Riemann–Finsler Geometry
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Landsberg Curvature, S-Curvature and Riemann Curvature

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Contents

1. Introduction

Roughly speaking, Finsler metrics on a manifold are regular, but not necessarily reversible, distance functions. In 1854, B. Riemann attempted to study a special class of Finsler metrics — Riemannian metrics — and introduced what is now called the Riemann curvature. This infinitesimal quantity faithfully reveals the local geometry of a Riemannian manifold and becomes the central concept of Riemannian geometry. It is a natural problem to understand general regular distance functions by introducing suitable infinitesimal quantities. For more than half a century, there had been no essential progress until P. Finsler studied the variational problem in a Finsler manifold. However, it was L. Berwald who first successfully extended the notion of Riemann curvature to Finsler metrics by introducing what is now called the Berwald connection. He also introduced some non-Riemannian quantities via his connection [Berwald 1926; 1928]. Since then, Finsler geometry has been developed gradually.

The Riemann curvature is defined using the induced spray, which is independent of any well-known connection in Finsler geometry. It measures the shape of the space. The Cartan torsion and the distortion are two primary geometric quantities describing the geometric properties of the Minkowski norm in each tangent space. Differentiating them along geodesics gives rise to the Landsberg curvature and the S-curvature. These quantities describe the rates of change of the "color pattern" on the space.

In this article, I am going to discuss the geometric meaning of the Landsberg curvature, the S-curvature, the Riemann curvature, and their relationship. I will give detailed proofs for several important local and global results.

2. Finsler Metrics

By definition, a Finsler metric on a manifold is a family of Minkowski norms on the tangent spaces. A *Minkowski norm* on a vector space V is a nonnegative function $F : \mathrm{V} \to [0, \infty)$ with the following properties:

(i) F is positively y-homogeneous of degree one, i.e., $F(\lambda y) = \lambda F(y)$ for any $y \in \mathrm{V}$ and any $\lambda > 0$;
(ii) F is C^∞ on $\mathrm{V} \setminus \{0\}$ and for any tangent vector $y \in \mathrm{V} \setminus \{0\}$, the following bilinear symmetric form $\boldsymbol{g}_y : \mathrm{V} \times \mathrm{V} \to \mathrm{R}$ is positive definite:

$$\boldsymbol{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} \big(F^2(y + su + tv) \big)|_{s=t=0}.$$

A Minkowski norm is said to be *reversible* if $F(-y) = F(y)$ for $y \in \mathrm{V}$. In this article, Minkowski norms are not assumed to be reversible. From (i) and (ii), one can show that $F(y) > 0$ for $y \neq 0$ and $F(u + v) \leq F(u) + F(v)$ for $u, v \in \mathrm{V}$. See [Bao et al. 2000] for a proof.

Let $\langle\ ,\ \rangle$ denote the standard inner product on R^n, defined by $\langle u, v \rangle := \sum_{i=1}^n u^i v^i$. Then $|y| := \sqrt{\langle y, y \rangle}$ is called the standard Euclidean norm on R^n. Let $b \in \mathrm{R}^n$ with $|b| < 1$, then $F = |y| + \langle b, y \rangle$ is a Minkowski norm on R^n, which is called a *Randers norm.*

Let M be a connected, n-dimensional, C^∞ manifold. Let $TM = \bigcup_{x \in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We denote a typical point in TM by (x, y), where $y \in T_xM$, and set $TM_0 := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x, 0) \mid x \in X,\, 0 \in T_xM\}$. A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:

(a) F is C^∞ on TM_0;
(b) At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

The pair (M, F) is called a *Finsler manifold*.

Let (M, F) be a Finsler manifold. Let (x^i, y^i) be a standard local coordinate system in TM, i.e., y^i's are determined by $y = y^i(\partial/\partial x^i)|_x$. For a vector $y = y^i(\partial/\partial x^i)|_x \neq 0$, let $g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}(x,y)$. The induced inner product $\boldsymbol{g}_y$ is given by

$$\boldsymbol{g}_y(u,v) = g_{ij}(x,y)u^iv^j,$$

where $u = u^i(\partial/\partial x^i)|_x$ and $v = v^i(\partial/\partial x^i)|_x$. By the homogeneity of F,

$$F(x,y) = \sqrt{\boldsymbol{g}_y(y,y)} = \sqrt{g_{ij}(x,y)y^iy^j}.$$

A Finsler metric $F = F(x,y)$ is called a *Riemannian metric* if the $g_{ij} = g_{ij}(x)$ are functions of $x \in M$ only.

There are three special Riemannian metrics.

Example 2.1 (Euclidean metric). The simplest metric is the Euclidean metric $\alpha_0 = \alpha_0(x,y)$ on R^n, which is defined by

$$\alpha_0(x,y) := |y|, \quad y = (y^i) \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{2–1}$$

We will simply denote (R^n, α_0) by R^n, which is called *Euclidean space*.

Example 2.2 (Spherical metric). Let $\mathrm{S}^n := \{x \in \mathrm{R}^{n+1} \mid |x| = 1\}$ denote the standard unit sphere in R^{n+1}. Every tangent vector $y \in T_x\mathrm{S}^n$ can be identified with a vector in R^{n+1} in a natural way. The induced metric α_{+1} on S^n is defined by $\alpha_{+1} = \|y\|_x$, for $y \in T_x\mathrm{S}^n \subset \mathrm{R}^{n+1}$, where $\|\cdot\|_x$ denotes the induced Euclidean norm on $T_x\mathrm{S}^n$. Let $\varphi : \mathrm{R}^n \to \mathrm{S}^n \subset \mathrm{R}^{n+1}$ be defined by

$$\varphi(x) := \left(\frac{x}{\sqrt{1+|x|^2}}, \frac{1}{\sqrt{1+|x|^2}}\right). \tag{2–2}$$

Then φ pulls back α_{+1} on the upper hemisphere to a Riemannian metric on R^n, which is given by

$$\alpha_{+1} = \frac{\sqrt{|y|^2 + (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+|x|^2}, \quad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n. \tag{2–3}$$

Example 2.3 (Hyperbolic metric). Let B^n denote the unit ball in R^n. Define

$$\alpha_{-1} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1-|x|^2}, \quad y \in T_x\mathrm{B}^n \cong \mathrm{R}^n. \tag{2–4}$$

We call α_{-1} the *Klein metric* and denote $(\mathrm{B}^n, \alpha_{-1})$ by H^n.

The metrics (2–1), (2–3) and (2–4) can be combined into one formula:

$$\alpha_\mu = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{1+\mu|x|^2}. \tag{2–5}$$

Of course, there are many non-Riemannian Finsler metrics on R^n with special geometric properties. We just list some of them below and discuss their geometric properties later.

Example 2.4 (Funk metric). Let

$$\Theta := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{1-|x|^2}, \quad y\in T_x\mathrm{B}^n\cong \mathrm{R}^n. \tag{2–6}$$

$\Theta=\Theta(x,y)$ is a Finsler metric on B^n, called the *Funk metric* on B^n.

For an arbitrary constant vector $a\in\mathrm{R}^n$ with $|a|<1$, let

$$\Theta_a := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{1-|x|^2}+\frac{\langle a,y\rangle}{1+\langle a,x\rangle}. \tag{2–7}$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. $\Theta_a=\Theta_a(x,y)$ is a Finsler metric on B^n. Note that $\Theta_0=\Theta$ is the Funk metric on B^n. We call Θ_a the *generalized Funk metric* on B^n [Shen 2003a].

Example 2.5 [Shen 2003b]. Let δ be an arbitrary number with $\delta<1$. Let

$$F_\delta := \frac{\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle}{2(1-|x|^2)}-\delta\frac{\sqrt{|y|^2-\delta^2(|x|^2|y|^2-\langle x,y\rangle^2)}+\delta\langle x,y\rangle}{2(1-\delta^2|x|^2)},$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. F_δ is a Finsler metric on B^n. Note that $F_{-1}=\alpha_{-1}$ is the Klein metric on B^n. Let Θ be the Funk metric on B^n defined in (2–6). We can express F_δ by

$$F_\delta=\tfrac{1}{2}\big(\Theta(x,y)-\delta\Theta(\delta x,y)\big).$$

Example 2.6 [Berwald 1929b]. Let

$$B := \frac{\big(\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}+\langle x,y\rangle\big)^2}{(1-|x|^2)^2\sqrt{|y|^2-(|x|^2|y|^2-\langle x,y\rangle^2)}}, \tag{2–8}$$

where $y\in T_x\mathrm{B}^n\cong\mathrm{R}^n$. Then $B=B(x,y)$ is a Finsler metric on B^n.

Example 2.7. Let ε be an arbitrary number with $|\varepsilon|<1$. Let

$$F_\varepsilon := \frac{\sqrt{\Psi\big(\frac{1}{2}(\sqrt{\Phi^2+(1-\varepsilon^2)|y|^4}+\Phi)\big)+(1-\varepsilon^2)\langle x,y\rangle^2}+\sqrt{1-\varepsilon^2}\langle x,y\rangle}{\Psi}, \tag{2–9}$$

where

$$\Phi:=\varepsilon|y|^2+(|x|^2|y|^2-\langle x,y\rangle^2), \quad \Psi:=1+2\varepsilon|x|^2+|x|^4.$$

$F_\varepsilon=F_\varepsilon(x,y)$ is a Finsler metric on R^n. Note that if $\varepsilon=1$, then $F_1=\alpha_{+1}$ is the spherical metric on R^n.

R. Bryant [1996; 1997] classified Finsler metrics on the standard unit sphere S^2 with constant flag curvature equal to $+1$ and geodesics being great circles. The Finsler metrics F_ε in (2–9) is a special family of Bryant's metrics expressed in a local coordinate system. See Example 12.7 for further discussion.

The examples of Finsler metrics above all have special geometric properties. They are locally projectively flat with constant flag curvature. Some belong to the class of (α,β)-*metrics*, that is, those of the form

$$F = \alpha\,\phi\Big(\frac{\beta}{\alpha}\Big), \tag{2–10}$$

where $\alpha = \alpha_x(y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = \beta_x(y) = b_i(x)y^i$ is a 1-form, and ϕ is a C^∞ positive function on some interval $I = [-r, r]$ big enough that $r \ge \beta/\alpha$ for all x and $y \in T_xM$. It is easy to see that any such F is positively homogeneous of degree one. Let $\|\beta\|_x := \sup_{y\in T_xM} \beta_x(y)/\alpha_x(y)$. Using a Maple program, we find that the Hessian $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i\alpha_j + b_j\alpha_i) + \rho_2\alpha_i\alpha_j,$$

where $\alpha_i = \alpha_{y^i}$ and

$$\rho = \phi^2 - s\phi\phi', \qquad \rho_0 = \phi\phi'' + \phi'\phi',$$
$$\rho_1 = -s(\phi\phi'' + \phi'\phi') + \phi\phi', \quad \rho_2 = s^2(\phi\phi'' + \phi'\phi') - s\phi\phi',$$

where $s := \beta/\alpha$ with $|s| \le \|\beta\|_x \le r$. Then

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}\big((\phi - s\phi') + (\|\beta\|_x^2 - s^2)\phi''\big)\det(a_{ij}).$$

If $\phi = \phi(s)$ satisfies

$$\phi(s) > 0, \quad \phi(s) - s\phi'(s) > 0, \quad \big((\phi - s\phi') + (b^2 - s^2)\phi''(s)\big) \ge 0 \tag{2–11}$$

for all s with $|s| \le b \le r$, then (g_{ij}) is positive definite; hence F is a Finsler metric.

Sabau and Shimada [2001] have classified (α,β)-metrics into several classes and computed the Hessian g_{ij} for each class. Below are some special (α,β)-metrics.

(a) $\phi(s) = 1 + s$. The defined function $F = \alpha + \beta$ is a Finsler metric if and only if the norm of β with respect to α is less than 1 at any point:

$$\|\beta\|_x := \sqrt{a^{ij}(x)b_i(x)b_j(x)} < 1, \quad x \in M.$$

A Finsler metric in this form is called a *Randers metric*. The Finsler metrics in Example 2.4 are Randers metrics. The Finsler metrics in Example 2.5 is the sum of two Randers metrics.

(b) $\phi(s) = (1 + s)^2$. The defined function $F = (\alpha + \beta)^2/\alpha$ is a Finsler metric if and only if $\|\beta\|_x < 1$ at any point $x \in M$. The Finsler metric in Example 2.6 is in this form.

By a *Finsler structure* on a manifold M we usually mean a Finsler metric. Sometimes, we also define a Finsler structure as a scalar function F^* on T^*M such that F^* is C^∞ on $T^*M \setminus \{0\}$ and $F^*_x := F^*|_{T^*_xM}$ is a Minkowski norm on

T_x^*M for $x \in M$. Such a function is called a *co-Finsler metric*. Given a co-Finsler metric, one can define a Finsler metric via the standard duality defined below.

Let $F^* = F^*(x,\xi)$ be a co-Finsler metric on a manifold M. Define a non-negative scalar function $F = F(x,y)$ on TM by

$$F(x,y) := \sup_{\xi \in T_x^*M} \frac{\xi(y)}{F^*(x,\xi)}.$$

Then $F = F(x,y)$ is a Finsler metric on M, said to be *dual to* F^*. In the same sense, $F^* = F^*(x,\xi)$ is also dual to F:

$$F^*(x,\xi) = \sup_{y \in T_xM} \frac{\xi(y)}{F(x,y)}.$$

Every vector $y \in T_xM \setminus \{0\}$ uniquely determines a covector $\xi \in T_x^*M \setminus \{0\}$ by

$$\xi(w) := \frac{1}{2}\frac{d}{dt}\big(F^2(x, y+tw)\big)|_{t=0}, \quad w \in T_xM.$$

The resulting map $\ell_x : y \in T_xM \to \xi \in T_x^*M$ is called the *Legendre transformation* at x. Similarly, every covector $\xi \in T_x^*M \setminus \{0\}$ uniquely determines a vector $y \in T_xM \setminus \{0\}$ by

$$\eta(y) := \frac{1}{2}\frac{d}{dt}\big(F^{*2}(x, \xi+t\eta)\big)|_{t=0}, \quad \eta \in T_x^*M.$$

The resulting map $\ell_x^* : \xi \in T_x^*M \to y \in T_xM$ is called the *inverse Legendre transformation* at x. Indeed, ℓ_x and ℓ_x^* are inverses of each other. Moreover, they preserve the Minkowski norms:

$$F(x,y) = F^*(x, \ell_x(y)), \quad F^*(x,\xi) = F(x, \ell_x^*(\xi)). \tag{2–12}$$

Let $\Phi = \Phi(x,y)$ be a Finsler metric on a manifold M and let $\Phi^* = \Phi^*(x,\xi)$ be the co-Finsler metric dual to Φ. By the above formulas, one can easily show that if $y \in T_xM \setminus \{0\}$ and $\xi \in T_x^*M \setminus \{0\}$ satisfy

$$\frac{d}{dt}\big(\Phi^*(x, \xi+t\eta)\big)|_{t=0} = \eta(y), \quad \eta \in T_x^*M.$$

Then

$$\Phi(x,y) = 1. \tag{2–13}$$

Let V be a vector field on M with $\Phi(x, -V_x) < 1$ and let $V^* : T^*M \to [0,\infty)$ denote the 1-form dual to V, defined by

$$V_x^*(\xi) = \xi(V_x), \quad \xi \in T_x^*M.$$

We have $\Phi^*(x, -V_x^*) = \Phi(x, -V_x) < 1$. Thus $F^* := \Phi^* + V^*$ is a co-Finsler metric on M. Define $F = F(x,y)$ by

$$F(x,y) := \sup_{\xi \in T_x^*M} \frac{\xi(y)}{F^*(x,\xi)}, \quad y \in T_xM. \tag{2–14}$$

F is a Finsler metric on M, called the *Finsler metric generated from* the pair (Φ, V). One can also define F in a different way without using the duality:

LEMMA 2.8. *Let $\Phi = \Phi(x, y)$ be a Finsler metric on M and let V be a vector field on M with $\Phi(x, -V_x) < 1$ for all $x \in M$. Then $F = F(x, y)$ defined in* (2–14) *satisfies*

$$\Phi\Big(x,\ \frac{y}{F(x,y)} - V_x\Big) = 1, \quad y \in T_xM. \tag{2–15}$$

Conversely, if $F = F(x, y)$ is defined by (2–15), *it is dual to the co-Finsler metric $F^* := \Phi^* + V^*$ as defined in* (2–14).

PROOF. For the co-Finsler metric $F^* = \Phi^* + V^*$, let $F = F(x, y)$ be defined in (2–14). Fix an arbitrary nonzero vector $y \in T_xM$. There is a covector $\xi \in T_x^*M$ such that

$$F(x,y) = \frac{\xi(y)}{F^*(x,\xi)}. \tag{2–16}$$

Let $\eta \in T_x^*M$ be an arbitrary covector. Consider the function

$$h(t) := \frac{\xi(y) + t\eta(y)}{\Phi^*(x, \xi + t\eta) + \xi(V_x) + t\eta(V_x)}.$$

Then $h(t) \le h(0) = F(x, y)$. Thus $h'(0) = 0$:

$$\eta(y)F^*(x,\xi) - \xi(y)\Big(\frac{d}{dt}\big(\Phi^*(x,\xi+t\eta)\big)|_{t=0} + \eta(V_x)\Big) = 0.$$

Dividing by $F^*(x, \xi)$ and using (2–16), one obtains

$$\eta(y) - F(x,y)\Big(\frac{d}{dt}\big(\Phi^*(x,\xi+t\eta)\big)|_{t=0} + \eta(V_x)\Big) = 0.$$

From this identity it follows that

$$\frac{d}{dt}\big(\Phi^*(x,\xi+t\eta)\big)|_{t=0} = \eta\Big(\frac{y}{F(x,y)} - V_x\Big), \quad \eta \in T_x^*M.$$

Thus $F(x, y)$ satisfies (2–15) as we have explained in (2–13).

Conversely, let $F = F(x, y)$ be defined by (2–15). Then for any $\xi \in T_x^*M$,

$$\Phi^*(x,\xi) = \sup_{y\in T_xM} \eta\Big(\frac{y}{F(x,y)} - V_x\Big).$$

One obtains

$$\sup_{y\in T_xM}\frac{\xi(y)}{F(x,y)} = \sup_{y\in T_xM}\xi\Big(\frac{y}{F(x,y)} - V_x\Big) + \xi(V_x) = \Phi^*(x,\xi) + V_x^*(\xi) = F^*(x,\xi).$$

Thus F^* is dual to F and so F is dual to F^*, that is, F is given by (2–14). □

Let $\Phi = \sqrt{\phi_{ij}(x)y^iy^j}$ be a Riemannian metric and let $V = V^i(x)(\partial/\partial x^i)$ be a vector field on a manifold M with

$$\Phi(x, -V_x) = \|V\|_x := \sqrt{\phi_{ij}(x)V^i(x)V^j(x)} < 1, \quad x \in M.$$

Solving (2–15) for $F = F(x, y)$, one obtains

$$F = \frac{\sqrt{(1 - \phi_{ij}V^iV^j)\phi_{ij}y^iy^j + (\phi_{ij}y^iV^j)^2} - \phi_{ij}y^iV^j}{1 - \phi_{ij}V^iV^j}. \tag{2–17}$$

Clearly, F is a Randers metric. It is easy to verify that any Randers metric $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$, can be expressed in the form (2–17). According to Lemma 2.8, any Randers metric $F = \alpha + \beta$ expressed in the form (2–17) can be constructed in the following way. Let $\Phi^* := \sqrt{\phi^{ij}(x)\xi_i\xi_j}$ be the Riemannian metric dual to $\Phi = \sqrt{\phi_{ij}(x)y^iy^j}$ and $V^* := \xi(V_x) = V^i(x)\xi_i$ be the 1-form dual to V. Then $F^* := \Phi^*(x, \xi) + V^*(\xi) = \sqrt{\phi^{ij}(x)\xi_i\xi_j} + V^i(x)\xi_i$ is a co-Finsler metric on M. Moreover, the dual Finsler metric F of F^* is given by (2–17). This is proved in [Hrimiuc and Shimada 1996].

It was discovered in [Shen 2003c; 2002] that if Φ is a Riemannian metric of constant curvature and V is a special vector field, the generated metric F is of constant flag curvature. This discovery opens the door to a classification of Randers metrics of constant flag curvature [Bao et al. 2003]. But Maple programs played an important role in the computations that led to it.

EXAMPLE 2.9. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and let

$$U_\phi := \big\{y \in \mathrm{R}^n \mid \phi(y) < 1\big\}.$$

Define

$$\Phi(x, y) := \phi(y), \quad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n.$$

$\Phi = \Phi(x, y)$ is called a *Minkowski metric* on R^n. Let $V_x := -x$, for $x \in \mathrm{R}^n$. V is a radial vector field pointing toward the origin. For any $x \in U_\phi$,

$$\Phi\big(x, -V_x\big) = \phi(x) < 1.$$

The pair (Φ, V) generates a Finsler metric $\Theta = \Theta(x, y)$ on U_ϕ by (2–15):

$$\Theta(x, y) = \phi\big(y + \Theta(x, y)x\big). \tag{2–18}$$

Differentiating with respect to x^k and y^k separately, one obtains

$$\big(1 - \phi_{w^l}(w)x^l\big)\Theta_{x^k}(x, y) = \phi_{w^k}(w)\Theta(x, y),$$
$$\big(1 - \phi_{w^l}(w)x^l\big)\Theta_{y^k}(x, y) = \phi_{w^k}(w),$$

where $w := y + \Theta(x, y)x$. It follows that

$$\Theta_{x^k}(x, y) = \Theta(x, y)\Theta_{y^k}(x, y). \tag{2–19}$$

This argument is from [Okada 1983].

A domain U_ϕ in R^n defined by a Minkowski norm ϕ is called a *strongly convex domain*. A Finsler metric $\Theta = \Theta(x, y)$ defined in (2–18) is called the *Funk metric* on a strongly convex domain in R^n. When $\phi = |y|$ is the standard Euclidean metric on R^n, $U_\phi = \mathrm{B}^n$ is the standard unit ball and $\Theta = \Theta(x, y)$ is given by (2–6). Equation (2–19) is the key property of Θ, from which one can derive other geometric properties of Θ.

Definition 2.10. A Finsler function $\Theta = \Theta(x, y)$ on an open subset in R^n is called a *Funk metric* if it satisfies (2–19).

Example 2.11. Let $\Phi(x, y) := |y|$ be the standard Euclidean metric on R^n and let $V = V(x)$ be a vector field on R^n defined by

$$V_x := |x|^2 a - 2\langle a, x\rangle x,$$

where $a \in \mathrm{R}^n$ is a constant vector. Note that

$$\Phi(x, -V_x) = \sqrt{\phi_{ij} V^i V^j} = |V_x| = |a||x|^2 < 1, \quad x \in \mathrm{B}^n(1/\sqrt{|a|}),$$

and that

$$\phi_{ij} y^i V^j = |x|^2 \langle a, y\rangle - 2\,\langle a, x\rangle\,\langle x, y\rangle.$$

Given the pair (Φ, V) above, one obtains, by solving (2–15) for F,

$$F = \frac{\sqrt{\big(|x|^2\langle a,y\rangle - 2\langle a,x\rangle\langle x,y\rangle\big)^2 + |y|^2\big(1-|a|^2|x|^4\big)} - \big(|x|^2\langle a,y\rangle - 2\langle a,x\rangle\langle x,y\rangle\big)}{1 - |a|^2|x|^4}. \tag{2–20}$$

This Randers metric F has very important properties. It is of scalar curvature and isotropic S-curvature. But the flag curvature and the S-curvature are not constant. See Example 11.2 below for further discussion.

3. Cartan Torsion and Matsumoto Torsion

To characterize Euclidean norms, E. Cartan [1934] introduced what is now called the Cartan torsion. Let $F = F(y)$ be a Minkowski norm on a vector space V. Fix a basis $\{\boldsymbol{b}_i\}$ for V. Then $F = F(y^i \boldsymbol{b}_i)$ is a function of (y^i). Let

$$g_{ij} := \tfrac{1}{2}[F^2]_{y^i y^j}, \quad C_{ijk} := \tfrac{1}{4}[F^2]_{y^i y^j y^k}(y), \quad I_i := g^{jk}(y) C_{ijk}(y),$$

where $(g^{ij}) := (g_{ij})^{-1}$. It is easy to see that

$$I_i = \frac{\partial}{\partial y^i}\big(\ln \sqrt{\det(g_{jk})}\,\big). \tag{3–1}$$

For $y \in \mathrm{V} \setminus \{0\}$, set

$$\boldsymbol{C}_y(u, v, w) := C_{ijk}(y) u^i v^j w^k, \quad \boldsymbol{I}_y(u) := I_i(y) u^i,$$

where $u := u^i \boldsymbol{b}_i$, $v := v^j \boldsymbol{b}_j$ and $w := w^k \boldsymbol{b}_k$. The family $\boldsymbol{C} := \{\boldsymbol{C}_y \mid y \in \mathrm{V} \setminus \{0\}\}$ is called the *Cartan torsion* and the family $\boldsymbol{I} := \{\boldsymbol{I}_y \mid y \in \mathrm{V} \setminus \{0\}\}$ is called the *mean Cartan torsion*. They are not tensors in a usual sense. In later sections, we will convert them to tensors on TM_0 and call them the (mean) Cartan tensor.

We view a Minkowski norm F on a vector space V as a *color pattern*. When F is Euclidean, the color pattern is *trivial* or *Euclidean*. The Cartan torsion $\boldsymbol{C}_y$ describes the non-Euclidean features of the color pattern in the direction $y \in \mathrm{V} \setminus \{0\}$. And the mean Cartan torsion $\boldsymbol{I}_y$ is the average value of $\boldsymbol{C}_y$.

A trivial fact is that a Minkowski norm F on a vector space V is Euclidean if and only if $\boldsymbol{C}_y = 0$ for all $y \in \mathrm{V} \setminus \{0\}$. This can be improved:

PROPOSITION 3.1 [Deicke 1953]. *A Minkowski norm is Euclidean if and only if* $\boldsymbol{I} = 0$.

To characterize Randers norms, M. Matsumoto introduces the quantity

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\big(I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\big), \tag{3–2}$$

where $h_{ij} := FF_{y^iy^j} = g_{ij} - g_{ip}y^p g_{jq} y^q/F^2$. For $y \in \mathrm{V} \setminus \{0\}$, set

$$\boldsymbol{M}_y(u,v,w) := M_{ijk}(y)u^i v^j w^k,$$

where $u = u^i \boldsymbol{b}_i$, $v = v^j \boldsymbol{b}_j$ and $w = w^k \boldsymbol{b}_k$. The family $\boldsymbol{M} := \{\boldsymbol{M}_y \mid y \in \mathrm{V}\setminus\{0\}\}$ is called the *Matsumoto torsion*. A Minkowski norm is called *C-reducible* if $\boldsymbol{M} = 0$.

LEMMA 3.2 [Matsumoto 1972b]. *Every Randers metric satisfies* $\boldsymbol{M} = 0$.

PROOF. Let $F = \alpha + \beta$ be an arbitrary Randers norm on a vector space V, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i y^i$ with $\|\beta\|_\alpha < 1$. By a direct computation, the $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ are given by

$$g_{ij} = \frac{F}{\alpha}\left(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha} + \frac{\alpha}{F}\Big(b_i + \frac{y_i}{\alpha}\Big)\Big(b_j + \frac{y_j}{\alpha}\Big)\right), \tag{3–3}$$

where $y_i := a_{ij}y^j$. The $h_{ij} = FF_{y^iy^j} = g_{ij} - g_{ip}y^p g_{jq}y^q/F^2$ are given by

$$h_{ij} = \frac{\alpha+\beta}{\alpha}\Big(a_{ij} - \frac{y_i y_j}{\alpha^2}\Big). \tag{3–4}$$

The inverse matrix $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$g^{ij} = \frac{\alpha}{F}\left(a^{ij} - (1-\|\beta\|^2)\frac{y^i}{F}\frac{y^j}{F} + \frac{\alpha}{F}\left(\Big(b^i - \frac{y^i}{\alpha}\Big)\Big(b^j - \frac{y^j}{F}\Big) - b^i b^j\right)\right). \tag{3–5}$$

The determinant $\det(g_{ij})$ is

$$\det(g_{ij}) = \Big(\frac{\alpha+\beta}{\alpha}\Big)^{n+1}\det(a_{ij}).$$

From this and (3–1), one obtains

$$I_i = \frac{n+1}{2(\alpha+\beta)}\Big(b_i - \frac{y_i}{\alpha}\frac{\beta}{\alpha}\Big). \tag{3–6}$$

Differentiating (3–3) yields

$$C_{ijk} = \frac{1}{n+1}\big(I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\big), \tag{3–7}$$

implying that $M_{ijk} = 0$. □

Matsumoto and Hōjō later proved that the converse is true as well, if $\dim \mathrm{V} \geq 3$.

PROPOSITION 3.3 [Matsumoto 1972b; Matsumoto and Hōjō 1978]. *If F is a Minkowski norm on a vector space* V *of dimension at least* 3, *the Matsumoto torsion of F vanishes if and only if F is a Randers norm.*

Their proof is long and I could not find a shorter proof which fits into this article.

4. Geodesics and Sprays

Every Finsler metric F on a connected manifold M defines a length structure L_F on oriented curves in M. Let $c : [a, b] \to M$ be a piecewise C^∞ curve. The *length* of c is defined by

$$L_F(c) := \int_a^b F\big(c(t), \dot{c}(t)\big)\, dt.$$

For any two points $p, q \in M$, define

$$d_F(p, q) := \inf_c L_F(c),$$

where the infimum is taken over all piecewise C^∞ curves c from p to q. The quantity $d_F = d_F(p, q)$ is a nonnegative function on $M \times M$. It satisfies

(a) $d_F(p, q) \geq 0$, with equality if and only if $p = q$; and
(b) $d_F(p, q) \leq d_F(p, r) + d_F(r, q)$ for any $p, q, r \in M$.

We call d_F the *distance function* induced by F. This is a weaker notion than the distance function of metric spaces, since d_F need not satisfy $d_F(p, q) = d_F(q, p)$ for $p, q \in M$. But if the Finsler metric F is reversible, that is, if $F(x, -y) = F(x, y)$ for all $y \in T_xM$, then d_F is symmetric.

A piecewise C^∞ curve $\sigma : [a, b] \to M$ is *minimizing* if it has least length:

$$L_F(\sigma) = d_F\big(\sigma(a), \sigma(b)\big).$$

It is *locally minimizing* if, for any $t_0 \in I := [a, b]$, there is a small number $\varepsilon > 0$ such that σ is minimizing when restricted to $[t_0 - \varepsilon,\, t_0 + \varepsilon] \cap I$.

DEFINITION 4.1. A C^∞ curve $\sigma(t)$, $t \in I$, is called a *geodesic* if it is locally minimizing and has constant speed (meaning that $F(\sigma(t), \dot{\sigma}(t))$ is constant).

LEMMA 4.2. *A C^∞ curve $\sigma(t)$ in a Finsler manifold (M, F) is a geodesic if and only if it satisfies the system of second order ordinary differential equations*

$$\ddot{\sigma}^i(t) + 2G^i\big(\sigma(t), \dot{\sigma}(t)\big) = 0, \tag{4–1}$$

where the $G^i = G^i(x,y)$ are local functions on TM defined by

$$G^i := \tfrac{1}{4} g^{il}(x,y)\big([F^2]_{x^k y^l}(x,y) y^k - [F^2]_{x^l}(x,y)\big). \tag{4–2}$$

This is shown by the calculus of variations; see, for example, [Shen 2001a; 2001b].

Let $\{\partial/\partial x^i, \partial/\partial y^i\}$ denote the natural local frame on TM in a standard local coordinate system, and set

$$G := y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{4–3}$$

where the $G^i = G^i(x,y)$, which are given in (4–2), satisfy the homogeneity property

$$G^i(x, \lambda y) = \lambda^2 G^i(x,y), \quad \lambda > 0. \tag{4–4}$$

G is a well-defined vector field on TM. Any vector field G on TM having the form (4–3) and satisfying the homogeneity condition (4–4) is called a *spray* on M, and the G^i are its *spray coefficients*.

Let

$$N^i_j = \frac{\partial G^i}{\partial y^j}, \quad \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j}.$$

Then $HTM := \operatorname{span}\{\delta/\delta x^i\}$ and $VTM := \operatorname{span}\{\partial/\partial y^i\}$ are well-defined and $T(TM_0) = HTM \oplus VTM$. That is, every spray naturally determines a decomposition of $T(TM_0)$.

For a Finsler metric on a manifold M and its spray G, a C^∞ curve $\sigma(t)$ in M is a geodesic of F if and only if the canonical lift $\gamma(t) := \dot\sigma(t)$ in TM is an integral curve of G. One can use this to define the notion of geodesics for sprays.

It is usually difficult to compute the spray coefficients of a Finsler metric. However, for an (α,β)-metric F, given by equation (2–10), the computation is relatively simple using a Maple program. Let $\bar G^i$ be the spray coefficients of the Riemannian metric α, with coefficients $\bar\Gamma^i_{jk}$, so that $\bar G^i = \frac{1}{2}\bar\Gamma^i_{jk}(x) y^j y^k$. These coefficients are called the *Christoffel symbols* of α. By (4–2), they are given by

$$\bar\Gamma^i_{jk} = \frac{a^{il}}{2}\Big(\frac{\partial a_{jl}}{\partial x^k} + \frac{\partial a_{kl}}{\partial x^j} - \frac{\partial a_{jk}}{\partial x^l}\Big).$$

To find a formula for the spray coefficients $G^i = G^i(x,y)$ of F in terms of α and β, we introduce the *covariant derivatives* of β with respect to α. Let $\theta^i := dx^i$ and ${\theta_j}^i := \bar\Gamma^i_{jk}\, dx^k$. We have

$$d\theta^i = \theta^j \wedge {\theta_j}^i, \quad da_{ij} = a_{kj}{\theta_i}^k + a_{ik}{\theta_j}^k.$$

Define $b_{i;j}$ by

$$b_{i;j}\theta^j := db_i - b_j {\theta_i}^j.$$

Let

$$r_{ij} := \tfrac{1}{2}\big(b_{i;j} + b_{j;i}\big), \qquad s_{ij} := \tfrac{1}{2}\big(b_{i;j} - b_{j;i}\big),$$
$${s^i}_j := a^{ih} s_{hj}, \quad s_j := b_i {s^i}_j, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i.$$

By (4–2) and using a Maple program, one obtains the following relationship:

Lemma 4.3. *The geodesic coefficients* G^i *are related to* $\bar{G}^i$ *by*

$$G^i = \bar{G}^i + \frac{\alpha\phi'}{\phi - s\phi'} s^i{}_0 + \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\big((\phi - s\phi') + (b^2 - s^2)\phi''\big)} \times \left(\frac{-2\alpha\phi'}{\phi - s\phi'} s_0 + r_{00}\right)\left(\frac{y^i}{\alpha} + \frac{\phi\phi''}{\phi\phi' - s(\phi\phi'' + \phi'\phi')} b^i\right), \quad (4\text{–}5)$$

where $s = \beta/\alpha$, $s^i{}_0 = s^i{}_j y^j$, $s_0 = s_i y^i$, $r_{00} = r_{ij} y^i y^j$ *and* $b^2 = a^{ij} b_i b_j$.

Consider the metric

$$F = \frac{(\alpha + \beta)^2}{\alpha},$$

where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form with $\|\beta\|_x < 1$ for every $x \in M$. By (4–5), we obtain a formula for the spray coefficients of F:

$$G^i = \bar{G}^i + \frac{2\alpha}{\alpha - \beta} \alpha s^i{}_0 + \frac{\alpha(\alpha - 2\beta)}{2\alpha^2 b^2 + \alpha^2 - 3\beta^2} \left(\frac{-4\alpha}{\alpha - \beta} \alpha s_0 + r_{00}\right)\left(\frac{y^i}{\alpha} + \frac{\alpha}{\alpha - 2\beta} b^i\right),$$

where $b = \|\beta\|_x$.

Given a spray G, we define the covariant derivatives of a vector field $X = X^i(t)(\partial/\partial x^i)|_{c(t)}$ along a curve c by

$$\begin{aligned} D_{\dot{c}} X(t) &:= \big(\dot{X}^i(t) + X^j(t) N^i_j(c(t), \dot{c}(t))\big) \frac{\partial}{\partial x^i}\Big|_{c(t)}, \\ \nabla_{\dot{c}} X(t) &:= \big(\dot{X}^i(t) + X^j(t) N^i_j(c(t), X(t))\big) \frac{\partial}{\partial x^i}\Big|_{c(t)}. \end{aligned} \quad (4\text{–}6)$$

$D_{\dot{c}} X(t)$ is the *linear covariant derivative* and $\nabla_{\dot{c}} X(t)$ the *covariant derivative* of $X(t)$ along c. The field X is *linearly parallel along* c if $D_{\dot{c}} X(t) = 0$, and *parallel along* c if $\nabla_{\dot{c}} X(t) = 0$. For linearly parallel vector fields $X = X(t)$ and $Y = Y(t)$ along a geodesic c, the expression $\boldsymbol{g}_{\dot{c}(t)}\big(X(t), Y(t)\big)$ is constant, and for a parallel vector field $X = X(t)$ along a curve c, $F\big(c(t), X(t)\big)$ is constant.

5. Berwald Metrics

Consider a Riemannian metric $F = \sqrt{g_{ij}(x) y^i y^j}$ on a manifold M. By (4–2), we obtain $G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$, where

$$\Gamma^i_{jk}(x) := \tfrac{1}{4} g^{il}(x) \left(\frac{\partial g_{lk}}{\partial x^j}(x) + \frac{\partial g_{jl}}{\partial x^k}(x) - \frac{\partial g_{jk}}{\partial x^l}(x)\right). \quad (5\text{–}1)$$

In this case the $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$ at any point $x \in M$. A Finsler metric $F = F(x, y)$ is called a *Berwald metric* if in any standard local coordinate system, the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$.

There are many non-Riemannian Berwald metrics.

EXAMPLE 5.1. Let $(\overline{M}, \bar{\alpha})$ and $(\underline{M}, \underline{\alpha})$ be Riemannian manifolds and let $M = \overline{M} \times \underline{M}$. Let $f : [0,\infty) \times [0,\infty) \to [0,\infty)$ be a C^∞ function satisfying $f(\lambda s, \lambda t) = \lambda f(s,t)$ for $\lambda > 0$ and $f(s,t) \neq 0$ if $(s,t) \neq 0$. Define

$$F(x,y) := \sqrt{f\big([\bar{\alpha}(\bar{x},\bar{y})]^2,\ [\underline{\alpha}(\underline{x},\underline{y})]^2\big)}, \tag{5–2}$$

where $x = (\bar{x}, \underline{x}) \in M$ and $y = \bar{y} \oplus \underline{y} \in T_xM \cong T_{\bar{x}}\overline{M} \oplus T_{\underline{x}}\underline{M}$.

We now find additional conditions on $f(s,t)$ under which the matrix $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$ is positive definite. Take standard local coordinate systems $(\bar{x}^{\bar{a}}, \bar{y}^{\bar{a}})$ in $T\overline{M}$ and $(\underline{x}^{\underline{a}}, \underline{y}^{\underline{a}})$ in $T\underline{M}$, so that $\bar{y} = \bar{y}^{\bar{a}}\,\partial/\partial\bar{x}^{\bar{a}}$ and $\underline{y} = \underline{y}^{\underline{a}}\,\partial/\partial\underline{x}^{\underline{a}}$. Then $(x^i, y^j) := (\bar{x}^{\bar{a}}, \underline{x}^{\underline{a}}, \bar{y}^{\bar{a}}, \underline{y}^{\underline{a}})$ is a standard local coordinate system in TM. Express $\bar{\alpha}$ and $\underline{\alpha}$ by

$$\bar{\alpha}(\bar{x},\bar{y}) = \sqrt{\bar{g}_{\bar{a}\bar{b}}(\bar{x})\bar{y}^{\bar{a}}\bar{y}^{\bar{b}}}, \quad \underline{\alpha}(\underline{x},\underline{y}) = \sqrt{\underline{g}_{\underline{a}\underline{b}}(\underline{x})\underline{y}^{\underline{a}}\underline{y}^{\underline{b}}},$$

We obtain

$$g_{ab} = 2f_{ss}\bar{y}_{\bar{a}}\bar{y}_{\bar{b}} + f_s\bar{g}_{\bar{a}\bar{b}}, \quad g_{a\underline{b}} = 2f_{st}\bar{y}_{\bar{a}}\underline{y}_{\beta}, \quad g_{\underline{a}\underline{b}} = 2f_{tt}\underline{y}_{\underline{a}}\underline{y}_{\underline{b}} + f_t\underline{g}_{\underline{a}\underline{b}}, \tag{5–3}$$

where $\bar{y}_{\bar{a}} := \bar{g}_{\bar{a}\bar{b}}\bar{y}^{\bar{b}}$ and $\underline{y}_{\underline{a}} := \underline{g}_{\underline{a}\underline{b}}\underline{y}^{\underline{b}}$. By an elementary argument, one can show that (g_{ij}) is positive definite if and only if $f(s,t)$ satisfies the conditions

$$f_s > 0, \quad f_t > 0, \quad f_s + 2sf_{ss} > 0, \quad f_t + 2tf_{tt} > 0, \quad f_sf_t - 2ff_{st} > 0.$$

In this case,

$$\det(g_{ij}) = h\big([\bar{\alpha}]^2,\ [\underline{\alpha}]^2\big)\det(\bar{g}_{\bar{a}\bar{b}})\det\big(\underline{g}_{\underline{a}\underline{b}}\big), \tag{5–4}$$

where

$$h := (f_s)^{\bar{n}-1}(f_t)^{\underline{n}-1}(f_sf_t - 2ff_{st}), \tag{5–5}$$

where $\bar{n} := \dim \overline{M}$ and $\underline{n} := \dim \underline{M}$.

By a direct computation, one can show that the spray coefficients of F split as the direct sum of the spray coefficients of $\bar{\alpha}$ and $\underline{\alpha}$:

$$G^{\bar{a}}(x,y) = \overline{G}^{\bar{a}}(\bar{x},\bar{y}), \quad G^{\underline{a}}(x,y) = \underline{G}^{\underline{a}}(\underline{x},\underline{y}), \tag{5–6}$$

where $\overline{G}^{\bar{a}}$ and $\underline{G}^{\underline{a}}$ are the spray coefficients of $\bar{\alpha}$ and $\underline{\alpha}$. From (5–6), one can see that the spray of F is independent of the choice of a particular function $f(s,t)$. In particular, the $G^i(x,y)$ are quadratic in $y \in T_xM$. Thus F is a Berwald metric.

A typical example of f is

$$f = s + t + \varepsilon\sqrt[k]{s^k + t^k},$$

where ε is a nonnegative real number and k is a positive integer. The Berwald metric obtained with this choice of f is discussed in [Szabó 1981].

Let (M, F) be a Berwald manifold and $p, q \in M$ be an arbitrary pair of points in M. Let $c : [0,1] \to M$ be a geodesic going from $p = c(0)$ to $q = c(1)$. Define a linear isomorphism $T : T_pM \to T_qM$ by $T(X(0)) := X(1)$, where $X(t)$ is a linearly parallel vector field along c, so $D_{\dot c}X(t) = 0$. Since F is a Berwald metric, the linear covariant derivative $\nabla_{\dot c}$ coincides with the covariant derivative $D_{\dot c}$ along c, by (4–6). Thus $X(t)$ is also parallel along c, that is, $\nabla_{\dot c}X(t) = 0$. Therefore, $F(c(t), X(t))$ is constant. This implies that $T : (T_pM, F_p) \to (T_qM, F_q)$ preserves the Minkowski norms. We have proved the following well-known result:

PROPOSITION 5.2 [Ichijyō 1976]. *On a Berwald manifold (M, F), all tangent spaces (T_xM, F_x) are linearly isometric to each other.*

On a Finsler manifold (M, F), we view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. As we mentioned early in Section 3, the Cartan torsion $\boldsymbol{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM \setminus \{0\}$. In the case when F is a Berwald metric on a manifold M, by Proposition 5.2, all tangent spaces (T_xM, F_x) are linearly isometric, and (M, F) is modeled on a single Minkowski space. More precisely, for any pair points $x, x' \in M$ and a geodesic from x to x', (linearly) parallel translation defines a linear isometry $T : (T_xM, F_x) \to (T_{x'}M, F_{x'})$. This linear isometry T maps the infinitesimal color pattern at x to that at x'. Thus the infinitesimal color patterns do not change over the manifold. If one looks at a Berwald manifold on a large scale, with the infinitesimal color pattern at each point shrunken to a single spot of color, one can only see a space with uniform color. The color depends on the Minkowski model.

A Finsler metric F on a manifold M is said to be *affinely equivalent* to another Finsler metric $\bar F$ on M if F and $\bar F$ induce the same sprays. By (5–6), one can see that the family of Berwald metrics in (5–2) are affinely equivalent.

PROPOSITION 5.3 [Szabó 1981]. *Every Berwald metric on a manifold is affinely equivalent to a Riemannian metric.*

Based on this observation, Z. I. Szabó [1981] determined the local structure of Berwald metrics.

6. Gradient, Divergence and Laplacian

Let $F = F(x, y)$ be a Finsler metric on a manifold M and let $F^* = F^*(x, \xi)$ be dual to F. Let f be a C^1 function on M. At a point $x \in M$, the differential $df_x \in T_x^*M$ is a 1-form. Define the dual vector $\nabla f_x \in T_xM$ by

$$\nabla f_x := \ell_x^*(df_x), \tag{6–1}$$

where $\ell_x^* : T_x^*M \to T_xM$ is the inverse Legendre transformation. By definition, ∇f_x is uniquely determined by

$$\eta(\nabla f_x) := \frac{1}{2}\frac{d}{dt}\big(F^{*2}(x, df_x + t\eta)\big)|_{t=0}, \quad \eta \in T_x^*M.$$

∇f_x is called the *gradient* of f at x. We have

$$F(x, \nabla f_x) = F^*(x, df_x).$$

If f is C^k ($k \geq 1$), then ∇f is C^{k-1} on $\{df_x \neq 0\}$ and C^0 at any point $x \in M$ with $df_x = 0$.

Given a closed subset $A \subset M$ and a point $x \in M$, let $d(A, x) := \sup_{z\in A} d(z, x)$ and $d(x, A) := \sup_{z\in A} d(x, z)$. The function ρ defined by $\rho(x) := d(x, A)$ is locally Lipschitz, hence differentiable almost everywhere. It is easy to verify [Shen 2001b] that

$$F(x, \nabla\rho_x) = F^*(x, d\rho_x) = 1 \quad \text{almost everywhere.} \tag{6–2}$$

Any function ρ satisfying satisfies (6–2) is called a *distance function* of the Finsler metric F; another example is the function ρ defined by $\rho(x) := -d(x, A)$. In general, a distance function on a Finsler manifold (M, F) is C^∞ on an open dense subset of M. For example, when $A = \{p\}$ is a point, the distance function ρ defined by $\rho(x) := d(x, p)$ or $rho(x) := d(p, x)$ is C^∞ on an open dense subset of M.

Let ρ be a distance function of a Finsler metric F on a manifold M. Assume that it is C^∞ on an open subset $U \subset M$. Then $\nabla\rho$ induces a Riemannian metric $\hat{F} := \sqrt{\boldsymbol{g}_{\nabla\rho}(v, v)}$ on U. Moreover, ρ is a distance function of $\hat{F}$ and $\nabla\rho = \hat{\nabla}\rho$ is the gradient of ρ with respect to $\hat{F}$. See [Shen 2001b].

Every Finsler metric F defines a volume form

$$dV_F := \sigma_F(x)\, dx^1 \cdots dx^n,$$

where

$$\sigma_F := \frac{\operatorname{Vol} \mathrm{B}^n}{\operatorname{Vol}\big\{(y^i) \in \mathrm{R}^n \mid F(x, y^i(\partial/\partial x^i)|_x) < 1\big\}}. \tag{6–3}$$

Here Vol denotes Euclidean volume in R^n. It was proved by H. Busemann [1947] that if F is reversible, the Hausdorff measure of the induced distance function d_F is represented by dV_F. When $F = \sqrt{g_{ij}(x)y^iy^j}$ is Riemannian,

$$\sigma_F = \sqrt{\det(g_{ij}(x))} \quad \text{and} \quad dV_F = \sqrt{\det(g_{ij}(x))}\, dx^1 \cdots dx^n.$$

For a vector field $X = X^i(x)(\partial/\partial x^i)|_x$ on M, the *divergence* of X is

$$\operatorname{div} X := \frac{1}{\sigma_F(x)} \frac{\partial}{\partial x^i}\big(\sigma_F(x) X^i(x)\big). \tag{6–4}$$

The *Laplacian* of a C^2 function f with $df \neq 0$ is

$$\Delta f := \operatorname{div} \nabla f.$$

Δ is a nonlinear elliptic operator. If there are points x at which $df_x = 0$, then ∇f is only C^0 at these points. In this case, Δf is only defined weakly in the sense of distributions.

For a C^∞ distance function ρ on an open subset $U \subset M$, $d\rho_x \neq 0$ at every point $x \in U$ and the level set $N_r := \rho^{-1}(r) \subset U$ is a C^∞ hypersurface in U. Thus $\Delta\rho$ can be defined by the above formula and its restriction to N_r, $H := \Delta\rho|_{N_r}$, is called the *mean curvature* of N_r with respect to the normal vector $\boldsymbol{n} = \nabla\rho|_{N_r}$.

7. S-Curvature

Consider an n-dimensional Finsler manifold (M, F). As mentioned in Section 5, we view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. The Cartan torsion $\boldsymbol{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM \setminus \{0\}$. The mean Cartan torsion $\boldsymbol{I}_y$ is the average value of $\boldsymbol{C}_y$. Besides the (mean) Cartan torsion, there is another geometric quantity associated with F_x. Take a standard local coordinate system (x^i, y^i) and let

$$\tau := \ln \frac{\sqrt{\det\bigl(g_{ij}(x,y)\bigr)}}{\sigma_F(x)}, \tag{7–1}$$

where σ_F is defined in (6–3). τ is called the *distortion* [Shen 1997; 2001b]. Intuitively, the distortion $\tau(x, y)$ is the directional *twisting number* of the infinitesimal color pattern at x. Observe that

$$\tau_{y^i} = \frac{\partial}{\partial y^i} \ln \sqrt{\det\bigl(g_{jk}(x,y)\bigr)} = \frac{1}{2} g^{jk} \frac{\partial g_{jk}}{\partial y^i} = g^{jk} C_{ijk} =: I_i. \tag{7–2}$$

Here σ_F does not occur in the first equality, because it is independent of y at each point x. If the distortion is isotropic at x, that is, if $\tau(x)$ is independent of the direction $y \in T_xM$, then $\tau(x) = 0$ and F_x is Euclidean (Proposition 3.1). In this case, the infinitesimal color pattern is in the simplest form at every point.

It is natural to study the rate of change of the distortion along geodesics. For $y \in T_xM \setminus \{0\}$, let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Let

$$\boldsymbol{S} := \frac{d}{dt} \tau\bigl(\sigma(t), \dot\sigma(t)\bigr)\Big|_{t=0}. \tag{7–3}$$

$\boldsymbol{S}$ is called the *S-curvature*. It is positively homogeneous of degree one in y:

$$\boldsymbol{S}(x, \lambda y) = \lambda \boldsymbol{S}(x, y), \quad \lambda > 0.$$

In a standard local coordinate system (x^i, y^i), let $G^i = G^i(x, y)$ denote the spray coefficients of F. Contracting (8–2) with g^{ij} yields

$$\frac{\partial G^m}{\partial y^m} = \tfrac{1}{2} g^{ml} \frac{\partial g_{ml}}{\partial x^i} y^i - 2 I_i G^i,$$

so

$$\boldsymbol{S} = y^i \frac{\partial \tau}{\partial x^i} - 2 \frac{\partial \tau}{\partial y^i} G^i = \tfrac{1}{2} g^{ml} \frac{\partial g_{ml}}{\partial x^i} y^i - 2 I_i G^i - y^m \frac{\partial}{\partial x^m} \ln \sigma_F$$

and finally

$$\boldsymbol{S} = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial}{\partial x^m} \ln \sigma_F. \tag{7–4}$$

PROPOSITION 7.1 [Shen 1997]. *For any Berwald metric*, $\boldsymbol{S} = 0$.

However, many metrics of zero S-curvature are non-Berwaldian.

Some comparison theorems in Riemannian geometry are still valid for Finsler metrics of zero S-curvature; see [Shen 1997; 2001b].

By definition, the S-curvature is the covariant derivative of the distortion along geodesics. Let $\sigma(t)$ be a geodesic and set

$$\tau(t) := \tau\big(\sigma(t), \dot\sigma(t)\big), \quad \boldsymbol{S}(t) := \boldsymbol{S}\big(\sigma(t), \dot\sigma(t)\big).$$

By (7–3), $\boldsymbol{S}(t) = \tau'(t)$, so if $\boldsymbol{S}$ vanishes, $\tau(t)$ is a constant. Intuitively, the distortion (twisting number) of the infinitesimal color pattern in the direction $\dot\sigma(t)$ does not change along any geodesic. However, the distortion might take different values along different geodesics.

A Finsler metric F is said to have *isotropic S-curvature* if

$$\boldsymbol{S} = (n+1)\, cF.$$

More generally, F is said to have *almost isotropic S-curvature* if

$$\boldsymbol{S} = (n+1)\,(cF + \eta),$$

where $c = c(x)$ is a scalar function and $\eta = \eta_i(x) y^i$ is a *closed* 1-form.

Differentiating the S-curvature gives rise to another quantity. Let

$$E_{ij} := \tfrac{1}{2} \boldsymbol{S}_{y^i y^j}(x,y). \tag{7–5}$$

For $y \in T_xM \setminus \{0\}$, the form $\boldsymbol{E}_y = E_{ij}(x,y)\, dx^i \otimes dx^j$ is a symmetric bilinear form on T_xM. We call the family $\boldsymbol{E} := \{\boldsymbol{E}_y \mid y \in TM \setminus \{0\}\}$ the *mean Berwald curvature*, or simply the *E-curvature* [Shen 2001a]. Let $\boldsymbol{h}_y := h_{ij}(x,y)\, dx^i \otimes dx^j$, where $h_{ij} := F F_{y^i y^j}$. We say that F has *isotropic E-curvature* if

$$\boldsymbol{E} = \tfrac{1}{2}(n+1)\, cF^{-1} \boldsymbol{h},$$

where $c = c(x)$ is a scalar function on M. Clearly, if the S-curvature is almost isotropic, the E-curvature is isotropic. Conversely, if the E-curvature is isotropic, there is a 1-form $\eta = \eta_i(x)\, dx^i$ such that $\boldsymbol{S} = (n+1)\,(cF + \eta)$. However, this η is not closed in general.

Finally, we mention another geometric meaning of the S-curvature. Let $\rho = \rho(x)$ be a C^∞ distance function on an open subset $U \subset M$ (see (6–2)). The gradient $\nabla\rho$ induces a Riemannian metric $\hat F = \hat F(z,v)$ on U by

$$\hat F(z,v) := \sqrt{\boldsymbol{g}_{\nabla\rho}(v,v)}, \quad v \in T_zU.$$

Let Δ and $\hat{\Delta}$ denote the Laplacians on functions with respect to F and $\hat{F}$. Then $H = \Delta\rho|_{N_r}$ and $\hat{H} = \hat{\Delta}\rho|_{N_r}$ are the *mean curvature* of $N_r := \rho^{-1}(r)$ with respect to F and $\hat{F}$, respectively. The S-curvature can be expressed by

$$\boldsymbol{S}(\nabla\rho) = \hat{\Delta}\rho - \Delta\rho = \hat{H} - H.$$

From these identities, one can estimate $\hat{\Delta}$ and obtain an estimate on $\Delta\rho$ under a Ricci curvature bound and an S-curvature bound. Then one can establish a volume comparison on the metric balls. See [Shen 2001b] for more details.

8. Landsberg Curvature

The (mean) Cartan torsion is a geometric quantity that characterizes the Euclidean norms among Minkowski norms on a vector space. On a Finsler manifold (M, F), one can view the Minkowski norm F_x on T_xM as an *infinitesimal color pattern* at x. The Cartan torsion $\boldsymbol{C}_y$ describes the non-Euclidean features of the pattern in the direction $y \in T_xM \setminus \{0\}$. The mean Cartan torsion $\boldsymbol{I}_y$ is the average value of $\boldsymbol{C}_y$. They reveal the non-Euclidean features which are different from that revealed by the distortion. Therefore, it is natural to study the rate of change of the (mean) Cartan torsion along geodesics.

Let (M, F) be a Finsler manifold. To differentiate the (mean) Cartan torsion along geodesics, we need linearly parallel vector fields along a geodesic. Recall that a vector field $U(t) := U^i(T)(\partial/\partial x^i)|_{\sigma(t)}$ along a geodesic $\sigma(t)$ is linearly parallel along σ if $D_{\dot\sigma}U(t) = 0$:

$$\dot{U}^i(t) + U^j(t) N^i_j(\sigma(t), \dot\sigma(t)) = 0. \tag{8–1}$$

On the other hand, by a direct computation using (4–2), one can verify that g_{ij} satisfy the following identity:

$$y^m \frac{\partial g_{ij}}{\partial x^m} - 2G^m \frac{\partial g_{ij}}{\partial y^m} = g_{im} N^m_j + g_{mj} N^m_i \tag{8–2}$$

Using (8–1) and (8–2), one can verify that for two linearly parallel vector fields $U(t), V(t)$ along σ,

$$\frac{d}{dt} \boldsymbol{g}_{\dot\sigma(t)}\big(U(t), V(t)\big) = 0.$$

In this sense, the family of inner products $\boldsymbol{g}_y$ does not change along geodesics. However, for linearly parallel vector fields $U(t), V(t)$ and $W(t)$ along σ, the functions $\boldsymbol{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big)$ and $\boldsymbol{I}_{\dot\sigma(t)}\big(U(t)\big)$ do change, in general. Set

$$\boldsymbol{L}_y(u, v, w) := \frac{d}{dt}\big(\boldsymbol{C}_{\dot\sigma(t)}(U(t), V(t), W(t))\big)|_{t=0} \tag{8–3}$$

and

$$\boldsymbol{J}_y(u) := \frac{d}{dt}\big(\boldsymbol{I}_{\dot\sigma(t)}(U(t))\big)|_{t=0},$$

where $u = U(0), v = V(0), w = W(0)$ and $y = \dot\sigma(0) \in T_xM$. The family $\boldsymbol{L} = \{\boldsymbol{L}_y \mid y \in TM \setminus \{0\}\}$ is called the *Landsberg curvature*, or *L-curvature*, and the family $\boldsymbol{J} = \{\boldsymbol{J}_y \mid y \in TM \setminus \{0\}\}$ is called the *mean Landsberg curvature*, or *J-curvature*. A Finsler metric is called a *Landsberg metric* if $\boldsymbol{L} = 0$, and a *weakly Landsberg metric* if $\boldsymbol{J} = 0$.

Let (x^i, y^i) be a standard local coordinate system in TM and set $C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k}$. From the definition, $\boldsymbol{L}_y = L_{ijk}\, dx^i \otimes dx^j \otimes dx^k$ is given by

$$L_{ijk} = y^m \frac{\partial C_{ijk}}{\partial x^m} - 2G^m \frac{\partial C_{ijk}}{\partial y^m} - C_{mjk} N^m_i - C_{imk} N^m_j - C_{ijm} N^m_k, \qquad (8\text{–}4)$$

and $\boldsymbol{J} = J_i dx^i$ is given by

$$J_i = y^m \frac{\partial I_i}{\partial x^m} - 2G^m \frac{\partial I_i}{\partial y^m} - I_m N^m_i. \qquad (8\text{–}5)$$

We have

$$J_i = g^{jk} L_{ijk}.$$

It follows from (4–2) that

$$g_{sm} G^m = \frac{1}{4}\Big(2\frac{\partial g_{sk}}{\partial x^m} - \frac{\partial g_{km}}{\partial x^s}\Big) y^k y^m.$$

Differentiating with respect to y^i, y^j, y^k and contracting the resulting identity by $\frac{1}{2} y^s$, one obtains

$$L_{ijk} = -\tfrac{1}{2} y^s g_{sm} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}. \qquad (8\text{–}6)$$

Thus if $G^m = G^m(x,y)$ are quadratic in $y \in T_xM$, then $L_{ijk} = 0$. This proves the following well-known result.

PROPOSITION 8.1. *Every Berwald metric is a Landsberg metric.*

By definition, the (mean) Landsberg curvature is the covariant derivative of the (mean) Cartan torsion along a geodesic. Let $\sigma = \sigma(t)$ be a geodesic and $U = U(t), V = V(t), W = W(t)$ be parallel vector fields along σ. Let

$$\boldsymbol{L}(t) := \boldsymbol{L}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big), \quad \boldsymbol{C}(t) := \boldsymbol{C}_{\dot\sigma(t)}\big(U(t), V(t), W(t)\big).$$

By (8–3),

$$\boldsymbol{L}(t) = \boldsymbol{C}'(t).$$

If F is Landsbergian, the Cartan torsion $\boldsymbol{C}_{\dot\sigma}$ in the direction $\dot\sigma(t)$ is constant along σ. Intuitively, the infinitesimal color pattern in the direction $\dot\sigma(t)$ does not change along σ. But the patterns might look different at neighboring points.

It is easy to see that in dimension two, a Finsler metric is Berwaldian if and only if $\boldsymbol{E} = 0$ (or $\boldsymbol{S} = 0$) and $\boldsymbol{J} = 0$. It seems that $\boldsymbol{E}$ and $\boldsymbol{L}$ are complementary to each other. So we may ask: *Is a Finsler metric Berwaldian if $\boldsymbol{E} = 0$ and $\boldsymbol{L} = 0$?* A more difficult problem is: *Is a Finsler metric Berwaldian if $\boldsymbol{L} = 0$?* So far, we do not know.

Finsler metrics with $\boldsymbol{L} = 0$ can be generalized as follows. Let F be a Finsler metric on an n-dimensional manifold M. We say that F has *relatively isotropic*

L-curvature $\boldsymbol{L} + cF\boldsymbol{C} = 0$, where $c = c(x)$ is a scalar function on M. We say that F has *relatively isotropic J-curvature* if $\boldsymbol{J} + cF\boldsymbol{I} = 0$.

There are many interesting Finsler metrics having isotropic L-curvature or (almost) isotropic S-curvature. We will discuss them in the following two sections.

9. Randers Metrics with Isotropic S-Curvature

We now discuss Randers metrics of isotropic S-curvature. Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M, with $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\beta = b_i(x)y^i$. Recall from page 307 that this is a special case of an (α, β)-metric, with $\phi(s) = 1 + s$. By (4–5), the spray coefficients G^i of F and $\bar{G}^i$ of α are related via

$$G^i = \bar{G}^i + Py^i + Q^i, \tag{9–1}$$

where

$$P := \frac{e_{00}}{2F} - s_0, \quad Q^i = \alpha s^i{}_0, \tag{9–2}$$

and $e_{00} := e_{ij}y^iy^j$, $s_0 := s_iy^i$, $s^i{}_0 := s^i{}_jy^j$. The formula above can be found in [Antonelli et al. 1993].

Let

$$\rho := \ln\sqrt{1 - \|\beta\|_x^2}.$$

The volume forms dV_F and dV_α are related by

$$dV_F = e^{(n+1)\rho(x)}dV_\alpha.$$

Since $s_{ij} = s_{ji}$, $s_{00} := s_{ij}y^iy^j = 0$ and $s^i{}_i = a^{ij}s_{ij} = 0$. Observe that

$$\frac{\partial(Py^m)}{\partial y^m} = \frac{\partial P}{\partial y^m}y^m + nP = (n{+}1)\,P, \qquad \frac{\partial Q^m}{\partial y^m} = \alpha^{-1}s_{00} + \alpha s^m_{\ m} = 0.$$

Since α is Riemannian, we have

$$\frac{\partial \bar{G}^m}{\partial y^m} = y^m\frac{\partial}{\partial x^m}\ln\sigma_\alpha.$$

Thus one obtains

$$\begin{aligned}\boldsymbol{S} &= \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}\ln\sigma_F \\ &= \frac{\partial \bar{G}^m}{\partial y^m} + \frac{\partial(Py^m)}{\partial y^m} + \frac{\partial Q^m}{\partial y^m} - (n{+}1)y^m\frac{\partial\rho}{\partial x^m} - y^m\frac{\partial}{\partial x^m}\ln\sigma_\alpha \\ &= (n{+}1)(P - \rho_0) = (n{+}1)\Big(\frac{e_{00}}{2F} - (s_0 + \rho_0)\Big),\end{aligned} \tag{9–3}$$

where $\rho_0 := \rho_{x^i}(x)y^i$.

LEMMA 9.1 [Chen and Shen 2003a]. *For a Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, the following conditions are equivalent*:

(a) *The S-curvature is isotropic, i.e.,* $\boldsymbol{S}=(n+1)cF$ *for a scalar function c on M.*

(b) *The S-curvature is almost isotropic, i.e.,* $\boldsymbol{S}=(n+1)(cF+\eta)$ *for a scalar function c and a closed 1-form η on M.*

(c) *The E-curvature is isotropic, i.e.,* $\boldsymbol{E}=\frac{1}{2}(n+1)cF^{-1}\boldsymbol{h}$ *for a scalar function c on M.*

(d) $e_{00}=2c(\alpha^2-\beta^2)$ *for a scalar function c on M.*

PROOF. (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are obvious.

(c) $\Longrightarrow$ (d). First, $\boldsymbol{S}=(n+1)(cF+\eta)$, where η is a 1-form on M. By (9–3), (c) is equivalent to the following $e_{00}=2cF^2+2\theta F$, where $\theta:=s_0+\rho_0+\eta$. This implies that

$$e_{00}=2c(\alpha^2+\beta^2)+2\theta\beta, \quad 0=4c\beta+2\theta.$$

Solving for θ in the second of these equations and plugging the result into the first, one obtains (d).

(d) $\Longrightarrow$ (a). Plugging $e_{00}=2c(\alpha^2-\beta^2)$ into (9–3) yields

$$\boldsymbol{S}=(n+1)\big(c(\alpha-\beta)-(s_0+\rho_0)\big). \tag{9–4}$$

On the other hand, contracting $e_{ij}=2c(a_{ij}-b_ib_j)$ with b^j gives $s_i+\rho_i+2cb_i=0$. Thus $s_0+\rho_0=-2c\beta$. Plugging this into (9–4) yields (a). □

EXAMPLE 9.2. Let $V=(A,B,C)$ be a vector field on a domain $U\subset \mathrm{R}^3$, where $A=A(r,s,t)$, $B=B(r,s,t)$ and $C=C(r,s,t)$ are C^∞ functions on U with

$$|V(x)|=\sqrt{A(x)^2+B(x)^2+C(x)^2}<1, \quad \forall x=(r,s,t)\in U.$$

Let $\Phi:=|y|$ be the standard Euclidean metric on R^3. Define $F=\alpha+\beta$ by (2–15) for the pair (Φ,V). α and β are given by

$$\alpha=\frac{\sqrt{\langle V(x),y\rangle^2+|y|^2(1-|V(x)|^2)}}{1-|V(x)|^2}, \qquad \beta=-\frac{\langle V(x),y\rangle}{1-|V(x)|^2},$$

where $y=(u,v,w)\in T_xU\cong \mathrm{R}^3$. One can easily verify that $\|\beta\|_x<1$ for $x\in U$. By a direct computation, one obtains

$$e_{11}=\frac{B^2(A_r-B_s)+C^2(A_r-C_t)-A_r+H}{1-A^2-B^2-C^2},$$
$$e_{22}=\frac{A^2(B_s-A_r)+C^2(B_s-C_t)-B_s+H}{1-A^2-B^2-C^2},$$
$$e_{33}=\frac{A^2(C_t-A_r)+B^2(C_t-B_s)-C_t+H}{1-A^2-B^2-C^2},$$
$$e_{12}=-\tfrac{1}{2}(A_s+B_r), \quad e_{13}=-\tfrac{1}{2}(A_t+C_r), \quad e_{23}=-\tfrac{1}{2}(B_t+C_s),$$

where $H:=2ABe_{12}+2ACe_{13}+2BCe_{23}$. Here as usual we write $A_r=\partial A/\partial r$, etc. On the other hand,

$$a_{ij}-b_ib_j=\frac{\delta_{ij}}{1-A^2-B^2-C^2}.$$

It is easy to verify that $e_{ij} = 2c(a_{ij} - b_i b_j)$ if and only if A, B, and C satisfy

$$A_r = B_s = C_t, \quad A_t + C_r = 0, \quad A_s + B_r = 0, \quad B_t + C_s = 0.$$

In this case,

$$c = -\tfrac{1}{2}A_r = -\tfrac{1}{2}B_s = -\tfrac{1}{2}C_t.$$

By Lemma 9.1, we know that $\boldsymbol{S} = 4cF$.

If $F = \alpha + \beta$ on an n-dimensional manifold M is generated from the pair (Φ, V), where $\Phi = \sqrt{\phi_{ij} y^i y^j}$ is a Riemannian metric and $V = V^i(\partial/\partial x^i)$ is a vector field on M with $\phi_{ij}(x)V^i(x)v^j(x) < 1$ for any $x \in M$, then F has isotropic S-curvature, $\boldsymbol{S} = (n+1)c(x)F$, if and only if

$$V_{i;j} + V_{j;i} = -4c\phi_{ij},$$

where $V_i = \phi_{ij}V^j$ and $V_{i;j}$ are the covariant derivatives of V with respect to Φ. This observation is made by Xing [2003]. It also follows from [Bao and Robles 2003b], although it is not proved there directly.

10. Randers Metrics with Relatively Isotropic L-Curvature

We now study Randers metrics with relatively isotropic (mean) Landsberg curvature. From its definition, the mean Landsberg curvature is the mean value of the Landsberg curvature. Thus if a Finsler metric has isotropic Landsberg curvature, it must have isotropic mean Landsberg curvature. I don't know whether the converse is true as well; no counterexample has been found. Nevertheless, for Randers metrics, "having isotropic mean Landsberg curvature" implies "having isotropic Landsberg curvature". According to Lemma 3.2, the Cartan torsion is given by (3–7). Differentiating (3–7) along a geodesic and using (8–4) and (8–5), we obtain

$$L_{ijk} = \frac{1}{n+1}(J_i h_{jk} + J_j h_{ik} + J_k h_{ij}). \tag{10–1}$$

Here we have used the fact that the angular form $\boldsymbol{h}_y$ is constant along geodesics. By (3–7) and (10–1), one can easily show that $J_i + cFI_i = 0$ if and only if $L_{ijk} + cFC_{ijk} = 0$. This proves the claim.

LEMMA 10.1 [Chen and Shen 2003a]. *For a non-Riemannian Randers metric $F = \alpha + \beta$ on an n-dimensional manifold M, these conditions are equivalent*:

(a) $\boldsymbol{J} + cF\boldsymbol{I} = 0$ (*or* $\boldsymbol{L} + cF\boldsymbol{C} = 0$).
(b) $\boldsymbol{S} = (n+1)cF$ *and* β *is closed.*
(c) $\boldsymbol{E} = \frac{1}{2}cF^{-1}\boldsymbol{h}$ *and* β *is closed.*
(d) $e_{00} = 2c(\alpha^2 - \beta^2)$ *and* β *is closed.*

Here $c = c(x)$ is a scalar function on M.

PROOF. By (10–1), to compute L_{ijk}, it suffices to compute J_i. First, the mean Cartan torsion is

$$I_i = \frac{1}{2}(n+1)F^{-1}\alpha^{-2}(\alpha^2 b_i - \beta y_i), \tag{10–2}$$

where $y_i := a_{ij}y^j$. By a direct computation using (8–5), one obtains

$$J_i = \tfrac{1}{4}(n+1)F^{-2}\alpha^{-2}\Big(2\alpha\big((e_{i0}\alpha^2 - y_i e_{00}) - 2\beta(s_i\alpha^2 - y_i s_0) + s_{i0}(\alpha^2+\beta^2)\big) \\ +\alpha^2(e_{i0}\beta - b_i e_{00}) + \beta(e_{i0}\alpha^2 - y_i e_{00}) - 2(s_i\alpha^2 - y_i s_0)(\alpha^2+\beta^2) + 4s_{i0}\alpha^2\beta\Big).$$

Using this and (10–2), one can easily prove the lemma. □

Thus, for any Randers metric $F = \alpha + \beta$, the J-curvature vanishes if and only if $e_{00} = 0$ and $d\beta = 0$. This is equivalent to $b_{i;j} = 0$, in which case, the spray coefficients of F coincide with that of α. This observation leads to the following result, first established by the collective efforts found in [Matsumoto 1974; Hashiguchi and Ichijyō 1975; Kikuchi 1979; Shibata et al. 1977].

PROPOSITION 10.2. *For a Randers metric $F = \alpha + \beta$, the following conditions are equivalent*:

(a) *F is a weakly Landsberg metric*, $\boldsymbol{J} = 0$.
(b) *F is a Landsberg metric*, $\boldsymbol{L} = 0$.
(c) *F is a Berwald metric.*
(d) *β is parallel with respect to α.*

EXAMPLE 10.3. Consider the Randers metric $F = \alpha + \beta$ on R^n defined by

$$\alpha := \frac{\sqrt{(1-\varepsilon^2)\langle x,y\rangle^2 + \varepsilon|y|^2(1+\varepsilon|x|^2)}}{1+\varepsilon|x|^2}, \qquad \beta := \frac{\sqrt{1-\varepsilon^2}\,\langle x,y\rangle}{1+\varepsilon|x|^2},$$

where ε is an arbitrary constant with $0 < \varepsilon \le 1$. Since β is closed, $s_{ij} = 0$ and $s_i = 0$. After computing $b_{i;j}$, one obtains

$$e_{ij} = \frac{\varepsilon\sqrt{1-\varepsilon^2}}{(1+\varepsilon|x|^2)(\varepsilon+|x|^2)}\delta_{ij}.$$

On the other hand, $a_{ij} - b_i b_j = \dfrac{\varepsilon}{1+\varepsilon|x|^2}\delta_{ij}$. Thus $e_{ij} = 2c(a_{ij} - b_i b_j)$ with

$$c := \frac{\sqrt{1-\varepsilon^2}}{2(\varepsilon+|x|^2)}.$$

By Lemma 10.1, F satisfies $\boldsymbol{L} + cF\boldsymbol{C} = 0$, $\boldsymbol{S} = (n+1)cF$, and $\boldsymbol{E} = \frac{1}{2}cF^{-1}\boldsymbol{h}$. See [Mo and Yang 2003] for a family of more general Randers metrics with nonconstant isotropic S-curvature.

11. Riemann Curvature

The Riemann curvature is an important quantity in Finsler geometry. It was first introduced by Riemann for Riemannian metrics in 1854. Berwald [1926; 1928] extended it to Finsler metrics using the Berwald connection. His extension of the Riemann curvature is a milestone in Finsler geometry.

Let (M, F) be a Finsler manifold and let $G = y^i(\partial/\partial x^i) - 2G^i(\partial/\partial y^i)$ be the induced spray. For a vector $y \in T_xM \setminus \{0\}$, set

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{11–1}$$

The local curvature functions $R^i{}_k$ and $R_{jk} := g_{ij}R^i{}_k$ satisfy

$$R^i{}_k y^k = 0, \quad R_{jk} = R_{kj}. \tag{11–2}$$

$\boldsymbol{R}_y = R^i{}_k(\partial/\partial x^i) \otimes dx^k : T_xM \to T_xM$ is a well-defined linear map. We call the family $\boldsymbol{R} = \{\boldsymbol{R}_y \mid y \in TM \setminus \{0\}\}$ the *Riemann curvature*. The Riemann curvature is actually defined for sprays, as shown in [Kosambi 1933; 1935]. When the Finsler metric is Riemannian, then

$$R^i{}_k(x, y) = R_j{}^i{}_{kl}(x)y^j y^l,$$

where $R(u, v)w = R_j{}^i{}_{kl}(x)w^j u^i v^j(\partial/\partial x^i)|_i$ denotes the Riemannian curvature tensor. Namely, $\boldsymbol{R}_y(u) = R(u, y)y$.

The geometric meaning of the Riemann curvature lies in the second variation of geodesics. Let $\sigma(t)$, for $a \le t \le b$, be a geodesic in M. Take a geodesic variation $H(t, s)$ of $\sigma(t)$, that is, a family of curves $\sigma_s(t) := H(t, s)$, $a \le t \le b$, each of which is a geodesic, with $\sigma_0 = \sigma$. Let

$$J(t) := \frac{\partial H}{\partial s}(t, 0).$$

Then $J(t)$ satisfies the *Jacobi equation*

$$D_{\dot\sigma}D_{\dot\sigma}J(t) + \boldsymbol{R}_{\dot\sigma(t)}(J(t)) = 0, \tag{11–3}$$

where $D_{\dot\sigma}$ is defined in (4–6). See [Kosambi 1933; 1935].

There is another way to define the Riemann curvature. Any vector $y \in T_xM$ can be extended to a nonzero C^∞ geodesic field Y in an open neighborhood U of x; a *geodesic field* is one for which every integral curve is a geodesic. Define

$$\hat F(z, v) := \sqrt{\boldsymbol{g}_{Y_z}(v, v)}, \quad v \in T_zU,\ z \in U.$$

Then $\hat F = \hat F(z, v)$ is a Riemannian metric on U. Let $\hat{\boldsymbol{g}} = \boldsymbol{g}_Y$ be the inner product induced by $\hat F$ and let $\hat{\boldsymbol{R}}$ be the Riemann curvature of $\hat F$. It is well-known in Riemannian geometry that

$$\hat{\boldsymbol{R}}_y(u) = 0, \quad \hat{\boldsymbol{g}}\big(\hat{\boldsymbol{R}}_y(u), v\big) = \hat{\boldsymbol{g}}\big(u, \hat{\boldsymbol{R}}_y(v)\big), \tag{11–4}$$

where $u, v \in T_xU$. An important fact is

$$\boldsymbol{R}_y(u) = \hat{\boldsymbol{R}}_y(u), \quad u \in T_xM. \tag{11–5}$$

See [Shen 2001b, Proposition 6.2.2] for a proof of (11–5). Note that $\hat{\boldsymbol{g}}_x = \boldsymbol{g}_y$. It follows from (11–4) and (11–5) that

$$\boldsymbol{R}_y(y) = 0, \quad \boldsymbol{g}_y(\boldsymbol{R}_y(u), v) = \boldsymbol{g}_y(u, \boldsymbol{R}_y(v)), \tag{11–6}$$

where $u, v \in T_xM$. In local coordinates, this equation is just (11–2). See [Shen 2001b] for the application of (11–5) in comparison theorems in conjunction with the S-curvature.

For a two-dimensional subspace $\Pi \subset T_xM$ and a nonzero vector $y \in \Pi$, define

$$\boldsymbol{K}(\Pi, y) := \frac{\boldsymbol{g}_y(\boldsymbol{R}_y(u), u)}{\boldsymbol{g}_y(y,y)\boldsymbol{g}_y(u,u) - \boldsymbol{g}_y(y,u)^2}, \tag{11–7}$$

where $u \in \Pi$ such that $\Pi = \text{span}\{y, u\}$. One can use (11–6) to show that $\boldsymbol{K}(\Pi, y)$ is independent of the choice of a vector u, but it is usually dependent on y. We call $\boldsymbol{K}(\Pi, y)$ the *flag curvature* of the *flag* (Π, y). When $F = \sqrt{g_{ij}(x)y^iy^j}$ is a Riemannian metric, $\boldsymbol{K}(\Pi, y) = \boldsymbol{K}(\Pi)$ is independent of $y \in \Pi$, in which case $\boldsymbol{K}(\Pi)$ is usually called the *sectional curvature* of the *section* $\Pi \subset T_xM$.

A Finsler metric F on a manifold M is said to be *of scalar curvature* $\boldsymbol{K} = \boldsymbol{K}(x,y)$ if for any $y \in T_xM \setminus \{0\}$ the flag curvature $\boldsymbol{K}(\Pi, y) = \boldsymbol{K}(x,y)$ is independent of the tangent planes Π containing y. From the definition, the flag curvature is a scalar function $\boldsymbol{K} = \boldsymbol{K}(x,y)$ if and only if in a standard local coordinate system,

$$R^i{}_k = \boldsymbol{K} F^2 h^i_k, \tag{11–8}$$

where $h^i_k := g^{ij}h_{jk} = g^{ij}FF_{y^jy^k}$. F is *of constant flag curvature* if this $\boldsymbol{K}$ is a constant. For a Riemannian metric, if the flag curvature $\boldsymbol{K}(\Pi, y) = \boldsymbol{K}(x,y)$ is a scalar function on TM, then $\boldsymbol{K}(x,y) = \boldsymbol{K}(x)$ is independent of $y \in T_xM$ and it is a constant when $n \geq 3$ by the Schur Lemma. In the next section we show that any locally projectively flat Finsler metric is of scalar curvature. Such metrics are for us a rich source of Finsler metrics of scalar curvature.

Classifying Finsler metrics of scalar curvature, in particular those of constant flag curvature, is one of the important problems in Finsler geometry. The local structures of projectively flat Finsler metrics of constant flag curvature were characterized in [Shen 2003b]. R. Bryant [1996; 1997; 2002] had earlier classified the global structures of projectively flat Finsler metrics of $\boldsymbol{K} = 1$ on S^n, and given some ideas for constructing non–projectively flat metrics of $\boldsymbol{K} = 1$ on S^n.

Very recently, some non–projectively flat metrics of constant flag curvature have been explicitly constructed; see [Bao–Shen 2002; Bejancu–Farran 2002; Shen 2002; 2003a; 2003b; 2003c; Bao–Robles 2003], for example. These are all Randers metrics. Therefore the classification of Randers metrics of constant flag curvature is a natural problem, first tackled in [Yasuda and Shimada 1977; Matsumoto 1989]. These authors obtained conditions they believed were necessary and sufficient for a Randers metric to be of constant flag curvature. Using their result strictly as inspiration, Bao and Shen [2002] constructed a family of Randers metrics on S^3 with $\boldsymbol{K} = 1$; these metrics do satisfy Yasuda and Shimada's conditions. Later, however, examples were found of Randers metrics of constant flag curvature not satisfying those conditions [Shen 2002; 2003c], showing that the earlier characterization was incorrect. Shortly thereafter, Randers metrics of constant flag curvature were characterized in [Bao–Robles 2003] using a system of PDEs, a result also obtained in [Matsumoto and Shimada 2002] by a different method. This subsequently led to a corrected version of the Yasuda–Shimada theorem. Finally, using the characterization in [Bao–Robles 2003], and motivated by some constructions in [Shen 2002; 2003c], Bao, Robles and Shen have classified Randers metrics of constant flag curvature with the help of formula (2–17):

THEOREM 11.1 [Bao et al. 2003]. *Let $\Phi = \sqrt{\phi_{ij} y^i y^j}$ be a Riemannian metric and let $V = V^i(\partial/\partial x^i)$ be a vector field on a manifold M with $\Phi(x, V_x) < 1$ for all $x \in M$. Let F be the Randers metric defined by* (2–17). *F is of constant flag curvature $\boldsymbol{K} = \lambda$ if and only if*

(a) *there is a constant c such that V satisfies $V_{i|j} + V_{j|i} = -4c\phi_{ij}$, where $V_i := \phi_{ij} V^j$, and*
(b) *Φ has constant sectional curvature $\tilde{\boldsymbol{K}} = \lambda + c^2$,*

where $_|$ denotes the covariant derivative with respect to Φ and c is a constant.

The equation $V_{i|j} + V_{j|i} = -4c\phi_{ij}$ of part (a) is by itself always equivalent to $\boldsymbol{S} = (n+1)cF$, for c a scalar function on M [Xing 2003].

An analogue of Theorem 11.1 still holds for Randers metrics of isotropic Ricci curvature, i.e., $\mathbf{Ric} = (n-1)\lambda F^2$, where $\lambda = \lambda(x)$ is a scalar function on M. See [Bao and Robles 2003b] in this volume.

We have not extended the result above to Randers metrics of scalar curvature. Usually, the isotropic S-curvature condition simplifies the classification problem. It seems possible to classify Randers metrics of scalar curvature and isotropic S-curvature. The following example is our first attempt to understand Randers metrics of scalar curvature and isotropic S-curvature.

EXAMPLE 11.2. Let $F = \alpha + \beta$ be the Randers metric defined in (2–20). Set $\Delta := 1 - |a|^2|x|^4$. We can write $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$, where

$$a_{ij} = \frac{\delta_{ij}}{\Delta} + \frac{\big(|x|^2 a^i - 2\langle a, x\rangle x^i\big)\big(|x|^2 a^j - 2\langle a, x\rangle x^j\big)}{\Delta^2}, \quad b_i = -\frac{|x|^2 a^i - 2\langle a, x\rangle x^i}{\Delta}.$$

Using Maple for the computation, we obtain, with the notations of Section 9,

$$e_{00} = \frac{2\langle a,x\rangle |y|^2}{\Delta} = 2\langle a,x\rangle(\alpha^2-\beta^2),$$
$$s_{j0} = 2\,\frac{\langle a,y\rangle x^j - \langle x,y\rangle a^j}{\Delta^2},$$
$$s_0 = b^i s_{i0} = 2\,\frac{|a|^2|x|^2\langle x,y\rangle + \langle a,x\rangle\langle a,v\rangle}{\Delta}.$$

By Lemma 9.1, we see that F has isotropic S-curvature:

$$\boldsymbol{S} = (n{+}1)\,\langle a,x\rangle F.$$

By (9–1), the spray coefficients $G^i = G^i(x,y)$ of F are

$$G^i = \overline{G}^i + Py^i + \alpha a^{ij} s_{j0},$$

where $P = e_{00}/(2F) - s_0 = \langle a,x\rangle(\alpha-\beta) - s_0$. Using the formulas for G^i and $R^i{}_k$ in (11–1), we can show that F is also of scalar curvature with flag curvature

$$\boldsymbol{K} = 3\frac{\langle a,y\rangle}{F} + 3\,\langle a,x\rangle^2 - 2|a|^2|x|^2.$$

12. Projectively Flat Metrics

A Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathrm{R}^n$ is *projectively flat* if every geodesic $\sigma(t)$ is straight in U, that is, if

$$\sigma^i(t) = x^i + f(t)y^i,$$

where $f(t)$ is a C^∞ function with $f(0)=0$, $f'(0)=1$ and $x=(x^i)$, $y=(y^i)$ are constant vectors. This is equivalent to $G^i = Py^i$, where $P=P(x,y)$ is positively y-homogeneous of degree one. P is called the *projective factor*.

It is generally difficult to compute the Riemann curvature, but for locally projectively flat Finsler metrics, the formula is relatively simple.

Consider a projectively flat Finsler metric $F = F(x,y)$ on an open subset $U \subset \mathrm{R}^n$. By definition, its spray coefficients are in the form $G^i = Py^i$. Plugging them into (11–1), one obtains

$$R^i{}_k = \Xi\delta^i_k + \tau_k y^i, \tag{12–1}$$

where

$$\Xi = P^2 - P_{x^k}y^k, \quad \tau_k = 3(P_{x^k} - PP_{y^k}) + \Xi_{y^k}.$$

Using (11–6), one can show that $\tau_k = -\Xi F^{-1}F_{y^k}$ and

$$R^i{}_k = \Xi\Big(\delta^i_k - \frac{F_{y^k}}{F}y^i\Big). \tag{12–2}$$

Thus F is of scalar curvature with flag curvature

$$\boldsymbol{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k}y^k}{F^2}. \tag{12–3}$$

Using (7–4), one obtains

$$\boldsymbol{S} = (n+1)P(x,y) - y^m \frac{\partial}{\partial x^m} \ln \sigma_F(x). \tag{12–4}$$

By (12–2), one immediately obtains the following result. (See also [Szabó 1977] and [Matsumoto 1980] for related discussions.)

PROPOSITION 12.1 [Berwald 1929a; 1929b]. *Every locally projectively flat Finsler metric is of scalar curvature.*

There is another way to characterize projectively flat Finsler metrics.

THEOREM 12.2 [Hamel 1903; Rapcsák 1961]. *Let $F = F(x,y)$ be a Finsler metric on an open subset $U \subset \mathrm{R}^n$. F is projectively flat if and only if F satisfies*

$$F_{x^k y^l} y^k - F_{x^l} = 0, \tag{12–5}$$

in which case, the spray coefficients are given by $G^i = Py^i$ with $P = \frac{1}{2} F_{x^k} y^k / F$.

PROOF. Let $G^i = G^i(x,y)$ denote the spray coefficients of F in the standard coordinate system in $TU \cong U \times \mathrm{R}^n$. One can rewrite (4–2) as

$$G^i = Py^i + Q^i, \tag{12–6}$$

where

$$P = \frac{F_{x^k} y^k}{2F}, \quad Q^i = \tfrac{1}{2} F g^{il} (F_{x^k y^l} y^k - F_{x^l}).$$

Thus F is projectively flat if and only if there is a scalar function $\tilde{P} = \tilde{P}(x,y)$ such that $G^i = \tilde{P} y^i$, i.e.,

$$Py^i + Q^i = \tilde{P} y^i. \tag{12–7}$$

Observe that

$$g_{ij} y^j Q^i = \tfrac{1}{2} F y^l (F_{x^k y^l} y^k - F_{x^l}) = 0.$$

Assume that (12–7) holds. Contracting with $y_i := g_{ij} y^j$ yields

$$P = \tilde{P}.$$

Then $Q^i = 0$ by (12–7). This implies (12–5). □

Since equation (12–5) is linear, if F_1 and F_2 are projectively flat on an open subset $U \subset \mathrm{R}^n$, so is their sum. If $F = F(x,y)$ is projectively flat on $U \subset \mathrm{R}^n$, so is its reverse $\bar{F} := F(x, -y)$. Thus the symmetrization

$$\tilde{F} := \tfrac{1}{2}\big(F(x,y) + F(x,-y)\big)$$

is projectively flat.

The Finsler metric $F = \alpha_\mu(x,y)$ in (2–5) satisfies (12–5), so it's projectively flat.

THEOREM 12.3. (Beltrami) *A Riemannian metric $F = F(x,y)$ on a manifold M is locally projectively flat if and only if it is locally isometric to the metric α_μ in* (2–5).

Using the formula (9–1), one can easily prove the following:

THEOREM 12.4. *A Randers metric $F = \alpha+\beta$ on a manifold is locally projectively flat if and only if α is locally projectively flat and β is closed.*

Besides projectively flat Randers metrics, we have the following examples.

EXAMPLE 12.5. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and let U be the strongly convex domain enclosed by the indicatrix of ϕ. Let $\Theta = \Theta(x, y)$ be the Funk metric on U (Example 2.9). By (2–19),

$$\Theta_{x^k y^l} y^k = (\Theta\Theta_{y^k})_{y^l} y^k = \tfrac{1}{2}(\Theta^2)_{y^k y^l} y^k = \tfrac{1}{2}[\Theta^2]_{y^l} = \Theta_{x^l}.$$

Thus Θ is projectively flat with projective factor

$$P = \frac{\Theta_{x^k} y^k}{2\Theta} = \frac{\Theta\Theta_{y^k} y^k}{2\Theta} = \tfrac{1}{2}\Theta.$$

By (12–3), the flag curvature is

$$\boldsymbol{K} = \frac{\Theta^2 - 2\Theta_{x^k} y^k}{4\Theta^2} = \frac{\Theta^2 - 2\Theta^2}{4\Theta^2} = -\frac{1}{4}.$$

EXAMPLE 12.6 [Shen 2003b]. Let $\phi = \phi(y)$ be a Minkowski norm on R^n and let U be the strongly convex domain enclosed by the indicatrix of ϕ. Let $\Theta = \Theta(x, y)$ be the Funk metric on U and define

$$F := \Theta(x, y)\big(1 + \Theta_{y^k}(x, y) x^k\big).$$

Since $F(0, y) = \Theta(0, y) = \phi(y)$ is a Minkowski norm, by continuity, F is a Finsler metric for x nearby the origin. By (2–19), one can verify that

$$F_{x^k y^l} y^k = F_{x^l}, \quad F_{x^k} y^k = 2\Theta F.$$

Thus F is projectively flat with projective factor $P = \Theta(x, y)$. By (2–19) and (12–3), we obtain

$$\boldsymbol{K} = \frac{\Theta^2 - \Theta_{x^k} y^k}{F^2} = \frac{\Theta^2 - \Theta\Theta_{y^k} y^k}{F^2} = 0.$$

Now we take a look at the Finsler metric $F = F_\varepsilon(x, y)$ defined in (2–9).

EXAMPLE 12.7. Let

$$F := \frac{\sqrt{\Psi\big(\frac{1}{2}(\sqrt{\Phi^2 + (1-\varepsilon^2)|y|^4} + \Phi)\big) + (1-\varepsilon^2)\langle x, y\rangle^2} + \sqrt{1-\varepsilon^2}\,\langle x, y\rangle}{\Psi}, \tag{12–8}$$

where

$$\Phi := \varepsilon|y|^2 + \big(|x|^2|y|^2 - \langle x, y\rangle^2\big), \quad \Psi := 1 + 2\varepsilon|x|^2 + |x|^4.$$

First, one can verify that $F = F_\varepsilon(x,y)$ satisfies (12–5). Thus F is projectively flat with spray coefficients $G^i = Py^i$, where $P = \frac{1}{2}F_{x^k}(x,y)y^k/F(x,y)$. A Maple computation gives

$$P = \frac{\sqrt{\Psi\big(\frac{1}{2}(\sqrt{\Phi^2+(1-\varepsilon^2)|y|^4}+\Phi)\big)-(1-\varepsilon^2)\langle x,y\rangle^2}-(\varepsilon+|x|^2)\langle x,y\rangle}{\Psi}. \tag{12–9}$$

Further, one can verify that P satisfies

$$P_{x^k}y^k = P^2 - F^2.$$

Thus

$$\boldsymbol{K} = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{P^2-(P^2-F^2)}{F^2} = 1.$$

That is, F has constant flag curvature $\boldsymbol{K} = 1$.

The projectively flat Finsler metrics constructed above are incomplete. They can be pulled back to S^n by (2–2) to form complete irreversible projectively flat Finsler metrics of constant flag curvature $\boldsymbol{K} = 1$. See [Bryant 1996; 1997].

13. The Chern Connection and Some Identities

The previous sections introduced several geometric quantities, such as the Cartan torsion, the Landsberg curvature, the S-curvature and the Riemann curvature. These quantities are not all independent. To reveal their relationships, we use the Chern connection to describe them as tensors on the slit tangent bundle, and use the exterior differentiation method to derive some important identities.

Let M be an n-dimensional manifold and TM its tangent bundle. As usual, a typical element of TM will be denoted by (x,y), with $y \in T_xM$. The natural projection $\pi : TM \to M$ pulls back the tangent bundle TM over M to a vector bundle π^*TM over the slit tangent bundle TM_0. The fiber of π^*TM at each point $(x,y) \in TM_0$ is a copy of T_xM. Thus we write a typical element of π^*TM as (x,y,v), where $y \in T_xM \setminus \{0\}$ and $v \in T_xM$. Let $\partial_{i|(x,y)} := \big(x,y,(\partial/\partial x^i)|_x\big)$. Then $\{\partial_i\}$ is a local frame for π^*TM. Let (x^i, y^i) be a standard local coordinate system in TM_0. Then $HT^*M := \mathrm{span}\{dx^i\}$ is a well-defined subbundle of $T^*(TM_0)$. Let

$$\delta y^i := dy^i - N^i_j dx^j,$$

where $N^i_j := \partial G^i/\partial y^j$. Then $VT^*M := \mathrm{span}\,\{\delta y^i\}$ is a well-defined subbundle of $T^*(TM_0)$, so that $T^*(TM_0) = HT^*M \oplus VT^*M$. The Chern connection is a linear connection on π^*TM, locally expressed by

$$DX = (dX^i + X^j{\omega_j}^i) \otimes \partial_i, \quad X = X^i\partial_i,$$

where the set of 1-forms $\{\omega_j{}^i\}$ is uniquely determined by

$$\begin{aligned} d\omega^i &= \omega^j \wedge \omega_j{}^i, \\ dg_{ij} &= g_{ik}\omega_j{}^k + g_{kj}\omega_i{}^k + 2C_{ijk}\omega^{n+k}, \end{aligned} \tag{13–1}$$

where $g_{ij} := \frac{1}{2}[F^2]_{y^iy^j}$, $C_{ijk} := \frac{1}{4}[F^2]_{y^iy^jy^k}$, $\omega^i := dx^i$, and $\omega^{n+i} := \delta y^i$. See [Bao and Chern 1993; Bao et al. 2000; Chern 1943; 1948; 1992]. Each 1-form $\omega_j{}^i$ is horizontal: $\omega_j{}^i = \Gamma^i_{jk}\,dx^k$. The coefficients $\Gamma^i_{jk} = \Gamma^i_{jk}(x,y)$ are called the *Christoffel symbols*. We have $N^i_j = y^k\Gamma^i_{jk}$. Thus

$$\omega^{n+i} = dy^i + y^j\omega_j{}^i. \tag{13–2}$$

Put

$$\Omega^i := d\omega^{n+i} - \omega^{n+j} \wedge \omega_j{}^i. \tag{13–3}$$

One can express Ω^i as

$$\Omega^i = \frac{1}{2}R^i{}_{kl}\omega^k \wedge \omega^l - L^i{}_{kl}\omega^k \wedge \omega^{n+l},$$

where

$$R^i{}_{kl} = \frac{\partial N^i_l}{\partial x^k} - \frac{\partial N^i_k}{\partial x^l} + N^s_l\frac{\partial N^i_k}{\partial y^s} - N^s_k\frac{\partial N^i_l}{\partial y^s},$$

and

$$L^i{}_{kl} := y^j\frac{\partial \Gamma^i_{jk}}{\partial y^l} = \frac{\partial N^i_k}{\partial y^l} - \Gamma^i_{kl}.$$

Let $R^i{}_k$ be defined in (11–1) and L_{ijk} be defined in (8–4). Then

$$R^i{}_k = R^i{}_{kl}y^l, \quad R^i{}_{kl} = \frac{1}{3}\Big(\frac{\partial R^i{}_k}{\partial y^l} - \frac{\partial R^i{}_l}{\partial y^k}\Big), \quad L^i{}_{kl} = g^{ij}L_{jkl}. \tag{13–4}$$

Put

$$\Omega_j{}^i := d\omega_j{}^i - \omega_j{}^k \wedge \omega_k{}^i.$$

One can express $\Omega_j{}^i$ as

$$\Omega_j{}^i = \frac{1}{2}R_j{}^i{}_{kl}\omega^k \wedge \omega^l + P_j{}^i{}_{kl}\omega^k \wedge \omega^{n+l}.$$

Differentiating (13–2) yields $\Omega^i = y^j\Omega_j{}^i$. Thus

$$R^i{}_{kl} = y^jR_j{}^i{}_{kl}, \quad L^i{}_{kl} = -y^jP_j{}^i{}_{kl}.$$

There is a canonical way to define the covariant derivatives of a tensor on TM_0 using the Chern connection. For the distortion τ on $TM \setminus \{0\}$, define $\tau_{|m}$ and $\tau_{.m}$ by

$$d\tau = \tau_{|i}\omega^i + \tau_{.i}\omega^{n+i}. \tag{13–5}$$

It follows from (7–2) that

$$\tau_{.i} = \frac{\partial \tau}{\partial y^i} = I_i. \tag{13–6}$$

For the induced Riemannian tensor, $g = g_{ij}\omega^i \otimes \omega^j$, define $g_{ij|k}$ and $g_{ij\cdot k}$ by

$$dg_{ij} - g_{kj}\omega_i{}^k - g_{ik}\omega_j{}^k = g_{ij|k}\omega^k + g_{ij\cdot k}\omega^{n+k}.$$

It follows from (13–1) that

$$g_{ij|k} = 0, \quad g_{ij\cdot k} = 2C_{ijk}.$$

Similarly, one can define $C_{ijk|l}$ at $I_{i|l}$. Equations (8–4) and (8–5) become

$$L_{ijk} = C_{ijk|m}y^m, \quad J_i = I_{i|m}y^m. \tag{13–7}$$

Differentiating (13–3) yields the Bianchi identity

$$d\Omega^i = -\Omega^j \wedge \omega_j{}^i + \omega^{n+j} \wedge \Omega_j{}^i. \tag{13–8}$$

It follows from (13–8) that

$$R_j{}^i{}_{kl} = R^i{}_{kl\cdot j} + L^i{}_{kj|l} - L^i{}_{lj|k} + L^i{}_{lm}L^m_{kj} - L^i{}_{km}L^m_{lj}. \tag{13–9}$$

We are going to find other relationships among curvature tensors. Differentiating (13–1) yields

$$0 = g_{ik}\Omega_j{}^k + g_{kj}\Omega_i{}^k + 2(C_{ijk|l}\omega^l + C_{ijk\cdot l}\omega^{n+l}) \wedge \omega^{n+k} + 2C_{ijk}\Omega^k.$$

It follows that

$$R_{jikl} + R_{ijkl} + 2C_{ijm}R^m_{kl} = 0, \tag{13–10}$$

where $R_{jikl} := g_{im}R_j{}^m{}_{kl}$, and

$$P_{jikl} + P_{ijkl} + 2C_{ijl|k} - 2C_{ijm}L^m_{kl} = 0,$$

where $P_{jikl} := g_{im}P_j{}^m{}_{kl}$. Then (13–9) can be expressed by

$$R_{jikl} = g_{im}R^m_{kl\cdot j} + L_{ikj|l} - L_{ilj|k} + L_{ilm}L^m_{kj} - L_{ikm}L^m_{lj}.$$

Plugging the formulas for R_{jikl} and R_{ijkl} into (13–10) yields

$$\begin{aligned} L_{ijk|l} - L_{ijl|k} &= -\tfrac{1}{2}(g_{im}R^m_{kl\cdot j} + g_{jm}R^m_{kl\cdot i}) - C_{ijm}R^m_{kl}, \\ I_{k|l} - I_{l|k} &= -R^m_{kl\cdot m} - I_mR^m_{kl}. \end{aligned} \tag{13–11}$$

The expression for $R^i{}_{kl}$ in (13–4) can be written as

$$R^i{}_{kl} = \tfrac{1}{3}(R^i{}_{k\cdot l} - R^i{}_{l\cdot k}). \tag{13–12}$$

LEMMA 13.1 [Mo 1999]. *L_{ijk} and $R^i{}_k$ are related by*

$$C_{ijk|p|q}y^py^q + C_{ijm}R^m_k = -\tfrac{1}{3}g_{im}R^m{}_{k\cdot j} - \tfrac{1}{3}g_{jm}R^m{}_{k\cdot i} - \tfrac{1}{6}g_{im}R^m{}_{j\cdot k} - \tfrac{1}{6}g_{jm}R^m{}_{i\cdot k}. \tag{13–13}$$

In particular,

$$I_{k|p|q}y^py^q + I_mR^m_k = -\tfrac{1}{3}(2R^m{}_{k\cdot m} + R^m{}_{m\cdot k}). \tag{13–14}$$

Proof. By (13–7), we have

$$L_{ijk|m}y^m = C_{ijk|p|q}y^p y^q, \quad J_{k|m}y^m = I_{k|p|q}y^p y^q.$$

Contracting the first line of (13–11) with y^l yields (13–13), and contracting (13–13) with g^{ij} yields (13–14). Here we have made use of (13–12). □

The equations above are crucial in the study of Finsler metrics of scalar curvature. Let $F = F(x, y)$ be a Finsler metric of scalar curvature with flag curvature $\boldsymbol{K} = \boldsymbol{K}(x, y)$. Then (11–8) holds. Plugging (11–8) into (13–13) and (13–14) yields

$$\begin{aligned} C_{ijk|p|q}y^p y^q + \boldsymbol{K}F^2 C_{ijk} &= -\tfrac{1}{3}F^2(\boldsymbol{K}_{\cdot i}h_{jk} + \boldsymbol{K}_{\cdot j}h_{ik} + \boldsymbol{K}_{\cdot k}h_{ij}), \\ I_{k|p|q}y^p y^q + \boldsymbol{K}F^2 I_k &= -\tfrac{1}{3}(n+1)F^2\boldsymbol{K}_{\cdot k}. \end{aligned} \tag{13–15}$$

Using the first of these equations, one shows that any compact Finsler manifold of negative constant flag curvature must be Riemannian [Akbar-Zadeh 1988].

It follows from (13–15) that for any Finsler metric F of scalar curvature with flag curvature $\boldsymbol{K}$, the Matsumoto torsion satisfies

$$M_{ijk|p|q}y^p y^q + \boldsymbol{K}F^2 M_{ijk} = 0. \tag{13–16}$$

One can use (13–16) to show that any Landsberg metric of scalar curvature with $\boldsymbol{K} \neq 0$ it is Riemannian, in dimension $n \geq 3$ [Numata 1975]. See also Corollary 17.4.

Using (13–16), one can easily prove this:

Theorem 13.2 [Mo and Shen 2003]. *Let (M, F) be a compact Finsler manifold of dimension $n \geq 3$. If F is of scalar curvature with negative flag curvature, F must be a Randers metric.*

Now we derive some important identities for the S-curvature. Differentiating (13–5) and using (13–3) and (13–6), one obtains

$$0 = d^2\tau = (\tau_{|k|l}\omega^l + \tau_{|k\cdot l}\omega^{n+l}) \wedge \omega^k + (I_{k|l}\omega^l + I_{k\cdot l}\omega^{n+l}) \wedge \omega^k + I_m\Omega^{n+m}.$$

This yields the Ricci identities

$$\tau_{|k|l} = \tau_{|l|k} + I_p R^p{}_{kl}, \tag{13–17}$$

$$\tau_{|k\cdot l} = I_{l|k} - I_p L^p{}_{kl}. \tag{13–18}$$

From the definition (7–3), the S-curvature can be regarded as

$$\boldsymbol{S} = \tau_{|m}y^m. \tag{13–19}$$

Contracting (13–17) with y^k yields

$$\boldsymbol{S}_{\cdot k} = (\tau_{|m}y^m)_{\cdot k} = \tau_{|m\cdot k}y^m + \tau_{|k} = I_{k|m}y^m - I_p L^p_{mk}y^m + \tau_{|k} = J_k + \tau_{|k},$$

where we have made use of (13–17) and (13–19). We restate this equation as

$$\boldsymbol{S}_{\cdot k} = \tau_{|k} + J_k. \tag{13–20}$$

LEMMA 13.3 [Mo 2002; Mo and Shen 2003]. *The S-curvature satisfies*

$$\boldsymbol{S}_{\cdot k|m}y^m - \boldsymbol{S}_{|k} = -\tfrac{1}{3}(2R^m{}_{k\cdot m} + R^m{}_{m\cdot k}). \tag{13–21}$$

PROOF. It follows from (13–20) that

$$\boldsymbol{S}_{\cdot k|l} = \tau_{|k|l} + J_{k|l}. \tag{13–22}$$

By (13–17) and (13–22), one obtains

$$\begin{aligned}\boldsymbol{S}_{\cdot k|m}y^m - \boldsymbol{S}_{|k} &= (\boldsymbol{S}_{\cdot k|m} - \boldsymbol{S}_{\cdot m|k})y^m = (\tau_{|k|m} - \tau_{|m|k})y^m + (J_{k|m} - J_{m|k})y^m \\ &= I_pR^p{}_{km}y^m + J_{k|m}y^m = I_pR^p{}_k - I_pR^p{}_k - \tfrac{1}{3}I_m(R^m{}_{k\cdot l} - R^m{}_{l\cdot k}) \\ &= -\tfrac{1}{3}I_m(R^m{}_{k\cdot l} - R^m{}_{l\cdot k}). \qquad \square\end{aligned}$$

14. Nonpositively Curved Finsler Manifolds

We now use some of the identities derived in the previous section to establish global rigidity theorems.

First, consider the mean Cartan torsion. Let (M, F) be an n-dimensional Finsler manifold. The norm of the mean Cartan torsion $\boldsymbol{I}$ at a point $x \in M$ is defined by

$$\|\boldsymbol{I}\|_x := \sup_{0\neq y\in T_xM} \sqrt{I_i(x,y)g^{ij}(x,y)I_j(x,y)}.$$

It is known that if $F = \alpha + \beta$ is a Randers metric, then

$$\|\boldsymbol{I}\|_x \le \frac{n+1}{\sqrt{2}}\sqrt{1-\sqrt{1-\|\beta\|_x^2}} < \frac{n+1}{\sqrt{2}}.$$

The bound in dimension two was suggested by B. Lackey. See [Shen 2001b, Proposition 7.1.2] or [Ji and Shen 2002] for a proof. Below is our first global rigidity theorem.

THEOREM 14.1 [Shen 2003d]. *Let (M, F) be an n-dimensional complete Finsler manifold with nonpositive flag curvature. Suppose that F has almost constant S-curvature $\boldsymbol{S} = (n+1)(cF+\eta)$ (with c a constant and η a closed 1-form) and bounded mean Cartan torsion $\sup_{x\in M}\|\boldsymbol{I}\|_x < \infty$. Then $\boldsymbol{J} = 0$ and $\boldsymbol{R}\circ\boldsymbol{I} = 0$. Moreover F is Riemannian at points where the flag curvature is negative.*

PROOF. It follows from (13–14) and (13–21) that

$$I_{k|p|q}y^py^q + I_mR^m{}_k = \boldsymbol{S}_{\cdot k|m}y^m - \boldsymbol{S}_{|k}. \tag{14–1}$$

Assume that the S-curvature is almost isotropic:

$$\boldsymbol{S} = (n+1)(cF+\eta),$$

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i dx^i$ is a closed 1-form on M. Observe that

$$\eta_{\cdot k|m}y^m - \eta_{|k} = (\eta_{k|m} - \eta_{m|k})y^m = \Big(\frac{\partial\eta_k}{\partial x^m} - \frac{\partial\eta_m}{\partial x^k}\Big)y^m = 0.$$

Thus

$$\begin{aligned} \boldsymbol{S}_{\cdot k|m} y^m - \boldsymbol{S}_{|k} &= (n+1)\,(c_{x^m} y^m F_{\cdot k} - c_{|k} F + \eta_{\cdot k|m} y^m - \eta_{|k}) \\ &= (n+1)\,(c_{x^m} y^m F_{\cdot k} - c_{|k} F). \end{aligned}$$

In this case, (13–21) becomes

$$2R^m_{\ k\cdot m} + R^m_{\ m\cdot k} = -3(n+1)\,(c_{x^m} y^m F_{\cdot k} - c_{|k} F) \tag{14–2}$$

and (14–1) becomes

$$I_{k|p|q} y^p y^q + I_m R^m_{\ k} = (n+1)\,(c_{x^m} y^m F_{\cdot k} - c_{|k} F).$$

By assumption, c is constant, so this last equation reduces

$$I_{k|p|q} y^p y^q + I_m R^m_{\ k} = 0. \tag{14–3}$$

Let $y \in T_x M$ be an arbitrary vector and let $\sigma(t)$ be the geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Since the Finsler metric is complete, one may assume that $\sigma(t)$ is defined on $(-\infty, \infty)$. The mean Cartan torsion $\boldsymbol{I}$ and the mean Landsberg curvature $\boldsymbol{J}$ restricted to $\sigma(t)$ are vector fields along $\sigma(t)$:

$$\boldsymbol{I}(t) := I^i(\sigma(t), \dot\sigma(t)) \frac{\partial}{\partial x^i}\Big|_{\sigma(t)}, \quad \boldsymbol{J}(t) := J^i(\sigma(t), \dot\sigma(t)) \frac{\partial}{\partial x^i}\Big|_{\sigma(t)}.$$

It follows from (8–5) or (13–7) that

$$D_{\dot\sigma} \boldsymbol{I}(t) = I^i_{\ |m}(\sigma(t), \dot\sigma(t)) \dot\sigma^m(t) \frac{\partial}{\partial x^i}\Big|_{\sigma(t)} = \boldsymbol{J}(t).$$

It follows from (14–3) that

$$D_{\dot\sigma} D_{\dot\sigma} \boldsymbol{I}(t) + \boldsymbol{R}_{\dot\sigma(t)}(\boldsymbol{I}_{\dot\sigma(t)}) = 0.$$

Setting

$$\varphi(t) := \boldsymbol{g}_{\dot\sigma(t)}\big(\boldsymbol{I}(t), \boldsymbol{I}(t)\big),$$

we obtain

$$\begin{aligned} \varphi''(t) &= 2\boldsymbol{g}_{\dot\sigma(t)}\big(\mathrm{D}_{\dot\sigma}\mathrm{D}_{\dot\sigma}\boldsymbol{I}(t), \boldsymbol{I}(t)\big) + 2\boldsymbol{g}_{\dot\sigma(t)}\big(\mathrm{D}_{\dot\sigma}\boldsymbol{I}(t), \mathrm{D}_{\dot\sigma}\boldsymbol{I}(t)\big) \\ &= -2\boldsymbol{g}_{\dot\sigma(t)}\big(\boldsymbol{R}_{\dot\sigma(t)}(\boldsymbol{I}(t)), \boldsymbol{I}(t)\big) + 2\boldsymbol{g}_{\dot\sigma(t)}\big(\boldsymbol{J}(t), \boldsymbol{J}(t)\big). \end{aligned} \tag{14–4}$$

By assumption, $\boldsymbol{K} \le 0$. Thus

$$\boldsymbol{g}_{\dot\sigma(t)}\big(\boldsymbol{R}_{\dot\sigma(t)}(\boldsymbol{I}(t)), \boldsymbol{I}(t)\big) \le 0.$$

It follows from (14–4) that

$$\varphi''(t) \ge 0.$$

Thus $\varphi(t)$ is convex and nonnegative. Suppose that $\varphi'(t_0) \ne 0$ for some t_0. By an elementary argument, $\lim_{t\to+\infty} \varphi(t) = \infty$ or $\lim_{t\to-\infty} \varphi(t) = \infty$. This implies that the mean Cartan torsion is unbounded, which contradicts the assumption. Therefore, $\varphi'(t) = 0$ and hence $\varphi''(t) = 0$. Since each term in (14–4) is nonnegative, one concludes that

$$\boldsymbol{g}_{\dot\sigma(t)}\big(\boldsymbol{R}_{\dot\sigma(t)}(\boldsymbol{I}(t)), \boldsymbol{I}(t)\big) = 0, \qquad \boldsymbol{J}(t) = 0.$$

Setting $t = 0$ yields

$$\boldsymbol{g}_y\big(\boldsymbol{R}_y(\boldsymbol{I}_y), \boldsymbol{I}_y\big) = 0 \tag{14–5}$$

and $\boldsymbol{J}_y = 0$. By (11–6), $\boldsymbol{R}_y(y) = 0$ and $\boldsymbol{R}_y$ is self-adjoint with respect to $\boldsymbol{g}_y$, i.e., $\boldsymbol{g}_y\big(\boldsymbol{R}_y(u), v\big) = \boldsymbol{g}_y(u, \boldsymbol{R}_y(v))$, for $u, v \in T_xM$. Thus there is an orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^n$ with $\boldsymbol{e}_n = y$ such that

$$\boldsymbol{R}_y(\boldsymbol{e}_i) = \lambda_i \boldsymbol{e}_i, \quad i = 1, \ldots, n,$$

with $\lambda_n = 0$. By assumption, the flag curvature is nonpositive. Then

$$\boldsymbol{g}_y(\boldsymbol{R}_y(\boldsymbol{e}_i), \boldsymbol{e}_i) = \lambda_i \le 0, \quad i = 1, \ldots, n-1.$$

Since $\boldsymbol{I}_y$ is perpendicular to y with respect to $\boldsymbol{g}_y$, one can express it as $\boldsymbol{I}_y = \mu_1 \boldsymbol{e}_1 + \cdots + \mu_{n-1}\boldsymbol{e}_{n-1}$. By (14–5), one obtains

$$0 = \boldsymbol{g}_y\big(\boldsymbol{R}_y(\boldsymbol{I}_y), \boldsymbol{I}_y\big) = \sum_{i=1}^{n-1} \mu_i^2 \lambda_i.$$

Since each term $\mu_i^2\lambda_i$ is nonpositive, one concludes that $\mu_i\lambda_i = 0$, or yet

$$\boldsymbol{R}_y(\boldsymbol{I}_y) = \sum_{i=1}^{n-1} \mu_i \lambda_i = 0. \tag{14–6}$$

Now suppose that F has negative flag curvature at a point $x \in M$. Then $\lambda_i < 0$ for $i = 1, \ldots, n-1$. By (14–6), one concludes that $\mu_i = 0$, $i = 1, \ldots, n-1$, namely, $\boldsymbol{I}_y = 0$. By Deicke's theorem [Deicke 1953], F is Riemannian. □

Corollary 14.2. *Every complete Berwald manifold with negative flag curvature is Riemannian.*

Proof. For a Berwald metric F on a manifold M, the Minkowski spaces (T_xM, F_x) are all linearly isometric (Proposition 5.2). Thus the Cartan torsion is bounded from above. Meanwhile, the S-curvature vanishes (Proposition 7.1). Thus F must be Riemannian. □

Example 14.3. Let $(\overline{M}, \bar\alpha)$ and $(\underline{M}, \underline{\alpha})$ be Riemannian manifolds and let $F = F(x, y)$ be the product metric on $M = \overline{M} \times \underline{M}$, defined in Example 5.1. We computed the spray coefficients of F in Example 5.1. Using (11–1), one obtains the Riemann tensor of F:

$$R^{\bar a}{}_{\bar b} = \overline{R}^{\bar a}{}_{\bar b}, \quad R^{\bar a}{}_{\underline{b}} = 0 = R^{\underline{a}}{}_{\bar b}, \quad R^{\underline{a}}{}_{\underline{b}} = \underline{R}^{\underline{a}}{}_{\underline{b}},$$

where $\bar{R}^{\bar{a}}{}_{\bar{b}}$ and $\underline{R}^{\underline{a}}{}_{\underline{b}}$ are the coefficients of the Riemann tensors of $\bar{a}$ and $\underline{a}$. Let $R_{ij} := g_{ik}R^k{}_j$ as usual, and define $\bar{R}_{\bar{a}\bar{b}}$ and $\underline{R}_{\underline{a}\underline{b}}$ similarly. Using (5–3), one obtains

$$R_{\bar{a}\bar{b}} = f_s\bar{R}_{\bar{a}\bar{b}}, \quad R_{\bar{a}\underline{b}} = 0 = R_{\underline{a}\bar{b}}, \quad R_{\underline{a}\underline{b}} = f_t\underline{R}_{\underline{a}\underline{b}}.$$

For any vector $v = v^i(\partial/\partial x^i)|_x \in T_xM$,

$$\boldsymbol{g}_y\big(\boldsymbol{R}_y(v), v\big) = f_s\bar{R}_{\bar{a}\bar{b}}v^{\bar{a}}v^{\bar{b}} + f_t\underline{R}_{\underline{a}\underline{b}}v^{\underline{a}}v^{\underline{b}}.$$

Thus if α_1 and α_2 both have nonpositive sectional curvature, F has nonpositive flag curvature.

Using (5–4), one can compute the mean Cartan torsion. First, observe that

$$I_i = \frac{\partial}{\partial y^i}\ln\sqrt{\det(g_{jk})} = \frac{\partial}{\partial y^i}\ln\sqrt{h\big([\alpha_1]^2, [\alpha_2]^2\big)},$$

where $h = h(s,t)$ is defined in (5–5). One obtains

$$I_{\bar{a}} = \frac{h_s}{h}\bar{y}_{\bar{a}}, \quad I_{\underline{a}} = \frac{h_t}{h}\bar{y}_{\underline{a}},$$

where $\bar{y}_{\bar{a}} := \bar{g}_{\bar{a}\bar{b}}y^{\bar{b}}$ and $\bar{y}_{\underline{a}} := \bar{g}_{\underline{a}\underline{b}}y^{\underline{b}}$. Since $\bar{y}_{\bar{a}}\bar{R}^{\bar{a}}{}_{\bar{b}} = 0$ and $\bar{y}_{\underline{a}}\bar{R}^{\underline{a}}{}_{\underline{b}} = 0$, one obtains

$$\boldsymbol{g}_y\big(\boldsymbol{R}_y(\boldsymbol{I}_y), \boldsymbol{I}_y\big) = I_iR^i{}_jI^j = \frac{h_s}{h}\bar{y}_{\bar{a}}\bar{R}^{\bar{a}}{}_{\bar{b}}I^{\bar{b}} + \frac{h_t}{h}\bar{y}_{\underline{a}}\bar{R}^{\underline{a}}{}_{\underline{b}}I^{\underline{b}} = 0.$$

Since $\boldsymbol{R}_y$ is self-adjoint and nonpositive definite with respect to $\boldsymbol{g}_y$, $\boldsymbol{R}_y(\boldsymbol{I}_y) = 0$. Therefore F satisfies the conditions and conclusions in Theorem 14.1.

The next example shows that completeness in Theorem 14.1 cannot be replaced by positive completeness.

Example 14.4. Let $\phi(y)$ be a Minkowski norm on R^n. Let $\Theta = \Theta(x,y)$ be the Funk metric on $U := \{y \in \mathrm{R}^n \mid \phi(y) < 1\}$ defined in (2–18). Let $a \in \mathrm{R}^n$ be an arbitrary constant vector. Let

$$F := \Theta(x,y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \quad y \in TU \cong U \times \mathrm{R}^n.$$

Clearly, F is a Finsler metric near the origin. By (2–19), one sees that the spray coefficients of F are given by $G^i = Py^i$, where

$$P := \frac{1}{2}\Big(\Theta(x,y) - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\Big).$$

Using this and (12–3), one obtains

$$\boldsymbol{K} = \frac{\frac{1}{4}\Big(\Theta - \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\Big)^2 - \frac{1}{2}\bigg(\Theta^2 + \Big(\frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\Big)^2\bigg)}{\Big(\Theta(x,y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\Big)^2} = -\frac{1}{4}.$$

Thus F has constant flag curvature $\boldsymbol{K} = -\frac{1}{4}$. See also [Shen 2003b, Example 5.3].

Now we compute the S-curvature of F. A direct computation gives

$$\frac{\partial G^m}{\partial y^m} = (n+1)\,P.$$

Let $dV = \sigma_F(x)\,dx^1 \cdots dx^n$ be the Finsler volume form on M. From (12–4), we obtain

$$\begin{aligned}\boldsymbol{S} &= \frac{n+1}{2}F(x,y) - (n+1)\frac{\langle a, y\rangle}{1+\langle a,x\rangle} - y^m \frac{\partial}{\partial x^m}\ln \sigma_F(x)\\ &= (n+1)\big(\tfrac{1}{2}F(x,y) + d\varphi_x(y)\big),\end{aligned}$$

where $\varphi(x) := -\ln\big((1+\langle a,x\rangle)\,\sigma_F(x)^{1/(n+1)}\big)$. Thus

$$\boldsymbol{E} = \tfrac{1}{4}(n+1)\,F^{-1}\boldsymbol{h},$$

where $\boldsymbol{h}_y = h_{ij}(x,y)\,dx^i \otimes dx^j$ is given by $h_{ij} = F(x,y)F_{y^iy^j}(x,y)$.

When $\phi(y) = |y|$ is the standard Euclidean norm, $U = \mathrm{B}^n$ is the standard unit ball in R^n and

$$\Theta = \frac{\sqrt{|y|^2 - \big(|x|^2|y|^2 - \langle x,y\rangle^2\big)}}{1-|x|^2}.$$

Thus

$$F = \frac{\sqrt{|y|^2 - \big(|x|^2|y|^2 - \langle x,y\rangle^2\big)}}{1-|x|^2} + \frac{\langle a,y\rangle}{1+\langle a,x\rangle}.$$

Assume that $|a| < 1$. It is easy to verify that F is a Randers metric defined on the whole B^n, with constant S-curvature $\boldsymbol{S} = \frac{1}{2}(n+1)F(x,y)$. One can show that F is positively complete on B^n, so that every geodesic defined on an interval (λ,μ) can be extended to a geodesic defined on $(\lambda,+\infty)$.

15. Flag Curvature and Isotropic S-Curvature

It is a difficult task to classify Finsler metrics of scalar curvature. All known Randers metrics of scalar curvature have isotropic S-curvature. Thus it is a natural idea to investigate Finsler metrics of scalar curvature which also have isotropic S-curvature.

PROPOSITION 15.1 [Chen et al. 2003]. *Let (M,F) be an n-dimensional Finsler manifold of scalar curvature with flag curvature $\boldsymbol{K} = \boldsymbol{K}(x,y)$. Suppose that the S-curvature is almost isotropic*,

$$\boldsymbol{S} = (n+1)\,(cF+\eta),$$

where $c = c(x)$ is a scalar function on M and $\eta = \eta_i(x)y^i$ is a closed 1-form. Then there is a scalar function $\sigma = \sigma(x)$ on M such that the flag curvature equals

$$\boldsymbol{K} = 3\,\frac{c_{x^m}y^m}{F} + \sigma. \tag{15–1}$$

Proof. By assumption, the flag curvature $\boldsymbol{K} = \boldsymbol{K}(x, y)$ is a scalar function on TM_0. Thus (11–8) holds. Plugging (11–8) into (14–2) yields

$$c_{x^m} y^m F_{\cdot k} - c_{x^k} F = -\tfrac{1}{3} \boldsymbol{K}_{y^k} F^2. \tag{15–2}$$

Rewriting (15–2) as

$$\left(\frac{1}{3} \boldsymbol{K} - \frac{c_{x^m} y^m}{F} \right)_{y^k} = 0,$$

one concludes that the quantity

$$\sigma := \boldsymbol{K} - \frac{3 c_{x^m} y^m}{F}$$

is a scalar function on M. This proves the proposition. □

Corollary 15.2 [Mo 2002]. *Let F be an n-dimensional Finsler metric of scalar curvature. If F has almost constant S-curvature, the flag curvature is a scalar function on M.*

From the definition of flag curvature, one can see that every two-dimensional Finsler metric is of scalar curvature. One immediately obtains the following:

Corollary 15.3. *Let F be a two-dimensional Finsler metric with almost isotropic S-curvature. Then the flag curvature is in the form* (15–1).

Let $F = F(x, y)$ be a two-dimensional Berwald metric on a surface M. It follows from Corollaries 15.3 and 15.2 that the Gauss curvature $\boldsymbol{K} = \boldsymbol{K}(x)$ is a scalar function of $x \in M$. Since F is a Berwald metric, the $G^i = \frac{1}{2} \Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i (\partial/\partial x^i)|_x \in T_x M$. By (11–1), the Riemann curvature, $R^i{}_k = R^i{}_k(x, y)$, are quadratic in y. This implies that the Ricci scalar $\mathbf{Ric} = R^m_{\ m}(x, y)$ is quadratic in y. Suppose that $\boldsymbol{K}(x_0) \neq 0$ at some point $x_0 \in M$. Then

$$F(x_0, y)^2 = \frac{\mathbf{Ric}(x_0, y)}{\boldsymbol{K}(x_0)}$$

is quadratic in $y \in T_{x_0} M$. Namely, $F_{x_0} = F|_{T_{x_0} M}$ is Euclidean at x_0. By Proposition 5.2, all tangent spaces $(T_x M, F_x)$ are linearly isometric to each other. One concludes that F_x is Euclidean for any $x \in M$ and F is Riemannian. Now we suppose that $\boldsymbol{K} \equiv 0$. Since F is Berwaldian, F must be locally Minkowskian. See [Szabó 1981] for a different argument.

16. Projectively Flat Metrics with Isotropic S-Curvature

Recall that a Finsler metric F on a manifold M is locally projectively flat if at any point $x \in M$, there is a local coordinate system (x^i) in M such that every geodesic $\sigma(t)$ is straight, i.e., $\sigma^i(t) = f(t) a^i + b^i$. This is equivalent to saying that in the standard local coordinate system (x^i, y^i), the spray coefficients G^i are in the form $G^i = P y^i$ with $P = F_{x^k} y^k / (2F)$. It is well-known that any locally projectively flat Finsler metric F is of scalar curvature, and its flag curvature equals $\boldsymbol{K} = (P^2 - P_{x^k} y^k)/F^2$ (see Proposition 12.1). Our goal is to characterize those with almost isotropic S-curvature.

First, by Beltrami's theorem and the Cartan classification theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Every Riemannian metric of constant sectional curvature μ is locally isometric to the metric α_μ on a ball in R^n, defined in (2–5). A Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is locally projectively flat (hence of constant sectional curvature) and β is closed. This follows directly from a result in [Bácsó and Matsumoto 1997] and the Beltrami theorem on projectively flat Riemannian metrics. If in addition, the S-curvature is almost isotropic, then β can be determined explicitly.

PROPOSITION 16.1 [Chen et al. 2003]. *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that F has almost isotropic S-curvature, $\boldsymbol{S} = (n+1)(cF + \eta)$, where c is a scalar function on M and η is a closed 1-form on M. Then*:

(a) *α is locally isometric to α_μ and β is a closed 1-form satisfying*

$$(\mu + 4c^2)\beta = -c_{x^k} y^k.$$

(b) *The flag curvature is given by*

$$\boldsymbol{K} = \frac{3c_{x^k} y^k}{\alpha + \beta} + 3c^2 + \mu = \tfrac{3}{4}(\mu + 4c^2)\frac{\alpha - \beta}{\alpha + \beta} + \frac{\mu}{4}. \tag{16–1}$$

(c) *If $\mu + 4c^2 \equiv 0$, then c is a constant and the flag curvature equals $-c^2$. In this case, $F = \alpha + \beta$ is either locally Minkowskian ($c = 0$) or, up to scaling ($c = \pm\frac{1}{2}$), locally isometric to the generalized Funk metric $\Theta_a = \Theta_a(x, y)$ of (2–7) or its reverse $\bar{\Theta}_a = \Theta_a(x, -y)$.*

(d) *If $\mu + 4c^2 \neq 0$, then $F = \alpha + \beta$ must be locally given by*

$$\alpha = \alpha_\mu(x, y), \quad \beta = -\frac{2c_{x^k}(x) y^k}{\mu + 4c^2} \tag{16–2}$$

where $c := c_\mu(x)$ is given by

$$c_\mu = \begin{cases} (\lambda + \langle a, x\rangle)\sqrt{\dfrac{\mu}{\pm(1 + \mu|x|^2) - (\lambda + \langle a, x\rangle)^2}}, & \mu \neq 0, \\[2ex] \dfrac{\pm 1}{2\sqrt{\lambda + 2\langle a, x\rangle + |x|^2}}, & \mu = 0, \end{cases}$$

for $a \in \mathrm{R}^n$ a constant vector and $\lambda \in \mathrm{R}$ a constant number.

PROOF. Let $\alpha_\mu = \sqrt{a_{ij}(x) y^i y^j}$ and $\beta = b_i(x) y^i$. We may assume that $\alpha = \alpha_\mu$ in a local coordinate system

$$a_{ij} = \frac{\delta_{ij}}{1 + \mu|x|^2} - \frac{\mu x^i x^j}{(1 + \mu|x|^2)^2}.$$

The Christoffel symbols of α are given by

$$\bar{\Gamma}^i_{jk} = -\mu \frac{x^j \delta^i_k + x^k \delta^i_j}{1+\mu|x|^2}.$$

Thus

$$\overline{G}^i = -\frac{\mu\langle x, y\rangle}{1+\mu|x|^2}\, y^i.$$

The spray coefficients of F are given by $G^i = \overline{G}^i + Py^i + Q^i$, where $P = e_{00}/(2F)$ and $Q^i = \alpha s^i{}_0$ are given by (9–1) and (9–2). Since β is closed, $s_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}) = 0$ and $s_i := b_j s^j{}_i = 0$. Thus $Q^i = 0$. By assumption, $\boldsymbol{S} = (n+1)(cF+\eta)$ and Lemma 9.1,

$$e_{00} = \beta_{|k} y^k = 2c(\alpha^2 - \beta^2). \tag{16–3}$$

Thus $P = e_{00}/(2F) - s_0 = c(\alpha - \beta)$ and

$$\beta_{x^k} y^k = \beta_{|k} y^k + 2\overline{G}^k \beta_{y^k} = 2c(\alpha^2 - \beta^2) - \frac{2\mu\langle x, y\rangle \beta}{1+\mu|x|^2}.$$

Then $G^i = \tilde{P} y^i$, where $\tilde{P} = -\dfrac{\mu\langle x, y\rangle}{1+\mu|x|^2} + c(\alpha - \beta)$. By (12–3), we obtain

$$\boldsymbol{K} F^2 = \tilde{P}^2 - \tilde{P}_{x^k} y^k = \mu\alpha^2 + c^2(3\alpha + \beta)(\alpha - \beta) - c_{x^k} y^k (\alpha - \beta).$$

On the other hand, by Theorem 15.1, the flag curvature is in the following form

$$\boldsymbol{K} = \frac{3c_{x^k} y^k}{\alpha + \beta} + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on M. It follows from the last two displayed equations that

$$2\big(2c_{x^k} y^k + (\sigma + c^2)\beta\big)\alpha + \big(2c_{x^k} y^k + (\sigma + c^2)\beta\big)\beta + (\sigma - 3c^2 - \mu)\alpha^2 = 0.$$

This gives

$$2c_{x^k} y^k + (\sigma + c^2)\beta = 0, \qquad \sigma - 3c^2 - \mu = 0.$$

Solving the second of these equations for σ and substituting into the first we get

$$(\mu + 4c^2)\beta = -2c_{x^k} y^k. \tag{16–4}$$

To prove part (c) of the Proposition, suppose that $\mu + 4c^2 \equiv 0$. Then c is constant. It follows from (16–1) that $\boldsymbol{K} = 3c^2 + \mu = -c^2$. The local structure of F can be easily determined [Shen 2003a].

Now suppose instead that $\mu + 4c^2 \neq 0$ on an open subset $U \subset M$. By (16–4),

$$\beta = -\frac{2c_{x^k} y^k}{\mu + 4c^2}. \tag{16–5}$$

Note that β is exact. It follows from (16–3) and (16–5) that

$$c_{x^ix^j} + \frac{\mu(x^i c_{x^j} + x^j c_{x^i})}{1+\mu|x|^2} = -c(\mu+4c^2)\Big(\frac{\delta_{ij}}{1+\mu|x|^2} - \frac{\mu x^i x^j}{(1+\mu|x|^2)^2}\Big) + \frac{12 c c_{x^i} c_{x^j}}{\mu+4c^2}. \quad (16\text{–}6)$$

We are going to solve for c. Let

$$f := \begin{cases} \dfrac{2c\sqrt{1+\mu|x|^2}}{\sqrt{\pm(\mu+4c^2)}}, & \mu \neq 0, \\ \dfrac{1}{c^2}, & \mu = 0, \end{cases}$$

where the sign is chosen so that the radicand $\pm(\mu+4c^2) > 0$. Then (16–6) reduces to

$$f_{x^ix^j} = \begin{cases} 0, & \mu \neq 0, \\ 8\delta_{ij}, & \mu = 0. \end{cases}$$

We obtain

$$f = \begin{cases} \lambda + \langle a, x\rangle, & \mu \neq 0, \\ 4\big(\lambda + 2\langle a, x\rangle + |x|^2\big), & \mu = 0, \end{cases}$$

where $a \in \mathrm{R}^n$ is a constant vector and λ is a constant. This gives part (d). □

By Proposition 16.1, one immediately obtains:

Corollary 16.2. *Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that F has almost constant S-curvature $\boldsymbol{S} = (n+1)(cF+\eta)$, where c is a constant. Then F is locally Minkowskian, or Riemannian with constant curvature, or up to a scaling, locally isometric to the generalized Funk metric in* (2–7).

Proof. Let μ be the constant sectional curvature of α. If $\mu + 4c^2 = 0$, by Proposition 16.1(c), $F = \alpha + \beta$ is either locally Minkowskian or, up to a scaling, locally isometric to the generalized Funk metric in (2–7). If $\mu + 4c^2 \neq 0$ instead, $F = \alpha + \beta$ is given by (16–2). Since $c_{x^k} = 0$, we get $\beta = 0$ and $F = \alpha$ is a Riemannian metric. □

Proposition 16.1 completely classifies projectively flat Randers metrics of almost isotropic S-curvature. If a Randers metric has almost isotropic S-curvature, its the E-curvature is isotropic. By Lemma 9.1, the S-curvature is isotropic. Thus a Randers metric is of almost isotropic S-curvature if and only if it is of isotropic S-curvature. This is not true for general Finsler metrics: if $\Theta(x, y)$ is the Funk metric on a strongly convex domain $U \subset \mathrm{R}^n$, the Finsler metric

$$F = \Theta(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \quad y \in T_xU \cong \mathrm{R}^n,$$

is projectively flat with almost isotropic S-curvature, according to Example 14.4. Thus it has isotropic E-curvature. However, this F is of isotropic S-curvature only for certain U's such as the standard unit ball.

A natural problem is whether there are other types of projectively flat Finsler metrics of almost isotropic S-curvature. Here is the answer:

PROPOSITION 16.3 [Chen and Shen 2003b]. *Let $F = F(x, y)$ be a projectively flat Finsler metric on a simply connected open subset $U \subset \mathrm{R}^n$. Suppose that F has almost isotropic S-curvature,*

$$\boldsymbol{S} = (n+1)c(F+\eta), \tag{16–7}$$

where c is a scalar function on M and η is a closed 1-form on U.

(a) *If $\boldsymbol{K}$ is not of the form $-c^2 + c_{x^m}y^m/F$ at every point $x \in U$, then $F = \alpha+\beta$ is a Randers metric on U. Further, α is of constant sectional curvature $\bar{\boldsymbol{K}} = \mu$ with $\mu + 4c^2 \neq 0$ and α and β are as in Proposition 16.1(c).*

(b) *If $\boldsymbol{K} \equiv -c^2 + c_{x^m}y^m/F$ on U, then c is a constant, and either F is locally Minkowskian ($c = 0$) or there exist a Funk metric Θ and a constant vector $a \in \mathrm{R}^n$ such that F has the form*

$$F = \frac{1}{2c}\Big\{\Psi + \frac{\langle a, y\rangle}{1+\langle a, x\rangle},$$

where $\Psi = \Theta(x, y)$ if $c = \frac{1}{2}$ and $\Psi = -\Theta(x, -y)$ if $c = -\frac{1}{2}$.

PROOF. Since F is projectively flat, the spray coefficients are given by $G^i = Py^i$, where

$$P := \frac{F_{x^k}y^k}{2F}.$$

Thus the S-curvature is given by (12–4) and the flag curvature of F is given by (12–3).

By assumption, $\boldsymbol{S}$ is of the form (16–7). Since η is closed on U, it can be written as $\eta(x, y) = dh_x(y)$, where $h = h(x)$ is a scalar function on U. Thus

$$P = cF + d\varphi_x, \tag{16–8}$$

where $\varphi(x) := h(x) + (\ln \sigma_F(x))/(n+1)$. It follows from the last two displayed equations that

$$F_{x^i}y^i = 2FP = 2F(cF + \varphi_{x^i}y^i).$$

Using this together with (16–8) and (12–3), one obtains

$$\begin{aligned}\boldsymbol{K} &= \frac{(cF+\varphi_{x^i}y^i)^2 - (c_{x^i}y^iF + cF_{x^i}y^i + \varphi_{x^ix^j}y^iy^j)}{F^2}\\ &= \frac{-c^2F^2 - c_{x^m}y^mF + (\varphi_{x^i}\varphi_{x^j} - \varphi_{x^ix^j})y^iy^j}{F^2}.\end{aligned} \tag{16–9}$$

On the other hand, since F is of scalar curvature, by Proposition 15.1, the flag curvature of F is given by (15–1). Comparing (16–9) with (15–1) yields

$$(\sigma + c^2)F^2 + 4c_{x^m}y^m F + (\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j})y^iy^j = 0. \tag{16–10}$$

Assume that $\boldsymbol{K} \neq -c^2 + c_{x^m}y^m/F$ at every point $x \in U$. Then, by (15–1), for any $x \in U$, there is a nonzero vector $y \in T_xU$ such that

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \neq 0.$$

We claim that $\sigma + c^2 \neq 0$ on U. If not, there is a point $x_0 \in U$ such that $\sigma(x_0) + c(x_0)^2 = 0$. The inequality above implies that $dc \neq 0$ at x_0. Then (16–10) at x_0 reduces

$$4c_{x^m}(x_0)y^m F(x_0, y) + \big(\varphi_{x^ix^j}(x_0) - \varphi_{x^i}(x_0)\varphi_{x^j}(x_0)\big)y^iy^j = 0. \tag{16–11}$$

Differentiating with respect to y^i, then restricting to the hyperplane

$$V := \{y \mid c_{x^m}(x_0)y^m = 0\},$$

one obtains

$$4c_{x^i}(x_0)F(x_0, y) + \big(\varphi_{x^ix^j}(x_0) - \varphi_{x^i}(x_0)\varphi_{x^j}(x_0)\big)y^j = 0.$$

In other words, $F(x_0, y)$ is a homogeneous linear function of $y \in V$. This is impossible, because $F(x_0, y)$ is always positive for $y \in V \setminus \{0\}$.

Now we may assume that $\sigma + c^2 \neq 0$ on U. One can solve the quadratic equation (16–10) for F,

$$F = \frac{\sqrt{(\sigma + c^2)(\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j})y^iy^j + 4(c_{x^m}y^m)^2} - 2c_{x^m}y^m}{\sigma + c^2}.$$

That is, F is expressed in the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_iy^i$ are given by

$$a_{ij} = \frac{(\sigma + c^2)(\varphi_{x^ix^j} - \varphi_{x^i}\varphi_{x^j}) + 4c_{x^i}c_{x^j}}{(\sigma + c^2)^2}, \quad b_i = -\frac{2c_{x^i}}{\sigma + c^2}.$$

Since F is a Randers metric, by Lemma 9.1, one concludes that $\boldsymbol{S}$ is isotropic, i.e., $\eta = 0$ and

$$\boldsymbol{S} = (n + 1)cF.$$

Since F is projectively flat, α is of constant sectional curvature $\bar{\boldsymbol{K}} = \mu$ and β is closed. Moreover, by Proposition 16.1, the flag curvature is given by (16–1). Note that $\sigma + c^2 \neq 0$ is equivalent to the inequality $\mu + 4c^2 \neq 0$. By Proposition 16.1(d), F is given by (16–2).

We now assume that $\boldsymbol{K} \equiv -c^2 + c_{x^i}y^i/F$. It follows from (15–1) that

$$\sigma + c^2 + \frac{2c_{x^m}y^m}{F} \equiv 0.$$

Suppose that $c_{x^m}(x_0)y^m \neq 0$ at some point x_0. From the preceding identity, one sees that $\sigma(x_0) + c(x_0)^2 \neq 0$. Thus

$$F(x_0, y) = -\frac{2c_{x^m}(x_0)y^m}{\sigma(x_0) + c(x_0)^2}$$

is a linear function. This is impossible. One concludes that $c_{x^m}y^m = 0$ on U, and hence c is a constant and $\sigma(x) = -c^2$ is a constant too. In this case, the flag curvature is given by $\boldsymbol{K} = -c^2$. Equation (16–10) reduces to

$$\varphi_{x^i x^j} - \varphi_{x^i}\varphi_{x^j} = 0,$$

which is easily solved to yield

$$\varphi = -\ln\bigl(1 + \langle a, x\rangle\bigr) + C,$$

where $a \in \mathrm{R}^n$ is a constant vector and C is a constant.

Assume that $c = 0$. Then $\boldsymbol{K} = -c^2 = 0$. It follows from (16–8) that the projective factor $P = d\varphi_x$ is a 1-form, hence the spray coefficients $G^i = Py^i$ are quadratic in $y \in T_xU$. By definition, F is a Berwald metric, and every Berwald metric with $\boldsymbol{K} = 0$ is locally Minkowskian (see [Bao et al. 2000] for a proof).

Assume that $c \neq 0$. By (16–8), $P = cF + d\varphi$. With $\Psi := P + cF = 2cF + d\varphi_x$, we have

$$F = \frac{1}{2c}(\Psi(x, y) - d\varphi_x) = \frac{1}{2c}\left(\Psi(x, y) + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}\right).$$

Since F is projectively flat and P is the projective factor,

$$F_{x^k} = (PF)_{y^k}, \quad P_{x^k} = PP_{y^k} + c^2 FF_{y^k}.$$

These equations imply that $\Psi_{x^i} = \Psi\Psi_{y^i}$. Let $\Theta := \Psi(x, y)$ if $c > 0$ and $\Theta := -\Psi(x, -y)$ if $c < 0$. Then Θ is a Funk metric and F is of the form stated in part (b) of the theorem. □

17. Flag Curvature and Relatively Isotropic L-Curvature

Although the relatively isotropic J-curvature condition is stronger than the isotropic S-curvature condition for Randers metrics (Lemma 10.1), it seems that there is no direct relationship between these two conditions. Nevertheless, for Finsler metrics of scalar curvature, the relatively isotropic J-curvature condition also implies that the flag curvature takes a special form in certain cases.

PROPOSITION 17.1 [Chen et al. 2003]. *Let F be an n-dimensional Finsler manifold of scalar curvature and of relatively constant J-curvature, so that*

$$\boldsymbol{J} + cF\boldsymbol{I} = 0, \tag{17–1}$$

for some constant c. Then

$$\boldsymbol{K} = -c^2 + \sigma e^{-3\tau/(n+1)},$$

where $\tau = \tau(x, y)$ is the distortion and $\sigma = \sigma(x)$ is a scalar function on M.

Proof. By assumption, $J_k = -cFI_k$. Using (13–6) and (13–7), one obtains

$$I_{k|p|q} y^p y^q = J_{k|m} y^m = -cFI_{k|m} y^m = c^2 F^2 \tau_{\cdot k}.$$

Plugging this into the second line of (13–15) yields

$$\tfrac{1}{3}(n+1)\boldsymbol{K}_{\cdot k} + (\boldsymbol{K} + c^2)\tau_{\cdot k} = 0.$$

This implies that

$$\big((\boldsymbol{K} + c^2)^{(n+1)/3} e^{\tau}\big)_{\cdot k} = (\boldsymbol{K} + c^2)^{(n-2)/3} e^{\tau} \big(\tfrac{1}{3}(n+1)\boldsymbol{K}_{\cdot k} + \boldsymbol{K}\tau_{\cdot k}\big) = 0.$$

Thus the function $(\boldsymbol{K} + c^2)^{(n+1)/3} e^{\tau}$ is independent of $y \in T_x M$. □

Proposition 17.1 in the particular case $c = 0$ was essentially achieved in the proof of [Matsumoto 1972a, Proposition 26.2], where it is assumed that F is a Landsberg metric, but what is needed is merely that $\boldsymbol{J} = 0$. Since the notion of distortion is not introduced in [Matsumoto 1972a], the result is stated in a local coordinate system.

Corollary 17.2. *Let F be a Finsler metric on a manifold M. Suppose that F has isotropic flag curvature not equal to $-c^2$ and that F has relatively constant J-curvature. Then F is Riemannian.*

Proof. By Proposition 17.1,

$$\boldsymbol{K}(x) = -c^2 + \sigma(x) e^{-3\tau/(n+1)}.$$

Since $\boldsymbol{K}(x) \neq -c^2$, one concludes that $\sigma(x) \neq 0$, so $\tau = \tau(x)$ is independent of $y \in T_x M$. It follows from (7–2) that $I_i = \tau_{y^i} = 0$. Thus F is Riemannian by Deicke's theorem [Deicke 1953]. □

Proposition 17.3. *Let F be a Finsler metric of scalar curvature on an n-dimensional manifold. Suppose that F has relatively isotropic L-curvature, so*

$$\boldsymbol{L} + cF\boldsymbol{C} = 0, \tag{17–2}$$

where c is a scalar function on M.

(a) *If c is constant, then*

$$\boldsymbol{K} = -c^2 + \sigma e^{-3\tau/(n+1)},$$

where σ is a scalar function on M.

(b) *If $n \geq 3$ and $\boldsymbol{K} \neq -c^2 + c_{x^m} y^m / F$ for almost all $y \in T_x M \setminus \{0\}$ at any point x in an open domain U of M, then $F = \alpha + \beta$ is a Randers metric in U.*

Proof. If F has relatively isotropic L-curvature, (17–1) holds by taking the average of (17–2) on both sides. Statement (a) then follows from Proposition 17.1.

Now we assume that $\boldsymbol{K} \neq -c^2 + c_{x^m}(x)y^m/F$ for almost all $y \in T_xM \setminus \{0\}$ at any point x in an open domain $U \subset M$. By assumption, $L_{ijk} = -cFC_{ijk}$, one obtains

$$C_{ijk|p|q}y^py^q = -c_{x^m}y^mFC_{ijk} - cFL_{ijk} = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2C_{ijk}.$$

Since $J_k = -cFI_k$ by (17–1), we have

$$I_{k|p|q}y^py^q = -c_{x^m}y^mFI_k - cFJ_k = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2I_k.$$

By the formula for M_{ijk} in (3–2), one obtains

$$M_{ijk|p|q}y^py^q = \Big(c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2M_{ijk}.$$

Since F is of scalar curvature, equation (13–16) holds. One obtains

$$\Big(\boldsymbol{K} + c^2 - \frac{c_{x^m}y^m}{F}\Big)F^2M_{ijk} = 0.$$

It follows that $M_{ijk} = 0$, so the Matsumoto torsion vanishes. By Proposition 3.3, $F = \alpha + \beta$ is a Randers metric on U. □

Proposition 17.3 was proved by H. Izumi [1976; 1977; 1982], The particular case $c = 0$ is proved by S. Numata [1975].

Corollary 17.4 [Numata 1975]. *Let F be a Finsler metric of scalar curvature on an n-dimensional manifold, with $n \geq 3$. Suppose that $\boldsymbol{L} = 0$ and $\boldsymbol{K} \neq 0$. Then F is Riemannian.*

Proof. By Proposition 17.3, $F = \alpha + \beta$ is a Randers metric with $\boldsymbol{L} = 0$. By Lemma 10.1, $\boldsymbol{S} = 0$ and β is closed. By Proposition 15.1, one concludes that $\boldsymbol{K} = \sigma(x)$ is a scalar function on M. It follows from (13–14) that $0 = -F^2\sigma(x)I_k$. By assumption, $\boldsymbol{K} = \sigma(x) \neq 0$. Thus $I_k = 0$ and F is Riemannian by Deicke's theorem. □

We may ask again: is there a non-Berwaldian Finsler metric satisfying $\boldsymbol{K} = 0$ and $\boldsymbol{L} = 0$ (or $\boldsymbol{J} = 0$)? If such a metric exists, it cannot be locally projectively flat and it cannot be a Randers metric. (Why?)

Example 17.5. Let $F = \alpha + \beta$ be the Randers metric on R^n defined by

$$F := |y| + \frac{\langle x, y\rangle}{\sqrt{1 + |x|^2}}, \quad y \in T_x\mathrm{R}^n \cong \mathrm{R}^n.$$

Note that

$$\|\beta\|^2 = \frac{|x|^2}{1 + |x|^2} < 1.$$

F is indeed a Randers metric on the whole of R^n. One can verify that F satisfies (12–5). Thus it is a projectively flat Randers metric on R^n. Further, the spray coefficients $G^i = Py^i$ are given by

$$P = c\Big(|y| - \frac{\langle x, y\rangle}{\sqrt{1+|x|^2}}\Big),$$

where $c = 1/\big(2\sqrt{1+|x|^2}\big)$. Let $\rho := \ln\sqrt{1-\|\beta\|^2} = -\ln\sqrt{1+|x|^2}$. By (9–3), one obtains $\boldsymbol{S} = (n+1)(P-\rho_0) = (n+1)cF$. Since β is closed, this is equivalent, by Proposition 10.1, to the identity $\boldsymbol{L} + cF\boldsymbol{C} = 0$.

Since F is projectively flat, it is of scalar curvature. Further computation yields the flag curvature:

$$\boldsymbol{K} = \frac{P^2 - P_{x^k}y^k}{F^2} = \frac{3}{4(1+|x|^2)} \cdot \frac{|y|\sqrt{1+|x|^2} - \langle x, y\rangle}{|y|\sqrt{1+|x|^2} + \langle x, y\rangle}.$$

Note that $\boldsymbol{K} \neq -c^2 + c_{x^k}(x)y^k/F(x,y)$ and that F is a Randers metric. This matches the conclusion in Proposition 17.3(b).

The Randers metric in Example 17.5 is locally projectively flat. There are non–projectively flat Randers metrics of scalar curvature and isotropic S-curvature; see Example 11.2. This example is a Randers metric generated by a special vector field on the Euclidean space by (2–15). In fact, we can determine all vector fields V on a Riemannian space form (M, α_μ) of constant curvature μ such that the generated Randers metric $F = \alpha + \beta$ by (α_μ, V) is of scalar curvature and isotropic S-curvature. This work will appear elsewhere.

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