Section 4
The Case of Algebra

In this section and the next, we "crank up the microscope" to take a closer look at the examination of (a) specific subject matter, and (b) what different kinds of assessments can reveal. The subject of this section is algebra — mathematical subject matter that has always been important, but that has taken on increased importance in the U.S. in recent years. This is due to the confluence of two important social movements. The first is the "algebra for all" movement, an attempt to equalize opportunity-to-learn in American schools. In the past, many high school students were placed in courses such as "business mathematics" instead of algebra, or stopped taking mathematics altogether. Lack of access to algebra narrowed their options, both in curriculum and in employment. Hence policymakers and teachers in the U.S. declared that every student should have access to high-quality mathematics instruction, including an introduction to algebra in the early high school years. The second movement is a trend toward "high-stakes" testing. In various states around the nation, students must now pass state-wide assessments in mathematics in order to graduate from high school. Given that algebraic skills are now considered to be a part of quantitative literacy (as in "algebra for all"), state exams often focus on algebra as the gateway to high school graduation. In California, for example, the state's High School Exit Examination (the CAHSEE) in mathematics focuses largely on algebra. Students who do not pass the exam will not be granted a high school diploma.[1] Thus, understanding and improving student performance in algebra is a critically important matter.

In Chapter 11, William McCallum lays out a framework for examining algebraic proficiency. As he notes, "in order to assess something, you have to have some idea of what that something is." Thus he begins by clarifying what he believes is important about thinking algebraically. His approach is grounded in the perspective outlined in the National Research Council's volume *Adding It Up: Helping Children Learn Mathematics*. That is, proficiency in any mathematical arena includes conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and having a productive disposition toward mathematics. He goes on to exemplify these.

[1] As of June 2006, the question of whether or not passing the CAHSEE can be required for graduation is being adjudicated in the courts. The requirement is in place as this introduction is being written.

In Chapter 12, David Foster provides a framework for thinking about big ideas in algebra and a set of assessment items to match. (This is one aspect of the Silicon Valley Mathematics Initiative (SVMI) agenda, which was described in Chapter 10.) Foster's first task was to settle on high priority themes for assessment in algebra. What really counts? What do you want to make sure gets teachers' and students' attention? This is an important in general, but it is especially important for the SVMI, in the context of California's state Standards. The Standards, captured in the 2006 *California Mathematics Framework for California Public Schools, Kindergarten Through Grade Twelve*, provide a long list of skills that students are to master. How is one to prioritize? Which items on the list reflect isolated skills; which ones reflect core mathematical ideas? How do you build assessments that get at the core ideas? Foster elaborates on five main algebraic themes: doing-undoing, building rules to represent functions, abstracting from computation, the concepts of function and variable, and the concepts of equality and equation. For each, he provides specific assessment items. As one reads the items, it becomes clear how much the devil is in the details. For example, if you want to see whether students can explain their thinking, you have to provide them opportunities on the assessment to do so. (And if you want them to get good at explaining their thinking, you need to provide them ample opportunities, in class, to practice this skill. Thus tests can be diagnostic and suggest modifications of classroom practices.)

In Chapter 13, Ann Shannon gives some examples that serve to "problematize" the issue of context. As we saw in Section 3, context makes a difference: "knowing" the mathematics and being able to use it in specific situations can be two different things. Shannon takes a close look at a number of tasks, showing that "context" is anything but a zero-or-one variable: small changes in problem statements can produce large differences in what people do or do not understand about the contexts that are represented in the problems. In her fine-grained analyses, Shannon points to the complexities of drawing inferences about what people understand from their responses on tests. This presages the discussion in the next section, on what assessments can reveal or obscure.

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Chapter 11
Assessing the Strands of Student Proficiency in Elementary Algebra

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Algebra and Functions

In order to assess something, you have to have some idea of what that something is. Algebra means many different things in today's schools (for a discussion, see [RAND 2003, Chapter 4]). In particular, the study of algebra is often blended with the study of functions. Although it is true that the notion of a function is lurking behind much of beginning algebra, and that there is an algebraic aspect to many of the tasks we want our students to carry out with functions, the conglomeration of algebra and functions has considerably muddied the waters in the teaching of algebra. Therefore, I'd like to spend some time clarifying my own stance before talking about how to assess proficiency. In the process, while acknowledging other possible uses, I will use the word "algebra" to mean the study of algebraic expressions and equations in which the letters stand for numbers. I include in this study consideration of the relationship between the algebraic form of expressions and equations and properties of their values and solutions. I make a distinction, however, between this study and the study of functions.

In the progression of ideas from arithmetic to algebra to functions, there is an increase in abstraction at each step, and the increase at the second step is at least as large as that at the first. In the step from arithmetic to algebra, we learn to represent numbers by letters, and calculations with numbers by algebraic expressions. In the step from algebra to functions we learn of a new sort of object "function" and represent functions themselves by letters. There are two complementary dangers in the teaching of algebra, each of which can cause students to miss the magnitude of this step.

The first danger is that functions and function notation appear to the students to be mostly a matter of using a sort of auxiliary notation, so that if a function f is defined by $f(x) = x^2 - 2x - 3$, say, then $f(x)$ is nothing more than a short-hand notation for the expression $x^2 - 2x - 3$. This results in a confusion between functions and the expressions representing them, which in turn leads to a broad area of confusion around the ideas of equivalent expressions and transforming expressions. For example, we want students to understand that $(x + 1)(x - 3)$ and $(x - 1)^2 - 4$ are equivalent expressions, each of which reveals different aspects of the same function. But without a strong notion of function as an object distinct from the expression defining it, the significance of equivalence and transformation is lost in a welter of equals signs following the $f(x)$. Students can't see the forest for the trees.

A complementary danger is that algebra appears as a sort of auxiliary to the study of functions. It is common to use multiple representations of functions in order to make concrete the notion of a function as an object in its own right. Functions are represented by graphs, tables, or verbal descriptions, in addition to algebraic expressions. By looking at the same object from different points of view, one aims to foster the notion of its independent existence. In this approach, algebra provides one way of looking at functions, but functions are not regarded as purely algebraic objects. This solves the problem of students not seeing the forest for the trees, but risks them not being able to see the trees for the forest. Overemphasizing functions when teaching algebra can obscure algebraic structure, and too much attention to graphical and numerical approaches can subvert the central goal of thinking about symbolic representation.

In considering assessment, I limit myself to the narrower meaning of algebra that I have been using, and interpret proficiency in algebra as largely a matter of proficiency with symbolic representations. I understand proficiency to include the five strands of mathematical competence described in [NRC 2001]:

- *conceptual understanding*: comprehension of mathematical concepts, operations, and relations
- *procedural fluency*: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- *strategic competence*: ability to formulate, represent, and solve mathematical problems
- *adaptive reasoning*: capacity for logical thought, reflection, explanation, and justification
- *productive disposition*: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Proficiency in Algebra

Consider the following common algebraic error, the like of which everybody who teaches calculus has seen at one time or another.

$$\begin{aligned}\int \frac{1}{2x^2+4x+4}\,dx &= \int \frac{1}{x^2+2x+2}\,dx \\ &= \int \frac{1}{(x+1)^2+1}\,dx \\ &= \arctan(x+1)+C\end{aligned}$$

The student has factored a 2 from the denominator, and then lost it. Students often regard such errors as minor slips, like a wobble while riding a bicycle, and instructors often indulge this point of view with partial credit. The suggested remediation for students who make this sort of error often is lots of drill; lots of practice riding the bicycle. The error may be regarded as a matter of *procedural fluency*; either the student does not know the rules of algebra, or is insufficiently practiced in their execution.

Of course, without further information, it is impossible to diagnose the error precisely. However, it is worth drawing on experience to consider possible diagnoses that involve other strands of proficiency. For example, is it possible that it could be an error of *conceptual understanding*? Experience talking to students about their mistakes suggests the possibility that this was not an unintentional procedural slip, but rather a confusion about what procedures are permissible under what circumstances. Indeed, there is a situation in which it is quite permissible to lose the factor of 2, namely in solving the equation

$$2x^2+4x+4=0.$$

For a student with a weak grasp of the difference between equations and expressions, the superficial procedural similarity between solving equations and transforming expressions is a snare. Both procedures involve doing something to expressions connected by equals signs. We should consider the possibility that the failure to appreciate the fundamental difference between the two situations is a failure of understanding, not a failure in manipulative skills.

As for *strategic competence*, there is an important aspect of this error which the partial credit mentality fails to weigh sufficiently: in addition to making the original slip, the student failed to correct it. Stepping back from the solution and contemplating it as a whole suggests an obvious strategy for checking the solution, namely differentiating the answer. Even without actually carrying out the differentiation, a student with a strategic frame of mind might well wonder where all the necessary 2s and 4s in the denominator will come from.

What of *adaptive reasoning*? Attention to the meaning of the equals sign and a tendency to reflect on the assertion being made each time you put one down on paper might have led the student to notice that the first line in the solution effectively declares a number to be half of itself.

Finally, there could be an overall failure of *productive disposition*. Experience suggests that erroneous solutions like the one described here are often the product of a lack of any purpose other than moving the symbols in the correct way. The bicyclist has a fundamental sense of purpose that corrects the wobble; this is often lacking in algebra students.

Assessing the Strands of Proficiency

Many algebra assessments concentrate on assessing procedural fluency. Indeed, the common view of algebra among students is that it is nothing but procedural fluency. The idea that there are ideas in algebra comes as a surprise to many. How would we go about assessing some of the other strands? One way is to ask more multi-step questions and word problems that involve all strands at once. But such questions rarely make their way onto standardized assessments. There is also a need for simple questions that assess the non-procedural aspects of algebraic proficiency. Here are a couple of suggested questions.

Conceptual understanding

Question: In the following problems, the solution to the equation depends on the constant a. Assuming a is positive, what is the effect of increasing a on the value of the solution? Does the solution increase, decrease, or remain unchanged? Give a reason for your answer that can be understood without solving the equation.

A. $x - a = 0$

B. $ax = 1$

C. $ax = a$

D. $x/a = 1$

Answer:

A. Increases. The larger a is, the larger x must be to give 0 when a is subtracted from it.

B. Decreases. The larger a is, the smaller x must be to give the product 1.

C. Remains unchanged. As a changes, the two sides of the equation change together and remain equal.

D. Increases. The larger a is, the larger x must be to give a ratio of 1.

The equations in this problem are all easy to solve. Nonetheless, students find this question difficult because they are not being asked to demonstrate a procedure, but rather to frame an explanation in terms of what it means for a number to be a solution to an equation. It is possible for students to learn the mechanics of solving equations without ever picking up the rather difficult concept of an equation as a statement of equality between two expressions whose truth is contingent upon the value of the variables, and the process of solving an equation as a series of logical inferences involving such statements. Asking students to reason about equations without solving them could assess this area of proficiency.

Strategic competence

Question: A street vendor of t-shirts finds that if the price of a t-shirt is set at $\$p$, the profit from a week's sales is

$$(p-6)(900-15p).$$

Which form of this expression shows most clearly the maximum profit and the price that gives that maximum?

A. $(p-6)(900-15p)$

B. $-15(p-33)^2+10935$

C. $-15(p-6)(p-60)$

D. $-15p^2+990p-5400$

Answer: B. Because $(p-33)^2$ is a square, it is always positive or zero, and it is only zero when $p=33$. In the expression for the profit, a negative multiple of this square is added to $10{,}935$. Thus the maximum profit is $\$10{,}935$, and the price which gives that profit is \$33.

In this problem the student is not asked to perform the manipulations that produce the different forms, but rather to show an understanding of why you might want to perform those manipulations, and which manipulations would be the best to choose for a given purpose. A similar question could be asked about form C, which shows prices that produce zero profit, and form A, which exhibits the profit as a product of two terms each of which has meaning in terms of the context of the original problem, allowing a student to infer that the production cost per item is 6\$, for example, or that the demand function is given by the expression $900-15p$.

Such questions invite students to contemplate the form of an algebraic expression and to formulate ideas about the possible purposes to which that form is adapted. Too often, students feel they must instantly do something to an expression, without having formulated a purpose.

Conclusion

Algebra is at core about proficiency with symbolic representations. Assessment of this proficiency should include fluency with symbolic manipulations, but should also include the other four strands of mathematical competence. Such a richer assessment is often unfortunately delayed until functions have been introduced, since the different ways of representing functions, and their numerous applications in real-world contexts, provide good ground for the formulation of more conceptual and strategic questions. Simple questions of this nature at the level of elementary algebra are hard to come by, with the result that it is often taught as a purely procedural skill, with its more conceptual and strategic aspects either ignored or veiled behind the more abstract concept of function.

References

[NRC 2001] National Research Council (Mathematics Learning Study: Center for Education, Division of Behavioral and Social Sciences and Education), *Adding it up: Helping children learn mathematics*, edited by J. Kilpatrick et al., Washington, DC: National Academy Press, 2001.

[RAND 2003] RAND Mathematics Study Panel, *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*, Santa Monica, CA: RAND, 2003.