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Cylindrical Partitions of Convex Bodies

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ABSTRACT. A cylindrical partition of a convex body in $\mathbb{R}^n$ is a partition of the body into subsets of smaller diameter, obtained by intersecting the body with a collection of mutually parallel convex-base cylinders. Convex bodies of constant width are characterized as those that do not admit a cylindrical partition. The main result is a finite upper bound, exponential in n, on the minimum number $b_c(n)$ of pieces needed in a cylindrical partition of every convex body of nonconstant width in $\mathbb{R}^n$. (A lower bound on $b_c(n)$, exponential in $\sqrt{n}$, is a consequence of the construction of Kalai and Kahn for counterexamples to Borsuk's conjecture.) We also consider cylindrical partitions of centrally symmetric bodies and of bodies with smooth boundaries.

1. Introduction and Preliminaries

Throughout this article, M denotes a compact subset of $\mathbb{R}^n$ containing at least two points. By $\operatorname{diam} M$ we denote the maximum distance between points of M, but *diameter of* M also means the line segment connecting any pair of points of M that realize this distance (ambiguity is always avoided by the context). A *Borsuk partition* of M is a family of subsets of M, each of diameter smaller than $\operatorname{diam} M$, whose union contains M. The *Borsuk partition number of* M, denoted by $b(M)$, is the minimum number of sets needed in a Borsuk partition of M. It is obvious that $b(M)$ is finite. It is also obvious that the maximum of $b(M)$ over all bounded sets M in $\mathbb{R}^n$ exists and is bounded above exponentially in n, since every set of diameter d is contained in a ball of radius d. Therefore the *n-th Borsuk partition number*, denoted by $b(n)$, and defined as the minimum number of sets needed for a Borsuk partition of any bounded set in $\mathbb{R}^n$, is finite. Since a Borsuk

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partition of a ball in $\mathbb{R}^n$ requires at least $n+1$ sets, it follows that $b(n) \geq n+1$. K. Borsuk [1933] conjectured that $b(n) = n+1$, to which a counterexample was found by G. Kalai and J. Kahn [1993] in dimension $n = 1325$. In fact, Kalai and Kahn proved that, for large n, $b(n)$ is bounded below exponentially in $1.2^{\sqrt{n}}$. (At the time of writing of this paper, the lowest dimension known for which Borsuk's conjecture fails is 298; see [Hinrichs and Richter 2003].)

In this paper we consider a special kind of Borsuk partitions, which we will call cylindrical partitions; we ask related questions and provide some answers.

DEFINITION. By a *cylinder* in the direction of a line l we understand a closed set that contains every line parallel to l that intersects the set. A cylinder's cross-section perpendicular to its direction is called the *base* of the cylinder. Let M be a compact set, let l be a line, and let $\{M_i\}$ be a Borsuk partition of M. We say that the partition is *cylindrical* and that l defines its direction, provided that each of the sets M_i is the intersection of M with a cylinder parallel to l. For brevity, we say "cylindrical partition" instead of "cylindrical Borsuk partition," assuming automatically that the pieces of M are of diameter smaller than $\operatorname{diam} M$.

For the purpose of studying the problem of existence and minimum cardinality of cylindrical partitions of M (over variable M) one can always replace M with its convex hull $\operatorname{Conv} M$. Therefore it will be assumed from now on that M is a convex body, unless otherwise specified.

The *width of M in the direction of line l* is the distance between the pair of hyperplanes of support of M that are perpendicular to l and enclose M between them. If M has the same width in every direction, then M is said to be a *body of constant width* (see [Heil and Martini 1993, p. 363]). It is easy to see that if segment d is of maximum length among all chords of M parallel to d, then there exist two parallel hyperplanes of support of M, each containing an end of d. It follows that a body of constant width has a diameter in every direction. Hence:

PROPOSITION 1. *If M is a body of constant width, then M does not admit a cylindrical partition in any direction (not even into an infinite number of pieces).*

The converse to the above is true as well:

PROPOSITION 2. *If M is a bounded set of nonconstant width, then there exists a direction in which M admits a finite cylindrical partition.*

PROOF. Denote the diameter of M by d. Let P_1 and P_2 be a pair of parallel hyperplanes supporting M from opposite sides and determining a width $d_1 < d$. Let M_1 be the perpendicular projection of M to P_1. There exists a finite partition of M_1 (in P_1) into sets of diameter smaller than $\sqrt{d^2 - d_1^2}$. This partition gives rise (perpendicularly to P_1) to a finite cylindrical partition of M. $\square$

COROLLARY 3. *Bodies of constant width are characterized among convex bodies as those that do not admit a finite cylindrical partition.*

The above characterization of bodies of constant width and the subsequent investigations were inspired by a previous result of A. Heppes [1959] characterizing curves of constant width in the plane.

DEFINITION. If M is a set of nonconstant width, let $b_c(M)$, the *Borsuk cylindrical partition number of M*, or the *cylindrical partition number of M* for short, denote the smallest number of pieces into which M can be cylindrically partitioned.

The notion defined below is analogous to the n-th Borsuk number:

DEFINITION. The maximum of $b_c(M)$ over all sets M of nonconstant width in $\mathbb{R}^n$ is called the *n-th cylindrical partition number* and is denoted by $b_c(n)$.

The above proof of Proposition 2 may seem to indicate that already in the plane the cylindrical partition number of a set of nonconstant width may be arbitrarily large. But in Section 3 we show that there is an upper bound for $b_c(M)$ depending on the dimension of the ambient space only, which justifies the definition of $b_c(n)$. Specifically, we show that $b_c(n)$ is bounded above exponentially in n. In our proof we apply the following classical result concerning bodies of constant width (see [Bonnesen and Fenchel 1974, p. 129]), obtained by E. Meissner [1911] for $n = 2$ and $n = 3$, and generalized to all n by B. Jessen [1928]:

THEOREM 4 (MEISSNER–JESSEN). *Every convex body in $\mathbb{R}^n$ can be enlarged, without increasing its diameter, to a body of constant width.*

Henceforth, the distance from point A to point B, the segment with ends A and B, and the line containing them are denoted by AB, $\overline{AB}$, and $\overleftrightarrow{AB}$, respectively.

DEFINITION. Let s be a segment and let l be a line containing neither of the two ends of s. The *angle at which s is seen from l* is the measure of the smallest dihedral angle with its edge on l and containing s. We denote this angle by $\angle(l, s)$.

Observe that the above definition is meaningful in every dimension $n \geq 2$, although for $n = 2$ the angle is always $0°$ or $180°$. In the next section we present a lemma concerning the minimum angle at which a diameter of a bounded set can be seen from the line of another diagonal.

2. Diameters of a Bounded Set

Here we relax the standing assumption that M is a convex body. We do not even require M to be compact, only to be bounded. In what follows, α_0 denotes the measure of the dihedral angle in a regular tetrahedron, $\alpha_0 = \arccos \frac{1}{3} = 70.52\ldots°$.

LEMMA 5. *Let M be a bounded set and let d_1 and d_2 be diameters of M that do not have a common end point. Then*

$$\angle(\overleftrightarrow{d_1}, d_2) \geq \alpha_0,$$

equality being attained if and only if the convex hull of $d_1 \cup d_2$ is a regular tetrahedron.

Proof. (F. Santos, private communication, 2003). Assume for simplicity that $\operatorname{diam} M = 1$ and let l denote the line of d_1. Observe that d_2 cannot have an end on l, hence $\angle(l, d_2)$ is well defined. If the segments d_1 and d_2 intersected, then we would have $\angle(l, d_2) = 180°$, hence we can assume otherwise. Then the lines of d_1 and d_2 cannot intersect at all, and, obviously, they cannot be parallel. Since any pair of skew lines in $\mathbb{R}^n$ with $n > 3$ determine a 3-dimensional flat containing them, we can now assume that M is a subset of $\mathbb{R}^3$. Hence all we need to prove is the following:

Assertion. Among all tetrahedra of diameter 1 and with two nonadjacent edges of length 1, the minimum of the dihedral angle at either one of the two edges is attained on, and only on, the regular tetrahedron.

Let $T = ABCD$ be a tetrahedron with $AB = CD = 1$, and with all four of its remaining edges of length at most 1. Obviously, T lies in a lens-like set L that is in the common part of two unit balls, one centered at A and the other at B. Let p be the plane containing $\overleftrightarrow{AB}$ and parallel to $\overleftrightarrow{CD}$, let h denote the distance between p and $\overleftrightarrow{CD}$, and let p_h be the plane parallel to p and containing $\overleftrightarrow{CD}$. The set $L_h = L \cap p_h$ is the common part of two circular disks (in p_h) of radius $\sqrt{1-h^2}$ each, their centers one unit apart. The boundary of L_h is the union of two circular arcs: the A-arc, consisting of points in L_h one unit away from A, and the B-arc, consisting of points one unit away from B. Clearly, edge $\overline{CD}$ of T lies in L_h; we will vary the position of that edge within L_h in order to minimize the dihedral angle at edge $\overline{AB}$.

Rotating $\overline{CD}$ (within L_h) about either of its ends to a position "more parallel to" $\overleftrightarrow{AB}$ (that is, to decrease the angle between $\overleftrightarrow{CD}$ and $\overleftrightarrow{AB}$) decreases the dihedral angle at $\overline{AB}$. By combining at most two such rotations we can bring $\overline{CD}$ to a position in which one of its ends, say C, lies on the B-arc, and the other one, D, on the A-arc of L_h. Then we have $AD = BC = 1$. Observe that unless the equality $AD = BC = 1$ held already before, this change requires at least one rotation, thus it actually decreases the dihedral angle at $\overline{AB}$.

Finally, if $AC < 1$, then increasing the length of $\overline{AC}$ while keeping the length of the remaining edges fixed results in a decrease of the dihedral angle at $\overleftrightarrow{AB}$ (and the same holds about lengthening the edge $\overline{BD}$). To prove this fact, consider a sufficiently small sphere centered at point B. The intersection of the sphere with T is an isosceles (spherical) triangle $t^* = A^*D^*C^*$ (labeling to reflect the correspondence to points A, D, and C) with legs A^*D^* and C^*D^* whose length remains constant, since triangles ABD and CBD remain rigid, hinged on their common edge $\overline{BD}$. Observe that the angle of t^* at A^* is the dihedral angle of T at the edge $\overline{AB}$. Now, as AC increases, the constant-length legs of the isosceles triangle t^* open wider, and the two equal angles at its base decrease.

This argument proves that the dihedral angle at $\overline{AB}$ attains its minimum when, and only when, T is regular, which completes the proof of the lemma. □

3. An Exponential Upper Bound for $b_c(n)$

In this section we prove our main result:

THEOREM 6. *There is a constant k such that $b_c(n) \leq k^n$ for every n.*

PROOF. Let K be a convex body of nonconstant width in $\mathbb{R}^n$ and let $\widehat{K}$ be a body of constant width containing K and of the same diameter as K, whose existence is provided by Theorem 4. Assume $\operatorname{diam} K = \operatorname{diam} \widehat{K} = 1$. Then there is a line l such that $d = l \cap \widehat{K}$ is of length 1, while the segment $l \cap K$ is shorter than 1, hence is a proper subset of d. Let H be the hyperplane perpendicular to l and let O denote the intersection point $l \cap H$. There exists a round cylinder C about l of radius r small enough so that $\operatorname{diam}(C \cap K) < 1$. Let B_r denote the base of C in H, which is an $(n-1)$-dimensional ball of radius r, centered at O. Let S denote the boundary of B_r, which is a sphere of dimension $n-2$.

Consider a covering of S with the smallest number s_{n-2} of congruent spherical caps C_i of angular diameter α slightly smaller than $\alpha_0 = 70.5\ldots^\circ$ (α_0 is the dihedral angle in a regular tetrahedron, as in Lemma 5). By a simple argument involving a saturated packing with caps and a rough volume estimate, s_{n-2} is bounded above exponentially in n. (C. A. Rogers [1963] gives a very good, specific upper bound obtained by a refined analysis.) Let V_i denote the cone (in H) composed of rays emanating from O and passing through C_i, and let W_i be the subset of V_i obtained by chopping off a small tip of V_i, say W_i is the convex hull of the closure of the set $V_i \setminus B_r$, a truncated cone. The family of $1 + s_{n-2}$ subsets of H consisting of B_r and the truncated cones W_i ($i = 1, 2, \ldots, s_{n-2}$) covers H. This covering gives rise to a family of cylinders in the direction of l whose intersections with K form a cylindrical partition of K. Indeed: every diagonal of K is a diagonal of $\widehat{K}$, and every diagonal of $\widehat{K}$ other than d, either:

(i) has a common end with d, in which case that end lies outside the union of the cylinders over the truncated cones W_i, or
(ii) has no common end with d and therefore is seen from l at an angle greater than or equal to α_0, which implies that it cannot be contained in any one of the cylinders over the sets W_i.

Since C does not contain any diameter of K by design and none of the cylinders over the truncated cones W_i does either, and since each of these cylindrical pieces is convex, we have a cylindrical partition of K, with an exponential (in n) upper bound on the number of pieces. □

REMARK. The construction in the proof of Theorem 6 actually demonstrates that $b_c(n) \leq 1 + s_{n-2}$. However, in case K is smooth, *i.e.*, at every point of the boundary of K the support hyperplane is unique, no diagonal of K has a common end with another diagonal of $\widehat{K}$. Then the cylinders over the cones V_i, $1 \le i \le s_{n-2}$, form a cylindrical partition of K already (the "central piece" C is not needed). Thus, $b_c(K) \le s_{n-2}$ for every n-dimensional smooth convex body K of nonconstant width.

The construction described above in the proof of Theorem 6, combined with the fact that the necessary number of pieces in a Borsuk partition of the n-dimensional regular simplex is $n+1$, yield:

COROLLARY 7. *A constant k exists such that the inequalities $n+1 \le b_c(n) \le k^n$ hold. In dimensions up to* 4, *we have, more specifically,* $b_c(2)=3$, $4 \le b_c(3) \le 7$, *and* $5 \le b_c(4) \le 15$. *Moreover, by virtue of the remark above, if K is a smooth convex body of nonconstant width in $\mathbb{R}^n$, then $b_c(K)=2$ for $n=2$, $b_c(K) \le 6$ for $n=3$, and $b_c(K) \le 14$ for $n=4$.*

The inequality $b_c(3) \le 7$ follows from $5\alpha_0 < 360° < 6\alpha_0$, *i.e.,* $s_1 = 6$. The inequality $b_c(4) \le 15$ is obtained by the fact that the 2-sphere (the boundary of the 3-dimensional ball) can be covered with 14 congruent spherical caps of spherical diameter smaller than α_0, *i.e.*, $s_2 = 14$. Indeed, the smallest diameter of 14 congruent spherical caps that can cover the 2-sphere is approximately 69.875° (see [Fejes Tóth 1969]), just a little bit less than α_0.

REMARK. Let A_n denote the counterexample of Kalai and Kahn [1993] to Borsuk's conjecture in $\mathbb{R}^n$, whose Borsuk partition number is bounded below exponentially by $\sqrt{n}$. Since each of the sets A_n is finite, the set $K_n = \operatorname{Conv} A_n$, being a polyhedron, is of nonconstant width. It follows that $b_c(K_n) \ge b(K_n)$.

Consequently, we have:

COROLLARY 8. *There exist constants $k_1 > 1$ and k_2 such that, for large n,*

$$k_1^{\sqrt{n}} \le b_c(n) \le k_2^n.$$

4. Cylindrical Partitions of Bodies With Central Symmetry

Under the assumption of central symmetry of the body to be cylindrically partitioned, the upper bound on the number of pieces needed is much lower than the bound obtained in the previous section:

THEOREM 9. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ other than a ball. Then $b_c(K) \le n$, and the inequality is sharp.*

PROOF. Assume that O is the symmetry center of K and that $\operatorname{diam} K = 1$. Then the ball B of radius $1/2$ and centered at O contains K as a proper subset. Therefore one of the diameters of B, say d, has both of its ends outside K. Let H be a hyperplane perpendicular to d and passing through O. The set $H \cap B$ is an $(n-1)$-dimensional ball (in H) of diameter 1, hence its boundary S, a sphere of dimension $n-2$, can be covered by n congruent caps c_i of diameter smaller than

180°, with centers placed at the vertices of a regular $(n-1)$-simplex inscribed in S. Let C_i be the cone composed of rays that emanate from O and pass through c_i. By erecting a cylinder D_i parallel to d with base C_i $(i = 1, 2, \dots, n)$, we obtain a cylindrical partition of K into n pieces, since neither of the convex sets $D_i \cap K$ contains a pair of antipodes of B.

The inequality is sharp, since any Borsuk partition of a ball in $\mathbb{R}^n$ with a pair of small antipodal congruent caps cut off requires n pieces. □

5. Final Remarks and Some Open Problems

Our Proposition 1, Proposition 2, and Corollary 3 can be generalized to n-dimensional Minkowski spaces by methods described in [Averkov and Martini 2002]. The classical Meissner–Jessen theorem (Theorem 4 here) has been generalized to convex bodies in n-dimensional Minkowski spaces by G. D. Chakerian and H. Groemer [1983]. Therefore it is perhaps possible that Theorem 6, our main result, can be so generalized as well, although the magnitude of the upper bound may depend on the unit ball in the Minkowski space.

One could generalize the concept of cylindrical partitions in $\mathbb{R}^n$ by considering "k-cylinders" obtained as a Cartesian product of a set lying in an $(n-k)$-dimensional flat with a k-dimensional flat (a 1-cylinder would then be a "usual" cylinder, *i.e.*, a product with a line). But, because of their connection to bodies of constant width, we decided to deal with cylindrical partitions based on the usual cylinders only.

M. Lassak [1982] proved that $b(n) \le 2^{n-1}+1$, and from a result of O. Schramm [1988] on covering a body of constant width with its smaller homothetic copies it follows that $b(n) \le 5n\sqrt{n}(4+\log n)\left(\frac{3}{2}\right)^{n/2}$, presenting an upper bound of order of magnitude $\left(\sqrt{1.5}\right)^n$. The precise asymptotic behavior of $b(n)$ remains unknown.

The problem of determining the precise asymptotic behavior of $b_c(n)$ as $n \to \infty$ (let alone the exact values), appears to be extremely difficult, just as, or perhaps even more so than, the similar problem for $b(n)$. But it seems reasonable to expect some improvements on the bounds given in Corollary 7 and in Theorem 8. In particular, we feel that the upper bound in $4 \le b_c(3) \le 7$ can be lowered, perhaps all the way down to 4. Also, one should be able to narrow the gap between the lower and upper bounds in $5 \le b_c(4) \le 15$.

And, finally, it seems strange that the seemingly natural inequality $b(n) \le b_c(n)$ is not obvious at all; perhaps it may even be false for some n. It is *a priori* conceivable that in some dimension n, the value of $b(n)$ is attained on a body (or bodies) of constant width only, and that in such dimension, $b_c(n)$, being defined by bodies of *nonconstant* width, is smaller than $b(n)$. Paradoxical as it may seem, thus far such possibility has not been excluded. However, it is quite obvious that $b(n) \le b_c(n)+1$, because every convex body of constant width can be reduced to a convex body of nonconstant width by separating from it one small piece.

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