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Extremal Problems Related to the Sylvester–Gallai Theorem

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Abstract. We discuss certain extremal problems in combinatorial geometry, including Sylvester's problem and its generalizations.

1. Introduction

Many interesting problems in combinatorial geometry have remained unsolved or only partially solved for a long time. From time to time breakthroughs are made. In this survey, we shall discuss the known results about some metric and nonmetric problems. In particular, we shall discuss the Sylvester–Gallai problem and the Dirac–Motzkin conjecture on the existence and number of ordinary lines, the Dirac conjecture on the number of connecting lines, and the problem of distinct and repeated distances. The main focus will be on versions of these problems in the Euclidean and real projective plane.

The method of allowable sequences will be described as a tool to give purely combinatorial solutions to extremal problems in combinatorial geometry.

2. Sylvester's Problem

Sylvester [1893] posed a question in the *Educational Times* that was to remain unsolved for 40 years until it was raised again by Erdős [1943]. Then it was soon solved by Gallai [1944], who gave an affine proof. More followed: Steinberg's proof in the projective plane and others by Buck, Grünwald and Steenrod, all collected in [Steinberg et al. 1944]; Kelly's Euclidean proof [1948], and others, including [Motzkin 1951; Lang 1955; Williams 1968].

We give the following definitions before we state the problem and its solutions.

Let P be a finite set of 3 or more noncollinear points in the plane. Let F be a finite collection of simple closed curves in the real projective plane which do not separate the plane, every two of which have exactly one point in common, where they cross. F is known as a *pseudoline arrangement*.

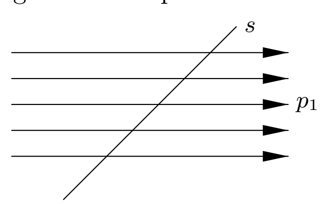
CONNECTING LINE: a line containing two or more points of P.
ORDINARY LINE: a connecting line which has exactly two points of P on it.
VERTEX: an intersection of two or more lines of a straight line arrangement or pseudolines of a pseudoline arrangement.
ORDINARY POINT: a vertex which is the intersection of exactly two lines or two pseudolines.

Sylvester asked for a proof of the statement that every set P of noncollinear points always determines an ordinary line. In the dual, one has to show that any straight line arrangement in which not all lines are concurrent has an ordinary point. By the principle of duality, proofs for point configurations carry over trivially into proofs for line arrangements and vice versa. The canonical correspondence maps the point (a, b) to the line $y = -ax + b$.

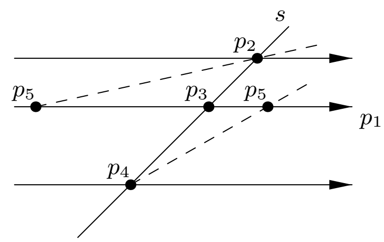
Levi [1926] introduced the notion of a pseudoline defined above. A natural question to ask is whether every pseudoline arrangement in which not all pseudolines are concurrent has an ordinary vertex. This is more general than the question of whether every straight line arrangement has an ordinary vertex, since every straight-line arrangement has an equivalent pseudoline arrangement, but there exist unstretchable pseudoline arrangements [Grünbaum 1970; Goodman and Pollack 1980b].

Solutions to Sylvester's problem. We now show some of the techniques used to solve Sylvester's problem in the both the primal and dual versions, and in the Euclidean as well as the projective plane.

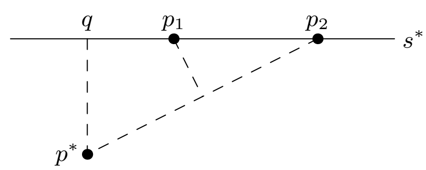
Gallai's proof (affine). Choose any point $p_1 \in P$. If p_1 lies on an ordinary line, we are done, so we may *assume that* p_1 *does not lie on any ordinary line*. Project p_1 to infinity and consider the set of lines containing p_1. These lines are parallel, and there are at least two such lines. Let s be a connecting line not through p_1 which forms the smallest angle with the parallel lines:



We assert that s is ordinary. If not, it must have at least 3 points p_2, p_3, p_4, as in the figure at the top of the next page. The connecting line through p_1 and p_3 has another point p_5, since it is not ordinary (this point is shown in two possible positions in the figure). Then, either p_5p_2 or p_5p_4 forms a smaller angle with the parallel lines than s, contradicting the hypothesis that s forms the smallest angle.

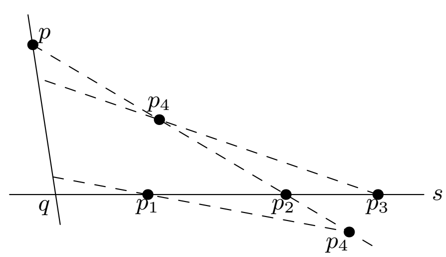


Kelly's proof (Euclidean). We have the set P of points not all collinear and the set S of connecting lines determined by P. Any point in P and any connecting line not through the point determine a perpendicular distance from the point to the line. The collection of all these distances is finite, because P and S are finite, so there is a smallest such distance. Let $p^* \in P$ and $s^* \in S$ be a nonincident pair realizing this smallest distance, and let q be the foot of the perpendicular line from p^* to s^*:



Then s^* is ordinary; otherwise it would contain three points of P, at least two of them lying on the same side of q. Let these two points be p_1 and p_2, with p_1 between q and p_2. Now the distance from p_1 to the connecting line p^*p_2 would be less than the distance from p^* to s^*, giving a contradiction.

Steinberg's proof (projective). With S and P as above, take any p in P. If p lies on an ordinary line we are done, so we may *assume that p lies on no ordinary line*. Let l be a line through p that is not a connecting line, that is, one that contains no point of P apart from p. Let Q be the set of intersections of l with lines in S, and take $q \in Q$ next to p (meaning that one of the open segments determined by p and q on the projective line l contains no element of Q). Let s be a line of S through q; then s must be ordinary. Otherwise, there would be three points of P on s, say p_1, p_2, p_3 (arranged in that order in $s \setminus \{q\}$; note that q is not in P, by our choice of l):



The line through p and p_2 would then contain another point of P, say p_4, since p lies on no ordinary line; then p_1p_4 or p_3p_4 would meet the forbidden segment pq (see the figure where two possibilities for p_4 are shown).

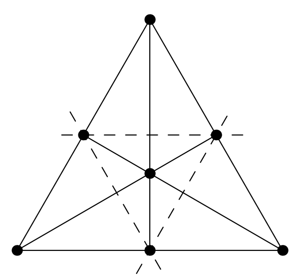
The Dirac–Motzkin conjecture. Having determined the existence of an ordinary line (or point, in the dual problem), attention was turned to the problem of establishing the number of ordinary lines (or points). For P an *allowable* set of points — one not all of whose elements are collinear — let $m(P)$ denote the number of ordinary lines determined by P. Define

$$m(n) = \min_{|P|=n} m(P),$$

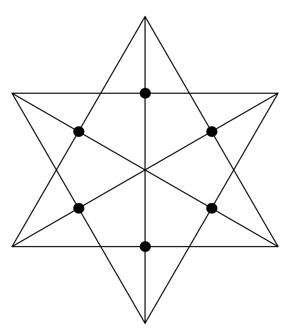
where P ranges over all allowable sets of points of cardinality $|P| = n$.

De Bruijn and Erdős [1948] proved that $m(n) \geq 3$, and this was proved again by Dirac [1951], who conjectured that there were at least $\lfloor n/2 \rfloor$ ordinary lines. In a different context, Melchior [1941] proved again the $m(n) \geq 3$ bound. Motzkin [1951] improved this to $m(n) > \sqrt{2n} - 2$. Kelly and Moser [1958] improved the lower bound to $3n/7$. Kelly and Rottenberg [1972] proved the same result for pseudoline arrangements. In 1980, Hansen gave a lengthy "proof" of Dirac's $\lfloor n/2 \rfloor$ conjecture, but it was found to be incorrect by Csima and Sawyer [1993], who nonetheless proved that there exist at least $6n/13$ ordinary lines.

Creating point configurations with few ordinary lines is hard. When n is odd, we know of configurations where the conjecture is tight only when $n = 7$ and $n = 13$. The former is shown by the Kelly–Moser configuration [1958]:



and the latter by the Crowe–McKee configuration [1968]. The Böröczky configurations [Crowe and McKee 1968] are valid for all even n; they are most easily visualized dually — here is the case $n = 12$, with dots marking ordinary vertices:



Solutions to the generalized problem. We now outline the techniques used in the progress towards settling the conjecture.

PENCIL: a collection of lines all of which intersect at a single point.
NEAR-PENCIL: a collection of lines all but one of which intersect at a single point.

According to the moment's convenience, we assume given either some arrangement L of lines not forming a pencil or near-pencil, or a configuration of points not all collinear. We seek to prove a lower bound for the number of ordinary points in the first case, and ordinary lines in the second.

Melchior's proof of the existence of 3 ordinary points. The lines of L partition the real projective plane into polygonal regions. Let V, E and F denote the number of vertices, edges and faces in the partition. By Euler's formula,

$$V - E + F = 1.$$

Let f_i denote the number of faces with exactly i sides and v_i the number of vertices incident with exactly i lines. Since the lines are not all concurrent, every face has at least three sides, so $f_2 = 0$. Then,

$$V = \sum_{i\geq 2} v_i, \quad F = \sum_{i\geq 3} f_i, \quad 2E = \sum_{i\geq 3} i f_i = 2\sum_{i\geq 2} i v_i.$$

This implies that

$$\begin{aligned} 3 = 3V - E + 3F - 2E &= 3\sum_{i\geq 2} v_i - \sum_{i\geq 2} i v_i + 3\sum_{i\geq 3} f_i - \sum_{i\geq 3} i f_i \\ &= \sum_{i\geq 2}(3-i)v_i + \sum_{i\geq 3}(3-i)f_i, \end{aligned}$$

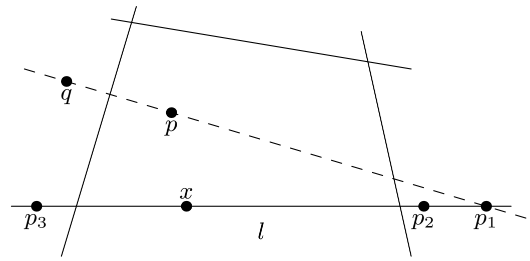
and hence that

$$v_2 = 3 + \sum_{i\geq 4}(i-3)v_i + \sum_{i\geq 4}(i-3)f_i \geq 3 + \sum_{i\geq 4}(i-3)v_i.$$

Thus, any finite set of nonconcurrent lines has at least 3 ordinary points.

Motzkin's proof of the existence of $O(\sqrt{n})$ ordinary lines. Consider a point $p \in P$ not lying on any ordinary line. (If there is no such point, there are at least $n/2$ ordinary lines and we are done.) Consider the set of connecting lines not passing through p. These partition the plane into regions, and p lies in one of these, which is called its *cell* C. If p has at least 3 lines on the boundary of its cell, then all the lines in the boundary of the region containing p must be ordinary.

It is easy to see that no point of P can lie on the edges of the cell C. Suppose one of the lines l on the boundary of the cell is not ordinary, that is, l has 3 points p_1, p_2, p_3 labeled so that p_1, x separate p_2, p_3, where x is a point on l not in P on the boundary of C (see figure on the next page). The line pp_1 is not ordinary by hypothesis, and therefore contains a point q of P. But then either



qp_2 or qp_3 cuts the cell C, contradicting the fact that C is the polygonal region containing p.

Thus, the ordinary lines partition the plane into polygonal regions, and all the points which do not lie an any ordinary line lie in one of these regions. It is easy to see that no region can have more than one point.

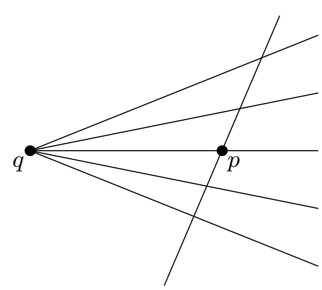
Now, m ordinary lines determine at most $\binom{m}{2}+1$ regions, and can have at most $2m$ points of P on them. Since every point is on an ordinary line or in a cell, we have $\binom{m}{2}+1+2m \geq n$, implying that $m \geq \sqrt{2n}-2$.

Kelly and Moser's proof of the existence of $3n/7$ ordinary lines. Let P be the set of points and S the set of connecting lines. We denote a generic point by p and a generic line by s. The set of lines of S which do not go through p subdivide the plane into polygonal regions. p is contained in one of these polygonal regions, which is called its *residence*.

Neighbor of p: a line of S containing the edges of the residence of p.
Order of p: the number of ordinary lines passing through p.
Rank of p: the number of neighbors of p which are ordinary lines.
Index of p: the sum of its order and rank.

Theorem 1. *If a point q has precisely one neighbor, then S is a near-pencil.*

This is because the neighboring line is the only line which does not pass through q, and all the other lines pass through q:



Theorem 2. *If a point p has precisely two neighbors, then S is a near-pencil.*

The lines of S that do not pass through p form a pencil, or else p would have at least three neighbors. Let q be the vertex of the pencil. Let s_i and s_j be two lines through q and p_i and p_j be points on s_i and s_j respectively, different from q. The connecting line through p_i and p_j does not pass through q and therefore passes through p. Thus, only one line of S passes through p, and all the rest pass through q, as in the previous figure.

As a consequence of the previous two theorems, we have:

THEOREM 3. *If S is not a near-pencil, each point of P has at least three neighbors.*

THEOREM 4. *If the order of p is zero, every neighbor of p is an ordinary line.*

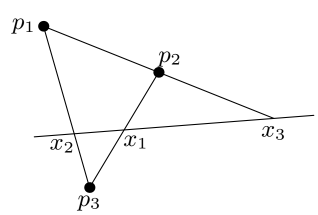
This was proved in [Motzkin 1951]; we gave the proof on page 483.

THEOREM 5. *Any point of P not of order two has index at least three.*

If the order is zero and S is not a near-pencil, the rank is at least three. If the order is at least three, there is nothing more to prove. If the order is one, the rank is at least two and the correct proof of this was given by Dirac in his review of Kelly and Moser's article [Dirac 1959].

THEOREM 6. *If a line s of S is a neighbor of three points p_1, p_2, p_3, then the points of P which lie on s are on the connecting lines determined by p_1, p_2, p_3.*

Three points that have a common neighbor cannot be collinear: if p_1, p_2 separate x, p_3, where x is the intersection of s with p_1p_2, then s cannot lie on the boundary of p_3's cell. Let the intersections of p_1p_2, p_2p_3, p_3p_1 with s be x_3, x_1, x_2 respectively. If p is a point of P on s such that x_ix_j separate x_kp, then pp_i and pp_j separate s from p_k. Here, i, j, k is some permutation of $1, 2, 3$.



This implies the following.

THEOREM 7. *A line l of S is a neighbor of at most four points.*

Suppose l was the neighbor of five points $p_1, \ldots, p_5$. Looking at p_1, p_2, p_3, we see that at least 2 of x_1, x_2, x_3 must be elements of P. Assume that x_2, x_3 are elements of P. However, neither x_2 nor x_3 can be on the lines p_1p_4 or p_1p_5. This means that one of the points of P on l is not on the connecting lines of the set p_1, p_4, p_5, implying that l is not a neighbor of one of the three points.

THEOREM 8. *If I_i is the index of the point p_i, then*

$$m \geq \frac{1}{6} \sum_{i=1}^{n} I_i$$

Since each ordinary line can be counted at most six times — four times as a neighbor and twice as being incident with each of its points — the sum of the index over all the points is greater than six times the number of ordinary lines.

THEOREM 9. $m \geq 3n/7$.

Let k be the number of points of order 2. Then

$$m \geq \frac{3(n-k)+2k}{6} = \frac{3n-k}{6},$$

which leads to $6m \geq 3n - k \geq 3n - m$ since $m \geq k$ (trivially). Hence $m \geq 3n/7$.

Proof by Csima and Sawyer. Csima and Sawyer improved upon Kelly and Moser by showing that except for pencils and the Kelly–Moser configuration the number s of ordinary points in a configuration of n lines is at least $\frac{6}{13}n$, with eqaulity occuring for the McKee configuration. They generalize the Kelly–Moser proof in the following way. In the Kelly–Moser proof, the sum of the indices of each point was compared to the six times the number of ordinary lines to get the desired bound. In the Csima–Sawyer result, the index is a weighted sum of the order and the rank. The following is a sketch of their proof for an arrangement of lines, and works for arrangements of pseudolines as well.

ATTACHED: An ordinary point which not on a line but associated to it, by proximity. For instance, in the proof of Kelly and Moser, the ordinary lines on the boundary of the cell of a point are attached to it.

TYPE of a line l: The pair $T(l) = (\mu, \nu)$, if there are exactly μ ordinary points on l and ν ordinary points attached to l.

α-WEIGHT of a line l of type (μ, ν): : the number $w_\alpha(l) = \alpha\mu + \nu$.

THEOREM 10. *Suppose* Γ *is a finite configuration of lines in the real projective plane having two lines of type* $(2, 0)$ *that intersect in an ordinary point. Then* Γ *is the Kelly–Moser configuration.*

THEOREM 11. *Apart from pencils, if* $T(l) \neq (2, 0)$, *then* $w_1(l) \geq 3$.

This is a restatement of a theorem of Kelly and Moser, which asserts that the index of a point which is not of order two is at least three.

THEOREM 12. *If* l_1 *and* l_2 *have an ordinary intersection in any configuration other than pencils, then* $w_1(l_1) + w_1(l_2) \geq 5$.

THEOREM 13. *Except for pencils and the Kelly–Moser configuration,* $s \geq \frac{6}{13}n$.

Partition the ordinary points into the sets

$$\sigma = \text{ordinary points that lie on a line of type } (2, 0),$$
$$\tau = \text{ordinary points that do not lie on a line of type } (2, 0).$$

and the lines into sets of bad, good and fair lines:

$$\mathcal{B} = \text{lines } l \text{ of type } (2,0),$$
$$\mathcal{G} = \text{lines } l \text{ that contain a point in } \sigma \text{ but } l \notin \mathcal{B},$$
$$\mathcal{F} = \text{lines } l \text{ that do not contain a point in } \sigma.$$

The set $\mathcal{G}$ is further partitioned into sets

$$\mathcal{G}_j = \text{lines } l \text{ in } \mathcal{G} \text{ which contain exactly } j \text{ points of } \sigma$$

Consider two lines l and m. If their intersection is in σ, we can assume without loss of generality that $l \in \mathcal{B}$. Then m has a 1-weight of at least three, and lies in $\mathcal{G}$. Thus, each point in σ appears on exactly one line from $\mathcal{B}$ and one line from $\mathcal{G}$. If $B = |\mathcal{B}|$, $G = |\mathcal{G}|$, $F = |\mathcal{F}|$, and $G_j = |\mathcal{G}_j|$, we have

$$G = \sum_j G_j \sum_{j \geq 1} G_j = |\sigma| = 2B.$$

If $l \in \mathcal{G}_1$, then $T(l) = (\mu, \nu) \geq (1,0)$, and $w_1(l) = \mu + \nu \geq 3$, and since $\alpha \geq 1$, we have $w_\alpha(l) = \alpha\mu + \nu \geq \alpha + 2$. If $l \in \mathcal{G}_2$, then $w_\alpha(l) \geq 2\alpha + 1$. If $l \in \mathcal{G}_j$ for $j \geq 3$, then $w_\alpha(l) \geq j\alpha$. If $l \in \mathcal{B}$, then $w_\alpha(l) = 2\alpha$, and if $l \in \mathcal{F}$, then $w_\alpha(l) \geq 3$.

Thus,

$$\begin{aligned}
\sum_{l \in \Gamma} w_\alpha(l) &= \sum_{l \in \mathcal{B}} w_\alpha(l) + \sum_j \sum_{m \in \mathcal{G}_j} + \sum_{l \in \mathcal{F}} w_\alpha(l) \\
&\geq 2\alpha B + (\alpha + 2)G_1 + (2\alpha + 1)G_2 + \sum_{j \geq 3} j\alpha G_j + 3F \\
&= 2\alpha B + \alpha(\sum_{j \geq 1} jG_j) + 2G_1 + G_2 + 3F \\
&= (4\alpha - 2)B + 3G_1 + 3G_2 + \sum_{j \geq 3} jG_j + 3F \\
&\geq (4\alpha - 2)B + 3G + 3F
\end{aligned}$$

Choosing $\alpha = \frac{5}{4}$ we get,

$$\sum_{l \in \Gamma} w_{5/4}(l) \geq 3B + 3G + 3F = 3n.$$

Consider a matrix with rows labeled by the lines l and columns labeled by the ordinary points. If the ith line is incident with the j-th ordinary point, the (i,j)-th entry of the matrix is $\frac{5}{4}$. If the j-th point is attached to the i-th line, the (i,j)-th entry is 1. All other entries are zero.

An ordinary point P is attached to at most four lines. Therefore, the column sum is at most $2(\frac{5}{4}) + 4 = \frac{13}{2}$. The sum over all the rows is exactly $\sum_{l \in \Gamma} w_{5/4}(l) \geq 3n$. Consequently,

$$3n \leq \sum_{l \in \Gamma} w_{5/4}(l) \leq \tfrac{13}{2} s.$$

3. Allowable Sequences

The notion of allowable sequences has proved very effective in determining the combinatorial classification of configurations of the plane.

A configuration of n points is an ordered n-tuple of distinct points in the plane. The points are labeled $1, 2, \ldots, n$. Given a configuration C and a directed line l which is not orthogonal to any line determined by two points of C, the orthogonal projection of C on l determines a permutation of $1, 2, \ldots, n$. As the line l rotates in a counterclockwise direction about a fixed point, we obtain a periodic sequence of permutations which is called the circular sequence of the configuration.

Allowable sequences are circular sequences constrained by the following properties:

1. Succesive permutations differ only by having the order of two or more adjacent numbers switched.
2. If a move results in the reversal of a pair ij then every other pair is reversed subsequently before i and j switch again.

Allowable sequences and the Sylvester problem. The point configurations encountered in the Sylvester problem must take into account highly degenerate cases. Since many points may be collinear, the corresponding circular sequence will have switches in which more than two adjacent numbers are reversed. The problem of showing the existence of an ordinary line is equivalent to the problem of determining whether a simple switch occurs.

History of the use of allowable sequences. Though the concept was introduced by Goodman and Pollack [1980a] to study the Erdős-Szekeres conjecture, it has been very useful in solving a range of problems which depend mainly on the order types of the point configuration. In particular, it has been used to show that

- every pseudoline arrangement of less than nine lines is stretchable [Goodman and Pollack 1980b];
- the number of directions determined by $2n$ points is at least $2n$ [Ungar 1982];
- the number of k-sets among a set of n points is $O(nk^{1/2})$ [Edelsbrunner and Welzl 1985];
- the maximum number of at most k-sets is $O(nk)$ [Welzl 1986];
- pseudoline arrangements are semispace equivalent if and only if they have the same allowable sequence modulo local equivalence [Goodman and Pollack 1984].

Properties determined by allowable sequences.

- $i_1, i_2, \ldots, i_k$ are collinear if and only if they switch simultaneously
- i is in the convex hull of $i_1, i_2, \ldots, i_k$ if and only if in every permutation in the sequence, i is preceded by one of $i_1, i_2, \ldots, i_k$.

- i is an extreme point if and only if some permutation begins with i
- ij is parallel to kl if and only if they both switch simultaneously
- ijk turn counterclockwise if and only if ij precedes ik, written as $ij \prec ik$.
- ij separates k from m if and only if when ij switches, k and m are on opposite sides of the substring ij in the permutation.

Using allowable sequences, Edelsbrunner and Welzl [1985] were able to derive improved upper bounds for the $k-$set problem viz. that the number of $k-$sets is $O(n\sqrt{k})$. Welzl [1986] generalized this result to bound the number of at most $k-$sets in a configuration of n points. Ungar [1982] was able to settle the conjecture regarding the number of directions determined by a configuration of points.

As an example of the power of allowable sequences, we give the following proof by Ungar.

Ungar's proof for the number of directions determined by $2n$ points. We pay special attention to switches which straddle the midpoint of a permutation. A switch in which some indices cross the midpoint is called a crossing move. The ith crossing move causes an increasing string straddling the midpoint to be reversed. If d_i denotes the distance from the midpoint to the nearest end of the string, then, at the ith crossing move, exactly $2d_i$ indices cross the midpoint.

Since every index must cross the midpoint, if there are t crossing moves in all, then

$$2d_1 + 2d_2 + \cdots + 2d_t \geq 2n$$

since some indices can cross more than once.

Between two crossing moves, there must be at least $d_i + d_{i+1} - 1$ noncrossing moves, since we must first tear down a decreasing string of length d_i and build an increasing string of length d_{i+1}, and a decreasing string can be shortened by at most one in a switch (an increasing string can be increased by at most one in a switch).

Thus, the total number of switches between the first crossing move and when this same crossing move occurs in reverse corresponds to a half period and has $\sum(d_i + d_{i+1} - 1 + 1) = \sum(2d_i) \geq 2n$.

This is a tight lower bound, since the regular $2n$-gon determines exactly $2n$ directions, as in the Böröczky configuration of page 482.

When the number of points is odd, say $2n+1$, the number of directions can be shown to be at least $2n$, since all but one point must cross the position $n+1$ in the permutation.

4. Colored Extensions of Sylvester's Problem

Let $\{P_i\}$ be a collection of sets of points, and let all points in the same set be assigned a color. A line is monochromatic if it passes through at least two points of the same color and no points of any other color. The following problem is attributed to Graham and Newman: Given a finite set of points in the plane colored either red or blue, and not all collinear, must there exist a monochromatic line? Motzkin [1967] solved the problem in the dual, showing there must exist a monochromatic point in an arrangement of colored lines. The proof is sketched in Section 4 (page 490). Chakerian [1970] and Stein gave additional proofs.

MONOCHROMATIC POINT: an intersection point in an arrangement of colored lines where all the lines intersecting at that point have the same color.

Consider the following question: Does there exist for every k a set of points in the plane so that if one colors the points by two colors in an arbitrary way, there should always be at least one line which contains at least k points, all of whose points have the same color? This is known to be true for $k = 3$, but nothing is known for larger values of k.

Various generalizations of this problem to higher dimensions have been proposed and solved [Chakerian 1970; Borwein 1982; Borwein and Edelstein 1983; Tingley 1975; Baston and Bostock 1978].

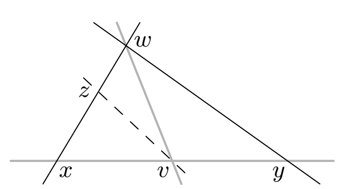
Clearly, we cannot insist that the monochromatic line be ordinary without additional restrictions. In the search for ordinary lines in the colored setting, Fukuda [1996] raised the following question. Let R be a set of red points and B be a set of blue points in the plane, not all on the same line. If R and B are separated by a line and their sizes differ by at most one, then there exists an ordinary bichromatic line, that is, a line with exactly one red point and one blue point. This conjecture is shown not to be true for small n in [Finschi and Fukuda 2003].

Pach and Pinchasi [2000] have shown that there exist bichromatic lines with few points.

Motzkin's solution of the existence of a bichromatic point.

THEOREM 14. *Let S and T be two sets of nonconcurrent lines in the real projective plane colored red and blue respectively. At least one of the intersection points in $S \cup T$ is monochromatic.*

Suppose S and T do not define any monochromatic vertex. Then, every intersection point w of two red lines has a blue line passing through it. These lines can be ordered so that the blue line lies in between the red lines. Since not all the blue lines are concurrent, there is some other blue line that does not pass through this intersection point. The new blue line forms a triangle wxy with the two red lines, as shown here (blue = gray):



Consider such a triangle that is minimal in the sense that it does not completely contain another such triangle. This must exist because there only finitely many triangles in the arrangement.

The intersection point v of the two blue lines must be monochromatic. If not, there must exist a red line through v producing a triangle vzx of the original type (two red lines and a blue line) which is contained in the minimal triangle, contradicting the assumption that wxy is minimal.

5. Connecting Lines and Dirac's Conjecture

Another interesting problem concerns the connecting lines of a set of P points. Define an $i-$line to be a connecting line containing exactly i points of P and let $t_i(P)$ denote the number of $i-$lines determined by P. Also, let $t(P) = \sum_{i\geq 2} t_i(P)$. Let $r(n)$ be the minimum over all configurations of n points of the maximum number of connecting lines from a single point. i.e.

$$r(n) = \min_{P\subset R^n} \max_{p\in P} t(p)$$

Dirac [1951] asked whether one of the n points must always be incident with at least $\lfloor \frac{n}{2} \rfloor$ of the connecting lines. He showed that this was the best possible by placing all the points evenly on two intersecting lines. He also proved a trivial lower bound of $\sqrt{n}$. In [Grünbaum 1972] a list of exceptions to this formulation is enumerated.

Erdős relaxed the problem by asking whether it could be shown that $r(n) \geq cn$. The more general question he raised was the following. Is it true that there exists an absolute constant c independent of k and n such that if $0 \leq k \leq 2$ and $t_i(P) = 0$ for $i > n - k$ then

$$ckn < t(P) < 1 + kn$$

The upper bound is trivial, and the lower bound was shown by Beck [1983] and Szemerédi and Trotter [1983], but with very small constants. Clarkson et al. [1990] improved the constant significantly.

The question of whether $t(P) \geq n$ was raised by Erdős [1943] and proved by various people including Erdős and Hanani [Hanani 1951]. Kelly and Moser [1958] were able to prove that

$$t(P) \geq kn - \tfrac{1}{2}(3k + 2)(k - 1)$$

if k is small compared to n and any connecting line contains at most $n-k$ points.

6. A Solution for Sylvester's Problem Using Allowable Sequences

We now look at a simple application of allowable sequences to solve Sylvester's problem.

Consider an allowable sequence of permutations of $1, \ldots, n$. In the first half-period, each permutation is obtained from the previous one by the switch of a substring that is monotonically increasing. We shall pay special attention to the switches involving 1 or n. We claim that the first switch involving a substring to the right of n or a substring to the left on 1 in the permutation is simple, thus proving the theorem.

Assume that n makes a switch before 1 makes a switch. Similar arguments hold for 1 if this not the case. This assumption implies that the first switch involving n does not involve 1.

Every substring switch involving n has n at the end of the substring before switch. After the switch, the right of n in the permutation consists of a concatenation of substrings, each of which is monotonically decreasing, since a switch turns an increasing substring into a decreasing one. Note that either n is involved in a simple switch, in which case there is nothing further to prove, or else each switch involving n has length at least three.

If there have been no switches to the right of n, the length of the longest monotonically increasing substring to the right of n is at most two, which can happen only at the end of one substring and the beginning of another formed by the switches involving n. Thus, the first switch involving elements to the right of n in the permutation has a length of exactly two.

There must be at least one such switch, since:

(i) n must switch at least twice as there is no switch of length n, which corresponds to the case when all the points are collinear, an excluded case.

(ii) We have assumed that n switches before 1, implying that 1 is not involved in the first switch involving n, which in turn implies that the elements to the right n are not always monotonically decreasing.

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