De Concini–Procesi Wonderful Arrangement Models: A Discrete Geometer's Point of View

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ABSTRACT. This article outlines the construction of De Concini–Procesi arrangement models and describes recent progress in understanding their significance from the algebraic, geometric, and combinatorial point of view. Throughout the exposition, strong emphasis is given to the combinatorial and discrete geometric data that lie at the core of the construction.

CONTENTS

1. An Invitation to Arrangement Models

The complements of coordinate hyperplanes in a real or complex vector space are easy to understand: The coordinate hyperplanes in $\mathbb{R}^n$ dissect the space into 2^n open orthants; removing the coordinate hyperplanes from $\mathbb{C}^n$ leaves the complex torus $(\mathbb{C}^*)^n$. Arbitrary subspace arrangements, i.e., finite families of linear subspaces, have complements with far more intricate combinatorics in the real case, and far more intricate topology in the complex case. Arrangement models improve this complicated situation locally—constructing an arrangement model means to alter the ambient space so as to preserve the complement and to replace the arrangement by a divisor with normal crossings, i.e., a collection of smooth hypersurfaces which locally intersect like coordinate hyperplanes. Almost a decade ago, De Concini and Procesi provided a canonical construction of arrangement models—*wonderful arrangement models*—that had significant impact in various fields of mathematics.

Why should a discrete geometer be interested in this model construction?

Because there is a wealth of *wonderful* combinatorial and discrete geometric structure lying at the heart of the matter. Our aim here is to bring these discrete pearls to light.

First, combinatorial data plays a descriptive role in various places: The combinatorics of the arrangement fully prescribes the model construction and a natural stratification of the resulting space. We will see details and examples in Section 2. In fact, the rather coarse combinatorial data reflect enough of the situation so as to, for instance, determine algebraic-topological invariants of the arrangement models (compare the topological interpretation of the algebra $D(\mathcal{L}, \mathcal{G})$ in Section 4.2).

Secondly, the combinatorial data put forward in the study of arrangement models invites purely combinatorial generalizations. We discuss such generalizations in Section 3 and show in Section 4 how this combinatorial generalization opens unexpected views when related back to geometry.

Finally, in Section 5 we propose arrangement models as a tool for resolving group actions on manifolds. Again, it is an open eye for discrete core data that enables the construction.

We have kept the exposition self-contained and illustrated it with many examples. We invite discrete geometers to discover an algebro-geometric context in which familiar discrete structures play a key role. We hope that yet many more bridges will be built between algebraic and discrete geometry — areas that, despite the differences in terminology, concepts, and methods, share what has inspired and driven mathematicians for centuries: a passion for geometry.

2. Introducing the Main Character

2.1. Basics on arrangements. We first need to fix some basic terminology, in particular as it concerns the combinatorial data of an arrangement. We suggest that the reader, who is not familiar with the setting, reads through the first part of this Section and compares the notions to the illustrations given for braid arrangements in Example 2.1.

An *arrangement* $\mathcal{A} = \{U_1, \ldots, U_n\}$ is a finite family of linear subspaces in a real or complex vector space V. The topological space most obviously associated to such an arrangement is its *complement* in the ambient space, $\mathcal{M}(\mathcal{A}) := V \setminus \bigcup \mathcal{A}$.

Having arrangements in real vector spaces in mind, the topology of $\mathcal{M}(\mathcal{A})$ does not look very interesting: the complement is a collection of open polyhedral cones, and apart from their number there is no significant associated topological data. In the complex case, however, already a single hyperplane in $\mathbb{C}^1$, the origin, has a nontrivial complement: it is homotopy equivalent to S^1, the 1-dimensional

sphere. The complement of two (for instance, coordinate) hyperplanes in $\mathbb{C}^2$ is homotopy equivalent to the torus $S^1 \times S^1$.

The combinatorial data associated with an arrangement is usually recorded in the *intersection lattice* $\mathcal{L} = \mathcal{L}(\mathcal{A})$, which is the set of intersections of subspaces in $\mathcal{A}$, partially ordered by reversed inclusion. We adopt terminology from the theory of partially ordered sets and often denote the unique minimum in $\mathcal{L}(\mathcal{A})$ (corresponding to the empty intersection, i.e., the ambient space V) by $\hat{0}$ and the unique maximum of $\mathcal{L}(\mathcal{A})$ (the overall intersection of subspaces in $\mathcal{A}$) by $\hat{1}$. In many situations, the elements of the intersection lattice are labeled by the codimension of the corresponding intersection. For arrangements of hyperplanes, this information is recorded in the rank function of the lattice — the codimension of an intersection X is the number of elements in a maximal chain in the half-open interval $(\hat{0}, X]$ in $\mathcal{L}(\mathcal{A})$.

As with any poset, we can consider the *order complex* $\Delta(\overline{\mathcal{L}})$ of the proper part, $\overline{\mathcal{L}} := \mathcal{L} \setminus \{\hat{0}, \hat{1}\}$, of the intersection lattice, i.e., the abstract simplicial complex formed by the linearly ordered subsets in $\overline{\mathcal{L}}$,

$$\Delta(\overline{\mathcal{L}}) = \left\{ X_1 < \ldots < X_k \mid X_i \in \mathcal{L} \setminus \{\hat{0}, \hat{1}\} \right\}.$$

The topology of $\Delta(\overline{\mathcal{L}})$ plays a prominent role for describing the topology of arrangement complements. For instance, it is the crucial ingredient for the explicit description of cohomology groups of $\mathcal{M}(\mathcal{A})$ by Goresky and MacPherson [1988, Part III].

For hyperplane arrangements, the homotopy type of $\Delta(\overline{\mathcal{L}})$ is well known: the complex is homotopy equivalent to a wedge of spheres of dimension equal to the codimension of the total intersection of $\mathcal{A}$. The number of spheres can as well be read from the intersection lattice, it is the absolute value of its Möbius function. For subspace arrangements however, the barycentric subdivision of any finite simplicial complex can appear as the order complex of the intersection lattice.

Besides $\Delta(\overline{\mathcal{L}})$, we will often refer to the cone over $\Delta(\overline{\mathcal{L}})$ obtained by extending the linearly ordered sets in $\overline{\mathcal{L}}$ by the maximal element $\hat{1}$ in $\mathcal{L}$. We will denote this complex by $\Delta(\mathcal{L} \setminus \{\hat{0}\})$ or $\Delta(\mathcal{L}_{>\hat{0}})$.

In order to have a standard example at hand, we briefly discuss braid arrangements. This class of arrangements has figured prominently in many places and has helped develop lots of arrangement theory over the last decades.

EXAMPLE 2.1 (BRAID ARRANGEMENTS). The arrangement $\mathcal{A}_{n-1}$ given by the hyperplanes

$$H_{ij} : \ x_i = x_j, \quad \text{for } 1 \le i < j \le n,$$

in real n-dimensional vector space is called the (*real*) *rank* $n-1$ *braid arrangement*. There is a complex version of this arrangement. It consists of hyperplanes H_{ij} in $\mathbb{C}^n$ given by the *same* linear equations. We denote the arrangement by $\mathcal{A}_{n-1}^{\mathbb{C}}$. Occasionally, we will use the analogous $\mathcal{A}_{n-1}^{\mathbb{R}}$ if we want to stress the real
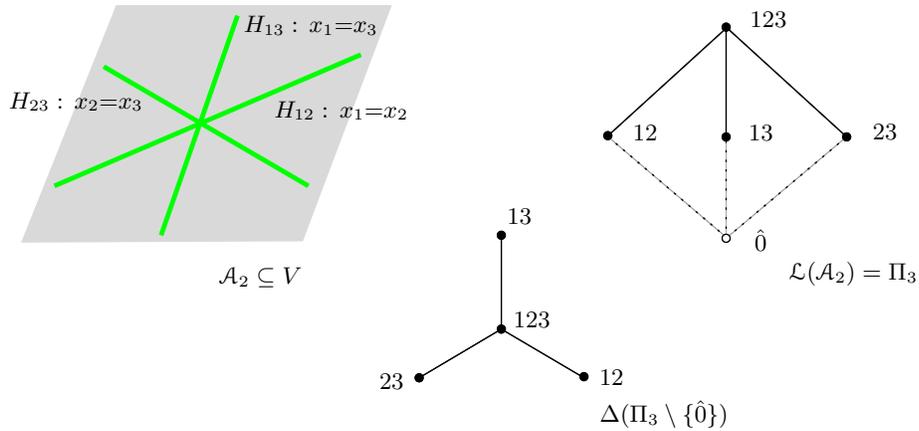
Figure 1. The rank 2 braid arrangement $\mathcal{A}_2$, its intersection lattice Π_3, and the order complex $\Delta(\Pi_3 \setminus \{\hat{0}\})$.

setting. In many situations a similar reasoning applies to the real and to the complex case. To simplify notation, we then use $\mathbb{K}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Observe that the diagonal $\Delta = \{x \in \mathbb{K}^n \mid x_1 = \ldots = x_n\}$ is the overall intersection of hyperplanes in $\mathcal{A}_{n-1}$. Without loosing any relevant information on the topology of the complement, we will often consider $\mathcal{A}_{n-1}$ as an arrangement in complex or real $(n{-}1)$-dimensional space $V = \mathbb{K}^n/\Delta \cong \{x \in \mathbb{K}^n \mid \sum x_i = 0\}$. This explains the indexing for braid arrangements, which may appear unusual at first sight.

The complement $\mathcal{M}(\mathcal{A}_{n-1}^{\mathbb{R}})$ is a collection of $n!$ polyhedral cones, corresponding to the $n!$ linear orders on n pairwise noncoinciding coordinate entries. The complement $\mathcal{M}(\mathcal{A}_{n-1}^{\mathbb{C}})$ is the classical configuration space of the complex plane

$$F(\mathbb{C}, n) = \big\{(x_1, \ldots, x_n) \in \mathbb{C}^n \;\big|\; x_i \neq x_j \text{ for } i \neq j \big\}.$$

This space is the classifying space of the pure braid group, which explains the occurrence of the term "braid" for this class of arrangements.

As the intersection lattice of the braid arrangement $\mathcal{A}_{n-1}$ we recognize the *partition lattice* Π_n, i.e., the set of set partitions of $\{1, \ldots, n\}$ ordered by reversed refinement. The correspondence to intersections in the braid arrangement can be easily described: The blocks of a partition correspond to sets of coordinates with identical entries, thus to the set of points in the corresponding intersection of hyperplanes.

The order complex $\Delta(\overline{\Pi}_n)$ is a pure, $(n{-}1)$-dimensional complex that is homotopy equivalent to a wedge of $(n{-}1)!$ spheres of dimension $n{-}1$.

In Figure 1 we depict the real rank 2 braid arrangement $\mathcal{A}_2$ in $V = \mathbb{R}^3/\Delta$, its intersection lattice Π_3, and the order complex $\Delta(\Pi_3 \setminus \{\hat{0}\})$. We denote partitions in Π_3 by their nontrivial blocks. The complex depicted is a cone over $\Delta(\overline{\Pi}_3)$, a union of three points, which indeed is the wedge of two 0-dimensional spheres.
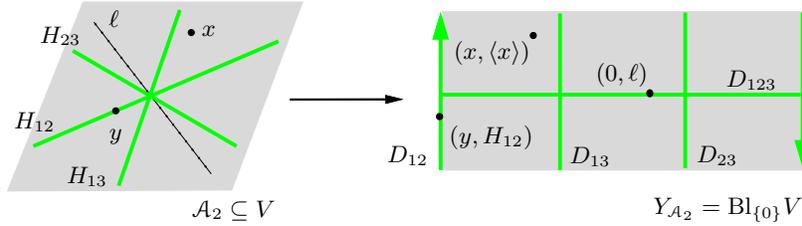
Figure 2. The maximal wonderful model for $\mathcal{A}_2$.

2.2. The model construction. We provide two alternative definitions for De Concini–Procesi arrangement models: the first one describes the models as closures of open embeddings of the arrangement complements. It comes in handy for technical purposes. By contrast, the second definition, which describes arrangement models as results of sequences of blowups, is much more intuitive and suitable for inductive constructions and proofs.

DEFINITION 2.2 (MODEL CONSTRUCTION I). Let $\mathcal{A}$ be an arrangement of real or complex linear subspaces in V. Consider the map

$$\Psi: \quad \mathcal{M}(\mathcal{A}) \to V \times \prod_{X \in \mathcal{L}_{>\hat{0}}} \mathbb{P}(V/X) \tag{2-1}$$
$$x \mapsto \big(x, (\langle x, X \rangle / X)_{X \in \mathcal{L}_{>\hat{0}}} \big);$$

it encodes the relative position of each point in the arrangement complement $\mathcal{M}(\mathcal{A})$ with respect to the intersection of subspaces in $\mathcal{A}$. The map Ψ is an open embedding; the closure of its image is called the *(maximal) De Concini–Procesi wonderful model for* $\mathcal{A}$ and is denoted by $Y_{\mathcal{A}}$.

DEFINITION 2.3 (MODEL CONSTRUCTION II). Let $\mathcal{A}$ be an arrangement of real or complex linear subspaces in V. Let $X_1, \ldots, X_t$ be a linear extension of the opposite order $\mathcal{L}_{>\hat{0}}^{\mathrm{op}}$ on $\mathcal{L}_{>\hat{0}}$. The *(maximal) De Concini–Procesi wonderful model for* $\mathcal{A}$ is the result $Y_{\mathcal{A}}$ of successively blowing up subspaces $X_1, \ldots, X_t$, respectively their proper transforms.

To avoid confusion with spherical blowups that have been appearing in model constructions as well [Gaiffi 2003], let us emphasize here that, also in the real setting, we think about blowups as substituting points by projective spaces. Before we list the main properties of arrangement models let us look at a first example.

EXAMPLE 2.4 (THE ARRANGEMENT MODEL $Y_{\mathcal{A}_2}$). Consider the rank 2 braid arrangement $\mathcal{A}_2$ in $V = \mathbb{R}^3/\Delta$. Following the description in Definition 2.3 we obtain $Y_{\mathcal{A}_2}$ by a single blowup of V at $\{0\}$. The result is an open Möbius band; the exceptional divisor $D_{123} \cong \mathbb{R}\mathbb{P}^1$ in $Y_{\mathcal{A}_2}$ intersects transversally with the proper transforms D_{ij} of the hyperplanes H_{ij}, $1 \le i < j \le 3$. We illustrate the blowup in Figure 2.

To recognize the Möbius band as the closure of the image of Ψ according to Definition 2.2, observe that the product on the right-hand side of (2–1) consists of two relevant factors, $V \times \mathbb{RP}^1$. A point x in $\mathcal{M}(\mathcal{A}_2)$ gets mapped to $(x, \langle x \rangle)$ and we observe a one-to-one correspondence between points in $\mathcal{M}(\mathcal{A}_2)$ and points in $Y_{\mathcal{A}_2} \setminus (D_{123} \cup D_{12} \cup D_{13} \cup D_{23})$. Points added when taking the closure are of the form (y, H_{ij}) for $y \in H_{ij} \setminus \{0\}$ and $(0, \ell)$ for ℓ some line in V.

Observe that the triple intersection of hyperplanes in V has been replaced by double intersections of hypersurfaces in $Y_{\mathcal{A}_2}$. Without changing the topology of the arrangement complement, the arrangement of hyperplanes has been replaced by a normal crossing divisor. Moreover, note that the irreducible divisor components D_{12}, D_{13}, D_{23}, and D_{123} intersect if and only if their indexing lattice elements form a chain in $\mathcal{L}(\mathcal{A}_2)$.

The observations we made for $Y_{\mathcal{A}_2}$ are special cases of the main properties of (maximal) De Concini–Procesi models that we list in the following:

THEOREM 2.5 [De Concini and Procesi 1995a, Theorems in §3.1 and §3.2]. (1) *The arrangement model $Y_{\mathcal{A}}$ as defined in 2.2 and 2.3 is a smooth variety with a natural projection map to the original ambient space, $\pi : Y_{\mathcal{A}} \longrightarrow V$, which is one-to-one on the arrangement complement $\mathcal{M}(\mathcal{A})$.*

(2) *The complement of $\pi^{-1}(\mathcal{M}(\mathcal{A}))$ in $Y_{\mathcal{A}}$ is a divisor with normal crossings; its irreducible components are the proper transforms D_X of intersections X in $\mathcal{L}$,*

$$Y_{\mathcal{A}} \setminus \pi^{-1}(\mathcal{M}(\mathcal{A})) = \bigcup_{X \in \mathcal{L}_{>\hat{0}}} D_X.$$

(3) *Irreducible components D_X for $X \in \mathcal{S} \subseteq \mathcal{L}_{>\hat{0}}$ intersect if and only if $\mathcal{S}$ is a linearly ordered subset in $\mathcal{L}_{>\hat{0}}$. If we think about $Y_{\mathcal{A}}$ as stratified by the irreducible components of the normal crossing divisor and their intersections, then the poset of strata coincides with the face poset of the order complex $\Delta(\mathcal{L}_{>\hat{0}})$.*

EXAMPLE 2.6 (THE ARRANGEMENT MODEL $Y_{\mathcal{A}_3}$). We now consider a somewhat larger and more complicated example, the rank 3 braid arrangement $\mathcal{A}_3$ in $V \cong \mathbb{R}^4/\Delta$. First note that the intersection lattice of $\mathcal{A}_3$ is the partition lattice Π_4, which we depict in Figure 3 for later reference. Again, we denote partitions by their nontrivial blocks.

Following again the description of arrangement models given in Definition 2.3, the first step is to blow up V at $\{0\}$. We obtain a line bundle over $\mathbb{RP}^2$; in Figure 4 we depict the exceptional divisor $D_{1234} \cong \mathbb{RP}^2$ stratified by the intersections of proper transforms of hyperplanes in $\mathcal{A}_3$.

This first step is now followed by the blowup of triple, respectively double intersections of proper transforms of hyperplanes in arbitrary order. In each such intersection the situation locally corresponds to the blowup of a 2-dimensional real vector space in a point as discussed in Example 2.4. Topologically, the arrangement model $Y_{\mathcal{A}_3}$ is a line bundle over a space obtained from a 7-fold punc-
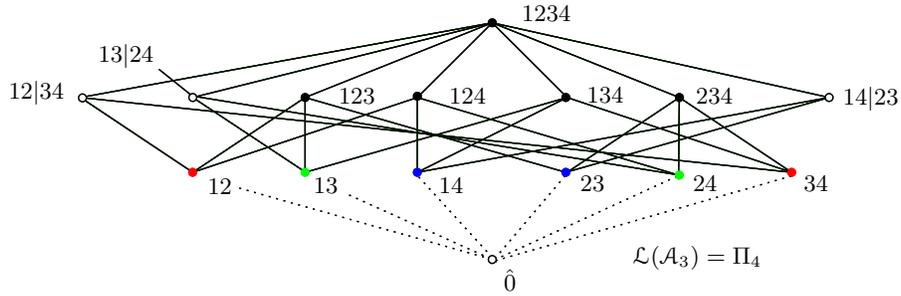
Figure 3. The intersection lattice of $\mathcal{A}_3$.

tured $\mathbb{RP}^2$ by gluing 7 Möbius bands along their boundaries into the boundary components.

We can easily check the statements of Theorem 2.5 for $Y_{\mathcal{A}_3}$. In particular, we see that intersections of irreducible divisors in $Y_{\mathcal{A}_3}$ are nonempty if and only if the corresponding index sets form a chain in $\mathcal{L}_{>\hat{0}}$. For instance, the 0-dimensional stratum of the divisor stratification that is encircled in Figure 4 corresponds to the chain $14 < 134 < 1234$ in $\Pi_4 \setminus \{\hat{0}\}$. For comparison, we depict the order complex of $\Pi_4 \setminus \{\hat{0}\}$ in Figure 5. Recall that the complex is a pure 2-dimensional cone with apex 1234 over $\Delta(\overline{\Pi}_4)$; we only draw its base.

If our only objective was to construct a model for $\mathcal{M}(\mathcal{A}_3)$ with a normal crossing divisor, it would be enough to blow up $\mathrm{Bl}_{\{0\}}V$ in the 4 triple intersections. The result would be a line bundle over a 4-fold punctured $\mathbb{RP}^2$ with 4 Möbius bands glued into boundary components.
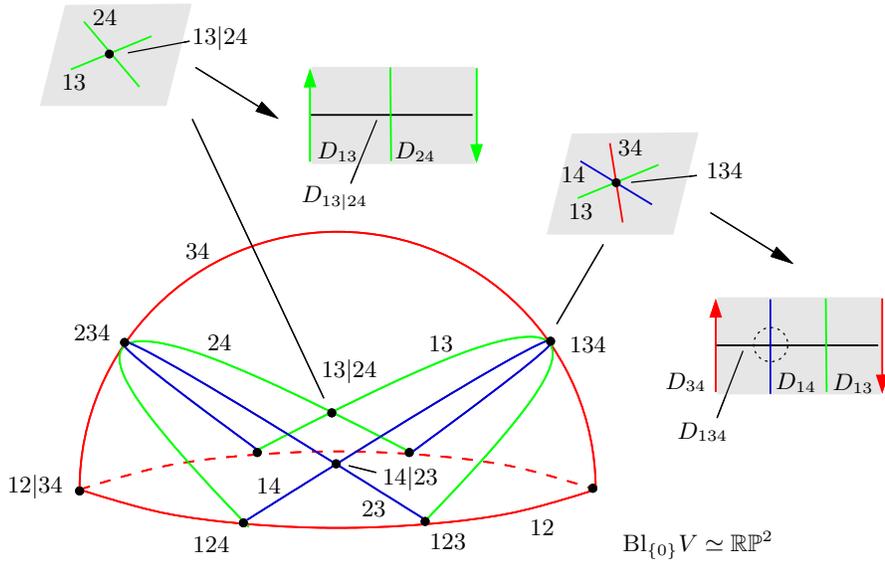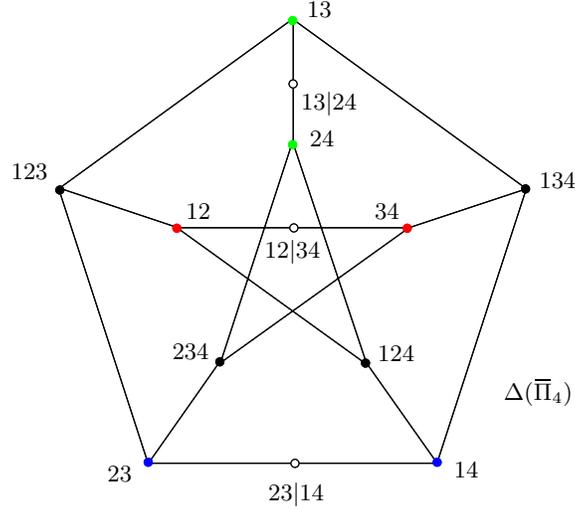

Figure 4. The construction of $Y_{\mathcal{A}_3}$.

Figure 5. The order complex $\Delta(\overline{\Pi}_4)$.

This observation leads to a generalization of the model construction presented so far: it is enough to do successive blowups on a specific *subset* of intersections in $\mathcal{A}$ to obtain a model with similar properties as those summarized in Theorem 2.5. In fact, appropriate subsets of intersections lattices, so-called *building sets*, were specified in [De Concini and Procesi 1995a]; all give rise to wonderful arrangement models in the sense of Theorem 2.5. The only reservation being that the order complex $\Delta(\mathcal{L}_{>\hat{0}})$ is no longer indexing nonempty intersections of irreducible divisors: chains in $\mathcal{L}_{>\hat{0}}$ are replaced by so-called *nested sets* — subsets of building sets that again form an abstract simplicial complex.

We will not give the original definitions of De Concini and Procesi for building sets and nested sets in this survey. Instead, we will present a generalization of these notions for arbitrary meet-semilattices in Section 3.1. This combinatorial abstraction has proved useful in many cases beyond arrangement model constructions. Its relation to the original geometric context will be explained in Section 4.1.

2.3. Some remarks on history. Before we proceed, we briefly sketch the historic background of De Concini–Procesi arrangement models. Moreover, we outline an application to a famous problem in arrangement theory that, among other issues, served as a motivation for the model construction.

Compactifications of configuration spaces due to Fulton and MacPherson [1994] have prepared the scene for wonderful arrangement models. Their work is concerned with classical configurations spaces $F(X, n)$ of smooth algebraic varieties X, i.e., spaces of n-tuples of pairwise distinct points in X:

$$F(X, n) = \{ (x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j \}.$$

A compactification $X[n]$ of $F(X, n)$ is constructed in which the complement of the original configuration space is a normal crossing divisor; in fact, $X[n]$ has properties analogous to those listed for arrangement models in Theorem 2.5. The relation to the arrangement setting can be summarized by saying that, on the one hand, the underlying spaces in the configuration space setting are incomparably more complicated — smooth algebraic varieties X rather than real or complex linear space; the combinatorics, on the other hand, is far simpler — it is the combinatorics of our basic Examples 2.4 and 2.6, the partition lattice Π_n. The notion of building sets and nested sets, which constitutes the defining combinatorics of arrangement models, has its roots in the Fulton–MacPherson construction for configuration spaces, hence is inspired by the combinatorics of Π_n.

Looking along the time line in the other direction, De Concini–Procesi arrangement models have triggered a number of more general constructions with similar spirit: compactifications of conically stratified complex manifolds by MacPherson and Procesi [1998], and model constructions for mixed real subspace and halfspace arrangements and real stratified manifolds by Gaiffi [2003] that use spherical rather than classical blowups.

As a first impact, the De Concini–Procesi model construction has yielded substantial progress on a longstanding open question in arrangement theory [De Concini and Procesi 1995a, Section 5], the question being whether combinatorial data of a complex subspace arrangement determines the cohomology algebra of its complement. For arrangements of hyperplanes, there is a beautiful description of the integral cohomology algebra of the arrangement complement in terms of the intersection lattice — the *Orlik–Solomon algebra* [1980]. Also, a prominent application of Goresky and MacPherson's Stratified Morse Theory states that cohomology of complements of (complex and real) subspace arrangements, as graded groups over $\mathbb{Z}$, are determined by the intersection lattice and its codimension labelling. In fact, there is an explicit description of cohomology groups in terms of homology of intervals in the intersection lattice [Goresky and MacPherson 1988, Part III]. However, whether *multiplicative* structure is determined as well remained an open question 20 years after it had been answered for arrangements of hyperplanes (see [Feichtner and Ziegler 2000; Longueville 2000] for results on particular classes of arrangements).

The De Concini–Procesi construction allows to apply Morgan's theory on rational models for complements of normal crossing divisors [Morgan 1978] to arrangement complements and to conclude that their *rational* cohomology algebras indeed are determined by the combinatorics of the arrangement. A key step in the description of the Morgan model is the presentation of cohomology of divisor components and their intersections in purely combinatorial terms [De Concini and Procesi 1995a, 5.1, 5.2]. For details on this approach to arrangement cohomology, see [De Concini and Procesi 1995a, 5.3].

Unfortunately, the Morgan model is fairly complicated even for small arrangements, and the approach is bound to rational coefficients. The model has been considerably simplified in work of Yuzvinsky [2002; 1999]. In [Yuzvinsky 2002] explicit presentations of cohomology algebras for certain classes of arrangements were given. However, despite an explicit conjecture of an integral model for arrangement cohomology in [Yuzvinsky 2002, Conjecture 6.7], extending the result to integral coefficients remained out of reach. Only years later, the question has been fully settled to the positive in work of Deligne, Goresky and MacPherson [Deligne et al. 2000] with a sheaf-theoretic approach, and parallely by de Longueville and Schultz [2001] using rather elementary topological methods: Integral cohomology algebras of complex arrangement complements are indeed determined by combinatorial data.

3. The Combinatorial Core Data: A Step Beyond Geometry

We will now abandon geometry for a while and in this section fully concentrate on combinatorial and algebraic gadgets that are inspired by De Concini–Procesi arrangement models.

We first present a combinatorial analogue of De Concini–Procesi resolutions on purely order theoretic level following [Feichtner and Kozlov 2004, Sections 2 and 3]. Based on the notion of building sets and nested sets for arbitrary lattices proposed therein, we define a family of commutative graded algebras for any given lattice.

The next Section then will be devoted to relate these objects to geometry — to the original context of De Concini–Procesi arrangement models and, more interestingly so, to different seemingly unrelated contexts in geometry.

3.1. Combinatorial resolutions. We will state purely combinatorial definitions of *building sets* and *nested sets*. Recall that, in the context of model constructions, building sets list the strata that are to be blown up in the construction process, and nested sets describe beforehand the nonempty intersections of irreducible divisor components in the final resolution.

Let $\mathcal{L}$ be a finite meet-semilattice, i.e., a finite poset such that any pair of elements has a unique maximal lower bound. In particular, such a meet-semilattice has a unique minimal element that we denote with $\hat{0}$. We will talk about semilattices for short. As a basic reference on partially ordered sets we refer to [Stanley 1997, Chapter 3].

DEFINITION 3.1 (COMBINATORIAL BUILDING SETS). A subset $\mathcal{G} \subseteq \mathcal{L}_{>\hat{0}}$ in a finite meet-semilattice $\mathcal{L}$ is called a *building set* if for any $X \in \mathcal{L}_{>\hat{0}}$ and $\max \mathcal{G}_{\leq X} = \{G_1, \ldots, G_k\}$ there is an isomorphism of posets

$$\varphi_X : \prod_{j=1}^{k} [\hat{0}, G_j] \xrightarrow{\cong} [\hat{0}, X] \qquad (3\text{--}1)$$

with $\varphi_X(\hat{0}, \ldots, G_j, \ldots, \hat{0}) = G_j$ for $j = 1, \ldots, k$. We call $F_{\mathcal{G}}(X) := \max \mathcal{G}_{\leq X}$ the *set of factors* of X in $\mathcal{G}$.

There are two extreme examples of building sets for any semilattice: we can take the full semilattice $\mathcal{L}_{>\hat{0}}$ as a building set. On the other hand, the set of elements X in $\mathcal{L}_{>\hat{0}}$ which do not allow for a product decomposition of the lower interval $[\hat{0}, X]$ form the unique minimal building set (see Example 3.3 below).

   Intuitively speaking, building sets are formed by elements in the semilattice that are the perspective factors of product decompositions.

   Any choice of a building set $\mathcal{G}$ in $\mathcal{L}$ gives rise to a family of so-called *nested sets*. These are, roughly speaking, subsets of $\mathcal{G}$ whose antichains are sets of factors with respect to the chosen building set. Nested sets form an abstract simplicial complex on the vertex set $\mathcal{G}$. This simplicial complex plays the role of the order complex for arrangement models more general than the maximal models discussed in Section 2.2.

DEFINITION 3.2 (NESTED SETS). Let $\mathcal{L}$ be a finite meet-semilattice and $\mathcal{G}$ a building set in $\mathcal{L}$. A subset $\mathcal{S}$ in $\mathcal{G}$ is called *nested* (or $\mathcal{G}$-*nested* if specification is needed) if, for any set of incomparable elements $X_1, \ldots, X_t$ in $\mathcal{S}$ of cardinality at least two, the join $X_1 \vee \ldots \vee X_t$ exists and does not belong to $\mathcal{G}$. The $\mathcal{G}$-nested sets form an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$, the *nested set complex* with respect to $\mathcal{L}$ and $\mathcal{G}$.

Observe that if we choose the full semilattice as a building set, then a subset is nested if and only if it is linearly ordered in $\mathcal{L}$. Hence, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{L}_{>\hat{0}})$ coincides with the order complex $\Delta(\mathcal{L}_{>\hat{0}})$.

EXAMPLE 3.3 (BUILDING SETS AND NESTED SETS FOR THE PARTITION LAT-TICE). Choosing the maximal building set in the partition lattice Π_n, we obtain the order complex $\Delta((\Pi_n) \setminus \{\hat{0}\})$ as the associated complex of nested sets. Topologically, it is a cone over a wedge of $(n-1)!$ spheres of dimension $n-1$.

   The minimal building set $\mathcal{G}_{\min}$ in Π_n is given by partitions with at most one block of size larger or equal 2, the so-called modular elements in Π_n. We can identify these partitions with subsets of $\{1, \ldots, n\}$ of size larger or equal 2. A collection of such subsets is nested, if and only if none of the pairs of subsets have a nontrivial intersection, i.e., for any pair of subsets they are either disjoint or one is contained in the other. Referring to a naive picture of such containment relation explains the choice of the term *nested*—it appeared first in the work of Fulton and MacPherson [1994] on compactifications of classical configuration spaces. As we noted earlier, the combinatorics they are concerned with is indeed the combinatorics of the partition lattice.

   For the rank 3 partition lattice Π_3, maximal and minimal building sets coincide, $\mathcal{G} = \Pi_3 \setminus \{\hat{0}\}$. The nested set complex $\mathcal{N}(\Pi_3, \mathcal{G})$ is the order complex $\Delta(\Pi_3 \setminus \{\hat{0}\})$ depicted in Figure 1.

For the rank 4 partition lattice Π_4, we have seen the nested set complex for the maximal building set $\mathcal{N}(\Pi_4, \mathcal{G}_{\max})$ in Figure 5. The nested set complex associated with the minimal building set $\mathcal{G}_{\min}$ in Π_4 is depicted in Figure 6. Again, $\mathcal{N}(\Pi_4, \mathcal{G}_{\min})$ is a cone with apex 1234, and we only draw its base, $\mathcal{N}(\overline{\Pi}_4, \mathcal{G}_{\min})$.
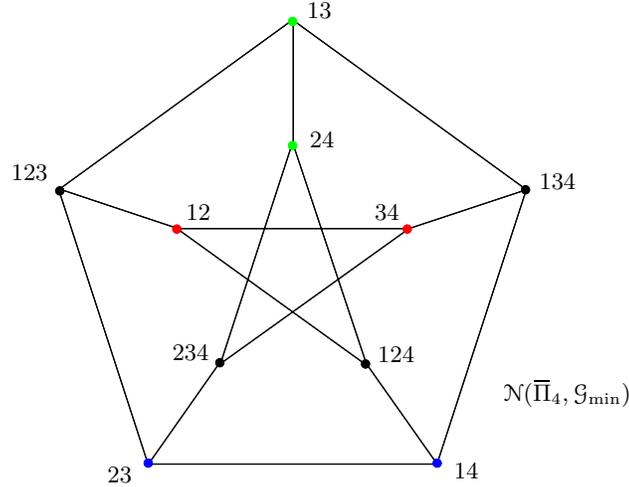
Figure 6. The nested set complex $\mathcal{N}(\overline{\Pi}_4, \mathcal{G}_{\min})$.

Adding one or two 2-block partitions to $\mathcal{G}_{\min}$ yields all the other building sets for Π_4. The corresponding nested set complexes are subdivisions of $\mathcal{N}(\Pi_4, \mathcal{G}_{\min})$.

When studying the (maximal) wonderful model $Y_{\mathcal{A}_3}$ in Example 2.6 we had observed that, if we only wanted to achieve a model with normal crossing divisors, it would have been enough to blow up the overall and the triple intersections. This selection of strata, respectively elements in $\mathcal{L}(\mathcal{A}_3) = \Pi_4$, exactly corresponds to the minimal building set $\mathcal{G}_{\min}$ in Π_4 — a geometric motivation for Definition 3.1.

We can also get a glimpse on the geometry that motivates the definition of nested sets: comparing simplices in $\mathcal{N}(\Pi_4, \mathcal{G}_{\min})$ with intersections of irreducible divisor components in the arrangement model resulting from blowups along subspaces in $\mathcal{G}_{\min}$, we see that there is a 1-1 correspondence. For instance, $\{12, 34\}$ is a nested set with respect to $\mathcal{G}_{\min}$, and divisor components D_{12} and D_{34} intersect in the model (compare Figure 4).

It is not a coincidence that, in the example above, one nested set complex is a subdivision of the other if one building set contains the other. In fact, the following holds:

THEOREM 3.4 [Feichtner and Müller 2005, Proposition 3.3, Theorem 4.2]. *For any finite meet-semilattice $\mathcal{L}$, and $\mathcal{G}$ a building set in $\mathcal{L}$, the nested set complex*

$\mathcal{N}(\mathcal{L}, \mathcal{G})$ *is homotopy equivalent to the order complex of* $\mathcal{L}_{>\hat{0}}$,

$$\mathcal{N}(\mathcal{L}, \mathcal{G}) \;\simeq\; \Delta(\mathcal{L}_{>\hat{0}}).$$

Moreover, if $\mathcal{L}$ *is atomic, i.e., any element is a join of a set of atoms, and* $\mathcal{G}$ *and* $\mathcal{H}$ *are building sets with* $\mathcal{G} \supseteq \mathcal{H}$, *then the nested set complex* $\mathcal{N}(\mathcal{L}, \mathcal{G})$ *is obtained from* $\mathcal{N}(\mathcal{L}, \mathcal{H})$ *by a sequence of stellar subdivisions. In particular, the complexes are homeomorphic.*

We now propose a construction on semilattices producing new semilattices: the *combinatorial blowup* of a semilattice in an element.

DEFINITION 3.5 (COMBINATORIAL BLOWUP). For a semilattice $\mathcal{L}$ and an element X in $\mathcal{L}_{>\hat{0}}$ we define a poset $(\mathrm{Bl}_X \mathcal{L}, \prec)$ on the set of elements

$$\mathrm{Bl}_X \mathcal{L} = \{\, Y \mid Y \in \mathcal{L}, Y \not\geq X \,\} \cup \{\, Y' \mid Y \in \mathcal{L}, Y \not\geq X, \text{ and } Y \vee X \text{ exists in } \mathcal{L} \,\}.$$

The order relation $<$ in $\mathcal{L}$ determines the order relation $\prec$ within the two parts of $\mathrm{Bl}_X \mathcal{L}$ described above,

$$Y \prec Z, \;\; \text{for} Y < Z \;\text{ in } \mathcal{L},$$
$$Y' \prec Z', \text{for} Y < Z \;\text{ in } \mathcal{L},$$

and additional order relations between elements of these two parts are defined by

$$Y \prec Z', \text{for} Y < Z \;\text{ in } \mathcal{L},$$

where in all three cases it is assumed that $Y, Z \not\geq X$ in $\mathcal{L}$. We call $\mathrm{Bl}_X \mathcal{L}$ the *combinatorial blowup* of $\mathcal{L}$ in X.

In fact, the poset $\mathrm{Bl}_X \mathcal{L}$ is again a semilattice. Figure 7 explains what is going on.

The construction does the following: it removes the closed upper interval on top of X from $\mathcal{L}$, and then marks the set of elements in $\mathcal{L}$ that are not larger or equal X, but have a join with X in $\mathcal{L}$. This subset of $\mathcal{L}$ (in fact, a lower ideal in the sense of order theory) is doubled and any new element Y' in the copy is
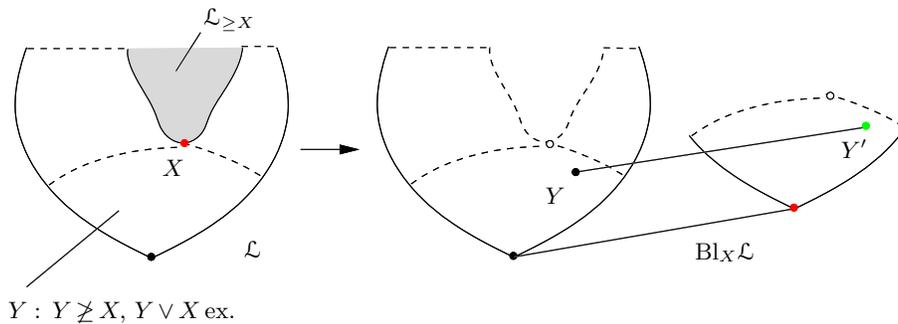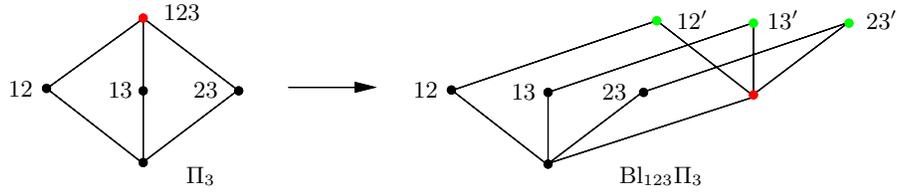

Figure 7. A combinatorial blowup.

Figure 8. The combinatorial blowup of Π_3 in 123.

defined to be covering the original element Y in $\mathcal{L}$. The order relations in the remaining, respectively the doubled, part of $\mathcal{L}$ stay the same as before.

In Figure 8 we give a concrete example: the combinatorial blowup of the maximal element 123 in Π_3, $\mathrm{Bl}_{123}\Pi_3$. The result should be compared with Figure 2. In fact, $\mathrm{Bl}_{123}\Pi_3$ is the face poset of the divisor stratification in $Y_{\mathcal{A}_2} = \mathrm{Bl}_{\{0\}}V$.

The following theorem shows that the three concepts introduced above — combinatorial building sets, nested sets, and combinatorial blowups — fit together so as to provide a combinatorial analogue of the De Concini–Procesi model construction.

THEOREM 3.6 [Feichtner and Kozlov 2004, Theorem 3.4]. *Let $\mathcal{L}$ be a semilattice, $\mathcal{G}$ a combinatorial building set in $\mathcal{L}$, and $G_1, \ldots, G_t$ a linear order on $\mathcal{G}$ that is nonincreasing with respect to the partial order on $\mathcal{L}$. Then, consecutive combinatorial blowups in $G_1, \ldots, G_t$ result in the face poset of the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$:*

$$\mathrm{Bl}_{G_t}(\ldots(\mathrm{Bl}_{G_2}(\mathrm{Bl}_{G_1}\mathcal{L}))\ldots) = \mathcal{F}(\mathcal{N}(\mathcal{L}, \mathcal{G})).$$

3.2. An algebra defined for atomic lattices. For any atomic lattice, we define a family of graded commutative algebras based on the notions of building sets and nested sets given above. Our exposition here and in Section 4.2 follows [Feichtner and Yuzvinsky 2004]. Restricting our attention to atomic lattices is not essential for the definition. However, for various algebraic considerations and for geometric interpretations (compare Section 4.2) it is convenient to assume that the lattice is atomic.

DEFINITION 3.7. Let $\mathcal{L}$ be a finite atomic lattice, $\mathfrak{A}(\mathcal{L})$ its set of atoms, and $\mathcal{G}$ a building set in $\mathcal{L}$. We define the algebra $D(\mathcal{L}, \mathcal{G})$ of $\mathcal{L}$ with respect to $\mathcal{G}$ as

$$D(\mathcal{L}, \mathcal{G}) := \mathbb{Z}\left[\{x_G\}_{G \in \mathcal{G}}\right] \Big/ \mathcal{I},$$

where the ideal of relations $\mathcal{I}$ is generated by

$$\prod_{i=1}^{t} x_{G_i} \qquad \text{for } \{G_1, \ldots, G_t\} \notin \mathcal{N}(\mathcal{L}, \mathcal{G}),$$

$$\sum_{G \geq H} x_G \qquad \text{for } H \in \mathfrak{A}(\mathcal{L}).$$

Observe that this algebra is a quotient of the face ring of the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$.

EXAMPLE 3.8 (ALGEBRAS ASSOCIATED TO Π_3 AND Π_4). For Π_3 and its only building set $\mathcal{G}_{\max} = \Pi_3 \setminus \{\hat{0}\}$, the algebra reads as follows:

$$D(\Pi_3, \mathcal{G}_{\max}) = \mathbb{Z}\left[x_{12}, x_{13}, x_{23}, x_{123}\right] \Big/ \left\langle \begin{matrix} x_{12}x_{13}, \ x_{12}x_{23}, \ x_{13}x_{23} \\ x_{12} + x_{123}, \ x_{13} + x_{123}, \ x_{23} + x_{123} \end{matrix} \right\rangle$$

$$\cong \mathbb{Z}\left[x_{123}\right] / \langle x_{123}^2 \rangle.$$

For Π_4 and its minimal building set $\mathcal{G}_{\min}$, we obtain the following algebra after simplifying slightly the presentation:

$$D(\Pi_4, \mathcal{G}_{\min}) \cong \mathbb{Z}\left[x_{123}, x_{124}, x_{134}, x_{234}, x_{1234}\right] \Big/$$

$$\left\langle \begin{matrix} x_{ijk}\, x_{1234} & \text{for all } 1 \leq i < j < k \leq 4 \\ x_{ijk}\, x_{i'j'k'} & \text{for all } ijk \neq i'j'k' \\ x_{ijk}^2 + x_{1234}^2 & \text{for all } 1 \leq i < j < k \leq 4 \end{matrix} \right\rangle.$$

There is an explicit description for a Gröbner basis of the ideal $\mathcal{I}$, which in particular yields an explicit description for a monomial basis of the graded algebra $D(\mathcal{L}, \mathcal{G})$.

THEOREM 3.9. (1) [Feichtner and Yuzvinsky 2004, Theorem 2] *The following polynomials form a Gröbner basis of the ideal $\mathcal{I}$:*

$$\prod_{G \in \mathcal{S}} x_G \qquad \text{for } \ \mathcal{S} \notin \mathcal{N}(\mathcal{L}, \mathcal{G}),$$

$$\prod_{i=1}^{k} x_{A_i} \left(\sum_{G \geq B} x_G \right)^{d(A,B)},$$

where $A_1, \ldots, A_k$ are maximal elements in a nested set $\mathcal{H} \in \mathcal{N}(\mathcal{L}, \mathcal{G})$, $B \in \mathcal{G}$ with $B > A = \bigvee_{i=1}^{k} A_i$, and $d(A, B)$ is the minimal number of atoms needed to generate B from A by taking joins.

(2) [Feichtner and Yuzvinsky 2004, Corollary 1] *The resulting linear basis for the graded algebra $D(\mathcal{L}, \mathcal{G})$ is given by the following set of monomials*:

$$\prod_{A \in \mathcal{S}} x_A^{m(A)},$$

where $\mathcal{S}$ is running over all nested subsets of $\mathcal{G}$, $m(A) < d(A', A)$, and A' is the join of $\mathcal{S} \cap \mathcal{L}_{<A}$.

Part (2) of Theorem 3.9 generalizes a basis description by Yuzvinsky [1997] for $D(\mathcal{L}, \mathcal{G})$ in the case of $\mathcal{G}$ being the minimal building set in an intersection lattice $\mathcal{L}$ of a complex hyperplane arrangement. Yuzvinsky's basis description has also been generalized in a somewhat different direction by Gaiffi [1997], namely for closely related algebras associated with complex subspace arrangements.

We will return to the algebra $D(\mathcal{L}, \mathcal{G})$ and discuss its geometric significance in Section 4.2.

4. Returning to Geometry

4.1. Understanding stratifications in wonderful models. We first relate the combinatorial setting of building sets and nested sets developed in Section 3.1 to its origin, the De Concini–Procesi model construction. Here is how to recover the original notion of building sets [De Concini and Procesi 1995a, Definition in §2.3], we call them *geometric building sets*, from our definitions:

DEFINITION 4.1 (GEOMETRIC BUILDING SETS). Let $\mathcal{L}$ be the intersection lattice of an arrangement of subspaces in real or complex vector space V and cd : $\mathcal{L} \to \mathbb{N}$ a function on $\mathcal{L}$ assigning the codimension of the corresponding subspace to each lattice element. A subset $\mathcal{G}$ in $\mathcal{L}$ is a *geometric building set* if it is a building set in the sense of 3.1, and for any $X \in \mathcal{L}$ the codimension of X is equal to the sum of codimensions of its factors, $F_\mathcal{G}(X)$:

$$\mathrm{cd}\,(X) = \sum_{Y \in F_\mathcal{G}(X)} \mathrm{cd}\,(Y).$$

An easy example shows that the notion of geometric building sets indeed is more restrictive than the notion of combinatorial building sets. For arrangements of hyperplanes, however, the notions coincide [Feichtner and Kozlov 2004, Proposition 4.5.(2)].

EXAMPLE 4.2 (GEOMETRIC VERSUS COMBINATORIAL BUILDING SETS). Let $\mathcal{A}$ denote the following arrangement of 3 subspaces in $\mathbb{R}^4$:

$$A_1 : \ x_4 = 0, \quad A_2 : \ x_1 = x_2 = 0, \quad A_3 : \ x_1 = x_3 = 0.$$

The intersection lattice $\mathcal{L}(\mathcal{A})$ is a boolean algebra on 3 elements; we depict the lattice with its codimension labelling in Figure 9. The set of atoms obviously is a combinatorial building set. However, any geometric building set must contain the intersection $A_2 \cap A_3$: its codimension is *not* the sum of codimensions of its (combinatorial) factors A_2 and A_3.
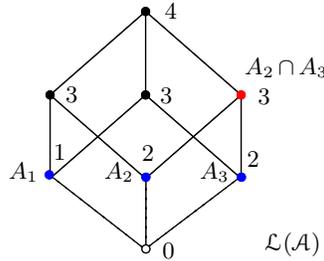
Figure 9. Geometric versus combinatorial building sets.

As we mentioned before, there are wonderful model constructions for arrangement complements $\mathcal{M}(\mathcal{A})$ that start from an arbitrary geometric building set $\mathcal{G}$ of the intersection lattice $\mathcal{L}(\mathcal{A})$ [De Concini and Procesi 1995a, 3.1]: In Definition 2.2, replace the product on the right hand side of (2–1) by a product over building set elements in $\mathcal{L}$, and obtain the wonderful model $Y_{\mathcal{A},\mathcal{G}}$ by again taking the closure of the image of $\mathcal{M}(\mathcal{A})$ under Ψ. In Definition 2.3, replace the linear extension of $\mathcal{L}_{>\hat{0}}^{\mathrm{op}}$ by a nonincreasing linear order on the elements in $\mathcal{G}$, and obtain the wonderful model $Y_{\mathcal{A},\mathcal{G}}$ by successive blowups of subspaces in $\mathcal{G}$, and of proper transforms of such.

The key properties of these models are analogous to those listed in Theorem 2.5, where in part (2), lattice elements are replaced by building set elements, and in part (3), chains in $\mathcal{L}$ as indexing sets of nonempty intersections of irreducible components of divisors are replaced by nested sets. Hence, the face poset of the stratification of $Y_{\mathcal{A},\mathcal{G}}$ given by irreducible components of divisors and their intersections coincides with the face poset of the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$. Compare Examples 2.6 and 3.3, where we found that nested sets with respect to the minimal building set $\mathcal{G}_{\mathrm{min}}$ in Π_4 index nonempty intersections of irreducible divisor components in the arrangement model $Y_{\mathcal{A}_3,\mathcal{G}_{\mathrm{min}}}$.

While the intersection lattice $\mathcal{L}(\mathcal{A})$ captures the combinatorics of the stratification of V given by subspaces of $\mathcal{A}$ and their intersections, the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$ captures the combinatorics of the divisor stratification of the wonderful model $Y_{\mathcal{A},\mathcal{G}}$. More than that: combinatorial blowups turn out to be the right concept to describe the incidence change of strata during the construction of wonderful arrangement models by successive blowups:

THEOREM 4.3 [Feichtner and Kozlov 2004, Proposition 4.7(1)]. *Let $\mathcal{A}$ be a complex subspace arrangement, $\mathcal{G}$ a geometric building set in $\mathcal{L}(\mathcal{A})$, and $G_1, \ldots, G_t$ a nonincreasing linear order on $\mathcal{G}$. Let $\mathrm{Bl}_i(\mathcal{A})$ denote the result of blowing up strata $G_1, \ldots, G_i$, for $i \le t$, and denote by $\mathcal{L}_i$ the face poset of the stratification of $\mathrm{Bl}_i(\mathcal{A})$ by proper transforms of subspaces in $\mathcal{A}$ and the exceptional divisors. Then the poset $\mathcal{L}_i$ coincides with the successive combinatorial blowups of $\mathcal{L}$ in $G_1, \ldots G_i$:*

$$\mathcal{L}_i \;=\; \mathrm{Bl}_{G_i}(\ldots(\mathrm{Bl}_{G_2}(\mathrm{Bl}_{G_1}\mathcal{L}))\ldots).$$

Combinatorial building sets, nested sets and combinatorial blowups occur in other situations and prove to be the right concept for describing stratifications in more general model constructions. This applies to the *wonderful conical compactifications* of MacPherson and Procesi [1998] as well as to models for mixed subspace and halfspace arrangements and for stratified real manifolds by Gaiffi [2003].

Also, combinatorial blowups describe the effect which stellar subdivisions in polyhedral fans have on the face poset of the fans. In fact, combinatorial blowups describe the incidence change of torus orbits for resolutions of toric varieties by

consecutive blowups in closed torus orbits. This implies, in particular, that for any toric variety and for any choice of a combinatorial building set in the face poset of its defining fan, we obtain a resolution of the variety with torus orbit structure prescribed by the nested set complex associated to the chosen building set. We believe that such combinatorially prescribed resolutions can prove useful in various concrete situations (see [Feichtner and Kozlov 2004, Section 4.2] for further details).

There is one more issue about nested set stratifications of maximal wonderful arrangement models that we want to discuss here, mostly in perspective of applications in Section 5. According to Definition 2.2, any point in the model Y_A can be written as a collection of a point in V and lines in V, one line for each element in $\mathcal{L}(\mathcal{A})$. There is a lot of redundant information in this rendering, e.g., points on the open stratum $\pi^{-1}(\mathcal{M}(\mathcal{A}))$ are fully determined by their first "coordinate entry", the point in $\mathcal{M}(\mathcal{A}) \subseteq V$.

Here is a more economic encoding of a point ω on Y_A [Feichtner and Kozlov 2003, Section 4.1]: we find that ω can be uniquely written as

$$\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t) = (x, \ell_1, \ell_2, \ldots, \ell_t), \qquad (4\text{--}1)$$

where x is a point in V, the $H_1, \ldots, H_t$ form a descending chain of subspaces in $\mathcal{L}_{>\hat{0}}$, and the ℓ_i are lines in V. More specifically, $x = \pi(\omega)$, and H_1 is the maximal lattice element that, as a subspace of V, contains x. The line ℓ_1 is orthogonal to H_1 and corresponds to the coordinate entry of ω indexed by H_1 in $\mathbb{P}(V/H_1)$. The lattice element H_2, in turn, is the maximal lattice element that contains both H_1 and ℓ_1. The specification of lines ℓ_i, i.e., lines that correspond to coordinates of ω in $\mathbb{P}(V/H_i)$, and the construction of lattice elements H_{i+1}, continues analogously for $i \geq 2$ until a last line ℓ_t is reached whose span with H_t is not contained in any lattice element other than the full ambient space V. Observe that the H_i are determined by x and the sequence of lines ℓ_i; we choose to include the H_i in order to keep the notation more transparent.

The full coordinate information on ω can be recovered from (4–1) by setting $H_0 = \bigcap \mathcal{A}$, $\ell_0 = \langle x \rangle$, and retrieving the coordinate ω_H indexed by $H \in \mathcal{L}_{>\hat{0}}$ as

$$\omega_H = \langle \ell_j, H \rangle / H \in \mathbb{P}(V/H),$$

where j is chosen from $\{1, \ldots, t\}$ such that $H \leq H_j$, but $H \nleq H_{j+1}$.

A nice feature of this encoding is that for a given point ω in Y_A we can tell the open stratum in the nested set stratification which contains it:

PROPOSITION 4.4 [Feichtner and Kozlov 2003, Proposition 4.5]. *A point ω in a maximal arrangement model Y_A is contained in the open stratum indexed by the chain $H_1 > H_2 > \ldots > H_t$ in $\mathcal{L}_{>\hat{0}}$ if and only if its point/line description* (4–1) *reads $\omega = (x, H_1, \ell_1, H_2, \ell_2, \ldots, H_t, \ell_t)$.*

4.2. A wealth of geometric meaning for $D(\mathcal{L}, \mathcal{G})$. We turn to the algebra $D(\mathcal{L}, \mathcal{G})$ that we defined for any atomic lattice $\mathcal{L}$ and combinatorial building set $\mathcal{G}$ in $\mathcal{L}$ in Section 3.2. We give two geometric interpretations for this algebra; one is restricted to $\mathcal{L}$ being the intersection lattice of a complex hyperplane arrangement and originally motivated the definition of $D(\mathcal{L}, \mathcal{G})$, the other applies to any atomic lattice and provides for a somewhat unexpected connection to toric varieties.

We comment briefly on the projective version of wonderful arrangement models that we need in this context (see [De Concini and Procesi 1995a, §4] for details). For any arrangement of linear subspaces $\mathcal{A}$ in V, a model for its projectivization $\mathbb{P}\mathcal{A} = \{\mathbb{P}A \mid A \in \mathcal{A}\}$ in $\mathbb{P}V$, i.e., for $\mathcal{M}(\mathbb{P}\mathcal{A}) = \mathbb{P}V \setminus \bigcup \mathbb{P}\mathcal{A}$, can be obtained by replacing the ambient space V by its projectivization $\mathbb{P}V$ in the model constructions 2.2 and 2.3. The constructions result in a smooth projective variety that we denote by $Y_{\mathcal{A}}^{\mathbb{P}}$. A model $Y_{\mathcal{A}, \mathcal{G}}^{\mathbb{P}}$ for a specific geometric building set $\mathcal{G}$ in $\mathcal{L}$ can be obtained analogously. In fact, under the assumption that $\mathbb{P}(\bigcap \mathcal{A})$ is contained in the building set $\mathcal{G}$, the affine model $Y_{\mathcal{A}, \mathcal{G}}$ is the total space of a (real or complex) line bundle over the projective model $Y_{\mathcal{A}, \mathcal{G}}^{\mathbb{P}}$ which is isomorphic to the divisor component in $Y_{\mathcal{A}, \mathcal{G}}$ indexed with $\bigcap \mathcal{A}$.

The most prominent example of a projective arrangement model is the minimal wonderful model for the complex braid arrangement, $Y_{\mathcal{A}_{n-2}^{\mathbb{C}}, \mathcal{G}_{\min}}$. It is isomorphic to the Deligne–Knudson–Mumford compactification $\overline{M}_{0,n}$ of the moduli space of n-punctured complex projective lines [De Concini and Procesi 1995a, 4.3].

Here is the first geometric interpretation of $D(\mathcal{L}, \mathcal{G})$ in the case of $\mathcal{L}$ being the intersection lattice of a complex hyperplane arrangement.

THEOREM 4.5 [De Concini and Procesi 1995b; Feichtner and Yuzvinsky 2004]. *Let $\mathcal{L} = \mathcal{L}(\mathcal{A})$ be the intersection lattice of an essential arrangement of complex hyperplanes $\mathcal{A}$ and $\mathcal{G}$ a building set in $\mathcal{L}$ which contains the total intersection of $\mathcal{A}$. Then, $D(\mathcal{L}, \mathcal{G})$ is isomorphic to the integral cohomology algebra of the projective arrangement model $Y_{\mathcal{A}, \mathcal{G}}^{\mathbb{P}}$:*

$$D(\mathcal{L}, \mathcal{G}) \;\cong\; H^*(Y_{\mathcal{A}, \mathcal{G}}^{\mathbb{P}}, \mathbb{Z}).$$

EXAMPLE 4.6 (COHOMOLOGY OF BRAID ARRANGEMENT MODELS). The projective arrangement model $Y_{\mathcal{A}_2}^{\mathbb{P}}$ is homeomorphic to the exceptional divisor in $Y_{\mathcal{A}_2} = \mathrm{Bl}_{\{0\}}\mathbb{C}^2$, hence to $\mathbb{CP}^1$. Its cohomology is free of rank 1 in degrees 0 and 2 and zero otherwise. Compare with $D(\Pi_3, \mathcal{G}_{\max})$ in Example 3.8.

The projective arrangement model $Y_{\mathcal{A}_3, \mathcal{G}_{\min}}^{\mathbb{P}}$ is homeomorphic to $\overline{M}_{0,5}$, whose cohomology is known to be free of rank 1 in degrees 0 and 4, free of rank 5 in degree 2, and zero otherwise. At least the coincidence of ranks is easy to verify in comparison with $D(\Pi_4, \mathcal{G}_{\min})$ in Example 3.8.

Theorem 4.5 in fact gives an elegant presentation for the integral cohomology of $\overline{M}_{0,n} \cong Y^{\mathbb{P}}_{\mathcal{A}_{n-2}, \mathcal{G}_{\min}}$ in terms of generators and relations:

$$H^*(\overline{M}_{0,n}) \cong D(\Pi_{n-1}, \mathcal{G}_{\min})$$

$$\cong \mathbb{Z}\left[\{x_S\}_{S \subseteq [n-1], |S| \geq 2}\right] \Big/ \left\langle \begin{array}{l} x_S\, x_T \quad \text{for } S \cap T \neq \varnothing,\, S \not\subseteq T,\, T \not\subseteq S, \\ \displaystyle\sum_{\{i,j\} \subseteq S} x_S \text{ for } 1 \leq i < j \leq n-1 \end{array} \right\rangle.$$

A lot of effort has been spent on describing the cohomology of $\overline{M}_{0,n}$ (see [Keel 1992]), but none of the presentations comes close to the simplicity of the one stated above.

A nice expression for the Hilbert function of $H^*(\overline{M}_{0,n})$ has been derived in [Yuzvinsky 1997] as a consequence of a monomial linear basis for minimal projective arrangement models presented there.

To propose a more general geometric interpretation for $D(\mathcal{L}, \mathcal{G})$, we start by describing a polyhedral fan $\Sigma(\mathcal{L}, \mathcal{G})$ for any atomic lattice $\mathcal{L}$ and any combinatorial building set $\mathcal{G}$ in $\mathcal{L}$.

DEFINITION 4.7 (A SIMPLICIAL FAN REALIZING $\mathcal{N}(\mathcal{L}, \mathcal{G})$). Let $\mathcal{L}$ be an atomic lattice with set of atoms $\mathfrak{A} = \{A_1, \ldots, A_n\}$, $\mathcal{G}$ a combinatorial building set in $\mathcal{L}$. For any $G \in \mathcal{G}$ define the characteristic vector v_G in $\mathbb{R}^n$ by

$$(v_G)_i := \begin{cases} 1 \text{ if } G \geq A_i, \\ 0 \text{ otherwise,} \end{cases} \quad \text{for } i = 1, \ldots, n.$$

The simplicial fan $\Sigma(\mathcal{L}, \mathcal{G})$ in $\mathbb{R}^n$ is the collection of cones

$$V_{\mathcal{S}} := \text{cone}\{v_G \mid G \in \mathcal{S}\}$$

for $\mathcal{S}$ nested in $\mathcal{G}$.

By construction, $\Sigma(\mathcal{L}, \mathcal{G})$ is a rational, simplicial fan that realizes the nested set complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$. The fan gives rise to a (noncompact) smooth toric variety $X_{\Sigma(\mathcal{L}, \mathcal{G})}$ [Feichtner and Yuzvinsky 2004, Proposition 2].

EXAMPLE 4.8 (THE FAN $\Sigma(\Pi_3, \mathcal{G}_{\max})$ AND ITS TORIC VARIETY). We depict $\Sigma(\Pi_3, \mathcal{G}_{\max})$ in Figure 10. The associated toric variety is the blowup of $\mathbb{C}^3$ in $\{0\}$ with the proper transforms of coordinate axes removed.
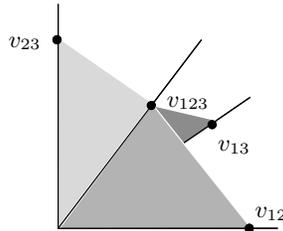

Figure 10. The simplicial fan $\Sigma(\Pi_3, \mathcal{G}_{\max})$.

The algebra $D(\mathcal{L}, \mathcal{G})$ here gains another geometric meaning, this time for *any* atomic lattice $\mathcal{L}$. The abstract algebraic detour of considering $D(\mathcal{L}, \mathcal{G})$ in this general setting is rewarded by a somewhat unexpected return to geometry:

THEOREM 4.9 [Feichtner and Yuzvinsky 2004, Theorem 3]. *For an atomic lattice* $\mathcal{L}$ *and a combinatorial building set* $\mathcal{G}$ *in* $\mathcal{L}$, $\mathcal{D}(\mathcal{L}, \mathcal{G})$ *is isomorphic to the Chow ring of the toric variety* $X_{\Sigma(\mathcal{L}, \mathcal{G})}$,

$$D(\mathcal{L}, \mathcal{G}) \;\cong\; \mathrm{Ch}^*(X_{\Sigma(\mathcal{L}, \mathcal{G})}).$$

5. Adding Arrangement Models to the Geometer's Toolbox

Let a differentiable action of a finite group Γ on a smooth manifold M be given. The goal is to modify the manifold by blowups so as to have the group act on the resolution $\widetilde{M}$ with abelian stabilizers—the quotient $\widetilde{M}/\Gamma$ then has much more manageable singularities than the original quotient. Such modifications for the sake of simplifying quotients have been of crucial importance at various places. One instance is Batyrev's work [1999] on stringy Euler numbers, which in particular implies a conjecture of Reid [1992], and constitutes substantial progress towards higher dimensional MacKay correspondence.

There are two observations that point to wonderful arrangement models as a possible tool in this context. First, the model construction is equivariant if the initial setting carries a group action: if a finite group Γ acts on a real or complex vector space V, and the arrangement $\mathcal{A}$ is Γ-invariant, then the arrangement model $Y_{\mathcal{A}, \mathcal{G}}$ carries a natural Γ-action for any Γ-invariant building set $\mathcal{G} \subseteq \mathcal{L}(\mathcal{A})$. Second, the model construction is not bound to arrangements. In fact, locally finite stratifications of manifolds which are local subspace arrangements, i.e., locally diffeomorphic to arrangements of linear subspaces, can be treated in a fully analogous way. In the complex case, the construction has been pushed to so-called *conical stratifications* in [MacPherson and Procesi 1998] with a real analogue in [Gaiffi 2003].

The significance of De Concini–Procesi model constructions for abelianizing group actions on complex varieties has been recognized by Borisov and Gunnells [2002], following work of Batyrev [1999; 2000]. Here we focus on the real setting.

5.1. Learning from examples: permutation actions in low dimension. Consider the action of the symmetric group $\mathfrak{S}_n$ on real n-dimensional space by permuting coordinates:

$$\sigma\,(x_1, \ldots, x_n) \;=\; (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \qquad \text{for } \sigma \in \mathfrak{S}_n, \; x \in \mathbb{R}^n.$$

Needless to say, we find a wealth of nonabelian stabilizers: For a point $x \in \mathbb{R}^n$ that induces the set partition $\pi = (B_1 | \ldots | B_t)$ of $\{1, \ldots, n\}$ by pairwise coinciding coordinate entries, the stabilizer of x with respect to the permutation action is the Young subgroup $\mathfrak{S}_\pi = \mathfrak{S}_{B_1} \times \ldots \times \mathfrak{S}_{B_t}$ of $\mathfrak{S}_n$, where $\mathfrak{S}_{B_i}$ denotes the symmetric subgroup of $\mathfrak{S}_n$ permuting the coordinates in B_i for $i = 1, \ldots, t$.

The locus of nontrivial stabilizers for the permutation action of $\mathfrak{S}_n$, in fact, is a familiar object: it is the rank $(n-1)$ braid arrangement $\mathcal{A}_{n-1}$. A natural idea that occurs when trying to abelianize a group action by blowups is to resolve the locus of *nonabelian stabilizers* in a systematic way. We look at some low-dimensional examples.

EXAMPLE 5.1 (THE PERMUTATION ACTION OF $\mathfrak{S}_3$). We consider $\mathfrak{S}_3$ acting on real 2-space $V \cong \mathbb{R}^3/\Delta$. The locus of nontrivial stabilizers consists of the 3 hyperplanes in $\mathcal{A}_2$: for $x \in H_{ij} \setminus \{0\}$, $\operatorname{stab} x = \langle (ij) \rangle \cong \mathbb{Z}_2$; in fact, 0 is the only point having a nonabelian stabilizer, namely it is fixed by all of $\mathfrak{S}_3$.

Blowing up $\{0\}$ in V according to the general idea outlined above, we recognize the maximal wonderful model for $\mathcal{A}_2$ that we discussed in Example 2.4.
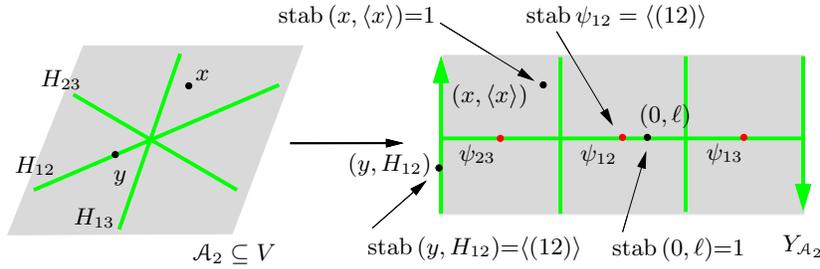

Figure 11. $\mathfrak{S}_3$ acting on $Y_{\mathcal{A}_2}$.

By construction, $\mathfrak{S}_3$ acts coordinate-wise on $Y_{\mathcal{A}_2}$. For points on proper transforms of hyperplanes $(y, H_{ij}) \in D_{ij}$, $1 \leq i < j \leq 3$, stabilizers are of order two: $\operatorname{stab}(y, H_{ij}) = \langle (ij) \rangle \cong \mathbb{Z}_2$. Otherwise, stabilizers are trivial, unless we are looking at one of the three points ψ_{ij} marked in Figure 11. E.g., for $\psi_{12} = (0, \langle (1, -1, 0) \rangle)$, $\operatorname{stab} \psi_{12} = \langle (12) \rangle \cong \mathbb{Z}_2$. Although the transposition (12) does not fix the line $\langle (1, -1, 0) \rangle$ point-wise, it fixes ψ_{12} as a point in $Y_{\mathcal{A}_2}$! We see that transpositions $(ij) \in \mathfrak{S}_3$ act on the open Möbius band $Y_{\mathcal{A}_2}$ by central symmetries in ψ_{ij}.

Observe that the nested set stratification is not fine enough to distinguish stabilizers: as the points ψ_{ij} show, stabilizers are not isomorphic on open strata.

EXAMPLE 5.2 (THE PERMUTATION ACTION OF $\mathfrak{S}_4$). We now consider $\mathfrak{S}_4$ acting on real 3-space $V \cong \mathbb{R}^4/\Delta$. The locus of nonabelian stabilizers consists of the triple intersections of hyperplanes in $\mathcal{A}_3$, i.e., the subspaces contained in the minimal building set $\mathcal{G}_{\min}$ in $\mathcal{L}(\mathcal{A}_3) = \Pi_4$. Our general strategy suggests to look at the arrangement model $Y_{\mathcal{A}, \mathcal{G}_{\min}}$.

We consider a situation familiar to us from Example 2.6. In Figure 12, we illustrate the situation after the first blowup step in the construction of $Y_{\mathcal{A}, \mathcal{G}_{\min}}$, i.e., the exceptional divisor after blowing up $\{0\}$ in V with the stratification induced by the hyperplanes of $\mathcal{A}_3$. To complete the construction of $Y_{\mathcal{A}, \mathcal{G}_{\min}}$, another 4 blowups in the triple intersections of hyperplanes are necessary, the result of
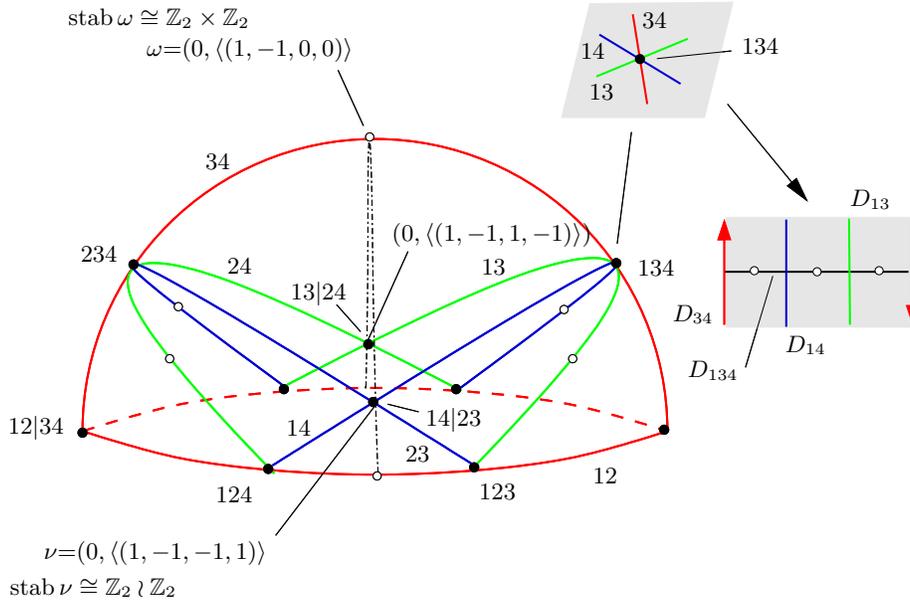
Figure 12. $\mathfrak{S}_4$ acting on $\mathrm{Bl}_{\{0\}}V$, where $V = \mathbb{R}^4/\Delta$.

which we illustrate locally for the triple intersection corresponding to 134. Triple intersections of hyperplanes in $\mathrm{Bl}_{\{0\}}V$ have stabilizers isomorphic to $\mathfrak{S}_3$ — the further blowups in triple intersections are indeed necessary to obtain an abelianization of the permutation action.

Again, we observe that the nested set stratification on $Y_{\mathcal{A},\mathcal{G}_{\min}}$ does not distinguish stabilizers: we indicate subdivisions of nested set strata resulting from nonisomorphic stabilizers by dotted lines, respectively unfilled points in Figure 12.

We now look at stabilizers of points on the model $Y_{\mathcal{A},\mathcal{G}_{\min}}$. We find points with stabilizers isomorphic to $\mathbb{Z}_2$ — any generic point on a divisor D_{ij} will be such. We also find points with stabilizers isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, e.g., the point ω on D_{1234} corresponding to the line $\langle(1,-1,0,0)\rangle$.

But, on $Y_{\mathcal{A},\mathcal{G}_{\min}}$ we also find points with *nonabelian* stabilizers! For example, the intersection of D_{14} and D_{23} on D_{1234} corresponding to the line $\langle(1,-1,-1,1)\rangle$ is stabilized by both (14) and $(12)(34)$ in $\mathfrak{S}_4$, which do not commute. In fact, the stabilizer is isomorphic to $\mathbb{Z}_2 \wr \mathbb{Z}_2$.

This observation shows that blowing up the locus of nonabelian stabilizers is not enough to abelianize the action! Further blowups in double intersections of hyperplanes are necessary, which suggests, contrary to our first assumption, the *maximal* arrangement model $Y_{\mathcal{A}_3}$ as an abelianization of the permutation action.

Some last remarks on this example: observe that stabilizers of points on $Y_{\mathcal{A}_3}$ all are elementary abelian 2-groups. We will later see that the strategy of resolving finite group actions on real vector spaces and even manifolds by constructing

a suitable maximal De Concini–Procesi model does not only abelianize the action, but yields stabilizers isomorphic to elementary abelian 2-groups.

Also, it seems we cannot do any better than that within the framework of blowups, i.e., we neither can get rid of nontrivial stabilizers, nor can we reduce the rank of nontrivial stabilizers any further. The divisors D_{ij} are stabilized by transpositions (ij) which supports our first claim. For the second claim, consider the point $\omega = (0, \ell_1)$ in $Y_{\mathcal{A}_3}$ with $\ell_1 = \langle (1, -1, 0, 0) \rangle$ (here we use the encoding of points on arrangement models proposed in (4–1)). We have seen above that $\mathrm{stab}\,\omega \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, in fact $\mathrm{stab}\,\omega = \langle (12) \rangle \times \langle (34) \rangle$. Blowing up $Y_{\mathcal{A}_3}$ in ω means to again glue in an open Möbius band. Points on the new exceptional divisor $D_\omega \cong \mathbb{RP}^1$ will be parameterized by tupels $(0, \ell_1, \ell_2)$, where ℓ_2 is a line orthogonal to ℓ_1 in V. A generic point on this stratum will be stabilized only by the transposition (12), specific points however, e.g., $(0, \ell_1, \langle (0, 0, 1, -1) \rangle)$ will still be stabilized by all of $\mathrm{stab}\,\omega \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

5.2. Abelianizing a finite linear action. Following the basic idea of proposing De Concini–Procesi arrangement models as abelianizations of finite group actions and drawing from our experiences with the permutation action on low-dimensional real space in Section 5.1 we here treat the case of finite linear actions.

Let a finite group Γ act linearly and effectively on real n-space $\mathbb{R}^n$. Without loss of generality, we can assume that the action is orthogonal [Vinberg 1989, Section 2.3, Theorem 1]; we fix the appropriate scalar product throughout.

Our strategy is to construct an arrangement of subspaces $\mathcal{A}(\Gamma)$ in real n-space, and to propose the maximal wonderful model $Y_{\mathcal{A}(\Gamma)}$ as an abelianization of the given action.

CONSTRUCTION 5.3 (THE ARRANGEMENT $\mathcal{A}(\Gamma)$). For any subgroup H in Γ, define a linear subspace

$$L(H) := \mathrm{span}\{\, \ell \mid \ell \text{ line in } \mathbb{R}^n \text{ with } H \circ \ell = \ell \,\}, \qquad (5\text{–}1)$$

the linear span of all lines in V that are invariant under the action of H.

Denote by $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma \circlearrowright \mathbb{R}^n)$ the arrangement of proper subspaces in $\mathbb{R}^n$ that are of the form $L(H)$ for some subgroup H in Γ.

Observe that the arrangement $\mathcal{A}(\Gamma)$ never contains any hyperplane: if $L(H)$ were a hyperplane for some subgroup H in Γ, then also its orthogonal line ℓ would be invariant under the action of H. By definition of $L(H)$, however, ℓ would then be contained in $L(H)$ which in turn would be the full ambient space.

THEOREM 5.4 [Feichtner and Kozlov 2005, Thm. 3.1]. *For any effective linear action of a finite group Γ on n-dimensional real space, the maximal wonderful arrangement model $Y_{\mathcal{A}(\Gamma)}$ abelianizes the action. Moreover, stabilizers of points on the arrangement model are isomorphic to elementary abelian 2-groups.*

The first example coming to mind is the permutation action of $\mathfrak{S}_n$ on real n-space. We find that $\mathcal{A}(\mathfrak{S}_n)$ is the rank 2 truncation of the braid arrangement,

$\mathcal{A}_{n-1}^{\mathrm{rk} \geq 2}$, i.e., the arrangement consisting of subspaces in $\mathcal{A}_{n-1}$ of codimension at least 2. For details, see [Feichtner and Kozlov 2005, Section 4.1]. In earlier work [Feichtner and Kozlov 2003], we had already proposed the maximal arrangement model of the braid arrangement as an abelianization of the permutation action. We proved that stabilizers on $Y_{\mathcal{A}_{n-1}}$ are isomorphic to elementary abelian 2-groups by providing explicit descriptions of stabilizers based on an algebraic-combinatorial set-up for studying these groups.

5.3. Abelianizing finite differentiable actions on manifolds. We now look at a generalization of the abelianization presented in Section 5.2. Assume that Γ is a finite group that acts differentiably and effectively on a smooth real manifold M. We first observe that such an action induces a linear action of the stabilizer $\mathrm{stab}\, x$ on the tangent space $T_x M$ at any point x in M. Hence, locally we are back to the setting that we discussed before: For any subgroup H in $\mathrm{stab}\, x$, we can define a linear subspace $L(x, H) := L(H)$ of the tangent space $T_x M$ as in (5–1), and we can combine the nontrivial subspaces to form an arrangement $\mathcal{A}_x := \mathcal{A}(\mathrm{stab} \circlearrowleft T_x M)$ in $T_x M$.

Combined with the information that a model construction in the spirit of De Concini–Procesi arrangement models exists also for local subspace arrangements, we need to stratify the manifold so as to locally reproduce the arrangement $\mathcal{A}_x$ in any tangent space $T_x M$. Here is how to do that:

CONSTRUCTION 5.5 (THE STRATIFICATION $\mathfrak{L}$). For any $x \in M$, and any subgroup H in $\mathrm{stab}\, x$, define a normal (!) subgroup $F(x, H)$ in H by

$$F(x, H) = \{h \in H \mid h \circ y = y \text{ for any } y \in L(x, H)\};$$

$F(x, H)$ is the subgroup of elements in H that fix all of $L(x, H)$ point-wise. Define $\mathfrak{L}(x, H)$ to be the connected component of the fixed point set of $F(x, H)$ in M that contains x. Now combine these submanifolds so as to form a locally finite stratification

$$\mathfrak{L} = (\mathfrak{L}(x, H))_{x \in M, \, H \leq \mathrm{stab}\, x}.$$

Observe that, as we tacitly did for stratifications induced by arrangements or by irreducible components of divisors, we only specify strata of proper codimension.

The stratification $\mathcal{L}$ locally coincides with the tangent space stratifications coming from our linear setting. Technically speaking: for any $x \in M$, there exists an open neighborhood U of x in M, and a $\mathrm{stab}\, x$-equivariant diffeomorphism $\Phi_x : U \to T_x M$ such that

$$\Phi_x(\mathfrak{L}(x, H)) = L(x, H) \tag{5–2}$$

for any subgroup H in $\mathrm{stab}\, x$. In particular, (5–2) shows that the stratification $\mathfrak{L}$ of M is a local subspace arrangement.

THEOREM 5.6 [Feichtner and Kozlov 2005, Theorem 3.4]. *Let a finite group Γ act differentiably and effectively on a smooth real manifold M. Then the wonderful*

model $Y_\mathfrak{L}$ induced by the locally finite stratification $\mathfrak{L}$ of M abelianizes the action. Moreover, stabilizers of points on the model $Y_\mathfrak{L}$ are isomorphic to elementary abelian 2-groups.

EXAMPLE 5.7 (ABELIANIZING THE PERMUTATION ACTION ON $\mathbb{RP}^2$). We take a small nonlinear example: the permutation action of $\mathfrak{S}_3$ on the real projective plane induced by $\mathfrak{S}_3$ permuting coordinates in $\mathbb{R}^3$.

We picture $\mathbb{RP}^2$ by its upper hemisphere model in Figure 13, where we agree to place the projectivization of $\Delta^\perp$ on the equator. The locus of nontrivial stabilizers of the $\mathfrak{S}_3$ permutation action consists of the projectivizations of hyperplanes $H_{ij}: x_i = x_j$, for $1 \leq i < j \leq 3$, and three additional points Ψ_{ij} on $\mathbb{P}\Delta^\perp$ indicated in Figure 13. The $\mathfrak{S}_3$ action can be visualized by observing that transpositions $(ij) \in \mathfrak{S}_3$ act as reflections in the lines $\mathbb{P}H_{ij}$, respectively.
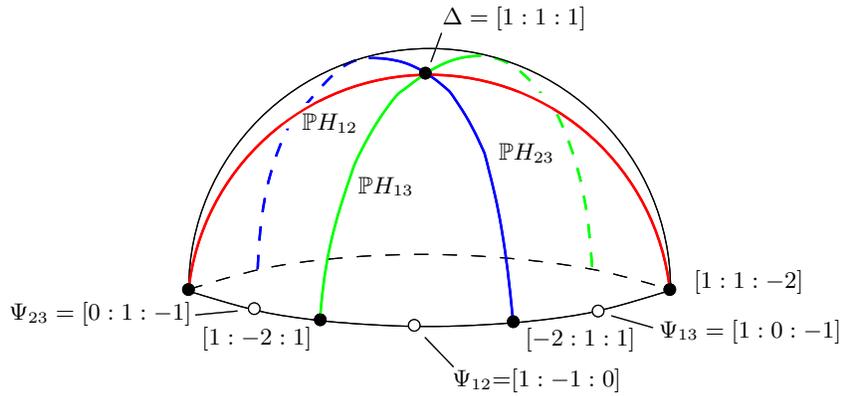

Figure 13. $\mathfrak{S}_3$ acting on $\mathbb{RP}^2$: the stabilizer stratification.

We find that the arrangements $\mathcal{A}_\ell$ in the tangent spaces $T_\ell \mathbb{RP}^2$ are empty, unless $\ell = [1{:}1{:}1]$. Hence, (5–2) allows us to conclude that the $\mathfrak{L}$-stratification of $\mathbb{RP}^2$ consists of a single point, $[1{:}1{:}1]$. Observe that the $\mathfrak{S}_3$-action on $T_{[1{:}1{:}1]}\mathbb{RP}^2$ coincides with the permutation action of $\mathfrak{S}_3$ on $\mathbb{R}^3/\Delta$.

The wonderful model $Y_\mathfrak{L}$ hence is a Klein bottle, the result of blowing up $\mathbb{RP}^2$ in $[1{:}1{:}1]$, i.e., glueing a Möbius band into the punctured projective plane.

Observe that the $\mathfrak{L}$-stratification is coarser than the codimension 2 truncation of the stabilizer stratification: The isolated points Ψ_{ij} on $\mathbb{P}\Delta^\perp$ have nontrivial stabilizers, but do not occur as strata in the $\mathfrak{L}$-stratification.

Acknowledgments

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