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A Conformal Energy for Simplicial Surfaces

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Abstract. A new functional for simplicial surfaces is suggested. It is invariant with respect to Möbius transformations and is a discrete analogue of the Willmore functional. Minima of this functional are investigated. As an application a bending energy for discrete thin-shells is derived.

1. Introduction

In the variational description of surfaces, several functionals are of primary importance:

- The area $\mathcal{A} = \int dA$, where dA is the area element, is preserved by isometries.
- The total Gaussian curvature $\mathcal{G} = \int K\, dA$, where K is the Gaussian curvature, is a topological invariant.
- The total mean curvature $\mathcal{M} = \int H\, dA$, where H is the mean curvature, depends on the external geometry of the surface.
- The Willmore energy $\mathcal{W} = \int H^2\, dA$ is invariant with respect to Möbius transformations.

Geometric discretizations of the first three functionals for simplicial surfaces are well known. For the area functional the discretization is obvious. For the local Gaussian curvature the discrete analog at a vertex v is defined as the angle defect

$$G(v) = 2\pi - \sum_i \alpha_i,$$

where the α_i are the angles of all triangles (see Figure 2) at vertex v. The total Gaussian curvature is the sum over all vertices $G = \sum_v G(v)$. The local mean

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curvature at an edge e is defined as

$$M(e) = l\theta,$$

where l is the length of the edge and θ is the angle between the normals to the adjacent faces at e (see Figure 6). The total mean curvature is the sum over all edges $M = \sum_e M(e)$. These discrete functionals possess the geometric symmetries of the smooth functionals mentioned above.

Until recently a geometric discretization of the Willmore functional was missing. In this paper we introduce a Möbius invariant energy for simplicial surfaces and show that it should be treated as a discrete Willmore energy.

2. Conformal Energy

Let S be a simplicial surface in 3-dimensional Euclidean space with set of vertices V, edges E and (triangular) faces F. We define a conformal energy for simplicial surfaces using the circumcircles of their faces. Each (internal) edge $e \in E$ is incident to two triangles. A consistent orientation of the triangles naturally induces an orientation of the corresponding circumcircles. Let $\beta(e)$ be the external intersection angle of the circumcircles of the triangles sharing e, which is the angle between the tangent vectors of the oriented circumcircles.

Definition 1. The local conformal (discrete Willmore) energy at a vertex v is the sum

$$W(v) = \sum_{e \ni v} \beta(e) - 2\pi$$

over all edges incident on v. The conformal (discrete Willmore) energy of a simplicial surface S without boundary is the sum

$$W(S) = \frac{1}{2} \sum_{v \in V} W(v) = \sum_{e \in E} \beta(e) - \pi |V|,$$

over all vertices; here $|V|$ is the number of vertices of S.

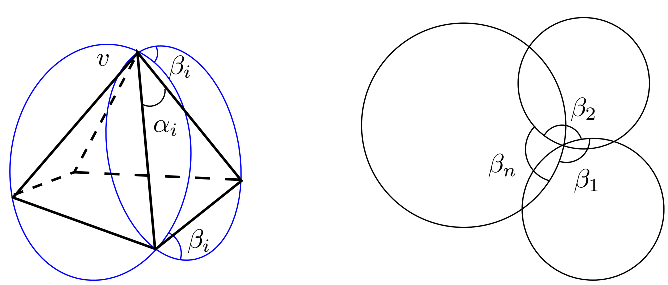

Figure 1. Definition of the conformal (discrete Willmore) energy.

Figure 1 presents two neighboring circles with their external intersection angle β_i as well as a view "from the top" at a vertex v showing all n circumcircles passing through v with the corresponding intersection angles $\beta_1, \dots, \beta_n$. For simplicity we will consider only simplicial surfaces without boundary.

The energy $W(S)$ is obviously invariant with respect to Möbius transformations. This invariance is an important property of the classical Willmore energy defined for smooth surfaces (see below).

Also, $W(S)$ is well defined even for nonoriented simplicial surfaces, because changing the orientation of both circles preserves the angle $\beta(e)$.

The star $S(v)$ of the vertex v is the subcomplex of S comprised by the triangles incident with v. The vertices of $S(v)$ are v and all its neighbors. We call $S(v)$ *convex* if for any its face $f \in F(S(v))$ the star $S(v)$ lies to one side of the plane of F, and *strictly convex* if the intersection of $S(v)$ with the plane of f is f itself.

PROPOSITION 2. *The conformal energy is nonnegative*:

$$W(v) \geq 0.$$

It vanishes if and only if the star $S(v)$ is convex and all its vertices lie on a common sphere.

The proof is based on an elementary lemma:

LEMMA 3. *Let $\mathcal{P}$ be a (not necessarily planar) n-gon with external angles β_i. Choose a point P and connect it to all vertices of $\mathcal{P}$. Let α_i be the angles of the triangles at the tip P of the pyramid thus obtained* (see Figure 2). *Then*

$$\sum_{i=1}^{n} \beta_i \geq \sum_{i=1}^{n} \alpha_i,$$

and equality holds if and only if $\mathcal{P}$ is planar and convex and the vertex P lies inside $\mathcal{P}$.

The pyramid obtained is convex in this case; note that we distinguish between convex and strictly convex polygons (and pyramids). Some of the external angles β_i of a convex polygon may vanish. The corresponding side-triangles of the pyramid lie in one plane.

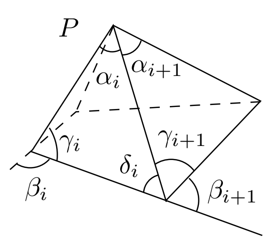

Figure 2. Toward the proof of Lemma 3.

Proof. Denote by γ_i and δ_i the angles of the side-triangles at the vertices of $\mathcal{P}$ (see Figure 2). The claim of Lemma 3 follows from adding over all $i = 1, \dots, n$ the two obvious relations

$$\beta_{i+1} \geq \pi - (\gamma_{i+1} + \delta_i), \qquad \pi - (\gamma_i + \delta_i) = \alpha_i.$$

All inequalities become equalities only in the case when $\mathcal{P}$ is planar, convex and contains P. $\square$

As a corollary we obtain a polygonal version of Fenchel's theorem [1929].

Corollary 4.

$$\sum_{i=1}^{n} \beta_i \geq 2\pi.$$

Proof. For a given $\mathcal{P}$ choose the point P varying on a straight line encircled by $\mathcal{P}$. There always exist points P such that the star at P is not strictly convex, and thus $\sum \alpha_i \geq 2\pi$. $\square$

Proof of Proposition 2. The claim of Proposition 2 is invariant with respect to Möbius transformations. Applying a Möbius transformation M that maps the vertex v to infinity, we make all circles passing through v into straight lines and arrive at the geometry shown in Figure 2, with $P = M(\infty)$. Now the result follows immediately from Corollary 4. $\square$

Theorem 5. *Let S be a simplicial surface without boundary. Then*

$$W(S) \geq 0,$$

and equality holds if and only if S is a convex polyhedron inscribed in a sphere.

Proof. Only the second statement needs to be proved. By, Proposition 2, the equality $W(S) = 0$ implies that all vertices and edges of S are convex (but not necessarily strictly convex). Deleting the edges that separate triangles lying in one plane one obtains a polyhedral surface S_P with circular faces and all strictly convex vertices and edges. Proposition 2 implies that for every vertex v there exists a sphere S_v with all vertices of the star $S(v)$ lying on it. For any edge (v_1, v_2) of S_P two neighboring spheres S_{v_1} and S_{v_2} share two different circles of their common faces. This implies $S_{v_1} = S_{v_2}$ and finally the coincidence of all the spheres S_v. $\square$

The discrete conformal energy W defined above is a discrete analogue of the Willmore energy [1993] for smooth surfaces, which is given by

$$\mathcal{W}(S) = \frac{1}{4}\int_{\mathcal{S}} (k_1 - k_2)^2 \, dA = \int_{\mathcal{S}} H^2 \, dA - \int_{\mathcal{S}} K \, dA.$$

Here dA is the area element, k_1, k_2 the principal curvatures, $H = \frac{1}{2}(k_1 + k_2)$ the mean curvature, $K = k_1 k_2$ the Gaussian curvature of the surface. Here we prefer a definition for $\mathcal{W}$ with a Möbius-invariant integrand. It differs from the one in the introduction by a topological invariant.

We mention two important properties of the Willmore energy:

- $\mathcal{W}(S) \geq 0$, and $\mathcal{W}(S) = 0$ if and only if $\mathcal{S}$ is the round sphere.
- $\mathcal{W}(S)$ (together with the integrand $(k_1 - k_2)^2\, dA$) is Möbius-invariant [Blaschke 1929; Willmore 1993].

Whereas the first statement follows almost immediately from the definition, the second is a nontrivial property. We have shown that the same properties hold for the discrete energy W; in the discrete case Möbius invariance is built into the definition, and the nonnegativity of the energy is nontrivial.

In the same way one can define conformal (Willmore) energy for simplicial surfaces in Euclidean spaces of higher dimensions and space forms.

The discrete conformal energy is well defined for polyhedral surfaces with circular faces (not necessarily simplicial).

3. Computation of the Energy

Consider two triangles with a common edge. Let $a, b, c, d \in \mathbb{R}^3$ be their other edges, oriented as in Figure 3. Identifying vectors in $\mathbb{R}^3$ with imaginary quaternions $\operatorname{Im}\mathbb{H}$ one obtaines for the quaternionic product

$$ab = -\langle a, b\rangle + a \times b, \tag{3–1}$$

where $\langle a, b\rangle$ and $a \times b$ are the scalar and vector products in $\mathbb{R}^3$.

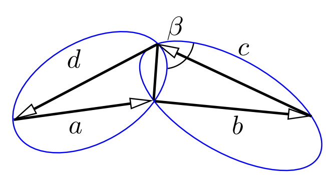

Figure 3. Formula for the angle between circumcircles.

Proposition 6. *The external angle $\beta \in [0, \pi]$ between the circumcircles of the triangles in Figure 3 is given by one of the equivalent formulas*:

$$\cos(\beta) = -\frac{\operatorname{Re} q}{|q|} = -\frac{\operatorname{Re} abcd}{|abcd|} = \frac{\langle a, c\rangle\langle b, d\rangle - \langle a, b\rangle\langle c, d\rangle - \langle b, c\rangle\langle d, a\rangle}{|a|\,|b|\,|c|\,|d|},$$

where $q = ab^{-1}cd^{-1}$ is the cross-ratio of the quadrilateral.

Proof. Since $\operatorname{Re} q$, $|q|$ and β are Möbius-invariant it is enough to prove the first formula for the planar case $a, b, c, d \in \mathbb{C}$, mapping all four vertices to a plane by a Möbius transformation. In this case q becomes the classical complex cross-ratio. Considering the arguments $a, b, c, d \in \mathbb{C}$ one easily arrives at $\beta = \pi - \arg q$.

The second representation follows from the identity $b^{-1} = -b/|b|$ for imaginary quaternions. Finally, applying (3–1) we obtain

$$\operatorname{Re} abcd = \langle a,b\rangle\langle c,d\rangle - \langle a\times b, c\times d\rangle = \langle a,b\rangle\langle c,d\rangle + \langle b,c\rangle\langle d,a\rangle - \langle a,c\rangle\langle b,d\rangle. \quad \square$$

4. Minimizing Discrete Conformal Energy

Similarly to the smooth Willmore functional $\mathcal{W}$, minimizing the discrete conformal energy W makes the surface as round as possible.

Let $\boldsymbol{S}$ denote the combinatorial data of S. The simplicial surface S is called a geometric realization of the abstract simplicial surface $\boldsymbol{S}$.

DEFINITION 7. Critical points of $W(S)$ are called *simplicial Willmore surfaces*. The conformal (Willmore) energy of an abstract simplicial surface is the infimum over all geometric realizations

$$W(\boldsymbol{S}) = \inf_{S\in\boldsymbol{S}} W(S).$$

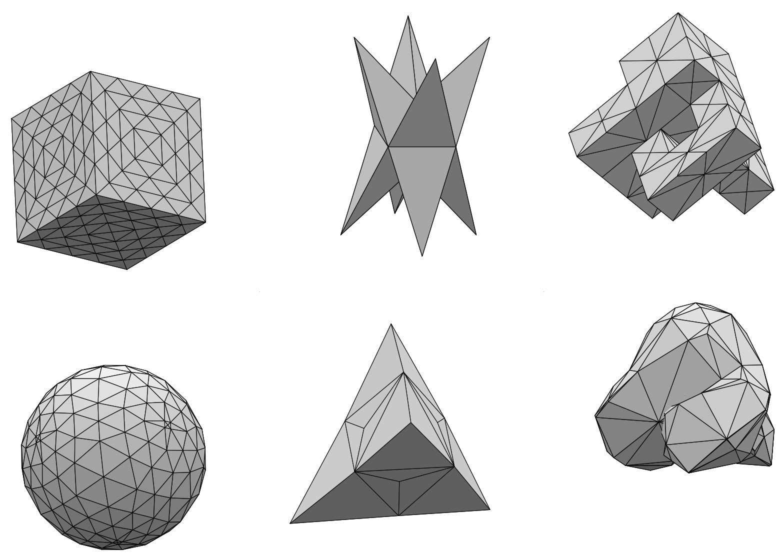

Figure 4. Discrete Willmore spheres of inscribable ($W = 0$) and noninscribable ($W > 0$) type, and discrete Boy surface.

Kevin Bauer implemented the proposed conformal functional with the Brakke's evolver [1992] and ran some numerical minimization experiments, whose results are exemplified in Figure 4. Corresponding entries in each row show initial configurations and the corresponding Willmore surfaces that minimize the conformal energy.

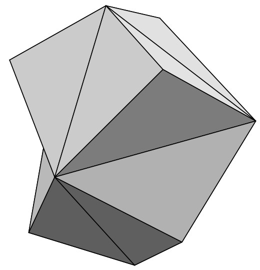

Figure 5. A discrete Willmore sphere of noninscribable type with 11 vertices and $W = 2\pi$.

Define the *discrete Willmore flow* as the gradient flow of the energy W. Under this flow the energy of the first simplicial sphere decreases to zero and the surface evolves into a convex polyhedron with all vertices lying on a sphere. The abstract simplicial surface of the central example is different and we obtain a simplicial Willmore sphere with positive conformal energy.

The rightmost example in the figure is a simplicial projective plane. The initial configuration is made from squares divided into triangles; see [Petit 1995]. We see that the minimum is close to the smooth Boy surface known (by [Karcher and Pinkall 1997]) to minimize the Willmore energy for projective planes.

The minimization of the conformal energy for simplicial spheres is related to a classical result of Steinitz [1928], who showed that there exist abstract simplicial 3-polytopes without geometric realizations all of whose vertices belong to a sphere. We call these combinatorial types *noninscribable*.

The noninscribable examples of Steinitz are constructed as follows [Grünbaum 2003]. Let $\boldsymbol{S}$ be an abstract simplicial sphere with vertices colored black and white. Denote the sets of white and black vertices by V_w and V_b respectively, so $V = V_w \cup V_b$. Assume that $|V_w| > |V_b|$ and that there are no edges connecting two white vertices. It is easy to see that $\boldsymbol{S}$ with these properties cannot be inscribed in a sphere. Indeed, assume that we have constructed such an inscribed convex polyhedron. Then the equality of the intersection angles at both ends of an edge (see left Figure 1) implies that

$$2\pi |V_b| \geq \sum_{e \in E} \beta(e) \geq 2\pi |V_w|.$$

This contradiction of the assumed inequality implies the claim.

To construct abstract polyhedra with $|V_w| > |V_b|$, take a polyhedron $\boldsymbol{P}$ whose number of vertices does not exceed the number of faces, $|\hat{F}| > |\hat{V}|$. Color all the vertices black, add white vertices at the faces and connect them to all black vertices of a face. This yields a polyhedron with black (original) edges and $|V_w| = |\hat{F}| > |V_b| = |\hat{V}|$. The example with minimal possible number of vertices $|V| = 11$ is shown in Figure 5. The starting polyhedron $\boldsymbol{P}$ here consists of two tetrahedra identified along a common face: $\hat{F} = 6$, $\hat{V} = 5$.

Hodgson, Rivin and Smith [Hodgson et al. 1992] have found a characterization of inscribable combinatorial types, based on a transfer to the Klein model of hyperbolic 3-space. It is not clear whether there exist noninscribable examples of non-Steinitz type.

Numerical experiments lead us to:

Conjecture 8. *The conformal energy of simplicial Willmore spheres is quantized*:

$$W = 2\pi N, \quad \text{for } N \in \mathbb{N}.$$

This statement belongs to differential geometry of discrete surfaces. It would be interesting to find a (combinatorial) meaning of the integer N. Compare also with the famous classification of smooth Willmore spheres by Bryant [1984], who showed that the energy of Willmore spheres is quantized by $\mathcal{W} = 4\pi N$, $N \in \mathbb{N}$.

The discrete Willmore energy is defined for ambient spaces ($\mathbb{R}^n$ or S^n) of any dimension. This leads to combinatorial Willmore energies

$$W_n(\boldsymbol{S}) = \inf_{S \in \boldsymbol{S}} W(S), \qquad S \subset S^n,$$

where the infimum is taken over all realizations in the n-dimensional sphere. Obviously these numbers build a nonincreasing sequence $W_n(\boldsymbol{S}) \geq W_{n+1}(\boldsymbol{S})$ that becomes constant for sufficiently large n.

Complete understanding of noninscribable simplicial spheres is an interesting mathematical problem. However the phenomenon of existence of such spheres might be seen as a problem in using of the conformal functional for applications in computer graphics, such as the fairing of surfaces. Fortunately the problem disappears after just one refinement step: all simplicial spheres become inscribable. Let $\boldsymbol{S}$ be an abstract simplicial sphere. Define its refinement $\boldsymbol{S}_R$ as follows: split every edge of $\boldsymbol{S}$ into two by putting additional vertices and connect these new vertices sharing a face of $\boldsymbol{S}$ by additional edges.

Proposition 9. *The refined simplicial sphere $\boldsymbol{S}_R$ is inscribable*, *and thus* $W(\boldsymbol{S}_R) = 0$.

Proof. Koebe's theorem (see [Ziegler 1995; Bobenko and Springborn 2004], for example) states that every abstract simplicial sphere $\boldsymbol{S}$ can be realized as a convex polyhedron S all of whose edges touch a common sphere S^2. Starting with this realization S it is easy to construct a geometric realization S_R of the refinement $\boldsymbol{S}_R$ inscribed in S^2. Indeed, choose the touching points of the edges of S with S^2 as additional vertices of S_R and project the original vertices of S (which lie outside of the sphere S^2) to S^2. One obtains a convex simplicial polyhedron S_R inscribed in S^2. $\square$

Another interesting variational problem involving the conformal energy is the optimization of triangulations of a given simplicial surface. Here one fixes the vertices and chooses an equivalent triangulation (abstract simplicial surface $\boldsymbol{S}$) minimizing the conformal functional. The minimum

$$W(V) = \min_{\boldsymbol{S}} W(S)$$

yields an "optimal" triangulation for a given vertex data. In the case of S^2 this optimal triangulation is well known.

PROPOSITION 10. *Let S be a simplicial surface with all vertices V on a two-dimensional sphere S^2. Then $W(S) = 0$ if and only if it is the Delaunay triangulation on the sphere*, *i.e.*, *S is the boundary of the convex hull of V.*

In differential geometric applications such as the numerical minimization of the Willmore energy of smooth surfaces (see [Hsu et al. 1992]) it is not natural to preserve the triangulation by minimizing the energy, and one should also change the combinatorial type decreasing the energy.

The discrete conformal energy W is not just a discrete analogue of the Willmore energy. One can show that it approximates the smooth Willmore energy, although the smooth limit is very sensitive to the refinement method and must be chosen in a special way. A computation (to be published elsewhere) shows that if one chooses the vertices of a curvature line net of a smooth surface $\mathcal{S}$ for the vertices of S and triangularizes it, $W(S)$ converges to $\mathcal{W}(S)$ by natural refinement. On the other hand, the infinitesimal equilateral triangular lattice gives in the limit and energy half again higher. Possibly the minimization of the discrete Willmore energy with vertices on the smooth surface could be used for the computation of the curvature line net. We will be investigating this interesting and complicated phenomenon.

5. Bending of Simplicial Surfaces

An accurate model for the bending of discrete surfaces is important for modeling in virtual reality.

Let $\mathcal{S}_0$ be a thin shell and $\mathcal{S}$ its deformation. The bending energy of smooth thin shells is given by the integral [Grinspun et al. 2003]

$$E = \int (H - H_0)^2 \, dA,$$

where H_0 and H are the mean curvatures of the original and deformed surface respectively. For $H_0 = 0$ it reduces to the Willmore energy.

To derive the bending energy for simplicial surfaces let us consider the limit of fine triangulation, i.e. of small angles between the normals of neighboring triangles. Consider an isometric deformation of two adjacent triangles. Let θ be the complement of the dihedral angle of the edge e, or, equivalently, the angle between the normals of these triangles (see Figure 6) and $\beta(\theta)$ the external intersection angle between the circumcircles of the triangles (see Figure 1) as a function of θ.

Proposition 11. *Assume that the circumcenters of the circumcircles of two adjacent triangles do not coincide. In the limit of small angles $\theta \to 0$, the angle β between the circles behaves as*

$$\beta(\theta) = \beta(0) + \frac{l}{L}\theta^2 + o(\theta^3),$$

where l is the length of the edge and $L \neq 0$ is the distance between the centers of the circles.

This proposition and our definition of conformal energy for simplicial surfaces motivate to suggest

$$E = \sum_{e \in E} \frac{l}{L}\theta^2$$

for the bending energy of discrete thin-shells.

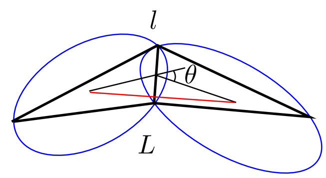

Figure 6. Toward the definition of the bending energy for simplicial surfaces.

In [Bridson et al. 2003; Grinspun et al. 2003] similar representations for the bending energy of simplicial surfaces were found empirically. They were demonstrated to give convincing simulations and good comparison with real processes. In [Grinspun et al. 2003] the distance between the barycenters is used for L in the energy expression but possible numerical advantages in using circumcenters are indicated.

Using the Willmore energy and Willmore flow is a hot topic in computer graphics. Applications include fairing of surfaces and surface restoration. We hope that our conformal energy will be useful for these applications and plan to work on them.

Acknowledgements

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