Section 2
Perspectives on Mathematical Proficiency

Definitions are important in education, as they are in mathematics. If one is to assess students' mathematical proficiency, then one had better start by defining the term. As the two essays in this section indicate, this is not as straightforward as it might seem. To echo R. Buckminster Fuller, a fundamental question is whether one considers mathematics be a noun or a verb. One's view makes a difference: what one defines mathematics to be has significant implications both for teaching and for assessment.

One way to view mathematics (the "noun" view) is as a wonderful and remarkably structured body of knowledge. From this perspective, the question becomes: How should that body of knowledge be organized so that students can best apprehend it? A second way to view mathematics is to think of it as What Mathematicians Do, with an emphasis on the verb. Even here, there are multiple levels of description. At the action level, for example, there are mathematical activities such as solving problems and proving theorems. At a deeper process level there are the activities of abstracting, generalizing, organizing, and reflecting (among others), which are called into service when one solves problems and proves theorems. In this section, R. James Milgram (in Chapter 4) and Alan H. Schoenfeld (in Chapter 5) explore aspects of the topics just discussed: the nature of mathematics and what it means to do mathematics, and implications of these views for both instruction and assessment. These two chapters, in combination with the three chapters in the previous section, establish the mathematical basis vectors for the space that is explored in the rest of this volume.

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Chapter 4
What is Mathematical Proficiency?

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In February of 2004 Alan Greenspan told the Senate Banking Committee that the threat to the standard of living in the U.S. isn't from jobs leaving for cheaper Asian countries. Much more important is the drop in U.S. educational standards and outcomes.

> "What will ultimately determine the standard of living of this country is the skill of the people," Greenspan pointed out .... "We do something wrong, which obviously people in Singapore, Hong Kong, Korea and Japan do far better. Teaching in these strange, exotic places seems for some reason to be far better than we can do it." [Mukherjee 2004]

Current estimates by Forrester Research (Cambridge, MA) are that over the next 15 years at least 3.3 million jobs and 136 billion dollars in wages will move to Asia.

Introduction

The first job of our education system is to teach students to read, and the majority of students do learn this. The second thing the system must do is teach students basic mathematics, and it is here that it fails. Before we can even think about fixing this — something we have been trying to do without success for many years — we must answer two basic questions.

- What does it mean for a student to be proficient in mathematics?
- How can we measure proficiency in mathematics?

These are hard questions. The initial question is difficult because mathematics is one of the most seriously misunderstood subjects in our entire K–12 educational system. The second question is hard for two reasons.

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(a) Too often test designers and textbook authors do not have a clear idea of what mathematics is. Indeed, something on the order of 25% of the questions on a typical state mathematics assessment are mathematically incorrect. (This will be expanded on in the last section, p. 55.)
(b) Proper assessment is difficult under the best conditions, but becomes essentially impossible when the people writing the tests do not adequately understand the subject.

The first question — what is mathematical proficiency? — will be discussed in what follows. Part of our theme will be to contrast practices in this country with those in the successful foreign countries that Alan Greenspan mentions. We will see that U.S. practices have relatively little connection with actual mathematics, but the programs in the high achieving foreign countries are closer to the mark.

We can understand this surprising assertion better when we understand that school mathematics instruction has drifted to the point where one simply cannot recognize much of actual mathematics in the subject as it is taught in too many of this country's schools. In fact this drift seems to be accelerating. It is fair to say that there has been more drift over the last twenty to thirty years than there was during the previous eighty.

Some schools in our country teach nothing but arithmetic, some nothing but something they call problem solving and mathematical reasoning. Both call what they teach "mathematics." Both are wrong. At this point and in this country, what is taught as mathematics is only weakly connected with actual mathematics, and typical curricula, whether "reform" or "traditional," tend to be off the point. Before continuing we must clarify this assertion.

What is Mathematics?

We must have some idea of what mathematics is in order to start our discussion. Unfortunately, a serious misconception already occurs here. Some things simply cannot be defined in ordinary language and mathematics is almost certainly one of them. This doesn't mean that we can't describe the subject in general terms. We just can't sharply limit it with a definition.

Over the years, a number of people have tried to define mathematics as "the study of patterns" or "the language of science," but professional mathematicians have avoided trying to define mathematics. As best I can recollect, the nearest that a research mathematician came to attempting a definition in print was Roy Adler in the mid-1960s who suggested the semi-serious "Mathematics is what mathematicians do."

A few years back a serious attempt at a short description of mathematics was given privately by Norman Gottlieb at Purdue. He suggested "Mathematics is the study of precisely defined objects.” A number of people participating in this discussion said, in effect, “Yes, that’s very close, but let’s not publicize it since it would not sound very exciting to the current MTV generation, and would tend to confirm the widely held belief that mathematics is boring and useless.”

Realistically, in describing what mathematics is, the best we can do is to discuss the most important characteristics of mathematics. I suggest that these are:

(i) Precision (precise definitions of all terms, operations, and the properties of these operations).
(ii) Stating well-posed problems and solving them. (Well-posed problems are problems where all the terms are precisely defined and refer to a single *universe* where mathematics can be done.)

It would be fair to say that virtually all of mathematics is problem solving in precisely defined environments, and professional mathematicians tend to think it strange that some trends in K–12 mathematics education isolate *mathematical reasoning and problem solving* as separate topics within mathematics instruction.

For Item 1 above, the rules of logic are usually considered to be among the basic operations. However, even here mathematicians explore other universes where the “rules of logic” are different. What is crucial is that the rules and operations being used be precisely defined and understood. Mathematics is a WYSIWYG field. There can be no hidden assumptions — *What you see is what you get*.

The Stages Upon Which Mathematics Plays

What we have talked about above is mathematics proper. However, most people do not talk about mathematics but algebra, geometry, fractions, calculus, etc. when they discuss mathematics, and we have not mentioned any of these topics. So where do numbers, geometry, algebra fit in? Mathematics typically plays out on a limited number of stages,[1] and there is often considerable confusion between the stages and the mathematics on these stages.

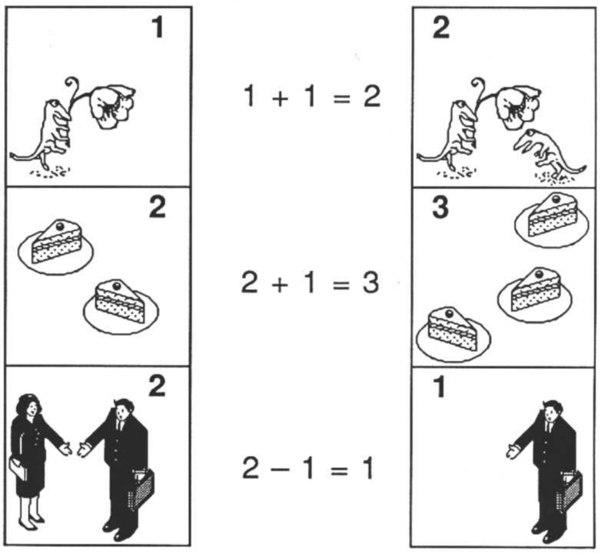
Here are some examples of stages. The integers build a stage, the rationals build a stage, and the reals build yet another a stage. These are the most important stages for mathematics by far, but you cannot limit mathematics to just these stages.

[1] By “stage” I mean — in mathematical terms — a category having objects, maps, maybe Cartesian products, and sufficient structure to do mathematics. However, stage, in the theatric sense, seems a very good description of these structures.

- Patterns, once one has a proper definition, build a stage. This is the theory of groups and group actions — a very advanced subject in mathematics.
- Geometry plays out on another stage. Geometry as we commonly know it, is the mathematics of the plane or space together with its points, lines, planes, distance, angles and transformations. However, the precise definitions here are even more difficult than is the case with patterns.

In practice, in school mathematics, some of the stages above are systematically but heuristically developed for students over a period of years and mathematics is played out to varying degrees. For example, here is a quick description of what happens during the first three years in a program that does things right — the Russian texts translated by the University of Chicago School Mathematics Project [UCSMP 1992a; 1992b; 1992c].

First grade. The stage on which first grade mathematics plays consists of the counting numbers from 1 to 100, addition, subtraction, and simple two- and three-dimensional geometric figures. There are very few definitions here and, since the stage is so small, the definitions can be quite different from definitions students will see later, though they should be present. For example, some of the definitions are given almost entirely via pictures, as is illustrated by the definition of adding and subtracting 1 on page 9:



Second grade. The stage is larger, the counting numbers at least from 1 to 1000, all four operations, time, beginning place value, small fractions, and a larger class of geometric figures. There are more definitions and they are more advanced than those in the first-grade universe. Here is the definition of multiplication. Note that it is the first paragraph in the chapter on multiplication and division on page 38:

Multiplication

143. 2 + 2 + 2 + 2 + 2 = 10

2 • 5 = 10

The addition of identical addends is called **multiplication**. The sum of the identical addends 2 + 2 + 2 + 2 + 2 can be written: 2 • 5.

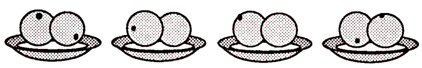
The multiplication sign is a dot (•). Multiplication problems are read as follows:

2 • 5 = 10 **If you take two five times, you get ten, or, two multiplied by five gives ten.**

The vertical bar above appears in these texts to indicate a definition. One can legitimately ask if this definition is sufficient, or if it is possible to do better. But here the main point is that there *is a definition* present, and its position in the exposition indicates that it is an important part of the sequence of instruction.

The definition of division is the first topic in the section on division that begins on page 42, immediately following the section on multiplication:

Division



162. 8 oranges were placed on dishes, 2 oranges to a dish. How many sets of 2 oranges were there? How many dishes were needed?

This problem is solved by the **division** operation. The division sign is two dots (:).

The solution to the problem can be written as follows:

8 : 2 = 4 Answer: 4 dishes

163. Grandmother had 10 carrots. She tied them into bunches of 5 carrots each. How many bunches did she get?

10 : 5 = 2 Answer: 2 bunches

In words, the translation of the definition (which is mostly visual) is that division of b by a is the number of equal groups of a objects making up b. It would likely be the teacher's obligation to see that each student understood the definition. For most, the example and practice would suffice. Some would require the entire verbal definition, and others might need a number of visual examples. This also illustrates an important point. There is no necessity that definitions be entirely verbal. For young learners, visual definitions may be more effective, and these Russian texts show a consistent progression from visual to verbal, as we will see shortly when we give the corresponding third-grade definitions.

As an example of how these definitions are applied, we have the discussion of even and odd numbers (from the section on multiplying and dividing by 2), page 90:

347. From the series of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, first give those which are divisible by 2, and then those which are not divisible by 2.

Numbers which are divisible by 2 are called **even**.
Numbers which are not divisible by 2 are called **odd**.

348. From the series of numbers 19, 10, 20, 8, 7, 1, 5, 4, 6, 18, 3, first write out all the odd numbers and then all the even numbers.

It is worth noting that the students have not yet seen fractions so the definition is actually unambiguous. (It has been pointed out to me that children are often ahead of the class instruction, and some children will be familiar with fractions, so that this definition, talking about *numbers* and not *whole numbers* is inappropriate. This is an interesting point, and illustrates one of the points of *tension* between rigorous mathematics and practical teaching. From the mathematical perspective, since the only *numbers* in the universe at the time of the definition are whole numbers, the definition is entirely correct. However, as a practical matter, the teacher must deal with children who want to test the definition on fractions.)

Third grade. The stage is much larger by third grade. Place value and the standard algorithms are fully developed.[2] Area and complex polygonal figures in the plane are present, weights and measures, velocity and the relationship between time and distance traveled at constant velocity have been given, fractions have been developed much further and are now represented on the number line.

[2] The details of the algorithms in the Russian texts are different from the details commonly taught in the United States. Nonetheless, the underlying mathematical algorithms are identical, so I use the term *standard algorithm* for the Russian algorithms as well as ours.

Again, there are new definitions appropriate to this new stage, some of which are quite sophisticated. Here is the definition of multiplication — as before the first item in the chapter on multiplication and division on page 89:

Multiplying by a One-Digit Number.

330. To multiply the number 18 by 3 is to take the number 18 as an addend 3 times:
$18 \cdot 3 = 18 + 18 + 18$.
To multiply a number k by 4 is to take it as an addend 4 times.
To multiply a number a by a number b is to take the number a as an addend b times.

331. Are the following equalities true?
$18 \cdot 5 = 18 + 18 + 18 + 18$
$c + c + c + c + c + c = c \cdot 8$
$12 + 12 + 12 + 11 = 12 \cdot 4$
$45 \cdot 6 = 45 + 45 + 45 + 45$

And here is the definition of division. Note the considerable advance on the second-grade definition. This definition will be unchanged in later grades, page 110:

Dividing by a One-Digit Number.

410. Division is related to multiplication; to divide 48 by 4 means to find a number which multiplied by 4 gives 48. This number is 12. That is, 48 : 4 = 12. What does it mean to divide 72 by 9? 100 by 25?

411. Check by multiplying whether the following divisions have been done correctly:

95 : 19 = 5 180 : 6 = 30 450 : 3 = 150

Just for illustration, here are some problems from the end of this chapter on page 137:

27. Solve the following problems by means of equations:
(1) Think of a number. If it is decreased 4 times and 180 is subtracted from the resulting number, 720 is obtained. What number is it?
(2) When the difference of some unknown number and the number 70 is divided by 9, 4 is obtained. Find the unknown number.
(3) 120 people volunteered to landscape a town. Several teams, each with 5 people, worked in a park, and the remaining 40 people planted trees along the streets. How many teams worked in the park?

This is a dramatic demonstration of just how far third-grade students can go with mathematics when the foundations are properly set up. Each concept used in these problems has been defined and discussed in this textbook.

We can compare this program and what is done in other successful foreign programs with what is typically done in this country.[3] For example, the problems above are very similar to those found in fourth-grade Singapore textbooks, so this level of expectation is normal for both countries. In the U.S., where precise definitions are not the norm, problems like these do not appear in textbooks until much later.

To a professional mathematician, one of the most glaring differences between the textbooks in high achieving countries and the United States is the care with which the distinctions above are made for the students and the precision (stage-appropriate, of course, but precision nonetheless) of the definitions in the programs of the successful countries. Moreover, this holds from the earliest grades onwards.

The tools of mathematics. Each stage has its own tools, the rules that we assume are valid there. There are also the overriding tools of mathematics, those tools that tend to have wide applicability over many stages. One of the most important of these tools is abstraction — focusing attention on the most important aspects of a situation or problem and excluding the extraneous.[4]

Problems in mathematics

A problem in mathematics (or a well-posed problem) is a problem where every term is precisely understood in the context of a single stage. It may have no answer, it may have an answer that contains many special subcases. The answer may be complex, as in the case of the problem

> Find all quadruples of whole numbers (n, a, b, c) that satisfy the equation $a^n + b^n = c^n$

— this is Fermat's "Last Theorem," of course, and the answer is $\{n = 1$ and $a+b = c$, or $n = 2$ and (a, b, c) forms a Euclidean triple, so $a = 2vw$, $b = v-w$, $c = v + w$ for two integers v, $w\}$; its proof, found by Andrew Wiles some ten years ago, is one of the greatest achievements of modern mathematics.

On the other hand, the answer might be simple, as for the problem "Is the square root of 2 a rational number?", whose answer is "No." The problem can be rephrased as

> Find all triples of integers that satisfy the two equations $a^2 + b^2 = c^2$ and $a = b$;

[3] To do this properly with any particular program would take much more space than we have available, but there is some discussion of a few of the issues with these programs on the next few pages.

[4] The type of abstraction being discussed here is part of problem solving. Abstraction appears in another form in mathematics, when processes that are common to a number of situations are generalized and given names. We do not discuss this aspect of abstraction here.

rigorously showing that the answer is $a = b = c = 0$, which implies that the square root of two is not rational, was one of the triumphs of Greek mathematics.

As long as all the terms are precisely understood and everything is included in a mathematical stage, we have a problem in mathematics. However, an item like

Find the next term in the sequence $3, 8, 15, 24, \ldots,$

which is extremely similar to items found on almost every state mathematics assessment that I've seen in this country is not a mathematics problem as stated.

Why not? The phrase "next term" has been given no meaning within the context of the question — presumably the stage of whole numbers and their operations. In order to answer this question one would have to *make a guess* as to what the phrase means.[5] At best this is a question in psychology. As such it is somewhat typical of the questions one used to find on IQ tests, or the now discontinued SAT analogies section, where a cultural or taught predisposition to understand "next term" in the same way as the person asking the question biases the results. In short, there most definitely tend to be hidden assumptions in problems of the type described.

It might be helpful to explain in more detail what is wrong with the above problem. Presumably what is wanted is to recognize that the n-th term is given by the rule $(n+1)^2 - 1 = n(n+2)$, but this only makes sense if you are told that the rule should be a polynomial in n of degree no more than 2. If you are not given this information then there is no reason that the following sequence is not equally correct (or equally incorrect):

$$3, 8, 15, 24, 3, 8, 15, 24, 3, 8, 15, 24 \ldots$$

with general term

$$\begin{cases} 3 & n \equiv 1 \bmod (4) \\ 8 & n \equiv 2 \bmod (4) \\ 15 & n \equiv 3 \bmod (4) \\ 24 & n \equiv 4 \bmod (4) \end{cases}$$

for $n \geq 1$. But that's only one possibility among an infinity of others. One could, for example, check that the polynomial

$$g(n) = n^4 - 10n^3 + 36n^2 - 48n + 24$$

has 3, 8, 15, 24 as its values at $n = 1, 2, 3, 4$ respectively, but $g(5) = 59$, and

[5] Guessing might be an appropriate strategy at some point when *solving* a problem, but not when *understanding* a problem. This is a real distinction between mathematics and some other disciplines. For example, it is perfectly sensible to make guesses about what it is you are trying to understand in science, but in mathematics, if the basic terms have not been defined, there can be no problem.

$g(6) = 168$. So the rule $g(n)$ would give an entirely different fifth term than would $n(n+2)$ though both give the same values at 1, 2, 3, and 4. Likewise, one could assume the rule is another repeating form, for example

$$3, 8, 15, 24, 15, 8, 3, 8, 15, 24, 15, 8, \ldots$$

where the value at $n + 6$ is the same as the value at n with the values at $n = 1, 2, 3, 4, 5, 6$ given above. In fact the way in which the original sequence could be continued is limited only by your imagination, and who is to say one of these *answers* is more correct than another?

Put yourself in the position of a student who has just been told that the correct answer to the "problem" $\{3, 8, 15, 24\}$ is 35. What will such a student think? There is no *mathematical* reason for such a claim, but the teacher is an authority figure, so this student will tend to accept the teacher's statement and revise any idea he or she might have about what mathematics is. The student will begin to arrive at the understanding that mathematics is, in fact, whatever the instructor wants it to be! Moreover, the student will, as a corollary, learn that answering a mathematical problem amounts to guessing what the person stating the problem wants. Once that happens the student is lost to mathematics. I cannot tell you the number of times that colleagues and even nonprofessionals from Russia, Europe, Japan, and China have mentioned these ubiquitous next-term problems on our tests to me and wondered how we managed to teach mathematics at all.

I wish I could say that these next-term questions are the only problem with K–12 mathematics in this country. But we've only skimmed the surface. For example, *the one thing that is most trumpeted by advocates of so-called reform math instruction in the United States is problem solving. We will see later that the handling of this subject in K–12 is every bit as bad as the next term questions.* To prepare for our discussion of problem solving we need some preliminaries.

School Mathematics as Lists

Perhaps the major reason for the pervasive collapse of instruction in the subject lies in the common view of many educators that learning mathematics consists of memorizing long lists of responses to various kinds of triggers — mathematics as lists.

Over the past eight years I have read a large number of K–8 programs both from this country and others and a number of math methods textbooks. None of the underlying foundations for the subject are ever discussed in this country's textbooks. Instead, it seems as if there is a checklist of disconnected topics. For example, there might be a chapter in a seventh- or eighth-grade textbook with the following sections:

- two sections on *solving one-step equations*,
- two sections on *solving two-step equations*,
- one section on *solving three-step equations*,

but no discussion of the general process of simplifying linear equations and solving them. Sometimes these check lists are guided by *state standards*, but the same structure is evident in older textbooks.

My impression is that too many K–8 teachers in this country, and consequently too many students as well, see mathematics as lists of disjoint, disconnected factoids, to be memorized, regurgitated on the proximal test, and forgotten, much like the dates on a historical time-line. Moreover, and more disturbing, since the isolated items are not seen as coherent and connected, there does not seem to be any good reason that other facts cannot be substituted for the ones that are out of favor.

A good example is long division. Many people no longer see the long-division algorithm as useful since readily available hand calculators and computers will do division far more quickly and accurately than we can via hand calculations. Moreover, it takes considerable class time to teach long division. Therefore many textbooks do not cover it. Instead, discussion of what is called *data analysis* replaces it. As far as I am able to tell, there was no consideration of the mathematical issues involved in this change, indeed no awareness that there are mathematical issues.[6]

The other concern that one has with the "mathematics as lists" textbooks is that it's like reading a laundry list. Each section tends to be two pages long. No section is given more weight than any other, though their actual importance may vary widely. Moreover, each section tends to begin with an example of a trivial application of this day's topic in an area that young students are directly experiencing. Cooking is common. Bicycles are common. But the deeper and more basic contributions of mathematics to our society are typically absent. How can students avoid seeing the subject as boring and useless?

[6]To mention but one, the algorithm for long division is quite different from any algorithm students have seen to this point. It involves a process of successive approximation, at each step decreasing the difference between the *estimate* and the exact answer by a factor of approximately 10. This is the first time that students will have been exposed to a convergent process that is not *exact*. Such processes become ever more important the further one goes in the subject. Additionally, this is the first time that estimation plays a major role in something students are doing. Aside from this, the sophistication of the algorithm itself should be very helpful in expanding students' horizons, and help prepare them for the ever more sophisticated algorithms they will see as they continue in mathematics. The algorithm will be seen by students again in polynomial long division, and from there becomes a basic support for the study of rational functions, with all their applications in virtually every technical area. In short, if students do not begin to learn these things using the long-division algorithm, they will have to get these understandings in some other way. These are aspects of mathematics that should not be ignored.

The Applications of Mathematics

The usual reasons given in school mathematics for studying mathematics are because it is “beautiful,” for “mental discipline,” or “a subject needed by an educated person.” These reasons are naive. It doesn’t matter if students find the subject beautiful or even like it. Doing mathematics isn’t like reading Shakespeare — something that every educated person should do, but that seldom has direct relevance to an adult’s everyday life in our society. The main reason for studying mathematics is that our society could not even function without the applications of a very high level of mathematical knowledge. Consequently, without a real understanding of mathematics one can only participate in our society in a somewhat peripheral way. Every student should have choices when he or she enters the adult world. Not learning real mathematics closes an inordinate number of doors.

The applications of mathematics are all around us. In fact, they are the underpinnings of our entire civilization, and this has been the case for quite a long time. Let us look at just a few of these applications. First there are buildings, aqueducts, roads. The mathematics used here is generally available to most people, but includes Euclidean geometry and the full arithmetic of the rationals or the reals.[7] Then there are machines, from the most primitive steam engines of three centuries back to the extremely sophisticated engines and mechanisms we routinely use today.

Sophisticated engines could not even be made until Maxwell’s use of differential equations in order to stop the engines of that time from flying apart, stopping, or oscillating wildly, so the mathematics here starts with advanced calculus. Today’s engines are far more sophisticated. Their designs require the solutions of complex nonlinear partial differential equations and very advanced work with linear algebra.

Today a major focus is on autonomous machines, machines that can do routine and even nonroutine tasks without human control. They will do the most repetitive jobs, for example automating the assembly line and the most dangerous jobs.

Such jobs would then be gone, to be replaced by jobs requiring much more sophisticated mathematical training. The mathematics needed for these machines, as was case with engines, has been the main impediment to actual widescale implementation of such robotic mechanisms. Recently, it has become clear that the key mathematics is available — mathematics of algebraic and geometric topology, developed over the last century — and we have begun to make dramatic progress in creating the programs needed to make such machines work.

[7] The need to build structures resistant to natural disasters like earthquakes requires much more advanced mathematics.

Because of this, we have to anticipate that later generations of students will not have the options of such jobs, and we will have to prepare them for jobs that require proportionately more mathematical education.

But this only touches the surface. Computers are a physical implementation of the rules of (mathematical) computation as described by Alan Turing and others from the mid-1930s through the early 1940s. Working with a computer at any level but the most superficial requires that you understand algorithms, how they work, how to show they are correct, and that you are able to construct new algorithms. The only way to get to this point is to *study* basic algorithms, understand why they work, and even why these algorithms are better (or worse) than others. The highly sophisticated "standard algorithms" of arithmetic are among the best examples to start. But one needs to know other algorithms, such as Newton's Method, as well. What is essential is real knowledge of and proficiency with *algorithms in general*, not just a few specific algorithms.

And we've still only touched the surface. Students have to be prepared to live effective lives in this world, not the world of five hundred years back. That world is gone, and it is only those who long for what never was who regret its passing. Without a serious background in mathematics one's options in our present society are limited and become more so each year. Robert Reich described the situation very clearly [2003]:

> The problem isn't the number of jobs in America; it's the quality of jobs. Look closely at the economy today and you find two growing categories of work — but only the first is commanding better pay and benefits. This category involves identifying and solving new problems.... This kind of work usually requires a college degree....
>
> The second growing category of work in America involves personal services.... Some personal-service workers need education beyond high school — nurses, physical therapists and medical technicians, for example. But most don't.

Mathematical Topics and Stages in School Mathematics

Historically, the choices of the mathematics played out on particular mathematical stages that is taught in K–12 have been tightly tied to the needs of our society. Thus, my own education in upstate New York and Minnesota, with learning to use logarithms and interpolation in fifth and sixth grade, exponentials and compound interest in seventh grade, and culminating in solid geometry and trigonometry was designed to prepare for the areas of finance, architecture, medicine, civil, and mechanical engineering. For example, exponentials are essential for figuring dosages of medicine and for dealing with instability, errors and vibrations in mechanisms.

In the countries most successful in mathematics education, these considerations routinely go into their construction of mathematics standards. People from all concerned walks of life set the *criteria* for the desired outcomes of the education system, and professional mathematicians then write the standards. In the U.S. the notion of overriding criteria and focused outcomes seems to virtually never play a role in writing mathematics standards, and the outcomes are generally chaotic.

Let us now return to our main theme — mathematical proficiency.

Definitions

We have talked about what mathematics is in general terms. The word that was most frequently used was precision. The first key component of mathematical proficiency is the ability to understand, use, and as necessary, create definitions.

> A definition selects a subset of the universe under discussion — the elements that satisfy the definition.

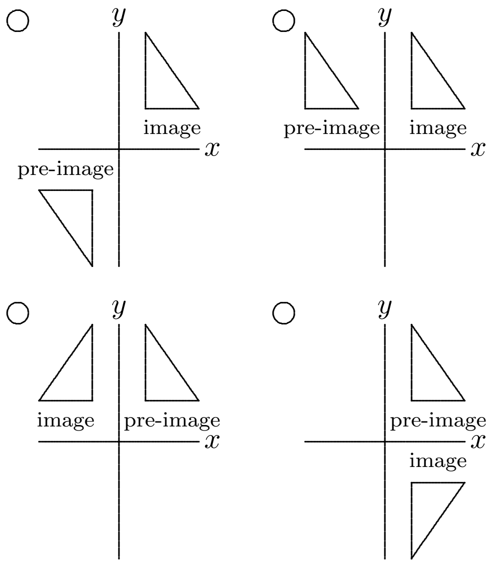
Once one has a definition one must understand it. This does not simply mean that one memorizes it and can repeat it verbatim on command. Rather, a student should understand why it is stated the way it is. It is necessary to apply at least the following three questions to every definition:

- What does the statement include?
- What does the statement exclude?
- What would happen if the definition were changed and why is the changed definition not used?

This is so basic that, once it became clear that my father could not dissuade me from becoming a mathematician, he gave me one key piece of advice. He said "Whenever you read a book or a paper in mathematics and you come to a definition, stop. Ask why the definition was given in the way it was, try variations and see what happens. You will not understand what is going on unless you do."

The lack of definitions in U.S. mathematics instruction. Definitions are the most problematic area in K–12 instruction in this country. First, they hardly ever appear in the early grades, and later, when people attempt to use definitions they get things wrong. Consider the following problem from a recent sample of released eighth-grade state assessment questions [Kentucky 2004, p. 5]:

4. Which diagram below best shows a rotation of the pre-image to the image?



The solution sheet tells us that the upper left choice is "the answer," but is it? Let us ignore the imprecision of the phrase "best shows," and assume that what is being asked is "Which diagram shows the effect of a rotation from the pre-image to the image?"[8] In fact, each of the answers is correct, depending on how well one knows the definition and properties of rotations.

A rotation in space always has an axis, a straight line fixed under the rotation. To see such a rotation take an orange, support it only at the top and bottom and spin the orange. When it stops you will have a rotation through the total angle of the spin. We obtain all but the picture at the upper right in this way, depending on the angle of the line connecting the top and bottom and where we put the triangle in the orange. But even the picture at the upper right can be obtained by a rotation from a more advanced mathematical perspective — that of projective geometry. This is a subject that was extensively studied about a century back, and is usually part of standard undergraduate geometry or applied linear algebra courses at the college level. It is the essential tool in computer graphics. One would hope that the people charged with writing a state math assessment would know mathematics at this level, but they should certainly be aware that rotations in space are easier to understand than rotations in the plane. Also, note the gratuitous use of the terms "image" and especially "pre-image" in this problem.

[8] As stated this problem rests on the undefined notion of "best." So, as discussed above, it is not a question in mathematics. But we might guess the intent of the question, and that is what I tried to do in rephrasing it.

It seems to be the norm in K–8 mathematics textbooks in this country that there are no definitions in the early grades, and this seems to also be the practice in virtually all our K–5 classrooms. For example, in the NCTM's *Principles and Standards* [NCTM 2000] we find

- develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
- use models, benchmarks, and equivalent forms to judge the size of fractions.

Note the strict use of models (or representations, in fact *multiple* representations), instead of definitions. We had a difficult time with this standard in California. If you are going to compare fractions you have to have a definition of fraction, and then a definition telling what it means for a fraction to be greater than another fraction.[9] So we changed it to

- Explain different interpretations of fractions, for example, parts of a whole, parts of a set, and division of whole number by whole numbers.

Even this was not entirely satisfactory since there was no explicit definition, but implicitly, the use of the term *interpretations* should give a hint that there is a single notion underlying all these different *representations*. At least that was what we hoped.[10]

Who gets hurt when definitions are not present? The emphasis on precision of language and definitions matters most for exactly the most vulnerable of our students. It is these students who must be given the most careful and precise foundations. The strongest students often seem able to fill in definitions for themselves with minimal guidance. On the other hand, foreign outcomes clearly show that with proper support along these lines, all students can get remarkably far in the subject.

Mathematical Problem Solving

We have seen that there are three basic components to mathematics: stages, definitions, and problem solving. We have discussed the first two. It is now time to discuss problem solving.

[9] Educators tend to look at one blankly when we say something like this. They typically respond that, intuitively, there is only one possible ordering. But this is not true. Orderings have been studied in advanced mathematics and it turns out that there are infinitely many *different* consistent meanings for less than or greater than for fractions and integers, each of which has its uses. The actual situation is very far from being intuitive.

[10] Underlying the lacuna in school math around definitions sometimes appears to be a belief, or perhaps a hope, that mathematics is innate, and that students, playing with manipulatives, will find all of mathematics already hiding in their memories. Mathematicians who teach mathematics for pre-service elementary school teachers often have to deal with such claims when students from the education schools come to them explaining that they do not have to learn what is currently being covered in the course since they will automatically know it when they need it.

To start, I think everyone needs to be aware of this basic truth:

PROBLEM SOLVING IS CURRENTLY AN ARCANE ART.

We do not know how to reliably teach problem solving. The most effective method I know is to have a mathematician stand in front of a class and solve problems. Many students seem to be able to learn something of this multifaceted area in this way, but, as we will see, the stage has to be carefully set before students can take advantage of this kind of experience.

What I will discuss now is what virtually all serious research mathematicians believe, and, as far as I've been able to ascertain, most research scientists. This is not what will be found in a typical math methods textbook. Other theories about mathematical problem solving are current there. It could be that the focus of the views on problem solving in these texts is concerned with routine problems where the biggest effort might be in understanding what the problem is asking. This can be a difficult step, but *here we are talking about solving a problem where the answer is not immediate and requires a novel idea from the student.* It is exactly this level of problem solving that should be the objective for every student, because, at a minimum, this is what virtually all nonroutine jobs will require.

For example, when I was young, dock work was brutal — lifting and carrying. Today, the vast majority of this work is done by huge robotic mechanisms, and the dock worker of today spends most of his or her time controlling a very expensive and complex machine or smaller forklifts. The usual requirement is two years of college to handle the big machines, because running these big machines entails extensive nonroutine problem solving.

The hidden part of problem solving. There is a hidden aspect to problem solving: something that happens behind the scenes, something that we currently do not know how to measure or explain. It is remarkable, when you read the biographies of great mathematicians and scientists that they keep saying of their greatest achievements, "I was doing something else and the answer to my problem just came to me."[11] This is not only true for the greatest, it is true for every serious research mathematician or scientist that I've ever talked to about this kind of issue.

Answers and ideas just seem to come out of the blue. But they don't! There are verbal and nonverbal aspects to problem solving. *Successful researchers have learned how to involve the nonverbal mechanisms in their brains in analyzing and resolving their problems, and it is very clear that these nonverbal regions are much more effective at problem solving than the verbal regions.* (My

[11] H.-H. Wu points out that the first example of this that he is aware of in print is due to H. Poincaré.

usual experience was to wake up suddenly at 2:00 a.m. or so, and the answer to a problem that I had been working on without success maybe two weeks back would be evident.)

In order to engage the nonverbal areas of the brain in problem solving, extensive training seems to be needed. This is probably not unlike the processes that one uses to learn to play a musical instrument.[12] Students must practice! One of the effects, and a clear demonstration that the process is working, is when students become fluent with the basic operations and don't have to think about each separate step.

For the common stages of school mathematics, students must practice with numbers. They must add them until basic addition is automatic. The same for subtraction and multiplication. They must practice until these operations are automatic. This is *not* so that they can amaze parents and friends with mathematical parlor tricks, but to facilitate the nonverbal processes of problem solving. At this time we know of no other way to do this, and I can tell you, from personal experience with students, that it is a grim thing to watch otherwise very bright undergraduates struggle with more advanced courses because they have to figure everything out at a basic verbal level. What happens with such students, since they do not have total fluency with basic concepts, is that — though they can often do the work — they simply take far too long working through the most basic material, and soon find themselves too far behind to catch up.

Skill and automaticity with numbers is only part of the story. Students must also bring abstraction into play. This is also very commonly an unconscious process. There are huge numbers of choices for what to emphasize and what to exclude in real problems so as to focus on the core of what matters. Indeed, it is often far from clear what the core actually is. As was the case before, one has to practice to facilitate abstraction. How?

One explores the situation, focusing on one area, then another, and accumulates sufficient data so that nonverbal tools in the brain can sort things out and focus on what matters. But in order to do this, the groundwork has to be laid. That is what algebra does (or is supposed to do). That is why students should practice with abstract problems and symbolic manipulation. Moreover, as we know, Algebra I and more particularly Algebra II are the gate keepers for college [Adelman 1999, p. 17]. When we think of problem solving in this way, that is not so surprising.

The need for further study. Our knowledge here is fragmentary and anecdotal. What I was saying above is highly plausible, and all the research mathematicians that I've discussed it with agree that it fits their experiences. However, it is not

[12] It is probably not a coincidence that an inordinate number of professional mathematicians are also skilled musicians.

yet possible to assert this knowledge as fact. Basic research needs to be done, much as was done for reading. The medical and psychological sciences almost certainly have the tools to begin such research now. But so far, such work is only in the earliest stages of development. In the meantime, I would suggest that the observations above not be ignored. It is clear that current approaches to problem solving in K–12 are not working as well as we would like.

We have not discussed the verbal aspects of problem solving. We will turn to them shortly, but first let us discuss one final aspect of mathematics, the interface between mathematics and the real world.

The Art of Creating Well-Posed Problems

This is another thing that we do not know how to teach. Rather, this is one of the most important things that our best Ph.D. students in mathematics actually are learning when they write a thesis. They are initially given small, reasonably well-posed problems to get their feet wet, and, if they survive this, then they are given a real problem, roughly posed, and some guidance.

What the students are then asked to do is to create sensible, appropriate, well-posed problems, that, when taken together, will give a satisfactory answer to the original question. And, of course, the students are expected to be able to resolve the questions they come up with.

It should be realized that not all real-world problems are amenable to mathematical analysis in this way — including those that talk about numbers. For example we have the following problem taken from the *California Mathematics Framework* [California 1992]:

> The 20 percent of California families with the lowest annual earnings pay an average of 14.1 percent in state and local taxes, and the middle 20 percent pay only 8.8 percent. What does that difference mean? Do you think it is fair? What additional questions do you have?

One can apply the processes we discussed earlier to create any number of well-posed questions, but it will be very difficult to find any that are highly relevant. A huge problem is how to give a precise but reasonable definition of “fair.” The idea of fairness is subject to much debate among social scientists, politicians, economists, and others. Then, when one attempts to see what the 14.1% and 8.8% might actually mean, further questions arise, including questions about the amounts spent by these two groups in other areas, and what the impact of these amounts might be. In fact, applying rigorous analysis of the type being discussed here with the objective of creating proper questions in mathematics shows just how poorly the question was actually phrased and prevents an educated person from taking such a question at face value. Moreover, it shows that one essentially cannot produce well-posed questions in mathematics that accurately reflect the objectives of this question. This example, even though numbers appear in it, is not a question that can be directly converted into questions in mathematics.

Let us look at one real-life example. A few years back I was asked to help the engineering community solve the basic problem of constructing algorithms that would enable a robotic mechanism to figure out motions without human intervention in order to do useful work in a region filled with obstacles.

The discussion below illustrates the key issues in creating well-posed problems from real-world problems, and the way in which mathematicians, scientists, engineers, and workers in other areas approach such problems.

The first step was to break the problem into smaller parts and replace each part by a precise question in mathematics. The physical mechanism was abstracted to something precise that modeled what we viewed as the most important features of the mechanism. Then the obstacles were replaced by idealized obstacles that could be described by relatively simple equations.

The initial problem was now replaced by an idealized problem that could be formulated precisely and was realistic in the sense that solutions of this new problem would almost always produce usable motions of the actual mechanism.

The second step was to devise a method to solve the mathematical problem. We tested the problem and soon realized that this first approximation was too big a step. A computer had to be able to plan motions when there were no obstacles before it could handle the idealized problem in a region with obstacles. The mathematics of even the problem with no obstacles had been a stumbling block for engineers, and the methods that are currently used for both regions with and without obstacles are quite crude — basically, create a large number of paths and see if any of them work! The difficulty with this approach is that it takes hours to compute relatively simple paths.

The plan was refined and revised. It turned out that the engineering community was not aware of a core body of mathematics that had been developed over the last hundred years. Within this work were basic techniques that could be exploited to completely resolve the problem of motion planning when no obstacles were present. Everything that was needed could be found in the literature. Of course, one needed extensive knowledge of mathematics to be able to read the literature and know where to look for what was known.

We could now resolve the simplified problem, but could we solve the original mathematical problem? With the solution of the first problem as background, we studied the problem with obstacles. The new techniques were applicable to this problem as well. But here new and very focused problem solving was needed, since the required results were not in the literature. It turned out that this, too, could be done. What was needed was real understanding and fluency with the mathematics being used.

Having solved the mathematical problem, could we apply the solution to the original real-world problem? Once the mathematical problem was solved, the solution and its meaning had to be communicated to the engineering community. Translating the mathematics into practical algorithms was the final step. It is currently being finished but already, programs have been written that do path planning for simple mechanisms in regions with lots of obstacles in fractions of a second.

What do we learn from this example? The first step is the key. One abstracts the problem, replacing it by problems that can be precisely stated within a common stage — hence are problems in mathematics — and that have a realistic chance of being solved. This requires real knowledge of the subject, and is a key reason why students have to learn a great deal of mathematics.[13] When solving or creating problems, knowledge of similar or related situations is essential.

But one also has to be sure that the resulting answers will be of use in the original problem. For this, one must be cognizant of what has been left out in the abstracted problem, and how the missing pieces will affect the actual results. *This is why understanding approximation and error analysis are so important.* When one leaves the precise arena of mathematics and does actual work with actual measurements, one has to know that virtually all measurements have errors and that error build up that has not been accounted for can make all one's work useless.

There is no one *correct problem* in this process, in the sense that there can be many different *families of mathematical problems* that one can usefully associate with the original real-world problem. However, it must be realized that each "mathematical problem" will have *only one answer*. This is another point where there has been confusion in school mathematics. There has been much discussion of real-world problems in current K–12 mathematics curricula, but then it is sometimes stated that these problems have many correct mathematical answers, a confounding of two separate issues.[14]

Here is a very elementary example. This is a popular third-grade problem.

> Two friends are in different classrooms. How do they decide which classroom is bigger?

[13] More precisely, I should add "carefully selected mathematics," where the selection criteria include the liklihood that the mathematics taught will be needed in targeted occupations and classes of problems.

[14] Perhaps this should be recognized as an example of what happens if one is not sufficiently precise in doing mathematics. Confounding *problems* with *answers* confused an entire generation of students and was one of the precipitating factors in the California math wars.

This is a real-world problem but not a problem in mathematics. The issue is that the word *bigger* does not have a precise meaning. Before the question above can be *associated to a problem in mathematics* this word must be defined. Each different definition gives rise to a different problem.

The next step is to solve these new problems. That's another story.

Mathematical Problem Solving II

We have discussed the general issues involved in solving problems in mathematics, the distinction between verbal processes and nonverbal processes. It is now time to talk about the verbal processes involved in problem solving.

Many people regard my former colleague, George Pólya, as the person who codified the verbal processes in problem solving.[15] He wrote five books on this subject, starting with *How to Solve It* [Pólya 1945], which was recently reprinted. We will briefly discuss his work on this subject.

One needs the context in which his books were written in order to understand what they are about. Pólya together with his colleague and long-time collaborator, G. Szegö, believed that their main mathematical achievement was the two-volume *Aufgaben und Lehrsätze aus der Analysis* [1971], recently translated as *Problems and Theorems in Analysis* and published by Springer-Verlag. These volumes of problems were meant to help develop the art of problem solving for graduate students in mathematics, and they were remarkably effective.[16] I understand that Pólya's main motivation in writing his problem-solving books was to facilitate and illuminate the processes he and Szegö hoped to see developed by students who worked through their two volumes of problems. (Pólya indicates in the two-volume *Mathematics and Plausible Reasoning* [1954] that graduate students were his main concern. But he also ran a special junior/senior seminar on problem solving at Stanford for years, so at many points he does discuss less advanced problems in these books.)

Another thing that should be realized is that the audience for these books was modeled on the only students Pólya really knew, the students at the Eidgenössische Technische Hochschule (Swiss Federal Institute of Technology) in Zürich, and the graduate and undergraduate mathematics majors at Stanford. Thus, when Pólya put forth his summary of the core verbal steps in problem solving:

[15] Other people, particularly Alan Schoenfeld, have studied and written on mathematical problem solving since, and I take this opportunity to acknowledge their work. However, the discussions in most mathematics methods books concentrate on Pólya's contributions, so these will be the focus of the current discussion.

[16] For example, my father helped me work through a significant part of the first volume when I was 18, an experience that completely changed my understanding of mathematics.

1. Understand the problem
2. Devise a plan
3. Carry out the plan
4. Look back

he was writing for very advanced students and he left out many critical aspects of problem solving like "check that the problem is well-posed," since he felt safe in assuming that his intended audience would not neglect that step. But, as we've seen in the discussion above, this first step cannot be left out for a more general audience. Indeed, for today's wider audience we have to think very carefully about what should be discussed here.

The other thing that appears to have been left out of Pólya's discussion is the fact that problem solving divides into its verbal and nonverbal aspects. This is actually not the case. Pólya was well aware of the distinction. Here is a core quote from [Pólya 1945, p. 9], where he talks about getting ideas, one of the key nonverbal aspects of problem solving:

> We know, of course, that it is hard to have a good idea if we have little knowledge of the subject, and impossible to have it if we have no knowledge. Good ideas are based on past experience and formerly acquired knowledge. Mere remembering is not enough for a good idea, but we cannot have any good idea without recollecting some pertinent facts; materials alone are not enough for constructing a house but we cannot construct a house without collecting the necessary materials. The materials necessary for solving a mathematical problem are certain relevant items of our formerly acquired mathematical knowledge, as formerly solved problems, or formerly proved theorems. Thus, it is often appropriate to start the work with a question: Do you know a related problem?

It is worth noting how these context difficulties have affected the importation of Pólya's ideas into the K–12 arena. Keep in mind that one of the key aspects of problem solving — illustrated by all the previous remarks — is the degree of *flexibility* that is needed in approaching a new problem. By contrast, today's math methods texts apply the rigidity of list-making even to Pólya's work. Thus we have the following expansion of Pólya's four steps taken from a widely used math methods book[17] that pre-service teachers are expected to learn as "problem solving."

1. Understanding the problem

   (i) Can you state the problem in your own words?

[17] The book's title and authors are not mentioned here because this represents a general failure and is not specific to this book.

(ii) What are you trying to find or do?

(iii) What are the unknowns?

(iv) What information do you obtain from the problem?

(v) What information, if any, is missing or not needed?

2. Devising a plan

(i) Look for a pattern

(ii) Examine related problems and determine if the same technique applied to them can be applied here

(iii) Examine a simpler or special case of the problem to gain insight into the solution of the original problem.

(iv) Make a table.

(v) Make a diagram.

(vi) Write an equation.

(vii) Use guess and check.

(viii) Work backward.

(ix) Identify a subgoal.

(x) Use indirect reasoning.

3. Carrying out the plan

(i) Implement the strategy or strategies in step 2 and perform any necessary actions or computations.

(ii) Check each step of the plan as you proceed. This may be intuitive checking or a formal proof of each step.

(iii) Keep an accurate record of your work.

4. Looking back

(i) Check the results in the original problem. (In some cases, this will require a proof.)

(ii) Interpret the solution in terms of the original problem. Does your answer make sense? Is it reasonable? Does it answer the question that was asked?

(iii) Determine whether there is another method of finding the solution.

(iv) If possible, determine other related or more general problems for which the technique will work.

Here are very similar expansions from another widely used math methods text, to reinforce the fact that the rigid expansion of Pólya's four steps above is the norm, rather than the exception:

Understand the problem

- Read the information
- Identify what to find or pose the problem
- Identify key conditions; find important data
- Examine assumptions.

Develop a plan

- Choose Problem-Solving Strategies
  (i) Make a model.
  (ii) Act it out.
  (iii) Choose an operation.
  (iv) Write an equation.
  (v) Draw a diagram.
  (vi) Guess-check-revise.
  (vii) Simplify the problem.
  (viii) Make a list.
  (ix) Look for a pattern.
  (x) Make a table.
  (xi) Use a specific case.
  (xii) Work backward.
  (xiii) Use reasoning.
- Identify subproblems.
- Decide whether estimation, calculation, or neither is needed.

Implement the plan

- If calculation is needed, choose a calculation method.
- Use Problem-Solving Strategies to carry out the plan.

Look back

- Check problem interpretation and calculations.
- Decide whether the answer is reasonable.
- Look for alternate solutions.
- Generalize ways to solve similar problems.

The consequences of the misunderstanding of problem solving in today's textbooks and tests. I mentioned at the beginning of this essay that state assessments in mathematics average 25% mathematically incorrect problems each. Indeed, with Richard Askey's help, I was responsible for guiding much of the development of the evaluation criteria for the mathematics portion of the recently released report on state assessments by Accountability Works [Cross et al. 2004]. In this role we had to read a number of state assessments, and 25% was consistent. These errors were not trivial typos, but basic misunderstandings usually centered around problem solving and problem construction.

Sadly, Pólya was fully aware of these risks, but could do nothing to prevent them even though he tried. It is not common knowledge, but when the School Mathematics Study Group (SMSG)[18] decided to transport Pólya's discussion to K–12, Pólya strenuously objected. Paul Cohen told me that at one of the annual summer meetings of SMSG at Stanford in the late 1960s, Pólya was asked to give a lecture, and in this lecture he explained why the introduction of "problem solving" as a key component of the SMSG program was a very bad mistake. Afterwards, Cohen told me that Pólya was well aware that his audience had applauded politely, but had no intention of following his advice. So Pólya asked Cohen, who had just won the Fields medal, if he would help. But it was not possible to deflect them.

In hindsight we can see just how accurate Pólya was in his concerns.

Summary

Mathematics involves three things: precision, stages, and problem solving. The awareness of these components and the ways in which they interact for basic stages such as the real numbers or the spaces of Euclidean geometry and the stages where algebra plays out are the essential components of mathematical proficiency. Perhaps the biggest changes in K–12 instruction that should be made to bring this to the forefront are in the use of definitions from the earliest grades onwards. Students must learn precision because if they do not, they will fail to develop mathematical competency. There is simply no middle ground here.

It is well known that early grade teachers are very concerned with making mathematics *accessible* to students, and believe that it is essential to make it fun. However, while many educators may believe that precision and accessibility are in direct opposition to each other, a study of the mathematics texts used in the programs of the successful foreign countries shows that this is not the case. Problems can be interesting and exciting for young students, and yet be precise.

Problem solving is a very complex process involving both verbal and nonverbal mental processes. There are traps around every corner when we attempt to codify problem solving, and current approaches in this country have not

[18]The School Mathematics Study Group was the new-math project that met in the summer at Stanford during the 1960s.

been generally successful. The concentration has been almost entirely on verbal aspects. But verbal problem solving skills, by themselves, simply do not get students very far. On the other hand, the only way mathematicians currently know to develop the nonverbal part involves hard work — practice, practice, more practice, and opportunities to see people who are able to work in this way actually solve problems.

Over a period of generations, teachers in the high-performing countries have learned many of these skills, and consequently I assume that students there are exposed to all the necessary ingredients. It seems to be remarkable how successful the results are. The percentages of students who come out of those systems with a real facility with mathematics is amazing.

There is no reason that our students cannot reach the same levels, but there is absolutely no chance of this happening overnight. Our entire system has to be rebuilt. The current generation of pre-service teachers must be trained in actual mathematics.

It is likely to take some time to rebuild our education system, and we cannot be misled by "false positives" into prematurely thinking that we've reached the goal. For example, right now, in California, the general euphoria over the dramatic rise in test outcomes through the middle grades puts us in the most dangerous of times, especially with limited resources and the fact that we have been unable to change the educations that our K–8 teachers receive.

Our current teachers in California have done a remarkable job of rebuilding their own knowledge. I'm in awe of them. I had assumed, when we wrote the current *California Mathematics Standards*, that the result would be a disaster since the new standards represented such a large jump from previous expectations and consequent teacher knowledge, but these teachers proved themselves to be far more resilient and dedicated than I had ever imagined.

As remarkable as our teachers have been, they could only go so far. But given the demonstrated quality of the people that go into teaching, if *mathematicians and mathematics educators* manage to do things right, I've every confidence that *they* will be able to go the rest of the way.

As Alan Greenspan's remarks at the beginning of this essay show, the stakes are simply too high for failure to be an option.

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