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Thinnest Covering of a Circle by Eight, Nine, or Ten Congruent Circles

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Abstract. Let r_n be the maximum radius of a circular disc that can be covered by n closed unit circles. We show that $r_n = 1 + 2\cos(2\pi/(n-1))$ for $n = 8$, $n = 9$, and $n = 10$.

1. Introduction

What is the maximum radius r_n of a circular disk which can be covered by n closed unit circles? The determination of r_n for $n \leq 4$ is an easy task: we have $r_1 = r_2 = 1$, $r_3 = 2/\sqrt{3}$ and $r_4 = \sqrt{2}$. The problem of finding r_5 has been motivated by a game popular on fairs around the turn of the twentieth century [Neville 1915; Ball and Coxeter 1987, pages 97–99]. The goal of the game was to cover a circular space painted on a cloth by five smaller circles equal to each other. The difficulty consisted in the restriction that an "on-line algorithm" had to be used, that is no circle was allowed to be moved once it had been placed. Neville [1915] conjectured that $r_5 = 1.64100446\ldots$ and this has been verified by K. Bezdek [1979; 1983] who also determined the value of $r_6 = 1.7988\ldots$. The proofs of these cases are complicated. The case $n = 7$ is again easy. We have $r_7 = 2$ and if 7 unit circles cover a circle C_7 of radius 2, then one of them is concentric with C_7 while the centers of the other circles lie in the vertices of a regular hexagon of side $\sqrt{3}$ concentric with C_7. In his thesis Dénes Nagy [1975] claimed without proof that $r_n = 1 + 2\cos\bigl(2\pi/(n-1)\bigr)$ for $n = 8$ and $n = 9$ and that, as for $n = 7$, the best arrangement has $(n-1)$-fold rotational symmetry. He conjectured the same for $n = 10$. Krotoszyński [1993] claimed to have proved this even for $n \leq 11$. Unfortunately, his proof contains some errors. In fact, Melissen and Schuur (see [Melissen 1997]) gave a counter example for $n = 11$.

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In this note we settle the cases $n = 8$, $n = 9$, and $n = 10$.

THEOREM. *Let r_n be the maximum radius of a circular disk which can be covered by n unit circles. Then we have for $n = 8, 9,$ and 10*

$$r_n = 1 + 2\cos\frac{2\pi}{n-1}.$$

Moreover, if for $n = 8$, $n = 9$, or $n = 10$, n unit circles cover a circle C_n of radius r_n, then one of them is concentric with C_n and the centers of the other circles are situated in the vertices of a regular $(n-1)$-gon at a distance $2\sin\bigl(\pi/(n-1)\bigr)$ from the center of C_n.

The analogous problems of the thinnest covering of a square and an equilateral triangle with a given number of equal circles, as well as the dual problem concerning the densest packing of a given number of equal circles in a circle, a square or an equilateral triangle have been investigated intensively. A comprehensive account can be found in [Melissen 1997].

Generally, given a compact set C in a metric space, one can consider the problems of the densest packing of n balls in C and the thinnest covering of C with n balls. In lack of similarity the problems are formulated in a dual form. Let $r_C(n)$ be the maximum number with the property that n balls of radius $r_C(n)$ can be packed in C and let $R_C(n)$ be the minimum number with the property that n balls of radius $r_C(n)$ can cover C. The basic task is, of course, to design effective algorithms determining the values of $r_C(n)$ and $R_C(n)$, as well as the corresponding arrangements. So far only the case of $r_C(n)$ for C a square has been solved, by an algorithm devised by Peikert [1994]; see also [Peikert et al. 1992]. Exact solutions are generally known only for small values of n. The only exception is the problem of densest packing of circles in an equilateral triangle. When C is an equilateral triangle, $r_C(n)$ is known for all n of the form $k(k+1)/2$, the triangular numbers; see [Groemer 1960; Oler 1961]. If C is an equilateral triangle with side-length 1, we have for such triangular numbers $r_C(n) = 1/2(k + \sqrt{3} - 1)$. The optimal arrangement is given by the regular triangular lattice.

Many conjectured best arrangements of circles, both for packing and for covering, have been constructed using different heuristic algorithms. The examples show that optimal arrangements quite often contain freely movable circles. This raises the following questions.

Does there exist a compact set C for which for infinitely many n an optimal packing of (covering with) n congruent circles contains a freely movable circle? Does there exist a C for which there is no n at all such that an optimal packing of (covering with) n congruent circles contains a freely movable circle? Is there a constant c, possibly depending on C but independent of n such that the number of freely movable circles in an optimal arrangement is at most c?

The densest packing of $n = k(k+1)/2$ circles in an equilateral triangle shows another interesting phenomenon. According a conjecture of Erdős and Oler [Croft et al. 1991, page 248] if n is a triangular number, then $r_C(n) = r_C(n-1)$, that is, the optimal arrangement for $n-1$ circles is obtained by removing one circle from the optimal arrangement of n circles. The conjecture has been confirmed for $n = 6$ and $n = 10$ [Melissen 1997]. There is a similar situation on the sphere: it is known (see [Rankin 1955], for example) that if $C = S^d$, the d-dimensional sphere, then $r_C(d+3) = r_C(d+4) = \ldots = r_C(2d+2)$. This suggests the following question.

For a given compact set C, are there natural numbers $k = k(C)$ and $K = K(C)$ such that $r_C(n) > r_C(n+k)$ and $R_C(n) > R_C(n+K)$ for every n?

I conjecture that the answer is yes if $C = S^d$ and also if C is a compact convex set in Euclidean or spherical space, but I would not be surprised if the answer turned out to be no for general compact sets, or even for convex bodies in hyperbolic geometry.

2. Proof of the Theorem

For the proof we modify the argument used by Schütte [1955] for the determination of the thinnest covering of the sphere by 5 and 7 congruent caps. Clearly, it suffices to show the second statement of the theorem, from that it follows immediately that no circle of radius greater than r_n can be covered by n unit circles ($n = 8$, $n = 9$, or 10). The proof of the three cases are similar, however the case $n = 10$ is more complicated. We shall leave two of the more involved discussions for $n = 10$ to Section 3. In the treatment of all three cases the functions $f_r(\alpha)$ and $F_r(\alpha)$ defined for $0 \le \alpha \le \pi$ by

$$f_r(\alpha) = 2 \arcsin \frac{\sin(\alpha/2)}{r}$$

and

$$F_r(\alpha) = r^2 \left(\arcsin \frac{\sin(\alpha/2)}{r} - \frac{1}{2} \sin\Bigl(2 \arcsin \frac{\sin(\alpha/2)}{r} \Bigr) \right) + \frac{\sin \alpha}{2}$$

play an important role. Here $r > 2$ is not a variable but a parameter.

The geometric meaning of $f_r(\alpha)$ and $F_r(\alpha)$ is the following: Let C be a circle of radius r centered at o and let $\widetilde{C}$ be a unit circle with center $\tilde{o} \in C$ such that $\operatorname{bd} C$ and $\operatorname{bd} \widetilde{C}$ intersect, say in a and b. If $\measuredangle a\tilde{o}b = \alpha$, then $f(\alpha) = \measuredangle aob$ and $F(\alpha)$ is the area of the domain bounded by the segments $\tilde{o}a$, $\tilde{o}b$ and the arc ab of $\operatorname{bd} C$.

We have

$$f_r'(\alpha) = \frac{\cos(\alpha/2)}{\bigl(r^2 - \sin^2(\alpha/2)\bigr)^{1/2}}, \qquad f_r''(\alpha) = -\frac{(r^2-1)\sin(\alpha/2)}{2\bigl(r^2 - \sin^2(\alpha/2)\bigr)^{3/2}},$$

$$F_r'(\alpha) = \frac{\sin\alpha\sin(\alpha/2)}{2\bigl(r^2-\sin^2(\alpha/2)\bigr)^{1/2}} + \frac{\cos\alpha}{2},$$
$$F_r''(\alpha) = -\Bigl(\sin\frac{\alpha}{2}\Bigr)\biggl(\frac{\cos\alpha\left(\sin^2(\alpha/2)-r^2\right)-r^2\cos^2(\alpha/2)}{2(r^2-\sin^2(\alpha/2))^{3/2}} + \cos\frac{\alpha}{2}\biggr).$$

Hence it is easily seen that $f_r(\alpha)$ is concave and strictly increasing for $0 \le \alpha \le \pi$. The concavity of $F_r(\alpha)$ needs some calculation. To check it, we have to show that

$$\cos\alpha\left(\sin^2\frac{\alpha}{2}-r^2\right) - r^2\cos^2\frac{\alpha}{2} + 2\cos\frac{\alpha}{2}\left(r^2-\sin^2\frac{\alpha}{2}\right)^{3/2} > 0.$$

Introducing the abbreviations $s=\sin(\alpha/2)$ and $c=\cos(\alpha/2)$, we have

$$\begin{aligned}\cos\alpha\left(s^2-r^2\right) - r^2c^2 + 2c\left(r^2-s^2\right)^{3/2} &\\ &> \cos\alpha\left(s^2-r^2\right)-r^2c^2+2c\left(r^2-s^2\right)\\ &= \left(2c^2-1\right)\left(1-c^2-r^2\right)-r^2c^2+2c\left(r^2-1+c^2\right)\\ &= (r^2-1)(1-c)(1+3c)+2c^3(1-c)>0.\end{aligned}$$

Let C_n be a circle of radius r_n centered at o and let $C_0,\dots,C_{n-1}$ be closed unit circles with centers $o_0,\dots,o_{n-1}$ covering C_n. We assume that for a circle C_i, $i=0,\dots,n-1$, for which $C_i\cap \operatorname{bd} C_n \neq \varnothing$ the centers of C_i and C_n lie on the same side of the radical axis of the circles C_i and C_n. Otherwise we reflect C_i in this radical axis and still get a covering of C_n. Let $\overline{C}_0,\dots,\overline{C}_{n-1}$ be unit circles in the position described in the theorem, that is so that $\overline{C}_0$ is concentric with C_n and the centers $\bar{o}_1,\dots,\bar{o}_{n-1}$ of $\overline{C}_1,\dots,\overline{C}_{n-1}$ are situated in the vertices of a regular $(n-1)$-gon at distance $2\sin\bigl(\pi/(n-1)\bigr)$ from o. We are going to show that the two arrangements of circles are congruent.

The following lemma claims that the two arrangements of circles $\{C_i\}_{i=0}^n$ and $\{\overline{C}_i\}_{i=0}^n$ have the same topological structure.

Lemma. *Exactly one of the circles* $\{C_i\}_{i=0}^{n-1}$ *is contained in* $\operatorname{int} C_n$. *Moreover, no three of the circles intersecting* $\operatorname{bd} C_n$ *can have a common point.*

Since $r_n>2$, there is a circle, say C_0, which is contained in $\operatorname{int} C_n$. Observe that an arc of $\operatorname{bd} C_n$ which is covered by a unit circle spans at o an angle not greater than $2\arcsin(1/r_n)$. Since

$$6\arcsin\frac{1}{r_8} = 2.76326081\ldots < \pi \quad\text{and}\quad 7\arcsin\frac{1}{r_9} = 2.989550105\ldots < \pi,$$

it follows that if $n=8$ or $n=9$, then $C_i\cap\operatorname{bd} C_n\neq\varnothing$ for $i=1,\dots,n-1$. We also observe that three unit circles with a common point cannot cover a part of $\operatorname{bd} C_n$ whose angle spanned at o exceeds $2\arcsin(2/r_n)$. Since

$$4\arcsin\frac{1}{r_8} + \arcsin\frac{2}{r_8} = 2.942412903\ldots < \pi$$

and

$$5 \arcsin \frac{1}{r_9} + \arcsin \frac{2}{r_9} = 3.111686536\ldots < \pi,$$

it follows that for $n = 8$ and 9 no three of the circles C_i, $i = 1, \ldots, n-1$, can have a common point.

This argument breaks down when $n = 10$. We have

$$7 \arcsin \frac{1}{r_{10}} = 2.841948021\ldots < \pi,$$

showing that at most two of the circles $\{C_i\}_{i=0}^{9}$ are contained in $\operatorname{int} C_{10}$; however

$$8 \arcsin \frac{1}{r_{10}} = 3.24794059\ldots > \pi,$$

so we cannot exclude in this way that two of the circles $\{C_i\}_{i=0}^{9}$ are contained in $\operatorname{int} C_{10}$. Also the proof that no three of the circles intersecting $\operatorname{bd} C_n$ can have a common point requires a different argument. Melissen [1997, pp. 108–111] proved the Lemma for $n = 10$ using an argument based on the investigation of distances. In Section 3 we repeat Melissen's argument for the proof of the first part of the Lemma and give an alternative proof for the second statement, using estimations of areas.

Let D_i, $i = 0, \ldots, n-1$, be the Dirichlet cell of C_i with respect to C_n. From the considerations above it follows that each vertex in the cell complex of the Dirichlet cells is trihedral, D_0 is an $(n-1)$-gon, while for $i = 1, \ldots, n-1$, D_i is a curved quadrilateral bounded by three line segments and an arc of $\operatorname{bd} C_n$. Thus the cell complex of the cells D_i is isomorphic to the cell complex of the Dirichlet cells $\bar{D}_i$ of the circles $\bar{C}_i$.

We introduce some notations. We describe them for the circles C_i. The same symbols with a bar will be used for the corresponding objects and quantities for the circles $\bar{C}_i$ (see Figure 1).

Let the vertices of D_0 be $p_1, \ldots, p_{n-1}$ and let the vertices of Dirichlet cells on $\operatorname{bd} C_n$ be $q_1, \ldots, q_{n-1}$. We write $p_n = p_1$, $q_n = q_1$ and assume, as we may without loss of generality, that the notation is chosen so that the vertices of D_i are $p_i,\ p_{i+1},\ q_{i+1},\ q_i$ for $i = 1, \ldots, n-1$. We write

$$\alpha_i = \measuredangle q_i o_i q_{i+1}, \quad \beta_i = \measuredangle p_i o_i q_i, \quad \gamma_i = \measuredangle p_{i+1} o_i q_{i+1},$$
$$\delta_i = \measuredangle p_i o_i p_{i+1}, \quad \varepsilon_i = \measuredangle p_i o_0 p_{i+1}.$$

We note that the assumption that o and o_i lie on the same side of the radical axis of C_n and C_i implies that

$$\alpha_i \le \pi$$

for $i = 1, \ldots, n-1$. It is easy to check that

$$\bar{\alpha}_i = \frac{6\pi}{n-1}, \quad \bar{\beta}_i = \bar{\gamma}_i = \frac{(n-5)\pi}{n-1}, \quad \text{and} \quad \bar{\delta}_i = \bar{\varepsilon}_i = \frac{2\pi}{n-1}.$$

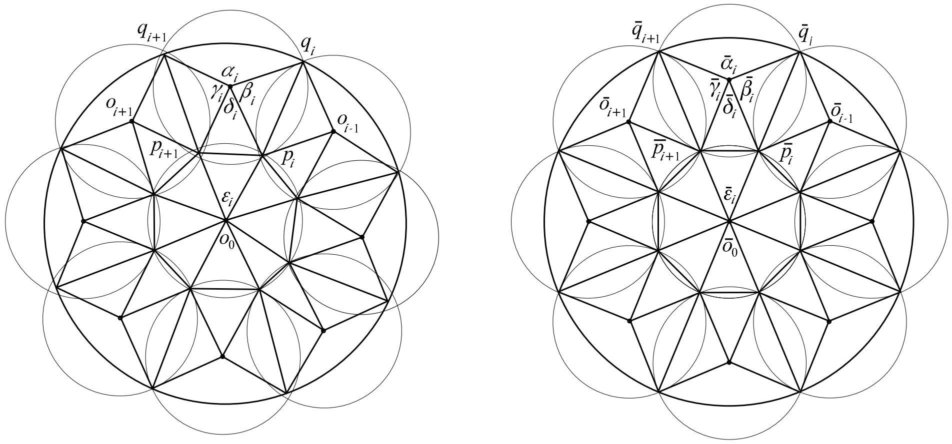

Figure 1.

Using the relation $\bar{\alpha}_i = 6\pi/(n-1)$ one can verify that

$$f\left(\frac{6\pi}{n-1}\right) = \frac{2\pi}{n-1}.$$

We dissect C_n into the triangles $T_i = p_i o_0 p_{i+1}$, $T_i^* = p_i o_i p_{i+1}$, $T_i^- = p_i o_i q_i$, $T_i^+ = p_{i+1} o_i q_{i+1}$ and into the regions R_i bounded by the segments $o_i q_i$, $o_i q_{i+1}$ and the arc $q_i q_{i+1}$ of $\operatorname{bd} C_n$, for $i = 1, \ldots, n-1$. We shall estimate the total area of these domains and show that it is less than the area of C_n unless the arrangement of the circles $C_0, \ldots, C_{n-1}$ is congruent to that of the circles $\bar{C}_0, \ldots, \bar{C}_{n-1}$.

Observe that the triangles T_i and T_i^* are congruent, so that

$$\sum_{i=1}^{n-1} \delta_i = \sum_{i=1}^{n-1} \varepsilon_i = 2\pi = \sum_{i=1}^{n-1} \bar{\delta}_i = \sum_{i=1}^{n-1} \bar{\varepsilon}_i \tag{1}$$

and

$$\sum_{i=1}^{n-1} (\beta_i + \gamma_i) = (n-1)2\pi - \sum_{i=1}^{n-1} (\alpha_i + \delta_i) = (n-2)2\pi - \sum_{i=1}^{n-1} \alpha_i. \tag{2}$$

We have

$$2\pi = \sum_{i=1}^{n-1} \measuredangle q_i o q_{i+1} \leq \sum_{i=1}^{n-1} f_{r_n}(\alpha_i) \leq (n-1) f_{r_n}\left(\frac{\sum_{i=1}^{n-1} \alpha_i}{n-1}\right).$$

Hence we get

$$\sum_{i=1}^{n-1} \alpha_i \geq (n-1) f_{r_n}^{-1}\left(\frac{2\pi}{n-1}\right) = (n-1)\frac{6\pi}{n-1} = \sum_{i=1}^{n-1} \bar{\alpha}_i. \tag{3}$$

Now we are in the position to estimate the total area of the parts of C_n. Using Jensen's inequality we get

$$\sum_{i=1}^{n-1}(a(T_i)+a(T_i^*)) \le \sum_{i=1}^{n-1}\sin\varepsilon_i \le (n-1)\sin\frac{\sum_{i=1}^{n-1}\varepsilon_i}{n-1}, \tag{4}$$

$$\sum_{i=1}^{n-1}(a(T_i^-)+a(T_i^+)) \le \frac{1}{2}\sum_{i=1}^{n-1}(\sin\beta_i+\sin\gamma_i) \le (n-1)\sin\frac{\sum_{i=1}^{n-1}(\beta_i+\gamma_i)}{n-1}, \tag{5}$$

$$\sum_{i=1}^{n-1}a(R_i) \le \sum_{i=1}^{n-1}F_{r_n}(\alpha_i) \le (n-1)F_{r_n}\left(\frac{\sum_{i=1}^{n-1}\alpha_i}{n-1}\right). \tag{6}$$

In view of (1) we have

$$(n-1)\sin\frac{\sum_{i=1}^{n-1}\varepsilon_i}{n-1} = \sum_{i=1}^{n-1}\sin\bar{\varepsilon}_i = \sum_{i=1}^{n-1}(a(\bar{T}_i)+a(\bar{T}_i^*)).$$

Write

$$\alpha = \sum_{i=1}^{n-1}\frac{\alpha_i}{n-1}.$$

Then we have, in view of (2),

$$\frac{\sum_{i=1}^{n-1}(\beta_i+\gamma_i)}{n-1} = \frac{2(n-2)\pi}{n-1} - \alpha;$$

hence, by (5) and (6),

$$\sum_{i=1}^{n-1}\big(a(T_i^-)+a(T_i^+)+a(R_i)\big) \le (n-1)\left(\sin\Big(\frac{2(n-2)\pi}{n-1}-\alpha\Big)+F_{r_n}(\alpha)\right).$$

The function

$$\sin\left(\frac{2(n-2)\pi}{n-1}-\alpha\right)+F_{r_n}(\alpha)$$

is concave for $0 \le \alpha \le \pi$ and, as it can be checked numerically, decreasing for $\alpha = 6\pi/(n-1)$. Therefore it is decreasing for $6\pi/(n-1) \le \alpha \le \pi$. Observing that

$$\frac{6\pi}{n-1} = \bar{\alpha}_i \quad\text{and}\quad \frac{2(n-2)\pi}{n-1} - \frac{6\pi}{n-1} = \frac{2(n-10)\pi}{n-1} = \bar{\beta}_i = \bar{\gamma}_i,$$

we deduce that

$$\begin{aligned}\sum_{i=1}^{n-1}\big(a(T_i^-)+a(T_i^+)+a(R_i)\big) &\le (n-1)\left(\sin\frac{2(n-10)\pi}{n-1}+F_{r_n}\left(\frac{6\pi}{n-1}\right)\right)\\ &= \frac{1}{2}\sum_{i=1}^{n-1}(\sin\bar{\beta}_i+\sin\bar{\gamma}_i)+\sum_{i=1}^{n-1}F_{r_n}(\bar{\alpha}_i)\\ &= \sum_{i=1}^{n-1}\big(a(\bar{T}_i^-)+a(\bar{T}_i^+)+a(\bar{R}_i)\big).\end{aligned} \tag{7}$$

Adding inequalities (4) and (7) we get

$$\begin{aligned} a(C_n) &= \sum_{i=1}^{n-1}(a(T_i)+a(T_i^*)+a(T_i^-)+a(T_i^+)+a(R_i)) \\ &\le \sum_{i=1}^{n-1}(a(\bar{T}_i)+a(\bar{T}_i^*)+a(\bar{T}_i^-)+a(\bar{T}_i^+)+a(\bar{R}_i)) = a(C_n). \end{aligned}$$

Therefore we have equality in all of the inequalities (5)–(7). This can only occur if the arrangements of the circles $C_0,\dots,C_{n-1}$ and $\bar{C}_0,\dots,\bar{C}_{n-1}$ are congruent.

3. Proof of the Lemma for $n = 10$

Let $C_0,\dots,C_9$ be closed unit circles covering the circle C_{10} of radius r_{10}. As in the previous section, we denote the center of C_i, $i=0,\dots,9$ by o_i and the center of C_{10} by o. We shall follow the argument of Melissen to show that no eight of the circles can cover $\operatorname{bd} C_{10}$. Suppose that $\operatorname{bd} C_{10} \subset \bigcup_{i=0}^{7} C_i$. Since the angular measure of an arc of $\operatorname{bd} C_{10}$ covered by a unit circle is at most $2\arcsin(1/r_{10})$ and

$$7\arcsin\frac{1}{r_{10}} = 2.841948021\ldots < \pi,$$

no proper subset of the circles C_i, $i=0,\dots,7$ covers $\operatorname{bd} C_{10}$, hence no three of the arcs $C_i \cap \operatorname{bd} C_{10}$, $0 \le i \le 7$ intersect. This property defines a cyclic order of the arcs $C_i \cap \operatorname{bd} C_{10}$. We assume that the notation is chosen so that this cyclic order coincides with the order of the indices, that is $C_0 \cap C_1 \cap \operatorname{bd} C_{10} \neq \varnothing,\dots,C_6 \cap C_7 \cap \operatorname{bd} C_{10} \neq \varnothing, C_7 \cap C_0 \cap \operatorname{bd} C_{10} \neq \varnothing$. We choose points $q_1,\dots,q_8$ from the sets $C_0 \cap C_1 \cap \operatorname{bd} C_{10},\dots,C_7 \cap C_0 \cap \operatorname{bd} C_{10}$, respectively.

Recall that the maximum angular measure of an arc of $\operatorname{bd} C_{10}$ covered by three unit circles with a common point is $2\arcsin(2/r_{10})$. Since

$$\arcsin\frac{2}{r_{10}} + 5\arcsin\frac{1}{r_{10}} = 2.940546309\ldots < \pi,$$

no three of the circles C_i, $i=0,\dots,7$ have a common point. Let

$$p_i = \operatorname{bd} C_{i-1} \cap C_i \cap \operatorname{int} C_{10}$$

for $i=1,\dots,7$, and $p_8 = C_7 \cap C_0 \cap \operatorname{int} C_{10}$ (see Figure 2).

The main observation of Melissen is that the points p_i, $i=1,\dots,8$, cannot be covered by two circles. This follows easily from the following result:

Proposition. *We have*

$$|p_i p_{i+3}| > 2 \quad \textit{and} \quad |p_i p_{i+4}| > 2$$

for $i = 1,\dots,8$, *with* $p_i = p_{i+8}$.

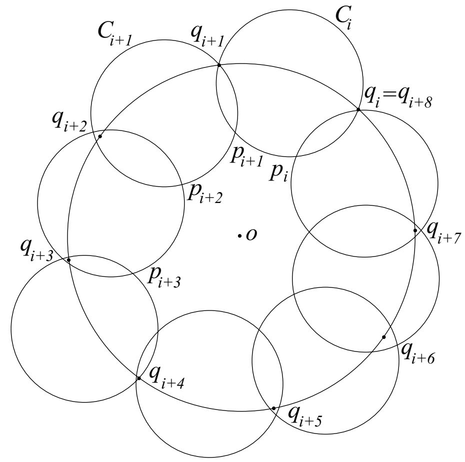

Figure 2.

Indeed, the first inequality readily implies that one of the circles C_8 and C_9 contains all points p_i with an odd subscript and the other contains all points with an even subscript. This, however, contradicts the second inequality.

In order to estimate the distances between the points p_i we need a lower bound for the distance $|op_i|$. Let $h(\vartheta)$ be the minimum distance between o and a point of intersection of the boundaries of two unit circles that cover an arc of angular measure ϑ from $\operatorname{bd} C_{10}$, with $0 \le \vartheta \le 4 \arcsin(1/r_{10})$. It is easy to see that this minimum distance is achieved in the symmetric position when each of the unit circles cover an arc of angular measure $\frac{\vartheta}{2}$ from $\operatorname{bd} C_{10}$. Using some trigonometry we calculate that

$$h(\vartheta) = r_{10} \cos \frac{\vartheta}{2} - 2\sqrt{1 - r_{10}^2 \sin^2 \frac{\vartheta}{4}} \cos \frac{\vartheta}{4}.$$

Writing $s = \sin^2 \frac{\vartheta}{4}$ we have

$$h'(\vartheta) = \frac{\sqrt{s}\left(\sqrt{1 - r_{10}^2 s^2} - r_{10} \cos \frac{\vartheta}{4}\right)^2}{\sqrt{1 - r_{10}^2 s^2}}$$

and

$$h''(\vartheta) = \frac{-2r_{10}(1 - r_{10}^2 s)^{3/2} \cos \frac{\vartheta}{2} + (1 + r_{10}^2 - 6r_{10}^2 s + 4r_{10}^4 s^2) \cos \frac{\vartheta}{4}}{8(1 - r_{10}^2 s^2)^{3/2}}.$$

It immediately follows that $h(\vartheta)$ is increasing. Observing that

$$1 + r_{10}^2 - 6r_{10}^2 s + 4r_{10}^4 s^2 > r_{10}^2 - \frac{5}{4}$$

we get

$$h''(\vartheta) > \frac{-2r_{10} \cos \frac{\vartheta}{2} + (r_{10}^2 - \frac{5}{4}) \cos \frac{\vartheta}{4}}{8} = \frac{-4r_{10} \cos^2 \frac{\vartheta}{4} + (r_{10}^2 - \frac{5}{4}) \cos \frac{\vartheta}{4} + 2r_{10}}{8}.$$

The minimum of the right side is $\frac{1}{8}\bigl(r_{10}^2 - 2r_{10} - \frac{5}{4}\bigr) > 0$, showing that $h(\vartheta)$ is convex.

Now we are in the position to estimate the distances between the points p_i. Write $\psi = \measuredangle p_i o p_{i+3}$ and $\xi = \measuredangle p_i o p_{i+4}$. Then we have, on the one hand,

$$\begin{aligned}\psi &= 2\pi - \measuredangle p_{i+3} o q_{i+4} - \sum_{j=4}^{6} \measuredangle q_{i+j} o q_{i+j+1} - \measuredangle q_{i+7} o p_i \\ &\geq 2\pi - 2 \arcsin \frac{2}{r_{10}} - 6 \arcsin \frac{1}{r_{10}} = \psi_{\min} = 2.026062\ldots > \frac{\pi}{2}\end{aligned}$$

and

$$\begin{aligned}\xi &= 2\pi - \measuredangle p_{i+4} o q_{i+5} - \sum_{j=45}^{6} \measuredangle q_{i+j} o q_{i+j+1} - \measuredangle q_{i+7} o p_i \\ &\geq 2\pi - 2 \arcsin \frac{2}{r_{10}} - 4 \arcsin \frac{1}{r_{10}} = \xi_{\min} = 2.838048136\ldots > \frac{\pi}{2},\end{aligned}$$

and on the other hand,

$$\begin{aligned}\psi &= \measuredangle p_i o q_{i+1} + \measuredangle q_{i+1} o q_{i+2} + \measuredangle q_{i+2} o p_{i+3} \\ &\leq 2 \arcsin \frac{2}{r_{10}} + 2 \arcsin \frac{1}{r_{10}} = 2.633152\ldots < \pi\end{aligned}$$

and

$$\begin{aligned}\xi &= \measuredangle p_i o q_{i+1} + \measuredangle q_{i+1} o q_{i+2} + \measuredangle q_{i+2} o p_{i+3} + \measuredangle p_{i+3} o p_{i+4} \\ &\leq 2 \arcsin \frac{2}{r_{10}} + 4 \arcsin \frac{1}{r_{10}} = 2\pi - \xi_{\min}.\end{aligned}$$

By the law of cosines we get

$$\begin{aligned}|p_i p_{i+3}| &= \sqrt{|op_i|^2 + |op_{i+3}|^2 - 2|op_i||op_{i+3}| \cos \psi}, \\ |p_i p_{i+4}| &= \sqrt{|op_i|^2 + |op_{i+4}|^2 - 2|op_i||op_{i+4}| \cos \xi}.\end{aligned}$$

Let ϑ_1, ϑ_2, and ϑ_3 be the angular measure of the arc of $\operatorname{bd} C_{10}$ covered by the pair of circles C_{i-1}, C_i, C_{i+2}, C_{i+3}, and C_{i+3}, C_{i+4}, respectively. As the triangles $p_i o p_{i+3}$ and $p_i o p_{i+4}$ are obtuse, we get lower bounds for $|p_i p_{i+3}|$ and $|p_i p_{i+4}|$ if we substitute for $|op_i|$, $|op_{i+3}|$, and $|op_{i+4}|$ their minimum values and for $\cos\psi$ and $\cos\xi$ their maximum values:

$$\begin{aligned}|p_i p_{i+3}| &\geq \sqrt{h^2(\vartheta_1) + h^2(\vartheta_2) - 2h(\vartheta_1)h(\vartheta_2) \cos \psi_{\min}}, \\ |p_i p_{i+4}| &\geq \sqrt{h^2(\vartheta_1) + h^2(\vartheta_3) - 2h(\vartheta_1)h(\vartheta_3) \cos \xi_{\min}}.\end{aligned}$$

We have, for $j = 2, 3$,

$$\vartheta_1 + \vartheta_j \leq 2\pi - 8 \arcsin \frac{1}{r_{10}}.$$

Since $h(\vartheta)$ is increasing and convex, therefore,

$$h(\vartheta_1) + h(\vartheta_j) \geq 2h\Bigl(\pi - 4 \arcsin \frac{1}{r_{10}}\Bigr).$$

The functions $\sqrt{h_1^2+h_2^2-2h_1h_2\cos\psi_{\min}}$ and $\sqrt{h_1^2+h_2^2-2h_1h_2\cos\xi_{\min}}$ are homogeneous of degree one in the variables h_1 and h_2, thus, they are convex. They are also increasing in both variables. Therefore

$$|p_ip_{i+3}| \geq \sqrt{2h^2(\pi-4\arcsin(1/r_{10}))(1-\cos\psi_{\min})} = \\ = 2h(\pi-4\arcsin(1/r_{10}))\sin\frac{\psi_{\min}}{2} = 2.02349\ldots > 2$$

and

$$|p_ip_{i+4}| \geq \sqrt{2h^2(\pi-4\arcsin(1/r_{10}))(1-\cos\xi_{\min})} = \\ = 2h(\pi-4\arcsin(1/r_{10}))\sin\frac{\xi_{\min}}{2} = 2.357538\ldots > 2.$$

This completes the proof of the Proposition and at the same time the proof of the first part of the Lemma.

It remains to show the second part of the Lemma, namely that if nine of the circles $C_0,\ldots,C_9$ intersect $\operatorname{bd} C_{10}$, then no three of them can have a common point. This part of the Lemma can be settled by estimating areas in a similar way as we did in the previous section.

Suppose that $C_0 \cap \operatorname{bd} C_{10} = \varnothing$ and $C_i \cap \operatorname{bd} C_{10} \neq \varnothing$ for $i = 1,\ldots,9$. We shall scrutinize the cell complex formed by the Dirichlet cells D_i of the circles C_i, $0 \leq i \leq 9$, with respect to C_{10}. We may assume that $D_i \cap \operatorname{bd} C_{10} \neq \varnothing$ for $i = 1,\ldots,9$, otherwise $\operatorname{bd} C_{10}$ is covered by eight circles, which we already excluded. Without loss of generality we may suppose that the arcs $D_i \cap \operatorname{bd} C_{10}$, $i = 1,\ldots,9$, are situated on $\operatorname{bd} C_{10}$ in their natural cyclic order.

We shall exclude the possibility that three of the Dirichlet cells $D_1,\ldots,D_9$ intersect. We note that three circles can intersect without their corresponding Dirichlet cells having a common point, however the case when no three of the cells $D_1,\ldots,D_9$ intersect has been already discussed in the previous section. Observe that

$$D_i \cap D_{i\pm j} = \varnothing \quad \text{for} \quad i = 1,\ldots,9, \quad j = 3,\, 4. \tag{8}$$

Otherwise the circles $C_i, C_{i\pm1}\ldots, C_{i\pm j}$ cover from $\operatorname{bd} C_{10}$ an arc whose angular measure is at most $2\arcsin(2/r_{10})$, while the angular measure of the arc covered by the other $9-j-1 \leq 5$ circles cannot exceed $10\arcsin(1/r_{10})$. Since $\arcsin(2/r_{10}) + 5\arcsin(2/r_{10}) = 2.940546309\ldots < \pi$, this is impossible.

Suppose that three of the cells $D_1,\ldots,D_9$ intersect. In view of (8) they must belong to consecutive indices. Assume, say, that $D_1 \cap D_2 \cap D_3 \neq \varnothing$. If no further triple of the cells $D_1,\ldots,D_9$ intersect, then the cells are arranged as depicted on the left side of Figure 3, where the Dirchlet cells are drawn by broken lines. We shall refer to this situation as Case 1.

If there is another intersecting triple, say D_i, D_{i+1}, and D_{i+2}, among the cells $D_1,\ldots,D_9$, then $\{1,2,3\} \cap \{i,i+1,i+2\} \neq \varnothing$. Otherwise the total angular

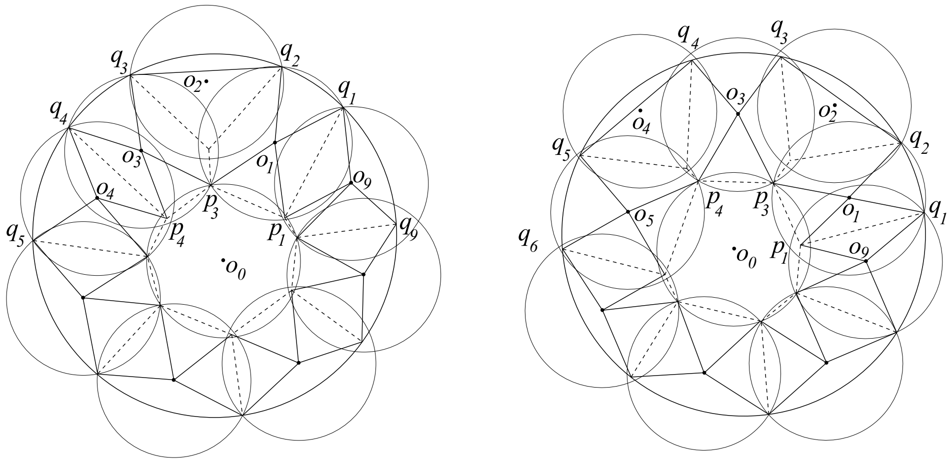

Figure 3.

measure of the arcs of $\operatorname{bd} C_{10}$ covered by the circles is at most $4\arcsin(2/r_{10}) + 6\arcsin(1/r_{10}) = 6.078289194\ldots < 2\pi$. D_2 cannot be the member of another intersecting triple of cells. For, if D_2, D_3, and D_4 or D_9, D_1, and D_2 intersect, then $D_1 \cap D_4 \neq \varnothing$ or $D_9 \cap D_3 \neq \varnothing$, which is impossible by (8). Thus, the only candidates for another triple of intersecting cells are $\{D_3, D_4, D_5\}$ and $\{D_8, D_9, D_1\}$. As these triples are disjoint, only one of them can have a nonempty intersection. Hence, the other case we have to investigate is that besides D_1, D_2, and D_3, say D_3, D_4, and D_5 have a common point. This is Case 2 which is represented on the right side of Figure 3.

As before, we denote by o_i, $i = 1, \ldots, 10$, the center of C_i, and by q_i, $i = 1, \ldots, 9$, the vertices of Dirichlet cells on $\operatorname{bd} C_{10}$ choosing the notation so that q_i is common to D_{i-1} and D_i. We denote the vertices of D_0 in their consecutive order for the two cases by p_1, p_3, p_4, $\ldots$, p_9 and p_1, p_3, p_5, $\ldots$, p_9, respectively, so that p_1q_1 is the side common to D_1 and D_9 (see Figure 3). We divide C_{10} into the following regions:

(i) the 16-gon $P_1 = p_1o_1p_3o_3p_4o_4p_5o_5p_6o_6p_7o_7p_8o_8p_9o_9$, the pentagon $P_2 = q_2o_1p_3o_3q_3$, the segment S_1 cut off from C_{10} by the chord q_2q_3, the quadrilaterals $Q_i = o_iq_{i+1}o_{i+1}p_{i+1}$, $3 \leq i \leq 9$, and the regions R_i, $1 \leq i \leq 9$, $i \neq 2$, bounded by the segments o_iq_i, o_iq_{i+1} and the arc q_iq_{i+1} of $\operatorname{bd} C_n$ in Case 1;

(ii) the 14-gon $P_1 = p_1o_1p_3o_3p_5o_5p_6o_6p_7o_7p_8o_8p_9o_9$, the pentagons

$$P_2 = q_2o_1p_3o_3q_3 \quad \text{and} \quad P_3 = q_4o_3p_5o_5q_5,$$

the two segments S_1 and S_2 cut off from C_{10} by the chords q_2q_3 and q_4q_5, respectively, the quadrilaterals $Q_i = o_iq_{i+1}o_{i+1}p_{i+1}$, $5 \leq i \leq 9$, and the regions R_i, $1 \leq i \leq 9$, $i \neq 2$, 4, bounded by the segments o_iq_i, o_iq_{i+1} and the arc q_iq_{i+1} of $\operatorname{bd} C_n$ in Case 2.

As we saw in the previous section, P_1 can be dissected into pairs of congruent triangles one half of the triangles making up the cell D_0. Hence we get

$$a(P_1) \leq 8 \sin \frac{\pi}{4} = 5.656854249\ldots \tag{9}$$

in Case 1 and

$$a(P_1) \leq 7 \sin \frac{2\pi}{7} = 5.472820377\ldots \tag{9'}$$

in Case 2.

The length of four sides of the pentagon P_2 (P_3) is bounded above by 1, while the length of the fifth side is at most 2. The area of such a pentagon cannot exceed the area of a pentagon with four sides of length 1 and one side of length 2 inscribed into a circle. The radius r of the circle is determined implicitly by the equation

$$4 \arcsin \frac{1}{2r} + \arcsin \frac{1}{r} = \pi.$$

$r_0 = 1.07326$ is an upper bound for r and

$$r_0^2 \left(\frac{1}{2} \sin 2 \arcsin \frac{1}{r_0} + 2 \sin \frac{\pi - \arcsin \frac{1}{r_0}}{2} \right) = 2.284572282\ldots$$

is an upper bound for the area of the pentagon. Thus, we have

$$a(P_2) \leq 2.284572282\ldots \tag{10}$$

and, in Case 2,

$$a(P_2) + a(P_3) \leq 4.5691944564\ldots. \tag{10'}$$

The area of S_1 (S_2) cannot exceed

$$r_{10}^2 \left(\arcsin \frac{1}{r_{10}} - \frac{1}{2} \sin 2 \arcsin \frac{1}{r_{10}} \right) = 0.27675335\ldots,$$

the area of a segment of C_{10} cut off by a chord of length 2. Hence

$$a(S_1) \leq 0.27675335\ldots \tag{11}$$

and

$$a(S_1) + a(S_2) \leq 0.533506699\ldots. \tag{11'}$$

Using the rough estimate $a(Q_i) \leq 1$ we get

$$\sum_{i=3}^{9} a(Q_i) \leq 7 \tag{12}$$

and

$$\sum_{i=5}^{9} a(Q_i) \leq 5, \tag{12'}$$

respectively.

We estimate the total area of the regions R_i using the method developed in the previous section. Let $x = \measuredangle q_2 o q_3$ in Case 1 and $x = \measuredangle q_2 o q_3 + \measuredangle q_4 o q_5$ in

Case 2. Then we have $x \le 2\arcsin(1/r_{10})$ and $x \le 4\arcsin(1/r_{10})$, respectively. Writing $\alpha_i = \measuredangle q_i o_i q_{i+1}$ we have

$$2\pi - x = \sum_{1\le i\le 9,\, i\ne 2} \measuredangle q_i o q_{i+1} \le \sum_{1\le i\le 9,\, i\ne 2} f_{r_{10}}(\alpha_i) \le 8 f_{r_{10}}\left(\frac{1}{8}\sum_{1\le i\le 9,\, i\ne 2} \alpha_i\right)$$

and

$$2\pi - x = \sum_{1\le i\le 9,\, i\ne 2,4} \measuredangle q_i o q_{i+1} \le \sum_{1\le i\le 9,\, i\ne 2,4} f_{r_{10}}(\alpha_i) \le 8 f_{r_{10}}\left(\frac{1}{8}\sum_{1\le i\le 9,\, i\ne 2,4} \alpha_i\right),$$

hence

$$\frac{1}{8}\sum_{1\le i\le 9,\, i\ne 2} \alpha_i \ge f_{r_{10}}^{-1}\Big(\frac{2\pi - x}{8}\Big) \ge f_{r_{10}}^{-1}\left(\frac{\pi - \arcsin\frac{1}{r_{10}}}{4}\right) = 2.028453422\ldots$$

and

$$\frac{1}{7}\sum_{1\le i\le 9,\, i\ne 2,\, 4} \alpha_i \ge f_{r_{10}}^{-1}\Big(\frac{\pi - x}{7}\Big) \ge f_{r_{10}}^{-1}\left(\frac{2\pi - 4\arcsin\frac{1}{r_{10}}}{7}\right) = 1.948256547\ldots,$$

respectively.

Observing that $F_{r_{10}}(\alpha)$ is decreasing for $\alpha \ge 1.9$ we get for the total area of the regions R_i the estimate

$$\begin{aligned}\sum_{1\le i\le 9,\, i\ne 2} a(R_i) &\le \sum_{1\le i\le 9,\, i\ne 2} F_{r_{10}}(\alpha_i) \le 8F_{r_{10}}\left(\frac{1}{8}\sum_{1\le i\le 9,\, i\ne 2}\alpha_i\right) \\ &\le 8F_{r_{10}}\left(f_{r_{10}}^{-1}\left(\frac{\pi - \arcsin\frac{1}{r_{10}}}{4}\right)\right) = 4.92397937\ldots \qquad (13)\end{aligned}$$

in Case 1 and

$$\begin{aligned}\sum_{1\le i\le 9,\, i\ne 2,4} a(R_i) &\le \sum_{1\le i\le 9,\, i\ne 2,4} F_{r_{10}}(\alpha_i) \le 7F_{r_{10}}\left(\frac{1}{7}\sum_{1\le i\le 9,\, i\ne 2,4}\alpha_i\right) \\ &\le 7F_{r_{10}}\left(f_{r_{10}}^{-1}\left(\frac{2\pi - 4\arcsin\frac{1}{r_{10}}}{7}\right)\right) = 4.332295377\ldots \qquad (13')\end{aligned}$$

in Case 2.

From inequalities (9)–(13) we conclude that

$$\begin{aligned}a(C_{10}) = a(P_1) + a(P_2) + a(S_1) + \sum_{i=3}^{9} a(Q_i) + \sum_{1\le i\le 9,\, i\ne 2} a(R_i) \le \\ \le 20.14216 < 20.1422 < a(C_{10})\end{aligned}$$

in Case 1, and it follows from (9′)–(13′) that

$$a(C_{10}) = a(P_1) + a(P_2) + a(P_3) + a(S_1) + a(S_2) + \sum_{i=5}^{9} a(Q_i) + \sum_{1 \le i \le 9,\, i \ne 2,4} a(R_i)$$
$$\le 20 < 20.1422 < a(C_{10})$$

in Case 2, yielding in both cases a contradiction.

This completes the proof of the Lemma.

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