pulling far away from the spectrum, it is intuitive that these potentials would have roughly the same effect.

That said, the reader will anticipate what comes next. The Riccati map immediately gives a second description of the limiting distribution of the largest β-ensemble eigenvalues in terms of the explosion question for the one dimensional diffusion $x \mapsto p(x)$

$$dp(x) = \frac{2}{\sqrt{\beta}}\, db(x) + (x - \lambda - p^2(x))\, dx. \tag{4-7}$$

To make things precise, return to the eigenvalue problem,

$$d\psi'(x) = \frac{2}{\sqrt{\beta}} \psi(x)\, db(x) + (x - \lambda)\psi(x)\, dx,$$

restricted to $[0, L]$ subject to $\psi(L) = 0$ as well as $\psi(0) = 0$. Denote by $\Lambda_0(L)$ the minimal Dirichlet eigenvalue, and take $\psi_0(x, \lambda)$ the solution of the initial value problem with $\psi_0(0) = 0$ and $\psi_0'(0) = 1$. As already mentioned, the event that

$\Lambda_0(L) \geq \lambda$ is the event that ψ_0 does not vanish before $x \leq L$. This is the classical "shooting method". Now make the Riccati move: $p(x,\lambda) = \psi_0'(x,\lambda)/\psi_0(x,\lambda)$ is the diffusion (4-7), and the event that $\psi_0(x,\lambda)$ has no root before $x = L$ is the event that the p motion, begun from $p(0,\lambda) = \psi'(0,\lambda)/\psi(0,\lambda) = +\infty$ at $x = 0$, fails to explode down to $-\infty$ before $x = L$. (While it is not customary to use the entrance/exit terminology for inhomogeneous motions, comparison with the homogeneous case will explain why $p(x,\lambda)$ may be started at $+\infty$ and leaves its domain only at $-\infty$.)

Granted Theorem 4.1, Λ_0, the ground state eigenvalue of the full line problem, exists, and it is obvious that $\Lambda_0(L)$ converges almost surely to that variable as $L \to \infty$. In other words,

$$P(\Lambda_0 > \lambda) = P(\psi(\,\cdot\,,\lambda) \text{ never vanishes}) = P_{+\infty}\big(p(\,\cdot\,,\lambda) \text{ does not explode}\big).$$

A description of $P(\Lambda_k > \lambda)$ is similar for all k. The probability that the second eigenvalue exceeds λ is the $P_{+\infty}$ probability that p explodes *at most* once to $-\infty$, and so on. All this, with its implications for the limiting largest eigenvalues in the β-ensembles is summarized in the next statement.

THEOREM 4.2. (*[Ramírez et al. 2006]*) *With* $x \mapsto p(x) = p(x,\lambda)$ *the motion* (4-7), *let* $P_\bullet$ *denote the measure on paths induced by* p *begun at* $p(0) = \bullet$ *and let* $\mathfrak{m}(\lambda,\beta)$ *denote the passage time of* p *to* $-\infty$. *Then,*

$$\lim_{n\to\infty} P\left(\sqrt{\frac{2}{\beta}}n^{1/6}(\lambda_{\beta,1} - \sqrt{2\beta n}) \leq \lambda\right) = P_{+\infty}\big(\mathfrak{m}(-\lambda,\beta) = +\infty\big),$$

and also

$$\lim_{n\to\infty} P\left(\sqrt{\frac{2}{\beta}}n^{1/6}(\lambda_{\beta,k} - \sqrt{2\beta n}) \leq \lambda\right)$$
$$= \sum_{\ell=1}^{k} \int_0^\infty \cdots \int_0^\infty P_{+\infty}\big(\mathfrak{m}(-\lambda,\beta) \in dx_1\big) P_{+\infty}\big(\mathfrak{m}(-\lambda + x_1,\beta) \in dx_2\big) \cdots$$
$$\cdots P_{+\infty}\big(\mathfrak{m}(-\lambda + x_1 + \cdots + x_{\ell-1},\beta) = +\infty\big),$$

for any fixed k.

Even at $\beta = 1, 2$, and 4, Theorems 4.1 and 4.2 provide yet another vantage point on the Tracy–Widom laws. Not only are these laws now tied to a much simpler mechanical model (1-d Schrödinger), the Riccati map has introduced a Markovian structure where none appeared to exist.

5. Questions for the future

Shape of Hill's ground state density. This is still in its infancy. In particular, the exact regularity of the potential at which one sees a transition between the white noise 3/2-heavy tail (Theorem 3.1) and the Gaussian tail (Theorem 3.2) is an interesting question.

Non-i.i.d. matrix ensembles. Little is known about the limiting scaled distribution of $\lambda_{\max}$ for Hermitian matrix ensembles with entries exhibiting correlations which do not vanish in the $n \uparrow \infty$ limit. For the sake of discussion, consider such a non-i.i.d. Gaussian matrix M. Given Theorem 4.1, it is believable that M has some random differential operator as its continuum limit. Further, if the correlations in M are strong enough, one might imagine that the white noise type potential of Stochastic Airy is replaced by a smoother Gaussian potential, and then Theorem 3.2 would become relevant.

Sample covariance ensembles. Of importance in statistics are ensembles of the form $X^T CX$ where X is comprised of say independent identically distributed real or complex Gaussians and C may be assumed diagonal. If $C = Id$, these are the classical null-Wishart or Laguerre ensembles at $\beta = 1$ (real) and $\beta = 2$ (complex), and the corresponding Tracy–Widom laws turn up at the spectral edge. In fact, Edelman and Dumitriu also have general $\beta > 0$ tridiagonal versions of these null ensembles to which the results of Section 4 apply. On the other hand, if C is not the identity the picture is rather murky. The possibility of phase transition away from Tracy–Widom if C is sufficiently "spiked" is proved in [Baik et al. 2005], while [El Karoui 2007] provides some conditions on C which will result in Tracy–Widom for $\lambda_{\max}(X^T CX)$. Both results however pertain only to the $\beta = 2$ case as the rely on the special structure of the eigenvalue density at that value of the parameter. Perhaps the strategy outlined above — scaling directly in the operator rather than in the spectral distribution — can be successfully employed in this direction.

Painlevé expressions. One hopes that either the random Airy operator or the associated diffusion will lead to explicit formulas in terms of Painlevé II for the limiting largest eigenvalue distributions at all $\beta > 0$. While we appear to be far from realizing this goal, here perhaps is a hint. By the Cameron–Martin formula: with $F_\beta(\lambda)$ the distribution function of $-\Lambda_0(\beta)$,

$$F_\beta(\lambda) = \lim_{L\to\infty} \lim_{a\to\infty} \int_{p(0)=a} e^{-\frac{\beta}{8}\int_0^L (\lambda+x-p^2(x))\,dp(x)} e^{-\frac{\beta}{8}\int_0^L (\lambda+x-p^2(x))^2 dx} \times \frac{e^{-\frac{\beta}{2}\int_0^L (p'(x))^2 dx}}{(2\pi 0^+)^{\infty/2}}\, dp^\infty.$$

The Itô factor

$$\int_0^L ((\lambda + x - p^2(x))\, dp(x)$$

only contributes boundary terms, leaving the integral to concentrate on minimizers of the functional

$$p \mapsto \int_0^L \big([\lambda + x - p^2(x)]^2 + (p'(x))^2\big)\, dx.$$

The associated Euler–Lagrange equation is Painlevé II.

Acknowledgements

We thank the referees for pointing out several important references.

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