On the Rank of a Tropical Matrix

MIKE DEVELIN, FRANCISCO SANTOS, AND BERND STURMFELS

ABSTRACT. This is a foundational paper in tropical linear algebra, which is linear algebra over the min-plus semiring. We introduce and compare three natural definitions of the rank of a matrix, called the Barvinok rank, the Kapranov rank and the tropical rank. We demonstrate how these notions arise naturally in polyhedral and algebraic geometry, and we show that they differ in general. Realizability of matroids plays a crucial role here. Connections to optimization are also discussed.

1. Introduction

The rank of a matrix M is one of the most important notions in linear algebra. This number can be defined in many different ways. In particular, the following three definitions are equivalent:

- *The rank of M is the smallest integer r for which M can be written as the sum of r rank one matrices. A matrix has rank 1 if it is the product of a column vector and a row vector.*
- *The rank of M is the smallest dimension of any linear space containing the columns of M.*
- *The rank of M is the largest integer r such that M has a nonsingular $r \times r$ minor.*

Our objective is to examine these familiar definitions over an algebraic structure which has no additive inverses. We work over the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$, whose arithmetic operations are

$$a \oplus b := \min(a, b) \qquad \text{and} \qquad a \odot b := a + b.$$

The set $\mathbb{R}^d$ of real d-vectors and the set $\mathbb{R}^{d\times n}$ of real $d \times n$-matrices are semi-modules over the semiring $(\mathbb{R}, \oplus, \odot)$. The operations of matrix addition and matrix multiplication are well defined. All our definitions of rank make sense over the tropical semiring $(\mathbb{R}, \oplus, \odot)$:

DEFINITION 1.1. The *Barvinok rank* of a matrix $M \in \mathbb{R}^{d\times n}$ is the smallest integer r for which M can be written as the tropical sum of r matrices, each of which is the tropical product of a $d \times 1$-matrix and a $1 \times n$-matrix.

DEFINITION 1.2. The *Kapranov rank* of a matrix $M \in \mathbb{R}^{d\times n}$ is the smallest dimension of any tropical linear space (to be defined in Definition 3.2) containing the columns of M.

DEFINITION 1.3. A square matrix $M = (m_{ij}) \in \mathbb{R}^{r \times r}$ is *tropically singular* if the minimum in

$$\det M := \bigoplus_{\sigma \in \mathcal{S}_r} m_{1\sigma_1} \odot m_{2\sigma_2} \odot \cdots \odot m_{r\sigma_r}$$
$$= \min \left\{ m_{1\sigma_1} + m_{2\sigma_2} + \cdots + m_{r\sigma_r} : \sigma \in \mathcal{S}_r \right\}$$

is attained at least twice. Here $\mathcal{S}_r$ denotes the symmetric group on $\{1, 2, \ldots, r\}$. The *tropical rank* of a matrix $M \in \mathbb{R}^{d\times n}$ is the largest integer r such that M has a nonsingular $r \times r$ minor.

These three definitions are easily seen to agree for $r = 1$, but in general they are not equivalent:

THEOREM 1.4. *If M is a matrix with entries in the tropical semiring $(\mathbb{R}, \oplus, \odot)$,*

$$tropical\ rank\,(M) \leq Kapranov\ rank\,(M) \leq Barvinok\ rank\,(M). \qquad (1–1)$$

Both of these inequalities can be strict.

The proof of Theorem 1.4 consists of Propositions 3.6, 4.1, 7.2 and Theorem 7.3 in this paper. As we go along, several alternative characterizations of the Barvinok, Kapranov and tropical ranks will be offered. One of them arises from the fact that every $d \times n$-matrix M defines a tropically linear map $\mathbb{R}^n \to \mathbb{R}^d$. The image of M is a polyhedral complex in $\mathbb{R}^d$. Following [Develin and Sturmfels 2004], we identify this polyhedral complex with its image in the tropical projective space $\mathbb{TP}^{d-1} = \mathbb{R}^d / \mathbb{R}(1, 1, \ldots, 1)$. This image is the *tropical convex hull* of (the columns of) M as in [Develin and Sturmfels 2004]. Equivalently, this *tropical polytope* is the set of all tropical linear combinations of the columns of M. We show in Section 4 that the tropical rank of M equals the dimension of this tropical polytope plus one, thus justifying the definition of the vanishing of the determinant given in Definition 1.3.

   The discrepancy between Definition 1.3 and Definition 1.2 comes from the crucial distinction between tropical polytopes and tropical linear spaces, as explained in [Richter-Gebert et al. 2005, § 1]. The latter are described in [Speyer

and Sturmfels 2004] where it is shown that they are parametrized by the tropical Grassmannian. That the two inequalities in Theorem 1.4 can be strict corresponds to two facts about tropical geometry which are unfamiliar from classical geometry. Strictness of the first inequality corresponds to the fact that a point configuration in tropical space can have a d-dimensional convex hull but not lie in any d-dimensional affine subspace. Strictness of the second inequality corresponds to the fact that a point configuration in a d-dimensional subspace need not lie in the convex hull of $d + 1$ points.

We start out in Section 2 by studying the Barvinok rank (Definition 1.1). This notion of rank arises in the context of combinatorial optimization [Barvinok et al. 1998; Butkovič 2003; Çela et al. 1998]. In Section 3 we study the Kapranov rank (Definition 1.2). This notion is the most natural one from the point of view of algebraic geometry, where tropical arithmetic arises as the "tropicalization" of arithmetic in a power series ring. It has good algebraic and geometric properties but is difficult to characterize combinatorially; for instance, it depends on the base field of the power series ring, which here we take to be the complex numbers $\mathbb{C}$, unless otherwise stated.

In Section 4 we study the tropical rank (Definition 1.3). This is the best notion of rank from a geometric and combinatorial perspective. For instance, it can be expressed in terms of regular subdivisions of products of simplices [Develin and Sturmfels 2004]. In Section 5, we use this characterization to show that the tropical and Kapranov ranks agree when either of them is equal to $\min(d, n)$.

Section 6 is devoted to another case where the Kapranov and tropical ranks agree, namely when either of them equals two. The set of $d \times n$-matrices enjoying this property is the space of trees with d leaves and n marked points. This space is studied in the companion paper [Develin 2004].

The second inequality of Theorem 1.4 is strict for many matrices (see Proposition 2.2 for examples), but it requires more effort to find matrices for which the first inequality is strict. Such matrices are constructed in Section 7 by relating Kapranov rank to realizability of matroids.

Our definition of "tropically nonsingular" is equivalent to what is called "strongly regular" in the literature on the min-plus algebra [Butkovič and Hevery 1985; Cuninghame-Green 1979]. The resulting notion of tropical rank, as well as the notion of Barvinok rank, have previously appeared in this literature. In fact, linear algebra in the tropical semiring has been called "the linear algebra of combinatorics" [Butkovič 2003]. In the final section of the paper we revisit some of that literature, which is concerned mainly with algorithmic issues, and relate it to our results. We also point out several (mostly algorithmic) open questions.

Summing up, the three definitions of rank studied in this paper generally disagree, and they have different flavors (combinatorial, algebraic, geometric). But they all share some of the familiar properties of matrix rank over a field. The following properties are easily checked for each of the three definitions of rank: the rank of a matrix and its transpose are the same; the rank of a minor
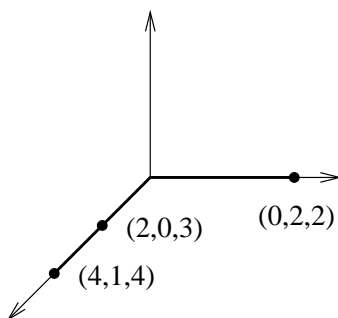
Figure 1. A tropical line in $\mathbb{TP}^2$, and a one-dimensional tropical polytope.

cannot exceed that of the whole matrix; the rank is invariant under (tropical) multiplication of rows or columns by constants, and under insertion of a row or column obtained as a combination of others; the rank of $M \oplus N$ is at most the sum of the ranks of M and N; the rank of $(M \,|\, N)$ is at least the ranks of M and of N and at most the sum of their ranks; and the rank of $M \odot N$ is at most the minimum of the ranks of M and N.

2. The Barvinok Rank

The Traveling Salesman Problem can be solved in polynomial time if the distance matrix is the tropical sum of r matrices of tropical rank one (with $\oplus$ as "max" instead of "min"). This result was proved by Barvinok, Johnson and Woeginger [Barvinok et al. 1998], building on earlier work of Barvinok. This motivates our definition of *Barvinok rank* as the smallest r for which $M \in \mathbb{R}^{d \times n}$ is expressible in this fashion. Since matrices of tropical rank one are of the form $X \odot Y^T$, for two column vectors $X \in \mathbb{R}^d$ and $Y \in \mathbb{R}^n$, this is equivalent to saying that M has a representation

$$M \quad = \quad X_1 \odot Y_1^T \ \oplus \ X_2 \odot Y_2^T \ \oplus \ \cdots \ \oplus \ X_r \odot Y_r^T. \tag{2–1}$$

For example, the following equation shows a 3×3-matrix which has Barvinok rank two:

$$\begin{pmatrix} 0 & 4 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \odot (0, 4, 2) \ \oplus \ \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \odot (2, 1, 0).$$

This matrix also has tropical rank 2 and Kapranov rank 2 because the matrix is tropically singular. The column vectors lie on the tropical line in $\mathbb{TP}^2 = \mathbb{R}^3/\mathbb{R}(1,1,1)$ defined by $2 \odot x_1 \oplus 3 \odot x_2 \oplus 0 \odot x_3$, depicted in Figure 1. Their convex hull, darkened, is a subset of the line and thus one-dimensional.

We next present two reformulations of the definition of Barvinok rank: in terms of tropical convex hulls as introduced in [Develin and Sturmfels 2004], and via a "tropical morphism" between matrix spaces.

PROPOSITION 2.1. *Let M be a real $d \times n$-matrix. The following properties are equivalent*:

(a) *M has Barvinok rank at most r.*
(b) *The columns of M lie in the tropical convex hull of r points in $\mathbb{TP}^{d-1}$.*
(c) *There are matrices $X \in \mathbb{R}^{d \times r}$ and $Y \in \mathbb{R}^{r \times n}$ such that $M = X \odot Y$. Equivalently, M lies in the image of the following tropical morphism, which is defined by matrix multiplication*:

$$\phi_r : \mathbb{R}^{d \times r} \times \mathbb{R}^{r \times n} \to \mathbb{R}^{d \times n}, \quad (X, Y) \mapsto X \odot Y. \qquad (2\text{--}2)$$

PROOF. Let $M_1, \ldots, M_n \in \mathbb{R}^d$ be the column vectors of M. Let $X_1, \ldots, X_r \in \mathbb{R}^d$ and $Y_1, \ldots, Y_r \in \mathbb{R}^n$ be the columns of two unspecified matrices $X \in \mathbb{R}^{d \times r}$ and $Y \in \mathbb{R}^{n \times r}$. Let Y_{ij} denote the jth coordinate of Y_i. The following three algebraic identities are easily seen to be equivalent:

(a) $M = X_1 \odot Y_1^T \oplus X_2 \odot Y_2^T \oplus \cdots \oplus X_r \odot Y_r^T$,
(b) $M_j = Y_{1j} \odot X_1 \oplus Y_{2j} \odot X_2 \oplus \cdots \oplus Y_{rj} \odot X_r$ for all $j = 1, \ldots, n$, and
(c) $M = X \odot Y^T$.

Statement (b) says that each column vector of M lies in the tropical convex hull of $X_1, \ldots, X_r$. The entries of the matrix Y are the multipliers in that tropical linear combination. This shows that the three conditions (a), (b) and (c) in the statement of the proposition are equivalent. □

Part (b) of Proposition 2.1 suggests that the Barvinok rank of a tropical matrix is more an analogue of the nonnegative rank of a matrix than of the usual rank. Recall (from [Cohen and Rothblum 1993], for instance) that the *nonnegative rank* of a real nonnegative matrix $M \in \mathbb{R}^{d \times n}$ is the smallest r for which M can be written as a product of nonnegative matrices of format $d \times r$ and $r \times n$. Equivalently, it is the smallest r for which the columns (or rows) of M lie in the positive hull of r nonnegative vectors. Compare this with the formulation of Barvinok rank given in Proposition 2.1 (b); this closer connection comes from the fact that tropical linear combinations yield an object more analogous to a "positive span" or "convex hull" [Develin and Sturmfels 2004; Richter-Gebert et al. 2005] than a linear span. For more information on nonnegative rank see [Cohen and Rothblum 1993], and for the connection to rank over other semigroup rings see [Gregory and Pullman 1983].

By Proposition 2.1, the set of all Barvinok matrices of rank $\leq r$ is the image of the tropical morphism ϕ_r. In particular, this set is a polyhedral fan in $\mathbb{R}^{d \times n}$. This fan has interesting combinatorial structure, even for $r = 2$. These fans are discussed in more detail in [Ardila 2004] and [Develin 2004].

We next present an example of a matrix which shows that the Barvinok rank can be much larger than the other two notions of rank. The matrix to be

considered is the *classical identity matrix*

$$
C_n \;=\;
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}. \tag{2–3}
$$

This looks like the unit matrix (in classical arithmetic) but it is far from being a unit matrix in tropical arithmetic, where 0 is the neutral element for $\odot$ and ∞ is the neutral element for $\oplus$. After obtaining the following result, we learned that the same calculation had already been done in [Çela et al. 1998].

PROPOSITION 2.2. *The Barvinok rank of the matrix C_n is the smallest integer r such that*

$$
n \;\le\; \binom{r}{\lfloor \frac{r}{2} \rfloor}.
$$

PROOF. Let r be an integer and assume that $n \le \binom{r}{\lfloor r/2 \rfloor}$. We first show that Barvinok rank$(C_n) \le r$. Let $S_1, \ldots, S_n$ be distinct subsets of $\{1, \ldots, r\}$ each having cardinality $\lfloor r/2 \rfloor$. For each $k \in 1, \ldots, r$, we define an $n \times n$-matrix $X_k = (x_{ij}^k)$ with entries in $\{0, 1, 2\}$ as follows:

$$
x_{ij}^k = \begin{cases} 0 & \text{if } k \in S_i \backslash S_j, \\ 2 & \text{if } k \in S_j \backslash S_i, \\ 1 & \text{otherwise.} \end{cases}
$$

The matrix X_k has tropical rank one. To see this, let $V_k \in \{0, 1\}^n$ denote the vector with ith coordinate equal to one or zero depending on whether k is an element of S_i or not. Then

$$
X_k \;=\; V_k^T \odot (1 \odot (-V_k)).
$$

To prove Barvinok rank$(C_n) \le r$, it now suffices to establish the identity

$$
C_n \;=\; X_1 \oplus X_2 \oplus \cdots \oplus X_r.
$$

Indeed, all diagonal entries of the matrices on the right hand side are 1, and the off-diagonal entries (for $i \neq j$) of the right hand side are $\min(x_{ij}^1, x_{ij}^2, \ldots, x_{ij}^r) = 0$, because $S_i \backslash S_j$ is nonempty.

To prove the converse direction, we consider an arbitrary representation

$$
C_n \;=\; Y_1 \oplus Y_2 \oplus \cdots \oplus Y_r
$$

where the matrices $Y_k = (y_{ij}^k)$ have tropical rank one. For each k we set $T_k := \{(i, j) : y_{ij}^k = 0\}$. Since the matrices Y_k are nonnegative and have tropical rank one, it follows that each T_k is a product $I_k \times J_k$, where I_k and J_k are subsets of

$\{1, \ldots, n\}$. Moreover, we have $I_k \cap J_k = \varnothing$ because the diagonal entries of Y_k are not zero. For each $i = 1, \ldots, n$ we set

$$S_i := \{k : i \in I_k\} \subseteq \{1, \ldots, r\}.$$

We claim that no two of the sets $S_1, \ldots, S_n$ are contained in one another. Sperner's Theorem [Aigner and Ziegler 1998] then proves that $n \leq \binom{r}{\lfloor r/2 \rfloor}$. To prove the claim, observe that if $S_i \subset S_j$ then the entry $y_{i,j}^k$ cannot be zero for any k. Indeed, if $k \in S_i \subseteq S_j$ then $j \in I_k$ implies $j \notin J_k$. And if $k \notin S_i$ then $i \notin I_k$.   □

For example, C_6 has Barvinok rank 4, as the following decomposition shows:

$$C_6 = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 & 2 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 \end{pmatrix}.$$

Similarly, C_{36} has Barvinok rank 8, even though all its 35×35 minors have Barvinok rank 7 (and its 8×8 minors have Barvinok rank at most 5). Asymptotically,

$$\text{Barvinok rank}(C_n) \sim \log_2 n.$$

We will see in Examples 3.5 and 4.4 that the Kapranov rank and tropical rank of C_n are both two.

3. The Kapranov Rank

The tropical semiring has a strong connection to power series rings and their algebraic geometry. We review the basic setup from [Speyer and Sturmfels 2004; Sturmfels 2002]. Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with complex coefficients. The elements in K are formal power series $f = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $a_1 < a_2 < \cdots$ are rational numbers that have a common denominator. Let $\deg : K^* \to \mathbb{Q}$ be the natural valuation sending a nonzero Puiseux series f to its degree a_1. For any two elements $f, g \in K$, we have $\deg(fg) = \deg f + \deg g = \deg f \odot \deg g$. In general we also have $\deg(f + g) = \min(\deg f, \deg g) = \deg f \oplus \deg g$, unless there is a cancellation of leading terms. Thus the tropical arithmetic is naturally induced from ordinary arithmetic in power series fields.

The field $K = \mathbb{C}\{\{t\}\}$ is algebraically closed of characteristic zero. If I is any ideal in $K[x_1, \ldots, x_d]$ then we write $V(I)$ for its variety in the d-dimensional algebraic torus $(K^*)^d$. Thus the elements of $V(I)$ are vectors $x(t) = (x_1(t), \ldots, x_d(t))$ where each $x_i(t)$ is a Puiseux series and $f(x(t)) = 0$ for each polynomial $f \in I$. Let us now enlarge the field K and allow all formal power series $f = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$ where the a_i can be real numbers, not just rationals. We denote this larger field by $\tilde{K}$ and we write $\tilde{V}(I)$ for the variety in $(\tilde{K}^*)^d$ defined by I.

The degree map can be applied coordinatewise, giving rise to a map which takes vectors of nonzero power series into $\mathbb{R}^d$:

$$\deg : (\tilde{K}^*)^d \to \mathbb{R}^d, \quad (f_1(t), \ldots, f_d(t)) \mapsto \big(\deg f_1, \ldots, \deg f_d\big).$$

We define the *tropical variety* of I, denoted $\mathcal{T}(I) \subset \mathbb{R}^d$, to be the image of $\tilde{V}(I)$ under the map deg. In [Speyer and Sturmfels 2004; Sturmfels 2002], the following alternative description of the tropical variety is given:

THEOREM 3.1. *The tropical variety $\mathcal{T}(I)$ is the set of vectors $w \in \mathbb{R}^n$ such that the initial ideal $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$ contains no monomial. The dimension of $\mathcal{T}(I)$ is the (topological) dimension of $V(I)$.*

The first statement in Theorem 3.1 is due to Misha Kapranov (in the special case when I is a principal ideal) and the third author (for arbitrary ideals I, in [Sturmfels 2002]). A complete proof can be found in [Speyer and Sturmfels 2004]. The second statement in Theorem 3.1 is due to Bieri and Groves [1984]. An elementary proof of this result, and the fact that $\mathcal{T}(I)$ is a polyhedral fan, appears in [Sturmfels 2002, § 9].

We defined Kapranov rank to be the smallest dimension of any tropical linear space containing the columns of M; now, we can make this precise by defining tropical linear spaces.

DEFINITION 3.2. A *tropical linear space* in $\mathbb{R}^d$ is any subset $\mathcal{T}(I)$ where I is an ideal generated by affine-linear forms $a_1 x_1 + \cdots + a_d x_d + b$ in $\tilde{K}[x] = \tilde{K}[x_1, \ldots, x_d]$. Its *dimension* is its topological dimension, which is equal to d minus the number of minimal generators of I.

Note that here the scalars $a_1, \ldots, a_n, b$ are power series in t with complex coefficients, the choice of the complex numbers being crucial. If I is the principal ideal generated by one affine-linear form $a_1 x_1 + \cdots + a_n x_n + b$, then $\mathcal{T}(I)$ is a *tropical hypersurface*. Tropical linear spaces were studied in [Speyer and Sturmfels 2004], where it was shown that they are parametrized by the *tropical Grassmannian*. Every tropical linear space L is a finite intersection of tropical hyperplanes, but not conversely, and the number of tropical hyperplanes needed is generally larger than the codimension of L.

Recall from Definition 1.2 that the Kapranov rank of a matrix $M \subset \mathbb{R}^{d \times n}$ is the smallest dimension of any tropical linear space containing the columns of M. It is not completely apparent in this definition that the Kapranov rank of a matrix and its transpose are the same, but this follows from our next result. Let J_r denote the ideal generated by all the $(r+1) \times (r+1)$-subdeterminants of a $d \times n$-matrix of indeterminates (x_{ij}). This is a prime ideal of dimension $rd + rn - r^2$, and the generating determinants form a Gröbner basis. The variety $V(J_r)$ consists of all $d \times n$-matrices with entries in K^* whose (classical) rank is at most r.

THEOREM 3.3. *For a real matrix $M = (m_{ij}) \in \mathbb{R}^{d \times n}$ the following statements are equivalent*:

(a) *The Kapranov rank of M is at most r.*
(b) *The matrix M lies in the tropical determinantal variety $\mathcal{T}(J_r)$.*
(c) *There exists a $d \times n$-matrix $F = (f_{ij}(t))$ with nonzero entries in the field $\tilde{K}$ such that the rank of F is less than or equal to r and $\deg(f_{ij}) = m_{ij}$ for all i and j.*

The power series matrix F in part (c) is called a *lift* of M. We abbreviate this as $\deg F = M$.

PROOF. The equivalence of (b) and (c) is simply our definition of tropical variety applied to the ideal J_r since, over the field $\tilde{K}$, lying in the variety of the determinantal ideal J_r is equivalent to having rank at most r. To see that (c) implies (a), consider the linear subspace of $\tilde{K}^d$ spanned by the columns of F. This is an r-dimensional linear space over a field, so it is defined by an ideal I generated by $d - r$ linearly independent linear forms in $\tilde{K}[x_1, \ldots, x_d]$. The tropical linear space $\mathcal{T}(I)$ contains all the column vectors of $M = \deg F$.

Conversely, suppose that (a) holds, and let L be a tropical linear space of dimension r containing the columns of M. Pick a linear ideal I in $\tilde{K}[x_1, \ldots, x_d]$ such that $L = \mathcal{T}(I)$. By applying the definition of tropical variety to the ideal I, we see that each column vector of M has a preimage in $\tilde{V}(I) \subset (\tilde{K}^*)^d$ under the degree map. Let F be the $d \times n$-matrix over $\tilde{K}$ whose columns are these preimages. Then the column space of F is contained in the variety defined by I, so we have $\mathrm{rank}(F) \leq r$, and $\deg F = M$ as desired. $\square$

COROLLARY 3.4. *The Kapranov rank of a matrix $M \in \mathbb{R}^{d \times n}$ is the smallest rank of any lift of M.*

The ideal J_1 is generated by the 2×2-minors $x_{ij}x_{kl} - x_{il}x_{kj}$ of the $d \times n$-matrix (x_{ij}). Therefore, a matrix of Kapranov rank one must certainly satisfy the linear equations $m_{ij} + m_{kl} = m_{il} + m_{kj}$. This happens if and only if there exist real vectors $X = (x_1, \ldots, x_d)$ and $Y = (y_1, \ldots, y_n)$ with

$$m_{ij} = x_i + y_j \text{ for all } i, j \quad \Longleftrightarrow \quad m_{ij} = x_i \odot y_j \text{ for all } i, j \quad \Longleftrightarrow \quad M = X^T \odot Y.$$

Conversely, if such X and Y exist, we can lift M to a matrix of rank one by substituting $t^{m_{ij}}$ for m_{ij}. Therefore, a matrix M has Kapranov rank one if and only if it has Barvinok rank one. In general, the Kapranov rank can be much smaller than the Barvinok rank, as the following example shows.

EXAMPLE 3.5. Let $n \geq 3$ and consider the classical identity matrix C_n. It does not have Kapranov rank one, so it has Kapranov rank at least two. Let

$a_3, a_4, \ldots, a_n$ be distinct nonzero complex numbers. Consider the matrix

$$F_n \;=\; \begin{pmatrix} t & 1 & t{+}a_3 & t{+}a_4 & \cdots & t{+}a_n \\ 1 & t & 1{+}a_3 t & 1{+}a_4 t & \cdots & 1{+}a_n t \\ t{-}a_3 & 1 & t & t{-}a_3{+}a_4 & \cdots & t{-}a_3{+}a_n \\ t{-}a_4 & 1 & t{-}a_4{+}a_3 & t & \cdots & t{-}a_4{+}a_n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t{-}a_n & 1 & t{-}a_n{+}a_3 & t{-}a_n{+}a_4 & \cdots & t \end{pmatrix}.$$

The matrix F_n has rank 2 because the i-th column (for $i \geq 3$) equals the first column plus a_i times the second column. Since $\deg F_n = C_n$, we conclude that C_n has Kapranov rank two.

The two-dimensional tropical plane containing the columns of C_n is the two-dimensional fan L in $\mathbb{R}^n$ which consists of the n cones $\{x_i \geq x_1 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_n\}$; this is the tropical variety defined by the ideal in $K[x_1, \ldots, x_n]$ generated by $n-2$ linear forms with generic coefficients in $\mathbb{C}$. Its image in $\mathbb{TP}^{n-1}$ is the line all of whose tropical Plücker coordinates are zero [Speyer and Sturmfels 2004].

The following proposition establishes half of Theorem 1.4.

PROPOSITION 3.6. *Every matrix $M \in \mathbb{R}^{d \times n}$ satisfies Kapranov rank$(M) \leq$ Barvinok rank(M), and this inequality can be strict.*

PROOF. Suppose that M has Barvinok rank r. Write $M = M_1 \oplus \cdots \oplus M_r$ where each M_i has Barvinok rank one. Then M_i has Kapranov rank one, so there exists a rank one matrix F_i over $\tilde{K}$ such that $\deg F_i = M_i$. Moreover, by multiplying the matrices F_i by random complex numbers, we can choose F_i such that there is no cancellation of leading terms in t when we form the matrix $F = F_1 + \cdots + F_r$. This means $\deg F = M$. Clearly, the matrix F has rank at most r. Theorem 3.3 implies that M has Kapranov rank at most r. Example 3.5 shows that the inequality can be strict. $\square$

A general algorithm for computing the Kapranov rank of a matrix M involves computing a Gröbner basis of the determinantal ideal J_r. Suppose we wish to decide whether a given real $d \times n$-matrix $M = (m_{ij})$ has Kapranov rank $> r$. To decide this question, we fix any term order $\prec_M$ on the polynomial ring $\mathbb{C}[x_{ij}]$ which refines the partial ordering on monomials given assigning weight m_{ij} to the variable x_{ij}, and we compute the reduced Gröbner basis $\mathcal{G}$ of J_r in the term order $\prec_M$. For each polynomial g in $\mathcal{G}$, we consider its leading form $\mathrm{in}_M(g)$ with respect to the partial ordering coming from M. As noted in [Sturmfels 1996, §1], we have $\mathrm{in}_{\prec_M}\big(\mathrm{in}_M(g)\big) = \mathrm{in}_{\prec_M}(g)$ for all $g \in \mathcal{G}$.

The ideal generated by the set of leading forms $\big\{\mathrm{in}_M(g) : g \in \mathcal{G}\big\}$ is the initial ideal $\mathrm{in}_M(J_r)$. Let x^{all} denote the product of all dn unknowns x_{ij}. The second step in our algorithm is to compute the saturation of the initial ideal with respect

to the coordinate hyperplanes:

$$\left(\mathrm{in}_M(J_r) : \langle x^{\mathrm{all}} \rangle^\infty \right) \ = \ \left\{ f \in \mathbb{C}[x_{ij}] : f \cdot (x^{\mathrm{all}})^s \in J_r \text{ for some } s \in \mathbb{N} \right\}. \quad (3\text{--}1)$$

Computing such an ideal quotient, given the generators $\mathrm{in}_M(g)$, is a standard operation in computational commutative algebra. It is a built-in command in software systems such as CoCoA [CoCoA 2000–], Macaulay 2 [Grayson and Stillman 1993–] or Singular [Greuel et al. 2001]. Here is a direct consequence of Theorems 3.1 and 3.3:

COROLLARY 3.7. *The matrix M has Kapranov rank $> r$ if and only if* (3–1) *is the unit ideal* $\langle 1 \rangle$.

In view of this, the (combinatorial) Theorem 5.5, Theorem 6.5 and Corollary 7.4 have the following commutative algebra implications. Recall from [Richter-Gebert et al. 2005] that a finite generating set S of an ideal I is a *tropical basis* if, for every weight vector $w \in \mathbb{R}^n$ for which the initial ideal $\mathrm{in}_w(I)$ contains a monomial, there is an $f \in S$ such that $\mathrm{in}_w(f)$ is a monomial. Every ideal I in $K[x_1, \ldots, x_n]$ has a tropical basis but tropical bases are often much larger than minimal generating sets.

COROLLARY 3.8. *The 3×3-minors of a matrix of indeterminates form a tropical basis. The same holds for the maximal minors of a matrix, but it does not hold for the 4×4-minors of a 7×7-matrix.*

We have defined Kapranov rank in terms of power series arithmetic over the complex field $\mathbb{C}$, which is a canonical choice for doing algebraic geometry. However, the same definition works over any field k. One can consider the Puiseux series field $K = k\{\{t\}\}$ with either rational or real exponents. Note that the former is not algebraically closed if k is algebraically closed of characteristic p, but this need not concern us. We denote the latter by $\tilde{K}$ as before. All we need is the degree map $(\tilde{K}^*)^d \to \mathbb{R}^d$. We make the following analogous definitions.

DEFINITION 3.9. Let $K = k\{\{t\}\}$. A *tropical linear space over k* is the image under "deg" of any linear subspace of the $\tilde{K}$-vector space $\tilde{K}^d$. Its *dimension* is equal to the dimension of that linear subspace. The *Kapranov rank over k* of a matrix $M \in \mathbb{R}^{n \times d}$ is the smallest dimension of a tropical linear space containing the columns of M.

Unless otherwise stated, we will concern ourselves only with Kapranov rank over the complex numbers. In the general setting, Theorem 3.3 is true over all fields, but Proposition 3.6 is true only over infinite fields because in its proof we needed to take random coefficients. Indeed, Example 6.6 in Section 6 shows a matrix whose Kapranov rank over the 2-element field $\mathbb{F}_2$ is greater than the Barvinok rank. Even over algebraically closed fields, the Kapranov rank of a matrix may depend on the characteristic of the field. We will discuss this further and give examples in Section 7.

4. The Tropical Rank

We begin by proving the first inequality in Theorem 1.4. To complete the proof of Theorem 1.4, it remains to be seen that the inequality can be strict. This will be done in Section 7.

PROPOSITION 4.1. *Every matrix* $M \in \mathbb{R}^{d \times n}$ *satisfies*

$$tropical\ rank\,(M)\ \leq\ Kapranov\ rank\,(M).$$

PROOF. If the matrix M has a tropically nonsingular $r \times r$ minor, then any lift of M to the power series field $\tilde{K}$ must have the corresponding $r \times r$-minor nonsingular over $\tilde{K}$, since the leading exponent of its determinant occurs only once in the sum. Consequently, no lift of M to $\tilde{K}$ can have rank less than r. By Theorem 3.3, this means that the Kapranov rank of M must be at least r. □

The set of all tropical linear combinations of a set of n vectors in $\mathbb{R}^d$ is a polyhedral complex. It has a 1-dimensional lineality space, spanned by the vector $(1, \ldots, 1)$, but upon quotienting out by this 1-dimensional space, we get a bounded subset in *tropical projective space* $\mathbb{TP}^{d-1} = \mathbb{R}^d/\mathbb{R}(1, \ldots, 1)$. This set is the *tropical convex hull* of the n given points in $\mathbb{TP}^{d-1}$, and it was investigated in depth in [Develin and Sturmfels 2004]. We review some relevant definitions and facts.

We fix a subset $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$. Given a point $x \in \mathbb{R}^d$, its *type* is the d-tuple of sets $S = (S_1, \ldots, S_d)$, where each $S_j \subset \{1, \ldots, n\}$ and $i \in S_j$ if $x_j - v_{ij} \geq x_k - v_{ik}$ for all $k \in \{1, \ldots, n\}$. Let X_S be the region consisting of points with type S; then according to [Develin and Sturmfels 2004, Theorem 15], the tropical convex hull of V equals the union of the bounded regions X_S, which are precisely those regions for which each S_j is nonempty. (If x is a point in the tropical convex hull with type S, then expressing x as a linear combination of the v_i's, we have $i \in S_j$ if the contribution of v_i is responsible for the j-th coordinate of x.) Indeed, (the topological closures of) these regions provide a polytopal decomposition of the tropical convex hull of V. Note that by definition, any type has the property that each $i \in \{1, \ldots, n\}$ is in some S_j.

The dimension of a particular cell X_S of the tropical polytope can be easily computed from the combinatorics of the d-tuple S: let G_S be the graph which has vertex set $1, \ldots, d$, with i and j connected by an edge if $S_i \cap S_j$ is nonempty. The dimension of X_S is one less than the number of connected components of the graph G_S.

Recall from Definition 1.3 that the tropical rank of a matrix is the size of the largest nonsingular square minor, and that an $r \times r$ matrix M is nonsingular if $\bigodot_{i=1}^{r} M_{\sigma(i),i} = \sum_{i=1}^{r} M_{\sigma(i),i}$ achieves its minimum only once as σ ranges over the symmetric group $\mathcal{S}_r$. Here is another characterization.

THEOREM 4.2. *Let $M \subset \mathbb{R}^{d \times n}$ be a matrix. Then the tropical rank of M is equal to one plus the dimension of the tropical convex hull of the columns of M, viewed as a tropical polytope in $\mathbb{TP}^{d-1}$.*

PROOF. Let $V = \{v_1, \ldots, v_n\}$ be the set of columns of M, and let $P = \mathrm{tconv}(V)$ be their tropical convex hull in $\mathbb{TP}^{d-1}$. Suppose that r is the tropical rank of M, that is, there exists a tropically nonsingular $r \times r$-submatrix of M, but all larger square submatrices are tropically singular.

We first show that $\dim P \geq r - 1$. We fix a nonsingular $r \times r$-submatrix M' of M. Deleting the rows outside M' means projecting P into $\mathbb{TP}^{r-1}$, and deleting the columns outside M' means passing to a tropical subpolytope P' of the image. Both operations can only decrease the dimension, so it suffices to show $\dim P' \geq r - 1$. Hence, we can assume that M is itself a tropically nonsingular $r \times r$-matrix. Also, without loss of generality, we can assume that the minimum over $\sigma \in \mathcal{S}_r$ of

$$f(\sigma) = \sum_{i=1}^{r} v_{i,\sigma(i)} \qquad (4\text{--}1)$$

is uniquely achieved when σ is the identity element $e \in \mathcal{S}_r$. We now claim that the cell $X_{(\{1\},\ldots,\{r\})}$ exists; to do this, we need to demonstrate that there exists a point with type $(\{1\}, \ldots, \{r\})$.

The inequalities which must be valid on this cell are $x_k - x_j \leq v_{jk} - v_{jj}$ for $j \neq k$. We claim that these inequalities define a full-dimensional region. Suppose not; then, by Farkas' Lemma, there exists a nonnegative linear combination of the inequalities $x_k - x_j \leq v_{jk} - v_{jj}$ which equals $0 \leq c$ for some nonpositive real c. This linear combination would imply that some other $\sigma \in \mathcal{S}_r$ has $f(\sigma) \leq f(e)$, a contradiction. So this cell is full-dimensional; it follows immediately that picking a point in its interior yields a point with type $(\{1\}, \ldots, \{r\})$, since because these inequalities are all strict, no other type-inducing inequalities can hold.

For the converse, suppose that $\dim P \geq r$. Pick a region X_S of dimension r, and assume by translating the points (which adds a constant to each row of X_S, not changing the rank of the matrix) that $(0, \ldots, 0)$ is in X_S, so that the only inequalities valid on 0 are those given by S. The graph G_S has $r + 1$ connected components, so we can pick $r + 1$ elements of $\{1, \ldots, n\}$ of which no two appear in a common S_j. Assume without loss of generality that this set is $\{1, \ldots, r+1\}$, and again without loss of generality rearrange the labeling of the coordinates so that $i \in S_j$ if and only if $i = j$, for $1 \leq i, j \leq r + 1$.

We now claim that the square submatrix consisting of the first $r + 1$ rows and columns of M is tropically nonsingular. Indeed, we have (using the definition of $f(\sigma)$ given in (4–1)):

$$f(\sigma) - f(e) = \sum_{i=1}^{r+1} v_{i,\sigma(i)} - \sum_{i=1}^{r+1} v_{ii} = \sum_{i=1}^{r+1} \left(v_{i,\sigma(i)} - v_{ii} \right),$$

but whenever $\sigma(i) \neq i$, $v_{i,\sigma(i)} - v_{ii} > 0$ since $i \in S_i$ and $i \notin S_{\sigma(i)}$ for the point 0. Therefore, if σ is not the identity, we have $f(\sigma) - f(e) > 0$, and e is the unique permutation in $\mathcal{S}_{r+1}$ minimizing the expression (4–1). So M has tropical rank at least $r + 1$. This is a contradiction, and we conclude that $\dim P = r - 1$. □

We next present a combinatorial formula for the tropical rank of a zero-one matrix, or any matrix which has only two distinct entries. We define the *support* of a vector in tropical space $\mathbb{R}^d$ as the set of its zero coordinates. We define the *support poset* of a matrix M to be the set of all unions of supports of column vectors of M. This set is partially ordered by inclusion.

PROPOSITION 4.3. *The tropical rank of a zero-one matrix with no column of all ones equals the maximum length of a chain in its support poset.*

The assumption that there is no column of all ones is needed for the statement to hold because a column of zeroes and a column of ones represent the same point in tropical projective space $\mathbb{TP}^{d-1}$.

PROOF. There is no loss of generality in assuming that every union of supports of columns of M is actually the support of a column. Indeed, the tropical sum of a set of columns gives a column whose support is the union of supports, and appending this column to M does not change the tropical rank since the tropical convex hull of the columns remains the same. Therefore, if there is a chain with r elements in the support poset we may assume that there is a set of r columns with supports contained in one another. Since there is no column of ones, from this we can easily extract an $r \times r$ minor with zeroes on and below the diagonal and 1's above the diagonal, which is tropically nonsingular.

Reciprocally, suppose there is a tropically nonsingular $r \times r$ minor N. We claim that the support poset of N has a chain of length r, from which it follows that the support poset of M also has a chain of length r. Assume without loss of generality that the unique minimum permutation sum is obtained in the diagonal. This minimum sum cannot be more than one, because if n_{ii} and n_{jj} are both 1 then changing them for n_{ij} and n_{ji} does not increase the sum. If the minimum is zero, orienting an edge from i to j if entry ij of N is zero yields an acyclic digraph, which admits an ordering. Rearranging the rows and columns according to this ordering yields a matrix with 1's above the diagonal and 0's on and below the diagonal. The tropical sum of the last i columns (which corresponds to union of the corresponding supports) then produces a vector with 0's exactly in the last i positions. Hence, there is a proper chain of supports of length r.

If the minimum permutation sum in N is 1, then let n_{ii} be the unique diagonal entry equal to 1. The i-th row in N must consist of all 1's: if n_{ij} is zero, then changing n_{ij} and n_{ji} for n_{ii} and n_{jj} does not increase the sum. Changing this row of ones to a row of zeroes does not affect the support poset of N (it just adds an element to every support), and yields a nonsingular zero-one matrix

with minimum sum zero to which we can apply the argument in the previous paragraph. □

EXAMPLE 4.4. The tropical rank of the classical identity matrix C_n equals two (for all n), since all of its 3×3 minors are tropically singular, while the principal 2×2 minors are not. The supports of its columns are all the sets of cardinality $n - 1$ and the support poset consists of them and the whole set $\{1, \ldots, n\}$. The maximal chains in the poset have indeed length two.

As with the matrices of Barvinok rank r, the $d \times n$ matrices of tropical rank at most r form a polyhedral fan given as the intersection of the tropical hypersurfaces $\mathcal{T}(f)$ where f runs over the **set of** $(r + 1) \times (r + 1)$-subdeterminants of a $d \times n$-matrix of unknowns (x_{ij}). Note that this is very similar to the Kapranov rank; by Theorem 3.3, the set of $d \times n$ matrices of tropical rank is the intersection of the tropical hypersurfaces $\mathcal{T}(f)$ where f runs over the **ideal generated by the** $(r + 1) \times (r + 1)$-subdeterminants of a $d \times n$-matrix of unknowns (x_{ij}).

However, these are not equal; matrices can have Kapranov rank strictly bigger than their tropical rank, as will be seen in Section 7. In this sense, the subdeterminants of a given size $r \geq 4$ do not form a tropical basis for the ideal they generate.

5. Mixed Subdivisions and Corank One

A useful tool in tropical convexity is the computation of tropical convex hulls by means of mixed subdivisions of the Minkowski sum of several copies of a simplex. We recall the definition of mixed subdivisions, adapted to the case of interest to us. See [Santos 2003] for more details.

DEFINITION 5.1. Let Δ^{d-1} be the standard $(d - 1)$-simplex in $\mathbb{R}^d$, with vertex set $A = \{e_1, \ldots, e_d\}$. Let $n\Delta^{d-1}$ denote its dilation by a factor of n, which we regard as the convex hull of the Minkowski sum $A + A + \cdots + A$ (n times). Let $M = (v_{ij}) \subset \mathbb{R}^{d \times n}$ be a matrix. Consider the lifted simplices

$$P_i := \operatorname{conv}\big\{(e_1, v_{1i}), \ldots, (e_d, v_{di})\big\} \subset \mathbb{R}^{d+1} \quad \text{for } i = 1, 2, \ldots, n.$$

The *regular mixed subdivision* of $n\Delta^{d-1}$ induced by M is the set of projections of the lower faces of the Minkowski sum $P_1 + \cdots + P_n$. Here, a face is called lower if its outer normal cone contains a vector with last coordinate negative.

It was shown in [Develin and Sturmfels 2004, § 4] that there is a bijection between the cells X_S in the convex hull of the columns of M and the interior cells in the regular subdivision of a product of simplices induced by M. Via the Cayley trick [Santos 2003], the latter biject to interior cells in the regular mixed subdivision defined above. Here we provide a short direct proof of the composition of these two bijections:

LEMMA 5.2. *Let $M \subset \mathbb{R}^{d \times n}$ and let $S = (S_1, \ldots, S_d)$, where each S_j is a subset of $\{1, \ldots, n\}$. Then, the following properties are equivalent:*

(1) *There exists a point in $\mathbb{R}^d$ of type S relative to the n points given by the columns of M.*
(2) *There is a nonnegative matrix M' such that M' is obtained from M by adding constants to rows or columns of M, and such that $M'_{ji} = 0$ precisely when $i \in S_j$.*
(3) *The regular mixed subdivision of $n\Delta^{d-1}$ induced by M has as a cell the Minkowski sum $\tau_1 + \cdots + \tau_n$ where $\tau_i = \mathrm{conv}(\{e_j : i \in S_j\})$.*

Moreover, if this happens, the cells referred to in parts (1) and (3) have complementary dimensions.

PROOF. Adding a constant to a row of M amounts to translating the set of n points in $\mathbb{TP}^{n-1}$, while adding a constant to a column leaves the point set unchanged. Consider a cell X_S in the tropical convex hull, let x be any point in the relative interior of X_S and let M' be the (unique) matrix obtained by translating the point set by a vector $-x$ and normalizing every column by adding a scalar so that its minimum coordinate equals 0. Conversely, for a matrix M' as in (2), consider the point x whose coordinates are the amounts added to the columns of M to obtain M'. The point x is in the tropical convex hull of the columns of M. Let S be its type. Then the modified matrix M' has zeroes precisely in entries (j, i) with $i \in S_j$, proving the equivalence of (1) and (2).

For the equivalence of (2) and (3), observe that adding a constant to a row or column of M does not change the mixed subdivision of $\sum P_i$. For a nonnegative matrix M' with at least a zero in every column, the positions of the zero entries define the face of $\sum P_i$ in the negative vertical direction. Conversely, for every cell of the regular mixed subdivision, we can apply a linear transformation changing only the last coordinate to give that cell height zero and all other vertices positive height (this is what it means to be in the lower envelope.) The resulting height function is precisely the matrix M' in (2), which proves the equivalence of (2) and (3). The assertion on dimensions is easy to prove. □

This lemma implies that the tropical convex hull is dual to the regular mixed subdivision.

COROLLARY 5.3. *Given a matrix M, the poset of types in the tropical convex hull of its columns and the poset of interior cells of the corresponding regular mixed subdivision are antiisomorphic.*

PROOF. From the proof of Lemma 5.2, it is clear that the poset of types (under $S < T$ if $S_j \subset T_j$ for each j) and the poset of cells in the regular mixed subdivision are antiisomorphic. Meanwhile, a type S is in the tropical convex hull of its columns if and only if each S_j is nonempty; this is the same condition categorizing when the corresponding cell is contained in the boundary of the

mixed subdivision (which occurs whenever there exists a vertex appearing in no summand.) □

COROLLARY 5.4. *Let $M \subset \mathbb{R}^{d \times n}$. The tropical rank of M equals d minus the minimal dimension of an interior cell in the regular mixed subdivision of $n\Delta^{d-1}$ induced by M.*

We can use these tools to prove that the tropical and Kapranov ranks of a matrix coincide if the latter is maximal.

THEOREM 5.5. *If a $d \times n$ matrix M has Kapranov rank equal to d, then it has tropical rank equal to d as well.*

PROOF. By Corollary 5.4, M has tropical rank d if and only if the corresponding regular mixed subdivision has an interior vertex. The theorem then follows from the next two lemmas. □

LEMMA 5.6. *A $d \times n$-matrix M has Kapranov rank less than d if and only if the corresponding regular mixed subdivision has a cell that intersects all facets of $n\Delta^{d-1}$.*

PROOF. If M has Kapranov rank less than d, then its column vectors lie in a tropical hyperplane. Since all tropical hyperplanes are translates of one another, there is no loss of generality in assuming that it is the hyperplane defined by $x_1 \oplus \cdots \oplus x_d$. That is, after normalization, all columns of M are nonnegative and have at least two zeroes. Then, by Lemma 5.2, the zero entries of M define a cell B in the regular mixed subdivision none of whose Minkowski summands are single vertices. In particular, for every facet F of Δ^{d-1} and for every $i \in \{1, \ldots, n\}$, the i-th summand of B is at least an edge and hence it intersects F. Hence, B intersects all facets of $n\Delta^{d-1}$. For the converse suppose the regular mixed subdivision has a cell B which intersects all facets of $n\Delta^{d-1}$. We may assume that M gives height zero to the points in that cell and positive height to all the others. The intersection of B with the j-th facet is given by the zero entries in M after deletion of the j-th row. In particular, B intersects the j-th facet if and only if every column has a zero entry outside of the j-th row, and so B intersects all facets if and only if all columns of M have at least two zeroes, implying that these all lie in the hyperplane defined by $x_1 \oplus \cdots \oplus x_d$. □

The cell in the preceding statement need not be unique. For example, if a tetrahedron is sliced by planes parallel to two opposite edges, then each maximal cell meets all the facets of the tetrahedron.

LEMMA 5.7. *In every polyhedral subdivision of a simplex which has no interior vertices, but arbitrarily many vertices on the boundary, there is a cell that intersects all of the facets.*

PROOF. Observe that there is no loss of generality in assuming that the polyhedral subdivision S is a triangulation. For a triangulation, we use Sperner's

Lemma [Aigner and Ziegler 1998]: "if the vertices of a triangulation of Δ are labeled so that (1) the vertices of Δ receive different labels and (2) the vertices in any face F of Δ receive labels among those of the vertices of F, then there is a fully labeled simplex".

Our task is to give our triangulation a Sperner labeling with the property that every vertex labeled i lies in the i-th facet of the simplex. The way to obtain this is: the vertex opposite to facet i is labeled $i + 1$. More generally, the label i of a vertex v is taken so that v is contained in facet i but not on facet $i - 1$. All labels are modulo d. $\square$

6. Matrices of Rank Two

By Theorem 4.2, if a matrix has tropical rank two, then the tropical convex hull of its columns is one-dimensional. Since it is contractible [Develin and Sturmfels 2004], this tropical polytope is a tree. Another way of showing this is via the corresponding regular mixed subdivision. Tropical rank 2 means that all the interior cells have codimension zero or one. Hence, the subdivision is constructed by slicing the simplex via a certain number of hyperplanes (which do not meet inside the simplex) and its dual graph is a tree. The special case when the matrix has Barvinok rank two is characterized by the following proposition.

PROPOSITION 6.1. *The following are equivalent for a matrix M:*

(1) *It has Barvinok rank* 2.
(2) *All its 3×3 minors have Barvinok rank* 2.
(3) *The tropical convex hull of its columns is a path.*

PROOF. $(1) \Longrightarrow (2)$ is trivial (the Barvinok rank of a minor cannot exceed that of the whole matrix) and $(3) \Longrightarrow (1)$ is easy: if a tropical polytope is a path, then it is the tropical convex hull of its two endpoints. Proposition 2.1 then implies that the Barvinok rank is two.

For $(2) \Longrightarrow (3)$ first observe that the case where M is 3×3 again follows from Proposition 2.1. We next prove the case where M is $d \times 3$ by contradiction: since the tropical convex hulls of rows and of columns of a matrix are isomorphic as cell complexes [Develin and Sturmfels 2004, Theorem 23], assume that the tropical convex hull of the *rows* of M is not a path. Then, there are three rows whose tropical convex hull is not a path, and their 3×3 minor has Barvinok rank 3. Finally, if M is of arbitrary size $d \times n$ and the tropical convex hull of its columns is not a path, consider three columns whose tropical convex hull is not a path and apply the previous case to them. $\square$

Our goal in this section is to show that if M has tropical rank 2 then it has Kapranov rank 2. Following Theorem 3.3 (c), this is done by constructing an explicit lift to a rank 2 matrix over $\tilde{K}$.

LEMMA 6.2. *Let M be a matrix of tropical rank two. Let x be a point in the tropical convex hull of the columns of M. Let M' be the matrix obtained by adding $-x$ to every column and then normalizing columns to have zero as their minimal entry. After possibly reordering the rows and columns, M' has the following block structure*:

$$M' := \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_k \end{pmatrix},$$

where the matrices A_i have all entries positive and every 2×2 minor has the property that the minimum of its four entries is achieved twice. Each $\mathbf{0}$ represents a matrix of zeroes of the appropriate size, and the first row and column blocks of M' may have size zero. Moreover, the tropical convex hull of the columns of M' is the union of the tropical convex hulls of the column vectors of the blocks augmented by the zero vector $\mathbf{0}$, and two of these k trees meet only at the point $\mathbf{0}$.

PROOF. First, adjoin the column x to our matrix if it does not already exist; since x is in the convex hull of M, this will not change the tropical convex hull of the columns of M. We can then simply remove it at the end, when it is transformed into a column of all zeroes. Thus, we can assume that one of the columns of the matrix M' consists of all zeroes.

The asserted block decomposition means that any two given columns of M' have either equal or disjoint cosupports, where the cosupport of a column is the set of positions where it does not have a zero. To prove that this holds, just observe that if it didn't then M' would have the following minor, where $+$ denotes a strictly positive entry. (Recall that each column has a zero in it.)

$$\begin{pmatrix} 0 & + & + \\ 0 & 0 & + \\ 0 & ? & 0 \end{pmatrix}$$

But this 3×3-matrix is tropically nonsingular. The assertion of the 2×2 minors follows from the fact that the nonnegative matrix

$$\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

is tropically singular if and only if the minimum of a, b, c and d is achieved twice.

Finally, the assertion about the convex hulls is trivial, since any linear combination of column vectors from a given block will have all zero entries except in the coordinates corresponding to that block. Any path joining two such points from different blocks will pass through the origin. $\square$

We next introduce a technical lemma for making a power series lifting sufficiently generic.

LEMMA 6.3. *Let A be a nonnegative matrix with no zero column and suppose that the smallest entry in A occurs most frequently in the first column. Let $\tilde{A}$ be the matrix*

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$$

obtained by adjoining a row and a column of zeroes. If $\tilde{A}$ has Kapranov rank two, then $\tilde{A}$ has a rank-2 lift $F \in \tilde{K}^{d \times n}$ in which every 2×2 minor is nonsingular and the i-th column can be written as a linear combination $\lambda_i u_1 + \mu_i u_2$ of the first two columns u_1 and u_2, with $\deg \lambda_i \geq \deg \mu_i = 0$.

PROOF. Starting with an arbitrary rank-2 lift $\tilde{F}$ of $\tilde{A}$, let F be obtained by adding to every column a $\tilde{K}$-linear combination of the first column of $\tilde{F}$ with coefficients of sufficiently high degree (so as to not change the degrees of the entries) but otherwise generic. This preserves the degree of every entry and thus the rank of the lift, but makes every 2×2 minor of F nonsingular; by "generic," all we require is that the ratio between the coefficients of two columns is not equal to the ratio between those two columns if they are scalar multiples of each other. No column of $\tilde{F}$ is a scalar multiple of its first column since no column of $\tilde{A}$ aside from the first is constant, so no column of F is a scalar multiple of the first column either.

Since the lift has rank two and the first two columns are linearly independent, the i-th column of F is now a $\tilde{K}$-linear combination $\lambda_i u_1 + \mu_i u_2$ of the first two columns. If the degrees of λ_i and μ_i are different, then their minimum must be zero in order to get a degree zero element in the first entry of column i. But then $\deg \mu_i > \deg \lambda_i = 0$ is impossible, because it would make the i-th column of A all zero. Hence $\deg \lambda_i > \deg \mu_i = 0$.

If the degrees are equal, then they are nonpositive in order to get degree zero for the first entry in $\lambda_i u_1 + \mu_i u_2$. But they cannot be equal and negative, or otherwise entries of positive degree in u_2 would produce entries of negative degree in u_i. Hence, $\deg \lambda_i = \deg \mu_i = 0$ in this case. $\square$

COROLLARY 6.4. *Let A and B be nonnegative matrices. Assume that the two matrices*

$$\tilde{A} := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \text{ and } \tilde{B} := \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

have Kapranov rank equal to 2. Then, the matrix

$$M := \begin{pmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix}$$

has Kapranov rank equal to 2 as well.

PROOF. We may assume that neither A nor B has a zero column. Hence Lemma 6.3 applies to both of them. We number the rows of M from $-k$ to k' and its columns from $-l$ to l', where $k \times l$ and $k' \times l'$ are the dimensions of

A and B respectively. In this way, A (respectively B) is the minor of negative (respectively, positive) indices. The row and column indexed zero consist of all zeroes. To further exhibit the symmetry between A and B the columns and rows in $\tilde{A}$ will be referred to "in reverse". That is to say, the first and second columns of it are the ones indexed 0 and -1 in M.

We now construct a lifting $F = (a_{i,j}) \in \mathbb{C}\{\{t\}\}^{d \times n}$ of M. We assume that we are given rank-2 lifts of $\tilde{A}$ and $\tilde{B}$ which satisfy the conditions of the previous lemma. Furthermore, we assume that the lift of the entry $(0,0)$ is the same in both, which can be achieved by scaling the first row in one of them.

We use exactly those lifts of $\tilde{A}$ and $\tilde{B}$ for the upper-left and bottom-right corner minors of M. Our task is to complete that with an entry $a_{i,j}$ for every i, j with $ij < 0$, such that $\deg(a_{i,j}) = 0$ and the whole matrix still has rank 2. We claim that it suffices to choose the entry $a_{-1,1}$ of degree zero and sufficiently generic. That this choice fixes the rest of the matrix is easy to see: The entry $a_{1,-1}$ is fixed by the fact that the 3×3 minor

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

needs to have rank 2. All other entries $a_{i,-1}$ and $a_{i,1}$ are fixed by the fact that the entries $a_{i,-1}$, $a_{i,0}$ and $a_{i,1}$ (two of which come from either $\tilde{A}$ or $\tilde{B}$) must satisfy the same dependence as the three columns of the minor above. For each $j = -l, \ldots, -2$ (respectively $j = 2, \ldots, l'$), let λ_j and μ_j be the coefficients in the expression of the j-th column of $\tilde{A}$ (respectively, of $\tilde{B}$) as $\lambda_j u_0 + \mu_j u_{-1}$ (respectively, $\lambda_j u_0 + \mu_j u_1$). Then, $a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,-1}$ (respectively, $a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,1}$).

What remains to be shown is that if $a_{-1,1}$ is of degree zero and sufficiently generic, all the new entries are of degree zero too. For this, observe that if $j \in \{-l', \ldots, 2\}$ then $a_{i,j}$ is of degree zero as long as the coefficient of degree zero in $a_{i,-1}$ are different from the degree zero coefficients in the quotient $-\lambda_j a_{i,0}/\mu_j$ (here we are using the assumption that $\deg \lambda_j \geq \deg \mu_j \geq 0$). The same is true for $j \in \{2, \ldots, l\}$, with $a_{i,1}$ instead of $a_{i,-1}$. In terms of the choice of $a_{-1,1}$, this translates to the following determinant having nonzero coefficient in degree zero:

$$\begin{pmatrix} a_{i,-1} & a_{i,0} & -\lambda_j a_{i,0}/\mu_j \\ a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \\ -\lambda_j a_{i,0}/\mu_j & a_{i,0} & a_{i,1} \end{pmatrix},$$

respectively for $j \in \{-l', \ldots, 2\}$ or $j \in \{2, \ldots, l\}$. That $a_{-1,1}$ and $a_{1,-1}$ sufficiently generic imply nonsingularity of these matrices follows from the fact that the following 2×2 minors come from the given lifts of $\tilde{A}$ and $\tilde{B}$, hence they are nonsingular:

$$\begin{pmatrix} a_{i,-1} & a_{i,0} \\ a_{0,-1} & a_{0,0} \end{pmatrix}, \qquad \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{i,0} & a_{i,1} \end{pmatrix}. \qquad \square$$

THEOREM 6.5. *Let M be a matrix of tropical rank* 2. *Then its Kapranov rank equals* 2 *as well.*

PROOF. The Kapranov rank of M is always at least the tropical rank, so we need only show that the Kapranov rank is less than or equal to 2. If the tropical convex hull P of the columns of M is a path, then M has Barvinok rank 2 (by Proposition 6.1) and thus Kapranov rank 2. Otherwise, let x be a node of degree at least three in the tree P. We apply the method of Lemma 6.2. Since x has degree at least three, it follows that there are at least three blocks A_i. In particular, M has at least three columns. We induct on the number of columns of M. If M has exactly three columns, then each block A_i is a single column, and every row of M has at most one positive entry. It is easy to construct an explicit lift of rank 2: in each row, lift the positive entry α as $-t^\alpha$ and the zero entries as -1 and $1 + t^\alpha$. If there are rows of zeroes, lift them as $(-1, -1, 2)$, for example.

Next, suppose that M has $m \geq 4$ columns. The two blocks with the smallest number of combined columns have at least 2 and at most $m-2$ rows all together. Possibly after adding a row and column of zeroes, this provides a decomposition of our matrix as

$$M \;=\; \begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix},$$

where both A and B have at least two columns (A is the union of these two blocks, B the union of all other blocks.) It then follows that the minors

$$\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

both have fewer columns than the original matrix. By the inductive hypothesis they have Kapranov rank 2. Applying Corollary 6.4 completes the inductive step of the theorem. $\qquad \square$

In the proof of Lemma 6.3 we again required the ability to pick generic field elements. Thus, Theorem 6.5 holds over any infinite coefficient field, but it may fail over finite fields. This is illustrated by the following example. Proposition 4.1 and Theorem 5.5 fail here too, as does the fact that Kapranov rank is invariant under insertion of a tropical combination of existing columns.

EXAMPLE 6.6. The matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

has Barvinok and tropical ranks equal to 2, but Kapranov rank 3 over the two-element field $\mathbb{F}_2$.

7. Matrices Constructed from Matroids

One of the important properties of rank in usual linear algebra is that it produces a matroid. Unfortunately, the definitions of tropical rank, Kapranov rank, and Barvinok rank all fail to do this. Consider the configuration of four points in the tropical plane $\mathbb{TP}^2$ given by the columns of

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

By any of our three definitions of rank, the maximal independent sets of columns are $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$. These do not all have the same size, and so they cannot be the bases of a matroid. The central obstruction here is that the sets $\{1,2,3\}$ and $\{1,2,4\}$ are (tropically) collinear, but the set $\{1,2,3,4\}$ is not. Despite this failure, there is a strong connection between tropical linear algebra and matroids.

The results in Sections 5 and 6 imply that any matrix whose tropical and Kapranov ranks disagree must be at least of size 5×5. The smallest example we know is 7×7. It is based on the *Fano matroid*. To explain the example, and to show how to construct many others, we prove a theorem about tropical representations of matroids. The reader is referred to [Oxley 1992] for matroid basics.

DEFINITION 7.1. Let $\mathcal{M}$ be a matroid. The *cocircuit matrix of* $\mathcal{M}$, denoted $\mathcal{C}(\mathcal{M})$, has rows indexed by the elements of the ground set of $\mathcal{M}$ and columns indexed by the cocircuits of $\mathcal{M}$. It has a 0 in entry (i,j) if the i-th element is in the j-th cocircuit and a 1 otherwise.

In other words, $\mathcal{C}(\mathcal{M})$ is the zero-one matrix whose columns have the cocircuits of $\mathcal{M}$ as supports. (As before, the support of a column is its set of zeroes.) As an example, the matrix C_n of Section 2 is the cocircuit matrix of the uniform matroid of rank 2 with n elements. Similarly, the cocircuit matrix of the uniform matroid $U_{n,r}$ has size $n \times \binom{n}{r-1}$ and its columns are all the zero-one vectors with exactly $r-1$ ones. The following results show that its tropical and Kapranov ranks equal r. The tropical polytopes defined by these matrices are the *tropical hypersimplices* studied in [Joswig 2005].

PROPOSITION 7.2. *The tropical rank of the cocircuit matrix* $\mathcal{C}(\mathcal{M})$ *is the rank of the matroid* $\mathcal{M}$.

PROOF. This is a special case of Proposition 4.3 because the rank of $\mathcal{M}$ is the maximum length of a chain of nonzero covectors, and the supports of covectors are precisely the unions of supports of cocircuits. Note that $\mathcal{C}(\mathcal{M})$ cannot have a column of ones because every cocircuit is nonempty. $\square$

THEOREM 7.3. *If the Kapranov rank of $\mathcal{C}(\mathcal{M})$ over the ground field k is equal to the rank of $\mathcal{M}$, then $\mathcal{M}$ is representable over k. If k is an infinite field, then the converse also holds.*

PROOF. Let $\mathcal{M}$ be a matroid of rank r on $\{1, \ldots, d\}$ which has n cocircuits and suppose that $F \in \tilde{K}^{d \times n}$ is a rank r lift of the cocircuit matrix $\mathcal{C}(\mathcal{M})$. For each row f_i of F, let $v_i \in k^d$ be the vector of constant terms in $f_i \in \tilde{K}^d$. We claim that $V = \{v_1, \ldots, v_d\}$ is a representation of $\mathcal{M}$. First note that V has rank at most r since every $\tilde{K}$-linear relation among the vectors f_i translates into a k-linear relation among the v_i. Our claim says that $\{i_1, \ldots, i_r\}$ is a basis of $\mathcal{M}$ if and only if $\{v_{i_1}, \ldots, v_{i_r}\}$ is a basis of V. Suppose $\{i_1, \ldots, i_r\}$ is a basis of $\mathcal{M}$. Then, as in the proof of Proposition 4.3, we can find a square submatrix of $\mathcal{C}(\mathcal{M})$ using rows $i_1, \ldots, i_r$ with 0's on and below the diagonal and 1's above it. This means that the lifted submatrix of constant terms is lower-triangular with nonzero entries along the diagonal. It implies that $v_{i_1}, \ldots, v_{i_r}$ are linearly independent, and, since $\mathrm{rank}(V) \leq r$, they must be a basis. We also conclude $\mathrm{rank}(V) = r$. If $\{i_1, \ldots, i_r\}$ is not a basis in $\mathcal{M}$, there exists a cocircuit containing none of them; this means that some column of $\mathcal{C}(\mathcal{M})$ has all 1's in rows $i_1, \ldots, i_r$. Therefore, $f_{i_1}, \ldots, f_{i_r}$ all have zero constant term in that coordinate, which means that $v_{i_1}, \ldots, v_{i_r}$ are all 0 in that coordinate. Since the cocircuit is not empty, not all vectors v_j have an entry of 0 in that coordinate, and so $\{v_{i_1}, \ldots, v_{i_r}\}$ cannot be a basis. This shows that V represents $\mathcal{M}$ over k, which completes the proof of the first statement in Theorem 7.3.

For the second statement, let us assume that $\mathcal{M}$ has no loops. This is no loss of generality because a loop corresponds to a row of 1's in $\mathcal{C}(\mathcal{M})$, which does not increase the Kapranov rank because every column has at least a zero. Assume $\mathcal{M}$ is representable over k and fix a $d \times n$-matrix $A \in k^{d \times n}$ such that the rows of A represent $\mathcal{M}$ and the sets of nonzero coordinates along the columns of A are the cocircuits of $\mathcal{M}$. Suppose $\{1, \ldots, r\}$ is a basis of $\mathcal{M}$ and let A' be the submatrix of A consisting of the first r rows. Write

$$A \quad = \quad \begin{pmatrix} \mathbf{I}_r \\ C \end{pmatrix} \cdot A'$$

where $\mathbf{I}_r$ is the identity matrix and $C \in k^{(d-r) \times r}$. Observe that A, hence C, cannot have a row of zeroes (because $\mathcal{M}$ has no loops). Since k is an infinite field, there exists a matrix $B' \in k^{r \times n}$ such that all entries of the $d \times r$-matrix $\begin{pmatrix} \mathbf{I}_r \\ C \end{pmatrix} \cdot B'$ are nonzero. We now define

$$F \quad = \quad \begin{pmatrix} \mathbf{I}_r \\ C \end{pmatrix} \cdot (A' + tB') \quad \in \quad \tilde{K}^{d \times n}.$$

This matrix has rank r and $\deg F = \mathcal{C}(\mathcal{M})$. This completes the proof of Theorem 7.3. $\qquad \square$

If k is representable over a finite field, its Kapranov rank (with respect to that field) may still exceed its tropical rank. It is easy to find examples; for instance, the matroid represented by $\{(0,1),(1,0),(1,1),(0,0)\}$ over $\mathbb{F}_2$ will work.

COROLLARY 7.4. *Let $\mathcal{M}$ be a matroid which is not representable over a given field k. Then the Kapranov rank with respect to k of the tropical matrix $\mathcal{C}(\mathcal{M})$ exceeds its tropical rank.*

This corollary furnishes many examples of matrices whose Kapranov rank exceeds their tropical rank. Consider, for example, the Fano and non-Fano matroids, depicted in Figure 2. They both have rank three and seven elements. The
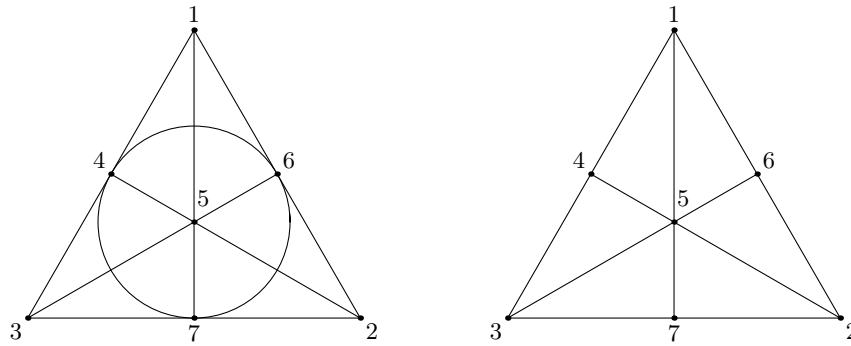
Figure 2. The Fano (left) and non-Fano (right) matroids.

first is only representable over fields of characteristic two, the second only over fields of characteristic different from two. In particular, Corollary 7.4 applied to these two matroids implies that over every field there are matrices with tropical rank equal to three and Kapranov rank larger than that. Also, it shows that the Kapranov rank of a matrix may be different over different fields k and k', even if k and k' are assumed to be algebraically closed. This is a more significant discrepancy than that of Example 6.6, which used a finite field.

More explicitly, the cocircuit matrix of the Fano matroid is

$$\mathcal{C}(\mathcal{M}) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is the smallest known example of a matrix whose Kapranov rank over $\mathbb{C}$ (four) is strictly larger than its tropical rank (three). Put differently, the seven columns of this matrix (in $\mathbb{TP}^6$) have as their tropical convex hull a two-dimensional cell complex which does not lie in any two-dimensional linear subspace of $\mathbb{TP}^6$, a feature decidedly absent from ordinary geometry.

Applied to nonrepresentable matroids, such as the *Vamos matroid* (rank 4, 8 elements, 41 cocircuits) or the *non-Pappus matroid* (rank 3, 9 elements, 20 cocircuits) [Oxley 1992], Corollary 7.4 yields matrices with different Kapranov and tropical ranks over *every* field. One can also get examples in which the difference of the two ranks is arbitrarily large. Indeed, given matrices A and B, we can construct the matrix

$$M := \begin{pmatrix} A & \infty \\ \infty' & B \end{pmatrix},$$

where ∞ and ∞' denote matrices of the appropriate dimensions and whose entries are sufficiently large. Appropriate choices of these large values (pick the extra columns to be points in the tropical convex hull of the columns of A and B and add large constants to each column) will ensure that the tropical and Kapranov ranks of M are the sums of those of A and of B. The difference between the Kapranov and tropical ranks of M is equal to the sum of this difference for A and for B.

The construction in Theorem 7.3 is closely related to the *Bergman complex* of the matroid $\mathcal{M}$. Ardila and Klivans [$\geq$ 2005] showed that this complex is triangulated by the order complex of the lattice of flats of $\mathcal{M}$. Since flats correspond to unions of cocircuits, the following result is easily derived:

PROPOSITION 7.5. *The Bergman complex of the matroid $\mathcal{M}$ is equal to the tropical convex hull of the rows of the modified cocircuit matrix $\mathcal{C}'(\mathcal{M})$, where the 1's in $\mathcal{C}(\mathcal{M})$ are replaced by ∞'s.*

For the Fano matroid, the Bergman complex is the cone over the incidence graph of points and lines in the matroid. It consists of 15 vertices, 35 edges and 21 triangles.

8. Related Work and Open Questions

As mentioned in the introduction, our definition of nonsingular square matrix corresponds to the notion of "strongly regular" in the literature on the max-plus (or min-plus) algebra. The definition of "regular matrix" in [Butkovič 1995; Butkovič and Hevery 1985; Cuninghame-Green 1979] is the following one, for which we prefer to use a different name:

DEFINITION 8.1. A square matrix M is *positively tropically regular* if, in the formula for its tropical determinant, the minimum over all even permutations equals the minimum over odd permutations. The *positive tropical rank* of a matrix is the maximum size of a positively tropically regular minor.

The reason for this terminology is that M is positively tropically regular if it lies outside the positive tropical variety defined by the determinant. For basics on positive tropical varieties and a detailed study of the positive tropical Grassmannian see [Speyer and Williams 2003]. The positive tropicalization of

determinantal varieties leads also to a notion of *positive Kapranov rank* that satisfies the inequalities

$$\text{pos. tropical rank}\,(M) \;\leq\; \text{pos. Kapranov rank}\,(M) \;\leq\; \text{Barvinok rank}\,(M).$$

Of course, the tropical and Kapranov ranks are less than or equal to their positive counterparts.

Our notion of tropical rank, however, appears in [Butkovič and Hevery 1985; Cuninghame-Green 1979] under a different name. Proposition 8.3 below was previously proved in [Butkovič and Hevery 1985]:

DEFINITION 8.2. The columns of a matrix $M \in \mathbb{R}^{d \times n}$ are *strongly linearly independent* if there is a column vector $b \in \mathbb{R}^d$ such that the tropical linear system $M \odot x = b$ has a unique solution $x \in \mathbb{R}^n$. A square matrix is *strongly regular* if its columns are strongly linearly independent.

PROPOSITION 8.3. *Strongly regular and tropically nonsingular are equivalent, for a square matrix.*

PROOF. Suppose an $r \times r$ matrix M is tropically nonsingular; then there is some $(r-1)$-dimensional cell X_S in the tropical convex hull of its columns in $\mathbb{TP}^{r-1}$. After relabeling we have $S_i = \{i\}$ for $i = 1, 2, \ldots, r$. Then taking a point in the relative interior of X_S yields a vector $b \in \mathbb{R}^r$ for which $M \odot x = b$ has a unique solution, each x_i being necessarily equal to $b_i - m_{ii}$.

Conversely, suppose the columns of an $r \times r$ matrix M are strongly linearly independent. Pick $b \in \mathbb{R}^r$ such that $M \odot x = b$ has a unique solution. Then, for each x_j, there exists a b_i for which the expression $\sum M_{ik} x_k$ is uniquely minimized for $k = j$ (otherwise we could increase x_j and get the same value for $M \odot x$). This is equivalent to b having type S, where $S_j = \{i\}$. □

COROLLARY 8.4. *The tropical rank of a matrix equals the largest size of a strongly linearly independent subset of its columns.*

We now discuss some algorithmic issues. Apart from Corollary 8.4, the main result in [Butkovič and Hevery 1985] is an $O(n^3)$ algorithm to check strong (i.e., tropical) regularity of an $n \times n$ matrix. The key step is to find a permutation that achieves the minimum in the determinantal tropical sum, which is the *assignment problem* in combinatorial optimization [Papadimitriou and Steiglitz 1982]. Similarly, it is shown in [Butkovič 1995] that the problem of testing positive tropical regularity of square matrices is equivalent to the problem of testing existence of even cycles in directed graphs.

For the Barvinok rank, we quote some results from [Çela et al. 1998]:

PROPOSITION 8.5. *The computation of the Barvinok rank of a matrix $M \in \{0,1\}^{d \times n}$ is an NP-complete problem. Deciding whether a matrix has Barvinok rank 2 can be done in time $O(dn)$.*

NP-completeness is proved by a reduction to the problem of covering a bipartite graph by complete bipartite subgraphs. For the case of rank 2, an algorithm is derived from the fact that matrices of Barvinok rank 2 are permuted *Monge matrices*. Çela et al. also prove that a matrix has Barvinok rank 2 if and only if all its 3×3 minors do (our Proposition 6.1) and that the Barvinok rank is bounded below by the maximum size of a strongly regular minor (i.e., by the tropical rank).

We finish by listing some open questions, most of them with an algorithmic flavor:

(1) Singularity of a single minor can be tested in polynomial time. But a naive algorithm to compute the tropical rank would need to check an exponential number of them. Can the tropical rank of a matrix be computed in polynomial time? In other words, is there a tropical analogue of Gauss elimination?

(2) Fix an integer k. The number of square minors of size at most $k+1$ of a $d \times n$ matrix M is polynomial in dn. Hence, there is a polynomial time algorithm for deciding whether M has tropical rank smaller or equal to k. Is the same true for the Barvinok rank? It is even open whether Barvinok rank equal to 3 can be tested in polynomial time.

(3) For a fixed k, a positive answer to either of the following two questions would imply a positive answer to the previous one:

(i) Is there a number $N(k)$ such that if all minors of M of size at most $N(k)$ have Barvinok rank at most k then M itself has Barvinok rank at most k? Proposition 2.2 shows that

$$N(k) \geq \binom{k+1}{\lfloor \frac{k+1}{2} \rfloor}.$$

(ii) Is there a polynomial time algorithm for the Barvinok rank of matrices with tropical rank bounded by k? (This is open even for $k = 2$).

(iii) Can we obtain a bound on the Kapranov rank given the tropical rank? That is, given a positive integer r, can we find a bound $N(r)$ so that all matrices of tropical rank r have Kapranov rank at most $N(r)$? The example of the classical identity matrix shows that the same cannot be done for Barvinok rank.

(4) Can the Barvinok rank of a matrix M be defined in terms of the regular mixed subdivision of $n\Delta^{d-1}$ produced by M? Ideally, we would like a "nice and simple" characterization such as the one given for the tropical rank in Corollary 5.4. But the question we pose is whether matrices producing the same mixed subdivision have necessarily the same Barvinok rank.

(5) All the questions above are open for the Kapranov rank, too.

(6) Is there a 5×5-matrix having tropical rank 3 but Kapranov rank 4?

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MIKE DEVELIN
AMERICAN INSTITUTE OF MATHEMATICS
360 PORTAGE AVE.
PALO ALTO, CA 94306
UNITED STATES
    develin@post.harvard.edu

FRANCISCO SANTOS
DEPTO. DE MATEMÁTICAS, ESTADÍSTICA Y COMPUTACIÓN
UNIVERSIDAD DE CANTABRIA
E-39005 SANTANDER
SPAIN
    francisco.santos@unican.es

BERND STURMFELS
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720
UNITED STATES
    bernd@math.berkeley.edu