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Finsler Geometry of Holomorphic Jet Bundles

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Introduction

A complex manifold X is *Brody hyperbolic* if every holomorphic map $f : \mathbb{C} \to X$ is constant. For compact complex manifolds this is equivalent to the condition that the *Kobayashi pseudometric* κ_1 (see (1.12)) is a positive definite Finsler metric. One may verify the hyperbolicity of a manifold by exhibiting a Finsler metric with negative *holomorphic sectional curvature*. The construction of such a metric motivates the use of parametrized jet bundles, as defined by Green–Griffiths. (The theory of these bundles goes back to [Ehresmann 1952].) We examine the algebraic-geometric properties (ample, big, nef, spanned and the dimension of the base locus) of these bundles that are relevant toward the metric's existence. To do this, we start by determining (and computing) basic invariants of jet bundles. Then we apply Nevanlinna theory, via the construction of an appropriate singular Finsler metric of logarithmic type, to obtain precise extensions of the classical Schwarz Lemma on differential forms toward jets. Particularly, this allows direct control over the analysis of the jets $j^k f$ of a holomorphic map $f : \mathbb{C} \to X$; namely, the image of $j^k f$ must be contained in the base locus of the jet differentials. For an algebraically nondegenerate holomorphic map we show by means of reparametrization that the algebraic closure of $j^k f$ is quite large while, under appropriate conditions, the base locus is relatively small. This contradiction shows that the map f must be algebraically degenerate. We apply this method to verify that a generic smooth hypersurface of $\mathbb{P}^3$, of degree $d \geq 5$, is hyperbolic (see Corollary 7.21). Using this we show also the existence of a smooth curve C of degree $d = 5$ in $\mathbb{P}^2$ such that $\mathbb{P}^2 \setminus C$ is *Kobayashi hyperbolic*.

In the classical theory of curves (Riemann surfaces) the most important invariant is the *genus*. The genus g of a curve is the number of independent global regular 1-forms: $g = h^0(\mathcal{K}_X) = \dim H^0(\mathcal{K}_X)$, where $\mathcal{K}_X = T^*X$ is the canonical bundle (which in the case of curves is also the cotangent bundle). A curve is hyperbolic if and only if $g \geq 2$. One way to see this is to take a basis $\omega_1, \dots, \omega_g$ of regular 1-forms and define a metric ρ on the tangent bundle by setting

$$\rho(v) = \left(\sum_{i=1}^{g} |\omega_i(v)|^2 \right)^{1/2}, \; v \in TX. \tag{$*$}$$

For $g = 1$ the metric is flat, that is, the Gaussian or *holomorphic sectional curvature* (hsc) is zero. Hence X is an elliptic curve. For $g \geq 2$ the curvature of this metric is strictly negative which, by the classical Poincaré–Schwarz Lemma, implies that X is hyperbolic. Algebraic geometers take the dual approach by interpreting ρ as defining a metric along the fibers of the dual $T^*X = \mathcal{K}_X$ and, for $g \geq 2$, the Chern form $c_1(\mathcal{K}_X, \rho)$ is positive, that is, the canonical bundle is ample. Indeed the following four conditions are equivalent:

(i) $g \geq 2$;
(ii) X is hyperbolic;
(iii) T^*X is *ample*; and
(iv) There exists a negatively curved metric.

For a complex compact manifold of higher dimension the number of independent 1-forms $g = h^0(T^*X)$ is known as the *irregularity* of the manifold. If $g \geq 1$, we may define ρ as in $(*)$. More generally, for each m, we may choose a basis $\omega_1, \dots, \omega_{g_m}$ of $H^0(\bigodot^m T^*X)$, where $\bigodot^m T^*X$ is the m-fold *symmetric* product, and define, if $g_m \geq 1$,

$$\rho(v) = \sum_{i=1}^{g_m} |\omega_i(v)|^{1/m}. \tag{$**$}$$

In dimension 2 or higher, ρ cannot, in general, be positive definite and it is only a *Finsler* rather than a *hermitian* metric. However the holomorphic sectional curvature may be defined for a Finsler metric and the condition that the curvature is negative implies that X is hyperbolic. It is known [Cao and Wong 2003] that the ampleness of T^*X is equivalent to the existence of a Finsler metric with negative *holomorphic bisectional curvature* (hbsc):

THEOREM [Aikou 1995; 1998; Cao and Wong 2003]. *T^*X is ample $\iff$ Finsler metric has negative hbsc $\implies$ Finsler metric has negative hsc $\implies$ X is hyperbolic.*

In our view the fundamental problems in hyperbolic geometry are the following.

PROBLEM 1. *Find an algebraic geometric characterization of the concept of negative hsc.*

PROBLEM 2. *Find an algebraic geometric and a differential geometric characterization of hyperbolicity.*

It is known that there are hyperbolic hypersurfaces in $\mathbb{P}^n$ (for each n). On the other hand, there are no global regular 1-forms on hypersurfaces in $\mathbb{P}^n$ for $n \geq 3$; indeed $h^0(\bigodot^m T^*X) = 0$ for all m. These hyperbolic hypersurfaces are discovered using, in one form or another, the Second Main Theorem of Nevanlinna Theory, which involves higher-order information; see for example [Wong 1989; Stoll and Wong 1994; Fujimoto 2001].

This leads us to the concept of the (parametrized) jet bundles [Ehresmann 1952], formalized (for complex manifolds) and studied by Green and Griffiths [1980]. Observe that a complex tangent v at a point x of a manifold may be represented by the first order derivative $f'(0)$ of a local holomorphic map $f : \Delta_r \to X, f(0) = x$ for some disc Δ_r of radius r in the complex plane $\mathbb{C}$ (more precisely, v is the equivalence class of such maps, as different maps may have the same derivative at the origin). A *k-jet* is defined as the equivalence class of the first k-th order derivatives of local holomorphic maps and the k-jet bundle, denoted J^kX, is just the collection of all (equivalence classes of) k-jets. Note that $J^1X = TX$. For $k \geq 2$ these bundles are $\mathbb{C}^*$ bundles but not vector bundles. The nonlinear structure is reflected in reparametrization. Namely, given a k-jet $j^kf(0) = (f(0), f'(0), \ldots, f^{(k)}(0))$ we obtain another k-jet by composing f with another local holomorphic self map ϕ in $\mathbb{C}$ that preserves the origin, then taking $j^k(f \circ \phi)(0)$. In particular, if ϕ is given by multiplication by a complex number λ we see that $j^k(f \circ \phi)(0) = (f(0), \lambda f'(0), \lambda^2 f''(0), \ldots, \lambda^k f^{(k)}(0))$. Equivalence under this action is denoted by $\lambda \cdot j^f(0)$ and this is the $\mathbb{C}^*$-action on J^kX; in general *there is no vector bundle structure on J^kX*. We write:

$$\lambda \cdot (v_1, \ldots, v_k) = (\lambda v_1, \lambda^2 v_2, \ldots, \lambda^k v_k), \ (v_1, \ldots, v_k) \in J^kX \qquad (***)$$

and assign the weight i to the variable v_i. A 1-form ω may be regarded as a holomorphic function on the tangent bundle $\omega : TX \to \mathbb{C}$ satisfying the condition $\omega(\lambda \cdot v) = \lambda\omega(v)$, that is, linearity along the fibers. More generally, an element $\omega \in H^0(\bigodot^m T^*X)$ is a holomorphic function on the tangent bundle $\omega : TX \to \mathbb{C}$ that is a homogeneous polynomial of degree m along the fibers. Analogously we define a *k-jet differential ω of weight m* to be a holomorphic function on the k-jet bundle $\omega : J^kX \to \mathbb{C}$ which is a *weighted* homogeneous polynomial of degree m along the fibers. The sheaf of k-jet differentials of weight m will be denoted by

$\mathcal{J}_k^m X$. Taking a basis $\omega_1, \dots, \omega_N$ of $H^0(\mathcal{J}_k^m X)$ we define

$$\rho(v_1, \dots, v_k) = \sum_{i=1}^{N} \bigl|\omega_i(v_1, \dots, v_k)\bigr|^{1/m},$$

and from (∗∗∗) we see (since each ω_i is a weighted homogeneous polynomial of degree m) that ρ is a Finsler pseudometric, that is,

$$\begin{aligned}\rho(\lambda \cdot (v_1, \dots, v_k)) &= \sum_{i=1}^{N} \bigl(|\lambda^m \omega_i(v_1, \dots, v_k)|\bigr)^{1/m} \\ &= |\lambda| \sum_{i=1}^{N} \bigl(|\omega_i(v_1, \dots, v_k)|\bigr)^{1/m} = |\lambda|\, \rho(v_1, \dots, v_k).\end{aligned}$$

The positive definiteness is a separate issue that one must deal with in higher dimension. The algebraic geometric concept that is equivalent to the positive definiteness of ρ is that the sheaf $\mathcal{J}_k^m X$ is *spanned* (meaning that global sections span the fiber at each point). Other relevant concepts here are whether such a sheaf is *ample*, *nef* (numerically effective), or *big*. These concepts are intimately related to the Chern numbers of the sheaf $\mathcal{J}_k^m X$ and the dimensions of the cohomology groups $h^i(\mathcal{J}_k^m X)$, $0 \le i \le n = \dim X$. The starting point here is the computation of the Euler characteristic in the case of a manifold of *general type* by the Riemann–Roch Formula. An asymptotic expansion of $\chi(\mathcal{J}_k^m X)$ was given in [Green and Griffiths 1980] with a sketch of the proof. Often in articles making reference to this result readers questioned the validity of the statement. A detailed proof, in the case of general type surfaces (complex dimension 2), of this formula was given in [Stoll and Wong 2002] using a different approach to that given in by Green and Griffiths. Indeed explicit formulas, not merely asymptotic expansions, were given for $\mathcal{J}_k^m X$, $k = 2$ and 3. The method of computation also shows that $\mathcal{J}_k^m X$ is not *semistable* (see Section 3, Remark 3.5) in the sense of Mumford–Takemoto despite the fact (see [Maruyama 1981; Tsuji 1987; 1988]) that all tensor products $\otimes T^* X$ and symmetric products $\bigodot^m T^* X$ are semistable if X is of general type. In this article we also introduce an analogue of semistability in the sense of Gieseker–Maruyama (see [Okonek et al. 1980]) and show that $\mathcal{J}_k^m X$ is not semistable (see Section 7) in this sense either.

We have (see Section 5 for the reason in choosing the weight $k!$ below)

$$\begin{aligned} T^* X \text{ is ample} &\implies \mathcal{J}_k^{k!} X \text{ is ample for all } k \\ &\implies \mathcal{J}_k^{k!} X \text{ is ample for some } k \\ &\iff \text{there exists Finsler metric on } J^k X \text{ with negative hbsc} \\ &\implies \text{there exists Finsler metric on } J^k X \text{ with negative hsc} \\ &\implies X \text{ is hyperbolic.} \end{aligned}$$

The condition that $\mathcal{J}_k^{k!}X$ is ample is much stronger than hyperbolicity of X. A weaker condition is that $\mathcal{J}_k^{k!}X$ is *big*; this says that

$$h^0(\mathcal{J}_k^{k!m}X) = \dim(X, \mathcal{J}_k^{k!m}X) = O(m^{n(k+1)-1}),$$

where $n = \dim X$. From the differential geometric point of view, this means that there is a pseudo-Finsler metric on J^kX that is generically positive definite and has generically negative hbsc (as defined wherever the metric is positive definite). The condition that $\mathcal{J}_k^{k!}X$ is big implies that, for any ample divisor D on X, there exists $m_0 = m_0(D)$ such that $\mathcal{J}_k^{k!m_0}X \otimes [-D]$ (the sheaf of k-jet differentials of weight $k!m_0$ vanishing along D) is big. This, however, is not quite enough to guarantee hyperbolicity; the problem is that the base locus of $\mathcal{J}_k^{k!}X \otimes [-D]$ may be "too big". As a natural correction, we verify, using the Schwarz Lemma for jet differentials (see Theorem 6.1, Corollaries 6.2 and 6.3), that the assumption $\mathcal{J}_k^{k!m_0}X \otimes [-D]$ is big and spanned (that is, the base locus is empty) does imply hyperbolicity.

However, the condition that a sheaf is big and spanned may be difficult to verify (unless, perhaps, it is already ample and we know of no hypersurfaces in $\mathbb{P}^3$ satisfying this condition). To alleviate this, we refine the form of Schwarz's Lemma (see Theorem 6.4 and Corollary 6.5) to establish the result that every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate if the dimension of the base locus of $\mathcal{J}_k^{k!m_0}X \otimes [-D]$ in the projectivized jet bundle $\mathbb{P}(J^kX)$ is no more than $n+k-1$. From this we show in Section 7, using the explicit computation of the invariants of the jet bundles in the first 3 sections (see Theorem 3.9 and Corollary 3.10), that the dimension of the base locus of a generic hypersurface of degree ≥ 5 in $\mathbb{P}^3$ is, indeed, at most $n+k-1 = k+1$ ($n = 2$ in this case) and consequently, hyperbolic. *The key ingredient is the extension of the inductive cutting procedure of the base locus, of* [Lu and Yau 1990] *and* [Lu 1991] (*see also* [Dethloff et al. 1995a; 1995b]) *in the case of* 1*-jets, to* k*-jets.* There is a delicate point in the cutting procedure, namely that *intersections of irreducible varieties may not be irreducible.* We show, again using the Schwarz Lemma, that *under the algebraically nondegenerate assumption on* f*, there is no loss of generality in assuming that the intersection is irreducible* (see the proof of Theorems 7.18 and 7.20).

The crucial analytic tools here are the Schwarz Lemma for jet differentials and its refined form. These are established using Nevanlinna Theory. We remark that jet differentials are used routinely in Nevanlinna Theory without a priori knowledge of whether regular jet differentials exist at all. The main idea of the proof of the Schwarz Lemma is to use jet differentials with logarithmic poles; to determine conditions under which the sheaf of such jet differentials is *spanned* and provides a *singular Finsler metric that is positive definite in the extended sense.* The classical Nevanlinna Theory is seen to work well with nonhermitian Finsler metrics with logarithmic poles on account of the fundamental principle (the Lemma of logarithmic derivatives of Nevanlinna) that logarithmic poles are relatively harmless (see the proof of Theorem 6.1 in Section 6 for details).

The article is organized as follows. We describe the parametrized jet bundles of Green–Griffiths, which differ from the usual jet bundles; for example, they are $\mathbb{C}^*$ bundles but in general not vector bundles. The definitions are recalled in Section 1. For the usual jet bundles there is the question of interpolation: Find all varieties with prescribed jets, say, at a finite number of points. This problem, for 1-jets, is equivalent to the Waring problem concerning when a general homogeneous polynomial is the sum of powers of linear forms. The Waring problem is related to the *explicit* construction (not merely existence) of hyperbolic hypersurfaces in $\mathbb{P}^n$ for any n. Limitation of space does not allow us to discuss this problem in this article. Solutions of the interpolation problem for a collection of points can be found in [Alexander and Hirschowitz 1992a; 1992b; 1995; Chandler 1995; 1998a; 2002]. The analogous problem concerning the Green–Griffiths jet bundles is still open.

In Section 2 we give a fairly detailed account of the jet bundles of curves. We calculate the Chern number $c_1(\mathcal{J}_k^m X)$ and the invariants $h^0(\mathcal{J}_k^m X)$ and $h^1(\mathcal{J}_k^m X)$. We show, by examples, how to construct jet differentials explicitly, in terms of the defining polynomial, in the case of curves of degree $d \geq 4$ in $\mathbb{P}^2$. Jet bundles may also be defined for varieties defined over fairly general fields (even in positive characteristic). The explicit construction of sections of powers of the canonical bundle, $\mathcal{K}_X^m$, was useful in the solution of the "strong uniqueness polynomial problem" (see Section 2 and the articles [An et al. 2004] in the complex case and [An et al. 2003a; 2003b] in the case of positive characteristic).

The formulas for invariants of the jet differentials for surfaces (the Chern numbers, the index, the Euler characteristic, the dimensions of cohomology groups) are given in Sections 3, 4 and 7. The calculations are similar to those over curves, though combinatorially much more complicated. We provide computations in special cases; the details are given in [Stoll and Wong 2002]. For example the explicit computation in Section 7 (see Theorem 7.7) shows that, for a smooth hypersurface in $\mathbb{P}^3$ the Euler characteristic $\chi(\mathcal{J}_2^m X)$ is big if and only if the degree is ≥ 16.

In Section 6 we prove a Schwarz Lemma for jet differentials. This is the generalization of the classical result for differential forms on curves: if

$$\omega \in H^0(X, \mathcal{K}_X \otimes [-D]),$$

that is, if ω is a regular 1-form vanishing on an effective ample divisor D in the curve X, then $f^*(\omega) \equiv 0$ for any holomorphic map $f : \mathbb{C} \to X$. This says that f' vanishes identically, that is, f is a constant. The proof given in Section 6 of the Schwarz Lemma for jets $j^k f$ has been in circulation since 1994 but was never formally published; it was used, for example, in the thesis of Jung [1995] and by Cherry–Ru in the context of p-adic jet differentials.

For surfaces of general type the fact that $\mathcal{J}_k^{k!}X$ is big for $k \gg 0$ is equivalent to the "hyperplane" line bundle $\mathcal{L}_k^{k!}$ being big on $\mathbb{P}(J^k X)$. (Note that $\mathcal{L}_k^m$ is locally free only if m is divisible by $k!$; see Section 5 for more details.) Schwarz's Lemma then implies that the lifting $[j^k f] : \mathbb{C} \to \mathbb{P}(J^k X)$ of a holomorphic map $f : \mathbb{C} \to X$ must be contained in some divisor $Y \subset \mathbb{P}(J^k X)$. The idea is to show that $\mathcal{L}_k^{k!}|_Y$ is again big so that the image of $[j^k f]$ is contained in a divisor Z of Y. Then we show that $\mathcal{L}_k^{k!}|_Z$ is big and continue until we reach the critical dimension $n+k-1 = k+1$. For surfaces of general type the sheaf of 1-jet differentials $\mathcal{L}_1^1$ is big if $c_1^2 > c_2$. In order for the restriction of $\mathcal{L}_1^1$ to subvarieties to be big, the condition that the index $c_1^2 - 2c_2$ is positive is required. The proof is based on the intersection theory of the projectivized tangent bundle $\mathbb{P}(TX)$ and the fact that the cotangent bundle T^*X of a surface of general type is semistable (in the sense of Mumford–Takemoto) relative to the canonical class. As remarked earlier the k-jet differentials are not semistable for any $k \geq 2$. However, by our explicit computation, for *minimal* surfaces of general type the index of $\mathcal{J}_k^{k!}X$ is positive for $k \gg 0$. Indeed, we may write the index as $\iota(\mathcal{J}_k^{k!}X) = c(\alpha_k c_1^2 - \beta_k c_2)$ where c, α_k and β_k are positive and we show that $\lim_{k\to\infty} \alpha_k/\beta_k = \infty$ (see Corollary 3.10). This is crucial in showing that $\mathcal{L}_k^{k!}|_Z$ is big in the cutting procedure. For example, for a smooth hypersurface of degree 5 in $\mathbb{P}^3, \iota(\mathcal{J}_k^{k!}X)$ is positive and the ratio α_k/β_k must be greater than 11 in order to establish the degeneracy of a map from $\mathbb{C}$ to X. Using the explicit expressions for α_k and β_k we show, with the aid of computer, that this occurs precisely for $k \geq 199$ (see the table at the end of Section 3 and Example 7.6 in Section 7). However, in order for the index of the restriction of the sheaf to subvarieties (in the cutting procedure) to be positive (verifying the hyperbolicity of X), k must be even larger. Using our formulas in the proof of Theorem 7.20, Professor B. Hu, using the computer, checked that $k \geq 2283$ is sufficient.

Note. *Experts who are familiar with parametrized jet bundles and are interested mainly in the proof of the Kobayashi conjecture may skip the first five sections* (*with the exception of Theorem* 3.9 *and Corollary* 3.10) *and proceed directly to Sections* 6 *and* 7.

1. Holomorphic Jet Bundles

Summary. *Two notions of jet bundles, the full and the parametrized bundles, are introduced. The parametrized jet bundle is only a $\mathbb{C}^*$-bundle, not a vector bundle in general. For the resolution of the Kobayashi conjecture, as dictated by analysis, it is necessary to work with the parametrized jet bundle.* (*See Section* 6 *on the Schwarz Lemma.*) *Some basic facts are recalled here, all of which may be found in* [Green and Griffiths 1980; Stoll and Wong 2002].

There are, in the literature, two different concepts of jet bundles of a complex manifold. The first is used by analysts (PDE), algebraic geometers [Chandler

1995; 1998a; 2002] and also by number theorists (see Faltings's work on rational points of an ample subvariety of an abelian variety and integral points of the complement of an ample divisor of an abelian variety [Faltings 1991]); it was used implicitly in [Ru and Wong 1991] (see also [Wong 1993b]) for the proof that there are only finitely many integral points in the complement of $2n+1$ hyperplanes in general position in $\mathbb{P}^n$. The second is the jet bundles introduced by Green and Griffiths [1980] (see also [Stoll and Wong 2002]). The first notion shall henceforth be referred to as the *full jet bundle* and these bundles are holomorphic vector bundles (locally free). The second notion of jet bundle shall be referred to as the *parametrized jet bundle*. These bundles are coherent sheaves that are holomorphic $\mathbb{C}^*$-bundles which, in general, are not locally free.

For a complex manifold X the (locally free) sheaf of germs of holomorphic tangent vector fields (differential operators of order 1) of X shall be denoted by T^1X or simply TX. An element of T^1X acts on the sheaf of germs of holomorphic functions by differentiation:

$$(D, f) \in T^1X \times \mathcal{O}_X \mapsto Df \in \mathcal{O}_X$$

and the action is linear over $\mathbb{C}$; in symbols, $D \in \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$. This concept may be extended as follows:

Definition 1.1. Let X be a complex manifold of dimension n. The *sheaf of germs of holomorphic k-jets* (differential operators of order k), denoted T^kX, is the subsheaf of the sheaf of germs of homomorphisms $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ consisting of elements (differential operators) of the form

$$\sum_{j=1}^{k} \sum_{i_j \in \mathbb{N}} D_{i_1} \circ \cdots \circ D_{i_j},$$

where $D_{i_j} \in T^1X$. In terms of holomorphic coordinates $z_1, \ldots, z_n$ an element of T^kX is expressed as

$$\sum_{j=1}^{k} \sum_{1 \le i_1, \ldots, i_j \le n} a_{i_1,\ldots,i_j} \frac{\partial^j}{\partial z_{i_1} \ldots \partial z_{i_j}},$$

where the coefficients $a_{i_1,\ldots,i_j}$ are symmetric in the indices $i_1, \ldots, i_j$. The bundle T^kX is locally free. One may see this by observing that $T^{k-1}X$ injects into T^kX and there is an exact sequence of sheaves:

$$0 \to T^{k-1}X \to T^kX \to T^kX/T^{k-1}X \to 0, \tag{1.1}$$

where $T^kX/T^{k-1}X \cong \bigodot^k T^1X$ is the sheaf of germs of k-fold symmetric products of T^1X. These exact sequences imply, by induction, that T^kX is locally free as each sheaf $\bigodot^k T^1X$, a symmetric product of the tangent sheaf, is locally free. A proof of (1.1) can be found in [Stoll and Wong 2002].

The parametrized k-jet bundles for complex manifolds are introduced by Green–Griffiths. (These are special cases of the general theory of jets due to Ehresmann [1952] for differentiable manifolds.) These bundles are defined as follows. Let $\mathcal{H}_x, x \in X$, be the sheaf of germs of holomorphic curves: $\{f : \Delta_r \to X,\ f(0) = x\}$ where Δ_r is the disc of radius r in $\mathbb{C}$. For $k \in \mathbb{N}$, define an equivalence relation $\sim_k$ by designating two elements $f, g \in \mathcal{H}_x$ as k-equivalent if $f_j^{(p)}(0) = g_j^{(p)}(0)$ for all $1 \leq p \leq k$, where $f_i = z_i \circ f$ and $z_1, \ldots, z_n$ are local holomorphic coordinates near x. The sheaf of *parametrized k-jets* is defined by

$$J^k X = \bigcup_{x \in X} \mathcal{H}_x / \sim_k . \tag{1.2}$$

Elements of $J^k X$ will be denoted by $j^k f(0) = (f(0), f'(0), \ldots, f^{(k)}(0))$. The fact that $J^k X, k \geq 2$, is in general not locally free may be seen from the nonlinearity of change of coordinates:

$$(w_j \circ f)' = \sum_{i=1}^{n} \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)',$$

$$(w_j \circ f)'' = \sum_{i=1}^{n} \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)'' + \sum_{i,k=1}^{n} \frac{\partial^2 w_j}{\partial z_i \partial z_k}(f)(z_i \circ f)'(z_k \circ f)'$$

and for each k,

$$(w_j \circ f)^{(k)} = \sum_{i=1}^{n} \frac{\partial w_j}{\partial z_i}(f)(z_i \circ f)^{(k)} + P\left(\frac{\partial^l w_j}{\partial z_{i_1} \ldots \partial z_{i_l}}(f),\ (w_j \circ f)^{(l)}\right),$$

where P is an integer-coefficient polynomial in $\partial^l w_j / \partial z_{i_1} \ldots \partial z_{i_l}$ and $(w_j \circ f)^{(l)}$ for $j = 1, \ldots, n$ and $l = 1, \ldots, k$. There is, however, a natural $\mathbb{C}^*$-action on $J^k X$ defined via parameterization. Namely, for $\lambda \in \mathbb{C}^*$ and $f \in \mathcal{H}_x$ a map $f_\lambda \in \mathcal{H}_x$ is defined by $f_\lambda(t) = f(\lambda t)$. Then $j^k f_\lambda(0) = (f_\lambda(0), f'_\lambda(0), \ldots, f_\lambda^{(k)}(0)) = (f(0), \lambda f'(0), \ldots, \lambda^k f^{(k)}(0))$. So the C^*-action is given by

$$\lambda \cdot j^k f(0) = (f(0), \lambda f'(0), \ldots, \lambda^k f^{(k)}(0)). \tag{1.3}$$

DEFINITION 1.2. The *parametrized k-jet bundle* is defined to be $J^k X$ together with the $\mathbb{C}^*$-action defined by (1.3) and shall simply be denoted by $J^k X$.

It is clear that, for a complex manifold of (complex) dimension $n, J^k X$ is a holomorphic $\mathbb{C}^*$-bundle of rank $r = kn$ and $T^k X$ is a holomorphic vector bundle of rank $r = \sum_{i=1}^{k} C_i^{n+i-1}$ where C_i^j are the usual binomial coefficients. Although $J^1 X = T^1 X = TX$ these bundles differ for $k \geq 2$. The nonlinearity of the change of coordinates formulas above shows that there is in general no natural way of injecting $J^{k-1} X$ into $J^k X$ as opposed to the case of $T^k X$ (see (1.1)). There is however a natural projection map (the *forgetting map*) $p_{kl} : J^k X \to J^l X$ for any $l \leq k$ defined simply by

$$p_{kl}(j^k f(0)) = j^l f(0), \tag{1.4}$$

which then respects the $\mathbb{C}^*$-action defined by (1.3) and so is a $\mathbb{C}^*$-bundle morphism. If $\Phi : X \to Y$ is a holomorphic map between the complex manifolds X and Y then the usual differential $\Phi_* : T^1X \to T^1Y$ is defined. More generally, the k-th order differential $\Phi_{k*} : T^kX \to T^kY$ is given by

$$\Phi_{k*} = (D_1 \circ \cdots \circ D_k)(g) \stackrel{\text{def}}{=} D_1 \circ \cdots \circ D_k(g \circ \Phi) \tag{1.5}$$

for any $g \in \mathcal{O}_Y$. The k-th order induced map for the parametrized jet bundle, denoted $J^k\Phi : J^kX \to J^kY$, can also be defined:

$$J^k\Phi(j^kf(0)) \stackrel{\text{def}}{=} (\Phi \circ f)^{(k)}(0) \tag{1.6}$$

for any $j^kf(0) \in J^kX$. For the parametrized jet bundle J^kX there is another notion closely related to the differential: the natural lifting of a holomorphic curve. Namely, given any holomorphic map $f : \Delta_r \to X (0 < r \leq \infty)$, the lifting $j^kf : \Delta_{r/2} \to J^kX$ is defined by

$$j^kf(\zeta) = j^kg(0), \ \zeta \in \Delta_{r/2} \tag{1.7}$$

where $g(\xi) = f(\zeta + \xi)$ is holomorphic for $\xi \in \Delta_{r/2}$.

Definition 1.3. The dual of the full jet bundles T^kX shall be called the *sheaf of germs of k-jet forms* and shall be denoted by T_k^*X. For $m \in \mathbb{N}$ the m-fold symmetric product shall be denoted by $\bigodot^m T_k^*X$ and its global sections shall be called *k-jet forms of weight m*.

In this article we shall focus on the dual of the parametrized jet bundles defined as follows.

Definition 1.4. The dual of J^kX (i.e., the sheaf associated to the presheaf consisting of holomorphic maps $\omega : j^kX|_U \to \mathbb{C}$ such that $\omega(\lambda \cdot j^kf) = \lambda^m\omega(j^kf)$ for all $\lambda \in \mathbb{C}^*$ and positive integer m) shall be referred to as the *sheaf of germs of k-jet differentials of weight m* and shall be denoted by $\mathcal{J}_k^mX$.

It follows from the definition of the $\mathbb{C}^*$-action on J^kX that a k-jet differential ω of weight m is of the form:

$$\omega(j^kf) = \sum_{|I_1|+2|I_2|+\cdots+k|I_k|=m} a_{I_1,\ldots,I_k}(z)(f')^{I_1}\ldots(f^{(k)})^{I_k}, \tag{1.8}$$

where $a_{I_1,\ldots I_k}$ are holomorphic functions, $I_j = (i_{1j}, \ldots, i_{nj}), n = \dim X$ are the multi-indices with each i_{lj} being a nonnegative integer and $|I_j| = i_{1j} + \cdots + i_{nj}$. In terms of local coordinates $(z_1, \ldots, z_n)$,

$$(f')^{I_1}\ldots(f^{(k)})^{I_k} = (f_1')^{i_{11}}\ldots(f_n')^{i_{n1}}\ldots(f_1^{(k)})^{i_{1k}}\ldots(f_n^{(k)})^{i_{nk}}.$$

Further, the coefficients $a_{I_1,\ldots I_k}(z)$ are symmetric with respect to the indices in each I_j. More precisely,

$$a_{(i_{\sigma_1(1)1},\ldots,i_{\sigma_1(n)1}),\ldots,(i_{\sigma_k(1)k},\ldots,i_{\sigma_k(n)k})} = a_{(i_{11},\ldots,i_{n1}),\ldots,(i_{1k},\ldots,i_{nk})},$$

where each σ_j, $j = 1, \ldots, n$, is a permutation on n elements. For example, $(f_1')^2(f_2'')^2 + f_1''' f_2' f_2'' + f_1'''' f_2'' + f_1''''' f_2'$ is a 5-jet differential of weight 6.

There are several important naturally defined operators on jet differentials; the first is a *derivation* $\delta : \mathcal{J}_k^m X \to \mathcal{J}_{k+1}^{m+1} X$ defined by

$$\delta\omega(j^{k+1} f) \stackrel{\text{def}}{=} (\omega(j^k f))'. \tag{1.9}$$

Note that in contrast to the exterior differentiation of differential forms $\delta \circ \delta \neq 0$ on jet differentials. In particular, given a holomorphic function ϕ defined on some open neighborhood U in X, the k-th iteration $\delta^{(k)}$ of δ,

$$\delta^{(k)} \phi(j^k f) = (\phi \circ f)^{(k)}, \tag{1.10}$$

is a k-jet differential of weight k.

Another difference between jet differentials and exterior differential forms is that a lower order jet differential can be naturally associated to a jet differential of higher order. The natural projection $p_{kl} : J^k X \to J^l X$ defined by $p_{kl}(j^k f) = j^l f$, for $k \geq l$, induces an injection $p_{kl}^* : \mathcal{J}_l^m X \to \mathcal{J}_k^m X$ defined by "forgetting" those derivatives higher than l:

$$p_{kl}^* \omega(j^k f) \stackrel{\text{def}}{=} \omega(p_{kl}(j^k f)) = \omega(j^l f). \tag{1.11}$$

We shall simply write $\omega(j^k f) = \omega(j^l f)$ if no confusion arises.

The wedge (exterior) product of differential forms is replaced by taking symmetric product; the symmetric product of a k-jet differential of weight m and a k'-jet differential of weight m' is a $\max\{k, k'\}$-jet differential of weight $m + m'$.

EXAMPLE 1.5. A 1-jet differential is a differential 1-form $\omega = \sum_{i=1}^n a_i(z) dz_i$. Let $f = (f_1, \ldots, f_n) : \Delta_r \to X$ be a holomorphic map. Then

$$\omega(j^1 f) = \sum_{i=1}^n a_i(f) dz_i(f') = \sum_{i=1}^n a_i(f) f_i'$$

and $\delta\omega$ is a 2-jet differential of weight 2, given by

$$\delta\omega(j^2 f) = (\omega(j^1 f))' = \left(\sum_{i=1}^n a_i(f) f_i' \right)' = \sum_{i,j=1}^n \frac{\partial a_i}{\partial z_j}(f) f_i' f_j' + \sum_{i=1}^n a_i(f) f_i''.$$

Analogously, $\delta^2\omega$ is a 3-jet differential of weight 3, given by

$$\delta^2\omega(j^3 f) = \sum_{i,j=1}^n \frac{\partial^2 a_i}{\partial z_j \partial z_k}(f) f_i' f_j' f_k' + 3 \sum_{ij=1}^n \frac{\partial a_i}{\partial z_j}(f) f_i'' f_j' + \sum_{i=1}^n a_i(f) f_i'''.$$

The concept of jet bundles extends also to singular spaces. Let us remark on how this may be defined. One may locally embed an open set U of X as a subvariety in a smooth variety $U \subset Y$. At a point $x \in U$ the stalk jet $(J^k Y)_x$ is then defined, as Y is smooth. The stalk $(J^k X)_x$ is defined as the subset

$$\left\{ j^k f(0) \in (J^k Y)_x \mid f : \Delta_r \to Y \text{ is holomorphic, } f(0) = x \text{ and } f(\Delta_r) \subset U \right\}.$$

From the differential geometric point of view, properties of the full jet bundle T_k^*X, as a vector bundle, are reflected by the curvatures of hermitian metrics along its fibers. The parametrized jet bundles, however, are only $\mathbb{C}^*$-bundles hence can only be equipped with *Finsler* metrics. A *Finsler pseudometric* (or a k-jet pseudometric) on X is a map $\rho = \rho_k : J^k X \to \mathbb{R}_{\geq 0}$ satisfying the condition

$$\rho(\lambda \cdot \boldsymbol{j}_k) = |\lambda| \rho(\boldsymbol{j}_k)$$

for all $\lambda \in \mathbb{C}$ and $\boldsymbol{j}_k \in J^k X$. It is said to be a *Finsler metric* if it is positive outside of the zero section. A $(k-1)$-jet (pseudo)-metric ($k \geq 2$) ρ_{k-1} can be considered as a k-jet (pseudo)-metric by the *forgetting map*:

$$\rho_{k-1}(\boldsymbol{j}_k) := \rho_{k-1}(\boldsymbol{j}_{k-1}).$$

where $\boldsymbol{j}_k = j^k f(0)$ and $\boldsymbol{j}_{k-1} = j^{k-1} f(0)$. Define, for $\boldsymbol{j}_k \in J^k M, k \geq 1$,

$$\kappa_k(\boldsymbol{j}_k) = \inf \{1/r\}, \tag{1.12}$$

where the infimum is taken over all r such that

$$\mathcal{H}_r^k(\zeta) = \{f : \Delta_r \to X \mid f \text{ is holomorphic and } j^k f(0) = \boldsymbol{j}_k\}$$

is nonempty. For $k = 1$ this is the usual Kobayashi–Royden pseudometric on $J^1 X = TX$. Henceforth we shall refer to κ_k as the *k-th infinitesimal Kobayashi–Royden pseudometric*. We shall also say that X is *k-jet hyperbolic* if κ_k is indeed a Finsler metric; that is, $\kappa_k(\boldsymbol{j}_k) > 0$ for each nonzero k-jet $\boldsymbol{j}_k$. Thus 1-jet hyperbolicity is the same as Kobayashi hyperbolicity. Since a holomorphic map $f : \Delta_r \to X$ such that $j^k f(0) = (z, \zeta_1, \ldots, \zeta_k)$ also satisfies $j^{k-1} f(0) = (z, \zeta_1, \ldots, \zeta_{k-1})$, we obtain:

$$\kappa_k(z, \zeta_1, \ldots, \zeta_k) \geq \kappa_{k-1}(z, \zeta_1, \ldots, \zeta_{k-1}). \tag{1.13}$$

From this we see that $(k-1)$-jet hyperbolicity implies k-jet hyperbolicity.

Remark 1.6. The notion introduced above is not to be confused with the k-dimensional ($1 \leq k \leq n = \dim X$) Kobayashi pseudometric in the literature (see [Lang 1987], for example); n-dimensional Kobayashi hyperbolicity is more commonly known as measure hyperbolicity.

In general the k-th Kobayashi–Royden metric does not have a good regularity property. It is well-known that κ_1 is upper-semicontinuous (see [Royden 1971] or [Kobayashi 1970]); a similar proof shows that the same is true for κ_k for any k. It is also known that κ_1, in general, is not continuous; however it is continuous if X is complete hyperbolic (that is, the distance function associated to the metric κ_1 is complete). In particular, κ_1 is continuous on a compact hyperbolic manifold X. On the other hand, using a partition of unity one may construct k-jet metrics that are continuous everywhere and smooth outside of the zero section. Consider first the space

$$\mathbb{C}^n_k = \underbrace{\mathbb{C}^n \times \ldots \times \mathbb{C}^n}_{k}$$

with $\mathbb{C}^*$-action $\lambda \cdot (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_k) \mapsto (\lambda \boldsymbol{z}_1, \lambda^2 \boldsymbol{z}_2, \ldots, \lambda^k \boldsymbol{z}_k)$. Define, for $Z = (\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \in \mathbb{C}^n_k$:

$$r_k(Z) = (|\boldsymbol{z}_1|^{2k!} + |\boldsymbol{z}_2|^{2k!/2} + \cdots + |\boldsymbol{z}_k|^{2k!/k})^{1/2k!} \tag{1.14}$$

where $|\boldsymbol{z}_i|$ is the usual Euclidean norm on $\mathbb{C}^n$. Observe that $r_k(\lambda \cdot Z) = |\lambda|\, r_k(Z)$ and that r_k is continuous on $\mathbb{C}^n_k$, smooth outside of the origin. Indeed $r_k^{2k!}$ is smooth on all of $\mathbb{C}^n_k$. Alternatively we can take

$$r_k(Z) = |\boldsymbol{z}_1| + |\boldsymbol{z}_2|^{1/2} + \cdots + |\boldsymbol{z}_k|^{1/k}, \tag{1.15}$$

which also satisfies $r_k(\lambda \cdot Z) = |\lambda|\, r_k(Z)$ and is continuous on $\mathbb{C}^n_k$, smooth outside of the set $[\boldsymbol{z}_1 \cdot \boldsymbol{z}_k = 0]$. On a local trivialization $J^k X|_U \cong U \times \mathbb{C}^n_k$ we define simply $\rho_k(z, Z) = r_k(Z)$ on $J^k X|_U$ so that a global k-jet metric is defined via a partition of unity subordinate to a locally finite trivialization cover. This general construction is of limited use as it does not take into account the geometry of the manifold.

In the case of a compact manifold a more useful construction can be carried out by taking a basis $\omega_1, \ldots, \omega_N$ of global holomorphic k-jet differentials (provided that these exist), and defining

$$\rho_k(j^k f) = \left(\sum_{i=1}^{N} |\omega_i(j^k f)|^2 \right)^{1/2}. \tag{1.16}$$

Then, since a jet differential is a linear functional on the k-jet bundle (that is, $\omega(\lambda \cdot j^k f) = \lambda \omega(j^k f)$), we see readily that $\rho_k(\lambda \cdot j^k f) = |\lambda| \rho_k(j^k f)$. It is clear from the definition that ρ_k is continuous on $J^k X$, real analytic on $J^k X \setminus \{\text{zero section}\}$; indeed, ρ_k^2 is real analytic on $J^k X$. For $k = 1$ use a basis of global holomorphic 1-forms. The number $N = h^0(T^* X)$ is the irregularity of X (for a Riemann surface this is just the genus of X). Thus the invariants $h^0(\mathcal{J}^m_k X)$ play an important role in the determination of hyperbolicity.

The jet bundles may be defined, in an analogous way, over fairly general fields. We conclude this section by introducing a very interesting problem:

Interpolation Problem. *Find all subvarieties in $\mathbb{P}^n = \mathbb{P}^n(\mathbb{K})$ (where $\mathbb{K}$ is an infinite field) of a given degree d with prescribed jet spaces at a finite number of points. More precisely, given subspaces $V_1 \subset T^k_{x_1} \mathbb{P}^n, \ldots, V_N \subset T^k_{x_N} \mathbb{P}^n$ (or $V_1 \subset J^k_{x_1} \mathbb{P}^n, \ldots, V_N \subset J^k_{x_N} \mathbb{P}^n$), find all varieties X of degree d such that $V_1 = T^k_{x_1} X, \ldots, V_N \subset T^k_{x_N} X$.*

At this time little is known about the problem for the bundle $J^k X$ however much is known in the case of $T^k X$. For example, the following is known (see [Chandler

1998a; 2002] or else [Alexander 1988; Alexander and Hirschowitz 1992a; 1992b; 1995]).

THEOREM 1.7. *Let Ψ be a general collection of d points in $\mathbb{P}^n$. The codimension in $H^0(\mathcal{O}_{\mathbb{P}^n}(3))$ of the space of sections singular on Ψ is* $\min\{(n+1)d, (n+3)!/3!\,n!\}$ *unless $n = 4$ and $d = 7$. More generally, the codimension in $H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ of the space of sections singular on Ψ is equal to* $\min\{(n+1)d, (n+m)!/m!\,n!\}$ *unless* $(n, m, d) = (2, 4, 5)$, $(3, 4, 9)$, $(4, 3, 7)$ *or* $(4, 4, 14)$.

The problem is related also to the *Waring problem* for linear forms: when can a general degree m form in $n+1$ variables be expressed as a sum of m-th powers of linear forms? Let $PS(n, m, d)$ be the space of homogeneous polynomials in $n+1$ variables expressible as $L_1^m + \cdots + L_d^m$, where $L_1, \ldots, L_d$ are linear forms. Then:

THEOREM 1.8. *With the notation above, we have*

$$\dim PS(n, m, d) = \min\{(n+1)d, (n+m)!/m!\,n!\}$$

unless $(n, m, d) = (2, 4, 5)$, $(3, 4, 9)$, $(4, 3, 7)$ *and* $(4, 4, 14)$.

For details, see the articles by Chandler and by Alexander and Hirschowitz in the references, as well as [Iarrobino and Kanev 1999].

2. Chern Classes and Cohomology Groups The Case of Curves

SUMMARY. *The theory of parametrized jet bundles is complicated by their not being vector bundles. This section discusses the case of curves to acquaint readers with the theory in the simplest situation. The theory is based on the fundamental result of Green and Griffiths on the filtration of the parametrized jet bundles (see Theorem* 2.3 *and Corollary* 2.4*). The explicit computations of this section have numerous applications (see for example* [An et al. 2003a; 2004; 2003b]).

In this section we compute the Chern numbers and the invariants $h^i(\mathcal{J}_k^m X)$, $i = 0, 1$, of the jet bundles for curves. In the case of curves in $\mathbb{P}^2$ we are interested in finding an explicit expression of a basis for $h^0(\mathcal{J}_k^m X)$. The procedure introduced here for the construction works as well for singular curves and in varieties defined over general differential fields. For applications in this direction to the *strong uniqueness polynomial problem* and the *unique range set problem*; see [An et al. 2004] in the complex case and [An et al. 2003a; 2003b] in the case of fields of positive characteristic.

For the full jet bundles the computation of Chern classes and cohomology groups is straightforward. Dualizing the defining sequence (1.1) we get an exact sequence

$$0 \to \bigodot^k T_1^* X \to T_k^* X \to T_{k-1}^* X \to 0. \tag{2.1}$$

For example, for $k=3$ the exact sequences

$$0 \to \bigodot^3 T_1^*X \to T_3^*X \to T_2^*X \to 0,$$
$$0 \to \bigodot^2 T_1^*X \to T_2^*X \to T_1^*X \to 0$$

and Whitney's Formula yields

$$c_1(T_3^*X) = c_1(T_2^*X) + c_1(\bigodot^3 T_1^*X) = c_1(T_1^*X) + c_1(\bigodot^2 T_1^*X) + c_1(\bigodot^3 T_1^*X).$$

In general, we have, by induction:

THEOREM 2.1. *The first Chern number of the bundle T_k^*X is given by*

$$c_1(T_k^*X) = \sum_{j=1}^{k} c_1(\bigodot^j T_1^*X).$$

In particular, if X is a Riemann surface,

$$c_1(T_k^*X) = \sum_{j=1}^{k} j c_1(T_1^*X) = \frac{k(k+1)}{2} c_1(\mathcal{K}_X) = k(k+1)(g-1)$$

where $\mathcal{K}_X = T_1^*X$ is the canonical bundle of X and g is the genus.

For a line bundle $\mathcal{L}$ and nonnegative integer i the i-fold tensor product is denoted by $\mathcal{L}^i$ and $\mathcal{L}^{-i}$ is the dual of $\mathcal{L}^i$. (Recall that tensor product and symmetric product on line bundles are equivalent.)

THEOREM 2.2. *Let X be a smooth curve of genus $g \geq 2$. Then $h^0(T_k^*X) = k^2(g-1)+1$ and $h^1(T_k^*X) = 1$.*

PROOF. By Riemann–Roch for curves,

$$\begin{aligned} h^0(\mathcal{K}_X^i) - h^1(\mathcal{K}_X^i) &= \chi(\mathcal{K}_X^i) = \chi(\mathcal{O}_X) + c_1(\mathcal{K}_X^i) \\ &= h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + 2(g-1)i = 1 - g + 2(g-1)i \end{aligned}$$

for any nonnegative integer i. Thus $h^0(\mathcal{K}_X^i) = h^1(\mathcal{K}_X^i) + (2i-1)(g-1) = h^0(\mathcal{K}_X^{1-i}) + (2i-1)(g-1)$. As $h^0(\mathcal{K}_X^{1-i}) = 1$ for $i=1$ and $h^0(\mathcal{K}_X^{1-i}) = 0$ for $i \geq 2$ we get

$$h^0(\mathcal{K}_X^i) = \begin{cases} 0, & i < 0, \\ 1, & i = 0, \\ g, & i = 1, \\ (2i-1)(g-1), & i \geq 2. \end{cases} \tag{2.2a}$$

By duality, $h^1(\mathcal{K}_X^i) = h^0(\mathcal{K}_X^{1-i})$; hence

$$h^1(\mathcal{K}_X^i) = \begin{cases} 0, & i \geq 2, \\ 1, & i = 1, \\ g, & i = 0, \\ (1-2i)(g-1), & i < 0. \end{cases} \tag{2.2b}$$

From the short exact sequence (2.1) we get the exact sequence

$$0 \to H^0(\mathcal{K}_X^k) \to H^0(T_k^*X) \to H^0(T_{k-1}^*X) \to H^1(\mathcal{K}_X^k) \to \\ \to H^1(T_k^*X) \to H^1(T_{k-1}^*X) \to 0.$$

From (2.2a,b) we deduce that, for $k \geq 2$, $H^1(T_k^*X) = H^1(T_{k-1}^*X)$ and that

$$h^0(T_k^*X) = h^0(T_{k-1}^*X) + h^0(\mathcal{K}_X^k).$$

These imply that

$$h^1(T_k^*X) = h^1(T^*X) = h^1(\mathcal{K}_X) = h^0(\mathcal{O}_X) = 1$$

for all $k \geq 1$ and that

$$h^0(T_k^*X) = \sum_{i=1}^{k} h^0(\mathcal{K}_X^i) = g + \sum_{i=2}^{k}(2i-1)(g-1) = k^2(g-1)+1. \qquad \square$$

The computation of Chern classes and cohomology groups for the parametrized jet bundles is somewhat more complicated. This depends on the fundamental filtration for these bundles due to Green–Griffiths. Let

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$$

be an exact sequence of sheaves. Then for any m there is a filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^m \subset \mathcal{F}^{m+1} = \bigodot^m \mathcal{S}$$

of the symmetric product $\bigodot^m \mathcal{S}$, such that $\mathcal{F}^i/\mathcal{F}^{i-1} \cong \bigodot^i \mathcal{S}' \otimes \bigodot^{m-i} \mathcal{S}''$. Analogously, for the exterior product $\bigwedge^m \mathcal{S}$ we have a filtration

$$0 = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^m \subset \mathcal{F}^{m+1} = \bigwedge^m \mathcal{S}$$

such that $\mathcal{F}^i/\mathcal{F}^{i-1} \cong \bigwedge^i \mathcal{S}' \otimes \bigwedge^{m-i} \mathcal{S}''$. These filtrations connect the cohomology groups of higher symmetric (resp. exterior) products to the cohomology groups of lower symmetric (resp. exterior) products. The analogue of these is the following theorem of Green and Griffiths (the proof can be found in [Stoll and Wong 2002]):

Theorem 2.3. *There exists a filtration of* $\mathcal{J}_k^m X$:

$$\mathcal{J}_{k-1}^m X = \mathcal{F}_k^0 \subset \mathcal{F}_k^1 \subset \cdots \subset \mathcal{F}_k^{[m/k]} = \mathcal{J}_k^m X$$

(*where* $[m/k]$ *is the greatest integer smaller than or equal to* m/k) *such that*

$$\mathcal{F}_k^i/\mathcal{F}_k^{i-1} \cong \mathcal{J}_{k-1}^{m-ki} X \otimes (\bigodot^i T^*X).$$

As an immediate consequence [Green and Griffiths 1980], we have:

COROLLARY 2.4. *Let X be a smooth projective variety. Then $\mathcal{J}_k^m X$ admits a composition series whose factors consist precisely of all bundles of the form*: $(\bigodot^{i_1} T^*X)\otimes\cdots\otimes(\bigodot^{i_k} T^*X)$ *where i_j ranges over all nonnegative integers satisfying $i_1+2i_2+\cdots+ki_k=m$. The first Chern number of $\mathcal{J}_k^m X$ is given by*

$$c_1(\mathcal{J}_k^m X)=\sum_{\substack{i_1+2i_2+\cdots+ki_k=m\\ i_j\in\mathbb{Z}_{\geq 0}}} c_1\bigl((\bigodot^{i_1} T^*X)\otimes\cdots\otimes(\bigodot^{i_k} T^*X)\bigr).$$

In particular, *if X is a curve then*

$$c_1(\mathcal{J}_k^m X)=\sum_{\substack{i_1+2i_2+\cdots+ki_k=m\\ i_j\in\mathbb{Z}_{\geq 0}}} (i_1+i_2+\cdots+i_k)c_1(T^*X).$$

EXAMPLE 2.5 [Stoll and Wong 2002]. It is clear that for $m<k$ the filtration degenerates and we have $\mathcal{J}_k^m X=\mathcal{J}_{k-1}^m X=\ldots=\mathcal{J}_m^m X$. In particular, $\mathcal{J}_2^1 X=\mathcal{J}_1^1 X=T^*X$. For $m=k=2$, the filtration is given by

$$\bigodot^2 T^*X=\mathcal{J}_1^2 X=\mathcal{S}_2^0\subset\mathcal{S}_2^1=\mathcal{J}_2^2 X,\ \ \mathcal{S}_2^1/\mathcal{S}_2^0\cong T^*X,$$

so we have the exact sequence

$$0\to\bigodot^2 T^*X\to\mathcal{J}_2^2 X\to T^*X\to 0.$$

Thus the first Chern numbers are related by the formula

$$c_1(\mathcal{J}_2^2 X)=c_1(\bigodot^2 T^*X)+c_1(T^*X).$$

Analogously, $\mathcal{J}_3^1 X=\mathcal{J}_2^1 X=\mathcal{J}_1^1 X=T^*X$ and $\mathcal{J}_3^2 X=\mathcal{J}_2^2 X$. The filtration of $\mathcal{J}_3^3 X$ is as follows:

$$\mathcal{J}_3^3 X=\mathcal{S}_3^1\supset S_3^0=\mathcal{J}_2^3 X,\ \mathcal{J}_3^3 X/\mathcal{J}_2^3 X=\mathcal{S}_3^1/\mathcal{S}_3^0\cong T^*X.$$

Hence we have an exact sequence

$$0\to\mathcal{J}_2^3 X\to\mathcal{J}_3^3 X\to T^*X\to 0.$$

Now the filtration of $\mathcal{J}_2^3 X$ is

$$\mathcal{J}_2^3 X=\mathcal{S}_2^1\supset\mathcal{S}_2^0=\mathcal{J}_1^3 X,\ \ \mathcal{J}_2^3 X/\mathcal{J}_1^3 X\cong T^*X\otimes T^*X$$

and, since $\mathcal{J}_1^3 X=\bigodot^3 T^*X$, we have an exact sequence

$$0\to\bigodot^3 T^*X\to\mathcal{J}_2^3 X\to T^*X\otimes T^*X\to 0.$$

From these two exact sequences we get

$$c_1(\mathcal{J}_3^3 X)=c_1(T^*X)+c_1(T^*X\otimes T^*X)+c_1(\bigodot^3 T^*X).$$

From basic representation theory (or linear algebra in this special case) we have $T^*X\otimes T^*X=\bigodot^2 T^*X\oplus\bigwedge^2 T^*X$ hence

$$c_1(\mathcal{J}_3^3 X)=c_1(T^*X)+c_1(\bigodot^2 T^*X)+c_1(\bigodot^3 T^*X)+c_1(\bigwedge^2 T^*X).$$

In representation theory $\bigwedge^2 T^*X$ is the *Weyl module* $W^*_{1,1}X$ associated to the partition $\{1,1\}$ (see [Fulton and Harris 1991]). Thus we have

$$c_1(\mathcal{J}^3_3X) = \sum_{j=1}^{3} c_1(\textstyle\bigodot^j T^*X) + c_1(W^*_{1,1}X).$$

In the special case of a Riemann surface $\bigwedge^2 T^*X$ is the zero-sheaf. Thus for a curve we have

$$c_1(\mathcal{J}^3_3X) = (1+2+3)c_1(T^*X) = 6c_1(T^*X).$$

For $m=k=4$, we have the filtrations $\mathcal{J}^4_4X = S^1_4 \supset S^0_4 = \mathcal{J}^4_3X$, $\mathcal{J}^4_4X/\mathcal{J}^4_3X = S^1_4/S^0_4 \cong T^*X$, $\mathcal{J}^4_3X = S^1_3 \supset S^0_3 = \mathcal{J}^4_2X$, $\mathcal{J}^4_3X/\mathcal{J}^4_2X = S^1_3/S^0_3 \cong T^*X \otimes T^*X$, and $\mathcal{J}^4_2X = S^2_2 \supset S^1_2 \supset S^0_2 = \mathcal{J}^4_1X$, with

$$\mathcal{J}^4_2X/\mathcal{S}^1_2 = \textstyle\bigodot^2 T^*X,\ S^1_2/S^0_2 \cong T^*X \otimes (\bigodot^2 T^*X).$$

Thus the Chern number is given by

$$\begin{aligned} c_1(\mathcal{J}^4_4X) = c_1(T^*X) + c_1(T^*X \otimes T^*X) + c_1(\textstyle\bigodot^2 T^*X)& \\ + c_1(T^*X \otimes (\textstyle\bigodot^2 T^*X)) + c_1(\textstyle\bigodot^4 T^*X).& \end{aligned}$$

From elementary representation theory we obtain

$$T^*X \otimes (\textstyle\bigodot^k T^*X) = W^*_{k,1}X \oplus (\bigodot^{k+1} T^*X)$$

where $W^*_{k,1}$ is the Weyl module associated to the partition $\{k,1\}$ so that

$$c_1(\mathcal{J}^4_4X) = c_1(\textstyle\bigodot^2 T^*X) + \sum_{i=1}^4 c_1(\bigodot^i T^*X) + \sum_{i=1}^2 c_1(W^*_{j,1}X).$$

In particular, if X is a curve,

$$c_1(\mathcal{J}^4_4X) = (1+2+2+3+4)c_1(T^*X) = 12c_1(T^*X).$$

The procedure can be carried out further; for instance,

$$\begin{aligned} c_1(\mathcal{J}^5_5X) &= \textstyle\sum_{j=2}^3 c_1(\bigodot^j T^*X) \\ &\qquad + \textstyle\sum_{j=1}^5 c_1(\bigodot^j T^*X) + \sum_{j=1}^2 c_1(W^*_{j,1}X) + \sum_{j=1}^3 c_1(W^*_{j,1}X), \\ c_1(\mathcal{J}^6_6X) &= c_1(T^*X) + 3c_1(T^*X \otimes T^*X) + 2c_1\big(T^*X \otimes (\textstyle\bigodot^3 T^*X)\big) \\ &\qquad + c_1(\textstyle\bigodot^2 T^*X) + c_1(\bigodot^3 T^*X) + c_1\big((\bigodot^2 T^*X) \otimes (\bigodot^2 T^*X)\big) \\ &\qquad + c_1\big(T^*X \otimes (\textstyle\bigodot^4 T^*X)\big) + c_1(\bigodot^6 T^*X). \end{aligned}$$

So if X is a curve we have

$$c_1(\mathcal{J}^5_5X) = (1+2+3+2+3+4+5)c_1(T^*X) = 20c_1(T^*X),$$

$$c_1(\mathcal{J}^6_6X) = (1+6+8+2+3+4+5+6)c_1(T^*X) = 35c_1(T^*X).$$

The calculation of the sum

$$\sum_{i_1+2i_2+\cdots+ki_k=m} i_1+\cdots+i_k \tag{2.3}$$

can be carried out using standard combinatorial results which we now describe.

Definition 2.6. (i) A *maximal set* of mutually conjugate elements of S_m (the symmetric group on m elements) is said to be a *class of* S_m.

(ii) A *partition* of a natural number m is a set of positive integers $i_1, \dots, i_q$ such that $m = i_1 + \cdots + i_q$.

The following asymptotic result concerning the number of partitions of a positive integer m is well-known in representation theory and in combinatorics [Hardy and Wright 1970]:

Theorem 2.7. *The number of partitions of m, the number of classes of S_m and the number of* (*inequivalent*) *irreducible representations of S_m are equal. This common number $p(m)$ is asymptotically approximated by the formula of Hardy–Ramanujan*:

$$p(m) \sim \frac{e^{\pi\sqrt{2m/3}}}{4m\sqrt{3}}.$$

The first few partition numbers are $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$, $p(7) = 15$, $p(8) = 22$, $p(9) = 30$, $p(10) = 42$, $p(11) = 56$, $p(12) = 77$, $p(13) = 101$. Consider first the case of partitioning a number by partitions of a *fixed* length k. Denote by $p_k(m)$ the number of *positive integral solutions* of the equation

$$x_1 + \cdots + x_k = m$$

with the condition that $1 \le x_k \le x_{k-1} \le \ldots \le x_1$. This number is equal to the number of *integer solutions* of the equation

$$y_1 + \cdots + y_k = m - k$$

with the condition that the solutions be *nonnegative* and $0 \le y_k \le y_{k-1} \le \ldots \le y_1$. If exactly i of the integers $\{y_1, \dots, y_k\}$ are positive then these are the solutions of $x_1 + \cdots + x_i = m - k$ and so there are $p_i(m-k)$ of such solutions. Consequently we have (see [Stoll and Wong 2002] for more details):

Lemma 2.8. *With the notation above we have*: $p(m) = \sum_{k=1}^{m} p_k(m)$, *where*

$$p_k(m) = \sum_{i=0}^{k} p_i(m-k),$$

for $1 \le k \le m$ and with the convention that $p_0(0) = 1, p_0(m) = 0$ if $m > 0$ and $p_k(m) = 0$ if $k > m$. Moreover, *the number $p_k(m)$ satisfies the following recursive relation*:

$$p_k(m) = p_{k-1}(m-1) + p_k(m-k).$$

Example 2.9. We shall compute $p(6)$ and $p(7)$ using the preceding lemma. We have $p_1(m) = p_m(m) = 1$ and $p_2(m) = m/2$ or $(m-1)/2$ according to m being even or odd; thus $p_1(6) = 1, p_2(6) = 3, p_6(6) = 1$. Analogously, we have:

$$p_3(m) = p_2(m-1) + p_3(m-3),$$
$$p_4(m) = p_3(m-1) + p_4(m-4),$$
$$p_5(m) = p_4(m-1) + p_5(m-5),$$

so that, for example:

$$p_3(6) = p_2(5) + p_3(3) = 2 + 1 = 3,$$
$$p_4(6) = p_3(5) + p_4(2) = p_2(4) = 2,$$
$$p_5(6) = p_4(5) = p_3(4) = p_2(3) = 1.$$

Since $p(m) = \sum_{k=1}^{m} p_k(m)$ we have

$$p(6) = \sum_{k=1}^{6} p_k(6) = 1 + 3 + 3 + 2 + 1 + 1 = 11.$$

For $m = 7$ we have $p_1(7) = 1$, $p_2(7) = 3$, $p_7(7) = 1$, $p_3(7) = p_2(6) + p_3(4) = p_2(6) + p_2(3) = 4$, $p_4(7) = p_3(6) = 3$, $p_5(7) = p_4(6) = 2$, $p_6(7) = p_5(6) = 1$; hence

$$p(7) = \sum_{k=1}^{7} p_k(7) = 1 + 3 + 4 + 3 + 2 + 1 + 1 = 15.$$

For $k \le m$ denote by $L_k(m)$ the sum of the lengths of all partitions λ of m whose length l_λ is at most k:

$$L_k(m) = \sum_{\lambda,\, l_\lambda \le k} l_\lambda.$$

The next lemma follows from the definitions [Wong 1999; Stoll and Wong 2002]:

Lemma 2.10. *With notation as above we have*

$$L_k(m) = \sum_{\lambda, l_\lambda \le k} l_\lambda = \sum_{j=1}^{k} j p_j(m) = \sum_{\lambda,\, l_\lambda \le k} \sum_{j=1}^{k} i_j,$$

where the sum on the right is taken over all partitions $\lambda = (\lambda_1, \ldots, \lambda_{\rho_\lambda})$ of m, $1 \le \lambda_{l_\lambda} \le \ldots \le \lambda_2 \le \lambda_1$, $l_\lambda \le k$ and i_j is the number of j's in $\{\lambda_1, \ldots, \lambda_{l_\lambda}\}$.

For $k = m, L(m) = L_m(m)$ is *the total length of all possible partitions of m*. For example if $m = 6$ then $L(6) = 1 + 6 + 9 + 8 + 5 + 6 = 35$ and for $m = 7, L(7) = 1 + 6 + 12 + 12 + 10 + 6 + 7 = 54$. Indeed we have:

Theorem 2.11. *If X is a nonsingular projective curve, the Chern number of $\mathcal{J}_m^m X$ is*

$$c_1(\mathcal{J}_m^m X) = L_m(m) c_1(\mathcal{K}_X) = \sum_{j=1}^{m} j p_j(m) c_1(\mathcal{K}_X),$$

where $\mathcal{K}_X$ is the canonical bundle of X.

There is a formula for the asymptotic behavior of $p_k(m)$:

THEOREM 2.12. *For k fixed and $m \to \infty$ the number $p_k(m)$ is asymptotically*

$$p_k(m) \sim \frac{m^{k-1}}{(k-1)!\,k!}.$$

We give below the explicit calculation of the above in the first few cases. For $m = k = 3$, we have $p(3) = 3$ and the possible indices are

	λ	l_λ	d_λ	i_1	i_2	i_3	$\sum_{j=1}^k i_j$
1	$(1,1,1)$	3	1	3	0	0	3
2	$(2,1)$	2	2	1	1	0	2
3	(3)	1	1	0	0	1	1

The cases cases correspond to the possible partitions of 3: $1+1+1 = 3$, $2+1 = 3$ and $3 = 3$, of respective lengths 3, 2, and 1. The Chern number $c_1(\mathcal{J}_3^3 X)$ of a curve X is obtained by summing the last column: $c_1(\mathcal{J}_3^3 X) = (1+2+3) \times c_1(T^*X) = 6c_1(T^*X)$.

For $m = k = 4$ the number of partitions is $p(4) = 5$ and we have

	λ	l_λ	d_λ	i_1	i_2	i_3	i_4	$\sum_{j=1}^k i_j$
1	$(1,1,1,1)$	4	1	4	0	0	0	4
2	$(2,1,1)$	3	3	2	1	0	0	3
3	$(3,1)$	2	3	1	0	1	0	2
4	$(2,2)$	2	2	0	2	0	0	2
5	(4)	1	1	0	0	0	1	1

and $c_1(\mathcal{J}_4^4 X) = 12c_1(T^*X)$.

For $m = k = 5$, $p(5) = 7$,

	λ	ρ_λ	d_λ	i_1	i_2	i_3	i_4	i_5	$\sum_{j=1}^k i_j$
1	$(1,1,1,1,1)$	5	1	5	0	0	0	0	5
2	$(2,1,1,1)$	4	4	3	1	0	0	0	4
3	$(3,1,1)$	3	6	2	0	1	0	0	3
4	$(2,2,1)$	3	5	1	2	0	0	0	3
5	$(4,1)$	2	4	1	0	0	1	0	2
6	$(3,2)$	2	15	0	1	1	0	0	2
7	(5)	1	1	0	0	0	0	1	1

and $c_1(\mathcal{J}_5^5 X) = 20c_1(T^*X)$.

For $m = k = 6$, $p(6) = 11$,

	λ	l_λ	d_λ	i_1	i_2	i_3	i_4	i_5	i_6	$\sum_{j=1}^k i_j$
1	$(1,1,1,1,1,1)$	6	1	6	0	0	0	0	0	6
2	$(2,1,1,1,1)$	5	5	4	1	0	0	0	0	5
3	$(3,1,1,1)$	4	10	3	0	1	0	0	0	4
4	$(2,2,1,1)$	4	9	2	2	0	0	0	0	4
5	$(4,1,1)$	3	10	2	0	0	1	0	0	3
6	$(3,2,1)$	3	36	1	1	1	0	0	0	3
7	$(2,2,2)$	3	5	0	3	0	0	0	0	3
8	$(5,1)$	2	30	1	0	0	0	1	0	2
9	$(4,2)$	2	9	0	1	0	1	0	0	2
10	$(3,3)$	2	5	0	0	2	0	0	0	2
11	(6)	1	1	0	0	0	0	0	1	1

and $c_1(\mathcal{J}_6^6 X) = 35c_1(T^*X)$.

The next few values of $L_k(k)$ are $L_7(7) = 54$, $L_8(8) = 86$, $L_9(9) = 128$, $L_{10}(10) = 192$, $L_{11}(11) = 275$, $L_{12}(12) = 399$, $L_{13}(13) = 556$, $L_{14}(14) = 780$, $L_{15}(15) = 1068$, $L_{16}(16) = 1463$.

Next we deal with the problem of computing the invariants: $h^i(\mathcal{J}_k^m X) = \dim H^i(\mathcal{J}_k^m X)$ for a curve X of genus $g \geq 2$. We have

$$h^0(\mathcal{J}_1^1 X) = h^0(\mathcal{K}_X) = g,$$
$$h^1(\mathcal{J}_1^1 X) = h^0(\mathcal{O}_X) = 1.$$

For curves the filtration of Green–Griffiths takes the form

$$\mathcal{J}_k^m X = \mathcal{S}_k^{[m/k]} \supset \cdots \supset \mathcal{S}_k^0 = \mathcal{J}_{k-1}^m X, \quad \mathcal{S}_k^i/\mathcal{S}_k^{i-1} = \mathcal{K}_X^i \otimes \mathcal{J}_{k-1}^{m-ki}(X).$$

Hence, for $k = 2$, $\mathcal{S}_2^i/\mathcal{S}_2^{i+1} = \mathcal{K}_X^i \otimes \mathcal{J}_1^{m-2i} = \mathcal{K}_X^i \otimes \mathcal{K}_X^{m-2i} = \mathcal{K}_X^{m-i}$. It is clear from the filtration that $\mathcal{J}_1^1 X \cong \mathcal{J}_2^1 X$ (the isomorphism is given by the forgetting map (1.11)). For $\mathcal{J}_2^2 X$ the filtration yields the short exact sequence

$$0 \to \mathcal{K}_X^2 = \bigodot^2 T^*X \to \mathcal{J}_2^2 X \to T^*X = \mathcal{K}_X \to 0,$$

from which we get the exact sequence

$$0 \to H^0(\mathcal{K}_X^2) \to H^0(\mathcal{J}_2^2 X) \to H^0(\mathcal{K}_X) \to H^1(\mathcal{K}_X^2) \to H^1(\mathcal{J}_2^2 X) \to H^1(\mathcal{K}_X) \to 0.$$

By (2.2b) we have $h^1(\mathcal{K}_X) = 1$ and $h^1(\mathcal{K}_X^2) = 0$ if $d \geq 2$. Hence, as $\mathcal{K}_X = \mathcal{O}_X(3-d)$,

$$h^0(\mathcal{J}_2^2 X) = h^0(\mathcal{K}_X) + h^0(\mathcal{K}_X^2) = 4g - 3,$$
$$h^1(\mathcal{J}_2^2 X) = h^1(T^*X) = h^0(\mathcal{O}_X) = 1.$$

For $\mathcal{J}_2^3 X$ we obtain the short exact sequence from the filtration,

$$0 \to \mathcal{K}_X^3 \to \mathcal{J}_2^3 X \to \mathcal{K}_X^2 \to 0,$$

and the exact cohomology sequence

$$0 \to H^0(\mathcal{K}_X^3) \to H^0(\mathcal{J}_2^3 X) \to H^0(\mathcal{K}_X^2) \to H^1(\mathcal{K}_X^3) \to H^1(\mathcal{J}_2^3 X) \to H^1(\mathcal{K}_X^2) \to 0.$$

Since $h^1(\mathcal{K}_X^2) = h^1(\mathcal{K}_X^3) = 0$ for $g \geq 2$ we find

$$\begin{aligned} &h^1(\mathcal{J}_2^3 X) = 0, \\ &h^0(\mathcal{J}_2^3 X) = h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X^3) = 8(g-1). \end{aligned}$$

For $\mathcal{J}_2^4 X$ the filtration is given by

$$\mathcal{J}_2^4 X = \mathcal{S}_2^2 \supset \mathcal{S}_2^1 \supset \mathcal{S}_2^0 = \mathcal{J}_1^4 X = \mathcal{K}_X^4$$

with $\mathcal{S}_2^2/\mathcal{S}_2^1 = \mathcal{K}_X^2, \mathcal{S}_2^1/\mathcal{S}_2^0 = \mathcal{K}_X^3$. From the filtration we have two short exact sequences,

$$0 \to \mathcal{S}_2^1 \to \mathcal{J}_2^4 X \to \mathcal{K}_X^2 \to 0 \quad \text{and} \quad 0 \to \mathcal{K}_X^4 \to \mathcal{S}_2^1 \to \mathcal{K}_X^3 \to 0.$$

For $g \geq 2$ we get from the second exact sequence and the fact that $h^1(\mathcal{K}_X^3) = h^1(\mathcal{K}_X^4) = 0$ that $h^1(\mathcal{S}_2^1) = 0$. This and the first exact sequence imply that

$$\begin{aligned} &h^1(\mathcal{J}_2^4 X) = h^1(\mathcal{K}_X^2) = 0 \text{ and} \\ &h^0(\mathcal{J}_2^4 X) = h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X^3) + h^0(\mathcal{K}_X^4) = 15(g-1). \end{aligned}$$

We get, inductively:

THEOREM 2.13. *For a smooth curve with genus $g \geq 2$ the following equalities hold*:

(i) $\mathcal{J}_2^1 X = \mathcal{K}_X$; *hence* $h^1(\mathcal{J}_2^1 X) = 1, h^0(\mathcal{J}_2^1 X) =$ *genus of* X;
(ii) $h^1(\mathcal{J}_2^2 X) = 1, \quad h^0(\mathcal{J}_2^2 X) = h^0(\mathcal{K}_X) + h^0(\mathcal{K}_X^2) = 4g-3$;
(iii) $h^1(\mathcal{J}_2^m X) = 0$, *and for* $m \geq 3$,

$$h^0(\mathcal{J}_2^m X) = \textstyle\sum_{j=0}^{[m/2]} h^0(\mathcal{K}_X^{m-j}) = (2m - [\frac{m}{2}] - 1)([\frac{m}{2}] + 1)(g-1);$$

(iv) *for* $i \geq 1$,

$$\begin{aligned} &h^0(\mathcal{J}_2^1 X \otimes \mathcal{K}_X^i) = h^0(\mathcal{K}_X^{i+1}) = (2i+1)(g-1), \\ &h^0(\mathcal{J}_2^2 X \otimes \mathcal{K}_X^i) = h^0(\mathcal{K}_X^{i+1}) + h^0(\mathcal{K}_X^{i+2}) = 4(i+1)(g-1), \end{aligned}$$

and for $m \geq 3$,

$$h^0(\mathcal{J}_2^m X \otimes \mathcal{K}_X^i) = \textstyle\sum_{j=0}^{[m/2]} h^0(\mathcal{K}_X^{m+i-j}) = (2m + 2i - [\frac{m}{2}] - 1)([\frac{m}{2}] + 1)(g-1).$$

PROOF. Parts (i), (ii) and (iii) are clear. For part (iv), tensoring the exact sequence $0 \to \mathcal{K}_X^2 \to \mathcal{J}_2^2 X \to \mathcal{K}_X \to 0$ by $\mathcal{K}_X^i$ yields the exact sequence $0 \to \mathcal{K}_X^{i+2} \to \mathcal{J}_2^2 X \otimes \mathcal{K}_X^i \to \mathcal{K}_X^{i+1} \to 0$. From the associated long exact cohomology sequence one sees that $h^0(\mathcal{J}_2^2 X\otimes\mathcal{K}_X^i) = h^0(\mathcal{K}_X^{i+1}) + h^0(\mathcal{K}_X^{i+2})$ as claimed. From the exact sequences

$$\begin{array}{lllll}
0\to & \mathcal{S}_{m,2}^{[m/2]-1} & \to \mathcal{J}_2^m X & \to \mathcal{K}^{[m/2]}\otimes\mathcal{J}_1^{m-2[m/2]}X = \mathcal{K}_X^{m-[m/2]} & \to 0,\\
0\to & \mathcal{S}_{m,2}^{[m/2]-2} & \to \mathcal{S}_{m,2}^{[m/2]-1} & \to \mathcal{K}_X^{[m/2]-1}\otimes\mathcal{J}_1^{m-2([m/2]-1)}X = \mathcal{K}_X^{m-[m/2]+1} & \to 0,\\
\cdots & \cdots & \cdots & \cdots & \cdots\\
0\to & \mathcal{J}_1^m X = \mathcal{K}_X^m & \to \mathcal{S}_{m,2}^1 & \to \mathcal{K}_X\otimes\mathcal{J}_1^{m-2}X = \mathcal{K}_X^{m-1} & \to 0,
\end{array}$$

we obtain, by tensoring with $\mathcal{K}_X^i$, $i\ge 0$, the exact sequences

$$\begin{array}{lllll}
0\to \mathcal{S}_{m,2}^{[m/2]-1}\otimes\mathcal{K}_X^i & \to & \mathcal{J}_2^m X\otimes\mathcal{K}_X^i & \to & \mathcal{K}_X^{m+i-[m/2]} \to 0,\\
0\to \mathcal{S}_{m,2}^{[m/2]-2}\otimes\mathcal{K}_X^i & \to & \mathcal{S}_{m,2}^{[m/2]-1}\otimes\mathcal{K}_X^i & \to & \mathcal{K}_X^{m+i-[m/2]+1} \to 0,\\
\cdots & & \cdots & & \cdots\\
0\to \mathcal{K}_X^{m+i} & \to & \mathcal{S}_{m,2}^1\otimes\mathcal{K}_X^i & \to & \mathcal{K}_X^{m+i-1} \to 0,
\end{array}$$

from which we deduce that $h^1(\mathcal{J}_2^m X\otimes\mathcal{K}_X^i) = h^1(\mathcal{S}_{m,2}^{[m/2]-j}\otimes\mathcal{K}_X^i) = 0$ for $0\le j\le [m/2]$ and that

$$h^0(\mathcal{J}_2^m X\otimes\mathcal{K}_X^i) = \sum_{j=0}^{[m/2]} h^0(\mathcal{K}_X^{m+i-j}),$$

as claimed. $\square$

The coefficient of $(g-1)$ in part (iv) of the preceding lemma may be expressed as

$$\alpha(m,i,2) = \begin{cases} \frac14(3m^2+4m(i+1)+8i-4) = \frac14(m+2)(3m+4i-2), & m \text{ even},\\ \frac14(3m^2+2m(2i+1)+4i-1) = \frac14(m+1)(3m+4i-1)4, & m \text{ odd}.\end{cases}$$

The coefficient of $(g-1)$ in part (iii) is $\alpha(m,2)=\alpha(m,0,2)$.

Corollary 2.14. *For a smooth curve of genus $g\ge 2$ we have, for $m\ge 3$,*

$$h^0(\mathcal{J}_2^m X) = \begin{cases} \frac14(3m^2+4m-4)(g-1), & m \text{ even},\\ \frac14(3m^2+2m-1)(g-1), & m \text{ odd},\end{cases}$$

and

$$c_1(\mathcal{J}_2^m X) = \begin{cases} \frac14(3m^2+6m)(g-1), & m \text{ even},\\ \frac14(3m^2+4m+1)(g-1), & m \text{ odd}.\end{cases}$$

Proof. The first formula is given by part (iii) of Theorem 2.13. The second formula is a consequence of the *Riemann–Roch for curves*:

$$h^0(\mathcal{J}_2^m X) - h^1(\mathcal{J}_2^m X) = c_1(\mathcal{J}_2^m X) - (\operatorname{rk}\mathcal{J}_2^m X)(g-1),$$

using the fact that $\operatorname{rk}\mathcal{J}_2^m X = [m/2]+1$ and that $h^1(\mathcal{J}_2^m X) = 0$ if $m\ge 3$. $\square$

We now deal with the case of general k. We shall be content with asymptotic formulas as the general formulas become complicated by the fact that the general formula for sums of powers is only given recursively. However the highest order term is quite simple:

$$\sum_{i=1}^{m} i^d = \frac{m^{d+1}}{d+1} + O(m^d). \tag{2.4}$$

The filtration theorem of Green–Griffiths implies that

$$\text{rk } \mathcal{J}_k^m X = \sum_{I \in \mathcal{I}_{k,m}} \text{rk } \mathcal{S}_I.$$

For a curve, $\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_{k-1}} T^*X \otimes \bigodot^{i_k} T^*X = \mathcal{K}_X^{|I|} = \mathcal{K}_X^{i_1+\cdots+i_k}$. Hence

$$\text{rk } \mathcal{J}_k^m X = \#\mathcal{I}_{k,m}, \;\; \mathcal{I}_{k,m} = \bigl\{ I = (i_1, \dots, i_k) \mid \textstyle\sum_{j=1}^k j i_j = m \bigr\}.$$

Alternatively, since $\mathcal{S}_I = \mathcal{S}_{I'} \otimes \bigodot^{i_k} T^*X$, where $I' = (i_1, \dots, i_{k-1}) \in \mathcal{I}_{k-1, m-ki_k}$, we have

$$\text{rk } \mathcal{J}_k^m X = \sum_{i_k=0}^{[m/k]} \text{rk } (\mathcal{J}_{k-1}^{m-ki_k} X \otimes \mathcal{K}_X^{i_k}) = \sum_{i_k=0}^{[m/k]} \text{rk } \mathcal{J}_{k-1}^{m-ki_k} X;$$

equivalently,

$$\#\mathcal{I}_{k,m} = \sum_{i_k=0}^{[m/k]} \#\mathcal{I}_{k-1, m-ki_k}.$$

THEOREM 2.15. *Let* $\mathcal{I}_{k,m} = \bigl\{ I = (i_1, \dots, i_k) \mid \sum_{j=1}^k j i_j = m \bigr\}$. *Then, for a curve* X,

$$rk\ \mathcal{J}_k^m X = \#\mathcal{I}_{k,m} = \frac{m^{k-1}}{k!(k-1)!} + O(m^{k-2}).$$

PROOF. It is clear that rk $\mathcal{J}_1^m X = 1$ and we have seen that rk $\mathcal{J}_2^m X = [m/2]+1$, thus writing rk $\mathcal{J}_k^m X = a_k m^{k-1} + O(m^{k-2})$ we get, via (2.4),

$$\begin{aligned} a_k m^{k-1} + O(m^{k-2}) &= a_{k-1} \sum_{i_k=0}^{[m/k]} (m - k i_k)^{k-2} + O(m^{k-2}) \\ &= a_{k-1} \sum_{i_k=0}^{[m/k]} (m - k i_k)^{k-2} + O(m^{k-2}) \end{aligned}$$

$$= a_{k-1}\sum_{j=0}^{k-2}(-1)^j\frac{(k-2)!}{j!(k-2-j)!}m^{k-2-j}k^j\sum_{i_k=0}^{[m/k]}i_k^j + O(m^{k-2})$$
$$= a_{k-1}\sum_{j=0}^{k-2}(-1)^j\frac{(k-2)!}{j!(k-2-j)!}m^{k-2-j}k^j\frac{m^{j+1}}{(j+1)k^{j+1}} + O(m^{k-2})$$
$$= \frac{a_{k-1}}{k}\sum_{j=0}^{k-2}\frac{(-1)^j}{j+1}\frac{(k-2)!}{j!(k-2-j)!}m^{k-1} + O(m^{k-2}).$$

The following formula is easily verified by double induction:

LEMMA 2.16. *For any positive integers $1 \le l \le k$, we have*

$$\sum_{j=0}^{k}\frac{(-1)^j}{j+l}\binom{k}{j} = \frac{(l-1)!\,k!}{(k+l)!}.$$

Using this lemma we obtain a recursive formula for $k \ge 2$:

$$a_k = \frac{a_{k-1}}{k(k-1)}, \quad a_1 = 1.$$

The first few values of a_k are $a_1 = 1$, $a_2 = 1/2$, $a_3 = 1/2^2 3$, $a_4 = 1/(2^4 3^2)$, $a_5 = 1/(2^6 3^2 5)$, $a_6 = 1/(2^7 3^3 5^2)$. The recursive formula also yields the general formula for a_k:

$$a_k = \frac{1}{\prod_{l=2}^{k}(l-1)l} = \frac{1}{(k-1)!\,k!}. \qquad \square$$

The filtration also yields a formula for a curve of genus g:

$$c_1(\mathcal{J}_k^m X) = \sum_{I\in\mathcal{I}_{k,m}} c_1(\mathcal{S}_I) = \sum_{I\in\mathcal{I}_{k,m}} |I| c_1(\mathcal{K}_X) = 2\sum_{I\in\mathcal{I}_{k,m}} |I|(g-1),$$

where $|I| = i_1 + \cdots + i_k$. On the other hand we have

$$c_1(\mathcal{J}_k^m X) = \sum_{i_k=0}^{[m/k]}\sum_{I'\in\mathcal{I}_{k-1,m-ki_k}}\big(c_1(\mathcal{S}_{I'}) + i_k c_1(\mathcal{K}_X)\big)$$
$$= 2\sum_{i_k=0}^{[m/k]}\sum_{I'\in\mathcal{I}_{k-1,m-ki_k}}(|I'| + i_k)(g-1).$$

It is clear that $c_1(\mathcal{J}_1^m X) = 2m(g-1)$ and we have seen that

$$c_1(\mathcal{J}_2^m X) = \begin{cases} \frac{1}{4}(3m^2+6m)(g-1), & m \text{ even},\\ \frac{1}{4}(3m^2+4m+1)(g-1), & m \text{ odd}.\end{cases}$$

THEOREM 2.17. *For a curve of genus $g \ge 2$ we have, for each $k \ge 2$,*

$$c_1(\mathcal{J}_k^m X) = \left(\frac{2(g-1)}{(k!)^2}\sum_{i=1}^{k}\frac{1}{i}\right)m^k + O(m^{k-1}).$$

PROOF. It is clear that asymptotically $c_1(\mathcal{J}_k^m X) = O(m^k)$. Hhence, writing

$$c_1(\mathcal{J}_k^m X) = 2b_k m^k(g-1) + O(m^{k-1}),$$

we get via Theorem 2.15 that, for $k \geq 3$,

$$\begin{aligned}
&2b_k m^k(g-1) + O(m^{k-1}) \\
&\qquad = 2\sum_{i_k=0}^{m/k} \sum_{I' \in \mathcal{I}_{k-1,m-ki_k}} (b_{k-1}(m-ki_k)^{k-1} + i_k)(g-1) + O(m^{k-1}) \\
&\qquad = 2(g-1)\sum_{i_k=0}^{m/k} \big(a_{k-1} i_k (m-ki_k)^{k-2} + b_{k-1}(m-ki_k)^{k-1}\big) + O(m^{k-1}).
\end{aligned}$$

By Lemma 2.16 we have

$$\sum_{i_k=0}^{m/k} i_k(m-ki_k)^{k-2} = \frac{1}{k^2}\sum_{j=0}^{k-2}\frac{(-1)^j}{j+2}\frac{(k-2)!}{j!(k-2-j)!}m^k = \frac{1}{k^3(k-1)}$$

and

$$\sum_{i_k=0}^{m/k}(m-ki_k)^{k-1} = \frac{1}{k}\sum_{j=0}^{k-1}\frac{(-1)^j}{j+1}\frac{(k-1)!}{j!(k-1-j)!}m^k = \frac{1}{k}\frac{(k-1)!}{k!} = \frac{1}{k^2}.$$

From these we obtain

$$2b_k m^k(g-1) + O(m^{k-1}) = 2(g-1)\left(\frac{a_{k-1}}{k^3(k-1)} + \frac{b_{k-1}}{k^2}\right) + O(m^{k-1})$$

and hence the recursive relation:

$$b_k = \frac{a_{k-1}}{k^3(k-1)} + \frac{b_{k-1}}{k^2} = \frac{1}{k^2 k!(k-1)!} + \frac{b_{k-1}}{k^2} = \frac{1}{k^2}(b_{k-1} + \frac{1}{k!(k-1)!})$$

with $a_1 = 1$ and $b_1 = 1$. An explicit formula is obtained by repeatedly using the recursion. More precisely, we first apply the recursive formula to b_{k-1}:

$$b_{k-1} = \frac{1}{(k-1)^2}\left(b_{k-2} + \frac{1}{(k-1)!(k-2)!}\right)$$

and substitution yields

$$b_k = \frac{1}{k^2}\left(\frac{1}{(k-1)^2}\left(b_{k-2} + \frac{1}{(k-1)!(k-2)!}\right) + \frac{1}{k!(k-1)!}\right).$$

The procedure above may be repeated until we reach $b_1 = 1$. Induction shows that

$$b_k = \frac{1}{(k!)^2}\sum_{i=1}^{k}\frac{1}{i}. \qquad \square$$

For general $m > k \geq 2$ we get from the filtrations the following $[m/k]$ exact sequences:

$$\begin{array}{lllll} 0 \to \mathcal{S}_{m,k}^{[m/k]-1} \to & \mathcal{J}_k^m X & \to & \mathcal{K}_X^{[m/k]} \otimes \mathcal{J}_{k-1}^{m-k[m/k]} X & \to 0 \\ 0 \to \mathcal{S}_{m,k}^{[m/k]-2} \to & \mathcal{S}_{m,k}^{[m/k]-1} & \to & \mathcal{K}_X^{[m/k]-1} \otimes \mathcal{J}_{k-1}^{m-k([m/k]-1)} X & \to 0 \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots \\ 0 \to \mathcal{J}_{k-1}^m X \to & \mathcal{S}_{m,k}^1 & \to & \mathcal{K}_X \otimes \mathcal{J}_{k-1}^{m-k} X & \to 0. \end{array} \tag{2.5}$$

Observe that $m - k[m/k] = 0$ or 1 depending on whether m is divisible by k. By induction, we see that $h^1(\mathcal{K}_X^i \otimes \mathcal{J}_{k-1}^{m-ki} X) = 0$ implies that $h^1(\mathcal{S}_{m,k}^i) = 0$, for $0 \leq i \leq [m/k]$. Hence $h^1(\mathcal{J}_k^m X) = 0$ for any $k \geq 2$, and as a result we also have

$$h^0(\mathcal{J}_k^m X) = h^0(\mathcal{J}_{k-1}^m X) + \sum_{i=1}^{[m/k]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_{k-1}^{m-ki} X) \quad \text{if } m > k. \tag{2.6}$$

Corollary 2.18. *Let X be a curve of genus $g \geq 2$. Then, $h^1(\mathcal{J}_k^m X) = 0$ if $m \geq k$, and for all $k \geq 2$ we have*

$$h^0(\mathcal{J}_k^m X) = \left(\frac{2(g-1)}{(k!)^2} \sum_{i=1}^{k} \frac{1}{i} \right) m^k + O(m^{k-1}).$$

Proof. By Riemann–Roch for curves, we have

$$h^0(\mathcal{J}_k^m X) - h^1(\mathcal{J}_k^m X) = c_1(\mathcal{J}_k^m X) - (\text{rk } \mathcal{J}_k^m X)(g-1).$$

As observed, $h^1(\mathcal{J}_k^m X)$ vanishes. By Theorem 2.15, $(\text{rk } \mathcal{J}_k^m X)(g-1)$ is of lower order, so $h^0(\mathcal{J}_k^m X) = c_1(\mathcal{J}_k^m X)$ and the result follows from Theorem 2.17. □

By induction we get from (2.6)

$$\begin{aligned} h^0(\mathcal{J}_k^m X) & \\ &= h^0(\mathcal{J}_2^m X) + \sum_{i=1}^{[m/3]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_2^{m-3i} X) + \cdots + \sum_{i=1}^{[m/k]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_{k-1}^{m-ki} X) \\ &= \sum_{i=0}^{[m/2]} h^0(\mathcal{K}_X^{m-i}) + \sum_{i=1}^{[m/3]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_2^{m-3i} X) + \cdots + \sum_{i=1}^{[m/k]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_{k-1}^{m-ki} X). \end{aligned}$$

Tensoring (2.5) with $\mathcal{K}_X^i$ yields exact sequences

$$\begin{array}{lllll} 0 \to \mathcal{S}_{m,k}^{[m/k]-1} \otimes \mathcal{K}_X^i \to & \mathcal{J}_k^m X \otimes \mathcal{K}_X^i & \to & \mathcal{K}_X^{[m/k]+i} \otimes \mathcal{J}_{k-1}^{m-k[m/k]} X & \to 0, \\ 0 \to \mathcal{S}_{m,k}^{[m/k]-2} \otimes \mathcal{K}_X^i \to & \mathcal{S}_{m,k}^{[m/k]-1} \otimes \mathcal{K}_X^i & \to & \mathcal{K}_X^{[m/k]+i-1} \otimes \mathcal{J}_{k-1}^{m-k([m/k]-1)} X & \to 0, \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots \\ 0 \to \mathcal{J}_{k-1}^m X \otimes \mathcal{K}_X^i \to & \mathcal{S}_{m,k}^1 & \to & \mathcal{K}_X^{i+1} \otimes \mathcal{J}_{k-1}^{m-k} X & \to 0. \end{array}$$

These imply that

$$h^0(\mathcal{J}_k^m X \otimes \mathcal{K}_X^i) = \sum_{j=0}^{[m/k]} h^0(\mathcal{K}_X^{i+j} \otimes \mathcal{J}_{k-1}^{m-kj} X).$$

Thus, for $k = 3$ we get by Theorem 2.13:

$$\begin{aligned} h^0(\mathcal{J}_3^m X) &= \sum_{i=0}^{[m/2]} h^0(\mathcal{K}_X^{m-i}) + \sum_{i=1}^{[m/3]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_2^{m-3i} X) \\ &= \sum_{i=0}^{[m/2]} h^0(\mathcal{K}_X^{m-i}) + \sum_{i=1}^{[m/3]} \sum_{j=0}^{[(m-3i)/2]} h^0(\mathcal{K}_X^{m-2i-j}). \end{aligned}$$

With this it is possible to write down the explicit formulas. In the case of $\mathcal{J}_2^m X$ there are only two cases depending on the parity of m. For 3-jets there are the following cases: (1a) $m = 3q$, q even; (1b) $m = 3q$, q odd; (2a) $m = 3q+1$, q even; (2b) $m = 3q+1$, q odd; (3a) $m = 3q+1$, q even; and (3b) $m = 3q+2$, q odd. For simplicity we shall only do this for case (1a). First we observe that the rank of $\mathcal{J}_3^m X$ is given by the number

$$\text{rk } \mathcal{J}_3^m X = \left[\frac{m}{2}\right] + 1 + \sum_{i=1}^{[m/3]} \left(\left[\frac{m-3i}{2}\right] + 1 \right) = O(m^2).$$

If m is divisible by 3! then

$$\begin{aligned} \text{rk } \mathcal{J}_3^m X &= \frac{m}{2} + 1 + \frac{m}{3} + \sum_{l=1}^{m/6} \frac{m-3(2l-1)}{2} + \sum_{l=1}^{m/6} \frac{m-3(2l)}{2} \\ &= \tfrac{1}{12}(m+3)(m+4). \end{aligned} \tag{2.7}$$

For $k = 3$ we get, by Theorem 2.13,

$$\begin{aligned} &h^0(\mathcal{J}_3^m X) \\ &= \sum_{i=0}^{[m/2]} h^0(\mathcal{K}_X^{m-i}) + \sum_{i=1}^{[m/3]} h^0(\mathcal{K}_X^i \otimes \mathcal{J}_2^{m-3i} X) \\ &= \sum_{i=0}^{[m/2]} h^0(\mathcal{K}_X^{m-i}) + \sum_{i=1}^{[m/3]} \sum_{j=0}^{[(m-3i)/2]} h^0(\mathcal{K}_X^{m-2i-j}) \\ &= (g-1) \times \\ &\quad \left(\left(2m-1-\left[\frac{m}{2}\right]\right)\left(\left[\frac{m}{2}\right]+1\right) + \sum_{i=1}^{[m/3]} \left(2m-4i-1-\left[\frac{m-3i}{2}\right]\right)\left(\left[\frac{m-3i}{2}\right]+1\right) \right). \end{aligned}$$

If m is divisible by 3!, both m and $[m/3]$ are even. Then, denoting the second sum above by S,

$$S = \sum_{l=1}^{m/6} \Big(2m - 4(2l-1) - 1 - \frac{m-1-3(2l-1)}{2}\Big)\Big(\frac{m-1-3(2l-1)}{2} + 1\Big) + \sum_{l=1}^{m/6} \Big(2m - 8l - 1 - \frac{m-6l}{2}\Big)\Big(\frac{m-6l}{2} + 1\Big)$$

$$= \frac{1}{2}\sum_{l=1}^{m/6}(3m^2 + 10m + 6 + 60l^2 - (28m+36)l) = \frac{1}{2^2 3^3} m(11m^2 - 18m - 18).$$

Thus for m divisible by 3! we have

$$h^0(\mathcal{J}_3^m X) = \left(\frac{1}{2^2 3^3} m(11m^2 - 18m - 18) + \frac{1}{2^2}(m+2)(3m-2)\right)(g-1),$$

and, by Riemann–Roch and (2.7),

$$c_1(\mathcal{J}_3^m X) = (g-1) \times \Big(\frac{1}{2^2 3^3} m(11m^2 - 18m - 18) + \frac{1}{2^2}(m+2)(3m-2) + \frac{1}{2^2 3}(m+3)(m+4)\Big).$$

EXAMPLE 2.19. The filtration of $\mathcal{J}_3^6 X$ is given by $\mathcal{J}_3^6 X = \mathcal{S}^2 \supset \mathcal{S}^1 \supset \mathcal{S}^0 = \mathcal{J}_2^6 X$, and the associated exact sequences are $0 \to \mathcal{S}^1 \to \mathcal{J}_3^6 X \to \mathcal{K}_X^2 \to 0$ and

$$0 \to \mathcal{J}_2^6 X \to \mathcal{S}^1 \to \mathcal{K}_X \otimes \mathcal{J}_2^3 X \to 0.$$

Hence $h^0(\mathcal{J}_3^6 X) = 0$ and $h^0(\mathcal{J}_3^6 X) = h^0(\mathcal{J}_2^6 X) + h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X \otimes \mathcal{J}_2^3 X)$. From the exact sequence $0 \to \mathcal{K}_X^3 \to \mathcal{J}_2^3 X \to \mathcal{K}_X^2 \to 0$ we obtain the exact sequence

$$0 \to \mathcal{K}_X^4 \to \mathcal{K}_X \otimes \mathcal{J}_2^3 X \to \mathcal{K}_X^3 \to 0$$

from which we conclude that

$$\begin{aligned} h^0(\mathcal{J}_3^6 X) &= h^0(\mathcal{J}_2^6 X) + h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X^3) + h^0(\mathcal{K}_X^4) \\ &= h^0(\mathcal{K}_X^2) + 2(h^0(\mathcal{K}_X^3) + h^0(\mathcal{K}_X^4)) + h^0(\mathcal{K}_X^5) + h^0(\mathcal{K}_X^6) \\ &= (3 + 2(5+7) + 9 + 11)(g-1) = 47(g-1). \end{aligned}$$

Next we consider the problem of constructing an explicit basis for $H^0(\mathcal{J}_k^m X)$. First we recall the construction of a basis for $H^0(\mathcal{J}_1^1 X) = H^0(\mathcal{K}_X)$. The procedure of this construction works in any algebraically closed field and has been used toward resolving the *uniqueness problem for rational and meromorphic functions*. (The reader is referred to [An et al. 2003a; 2004; 2003b] for details.) Let z_0, z_1, z_2 be the homogeneous coordinates on $\mathbb{P}^2$. Then

$$d\Big(\frac{z_i}{z_j}\Big) = \frac{z_j dz_i - z_i dz_j}{z_j^2} = \frac{\begin{vmatrix} z_1 & z_2 \\ dz_1 & dz_2 \end{vmatrix}}{z_j^2} \tag{2.8}$$

is a well-defined rational 1-form on $\mathbb{P}^n$. Let $P(z_0, z_1, z_2)$ be a homogeneous polynomial of degree d and

$$X = \{[z_0, z_1, z_2] \in \mathbb{P}^2(\mathbb{C}) \mid P(z_0, z_1, z_2) = 0\}.$$

Then, by Euler's Theorem, for $[z_0, z_1, z_2] \in X$, we have

$$z_0 \frac{\partial P}{\partial z_0}(z_0, z_1, z_2) + z_1 \frac{\partial P}{\partial z_1}(z_0, z_1, z_2) + z_2 \frac{\partial P}{\partial z_2}(z_0, z_1, z_2) = 0.$$

The tangent space of X is defined by the equation $P(z_0, z_1, z_2) = 0$ and

$$dz_0 \frac{\partial P}{\partial z_0}(z_0, z_1, z_2) + dz_1 \frac{\partial P}{\partial z_1}(z_0, z_1, z_2) + dz_2 \frac{\partial P}{\partial z_2}(z_0, z_1, z_2) = 0.$$

These may be expressed as

$$\begin{aligned} z_0 \frac{\partial P}{\partial z_0}(z_0, z_1, z_2) + z_1 \frac{\partial P}{\partial z_1}(z_0, z_1, z_2) &= -z_2 \frac{\partial P}{\partial z_2}(z_0, z_1, z_2), \\ dz_0 \frac{\partial P}{\partial z_0}(z_0, z_1, z_2) + dz_1 \frac{\partial P}{\partial z_1}(z_0, z_1, z_2) &= -dz_2 \frac{\partial P}{\partial z_2}(z_0, z_1, z_2). \end{aligned}$$

Then by Cramer's rule, we have on X

$$\frac{\partial P}{\partial z_0} = \frac{W(z_1, z_2)}{W(z_0, z_1)} \frac{\partial P}{\partial z_2}, \ \frac{\partial P}{\partial z_1} = \frac{W(z_2, z_0)}{W(z_0, z_1)} \frac{\partial P}{\partial z_2}$$

provided that the *Wronskian* $W(z_0, z_1) = z_0 dz_1 - z_1 dz_0 \not\equiv 0$ on any component of X; that is, the defining homogeneous polynomial of X has no linear factor of the form $az_0 + bz_1$. Thus

$$\frac{W(z_1, z_2)}{\frac{\partial P}{\partial z_0}(z_0, z_1, z_2)} = \frac{W(z_2, z_0)}{\frac{\partial P}{\partial z_1}(z_0, z_1, z_2)} = \frac{W(z_0, z_1)}{\frac{\partial P}{\partial z_2}(z_0, z_1, z_2)} \tag{2.9}$$

is a globally well-defined rational 1-form on any component of $\pi^{-1}(X) \subset \mathbb{C}^3 \setminus \{0\}$, where ($\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2(\mathbb{C})$ is the Hopf fibration), *provided that the expressions make sense* (*that is, the denominators are not identically zero when restricted to a component of* $\psi^{-1}(X)$). For our purpose, we also require that the form given by (2.9) *is not identically trivial when restricted to a component of* $\pi^{-1}(X)$. This is equivalent to the condition that *the Wronskians in the formula above are not identically zero; in other words, the defining homogeneous polynomial of* X *has no linear factor of the form* $az_i + bz_j$ *where* $a, b \in \mathbb{C}$, $0 \le i, j \le 2$ *and* $i \ne j$. If P, $\partial P/\partial z_0$, $\partial P/\partial z_1$, $\partial P/\partial z_2$ never vanish all at once (that is, X is smooth) then, at each point, one of the expressions in (2.9) is regular at the point. Hence so are the other expressions. This means that

$$\eta = \frac{\begin{vmatrix} z_1 & z_2 \\ dz_1 & dz_2 \end{vmatrix}}{\partial P/\partial z_0} \tag{2.10}$$

is regular on $\pi^{-1}(X)$. (Note that the form η is not well-defined on X unless $n=3$; see (2.8)). The form

$$\omega = \frac{\begin{vmatrix} z_1 & z_2 \\ dz_1 & dz_2 \end{vmatrix}}{z_0^2} \frac{z_0^{n-1}}{\partial P/\partial z_0} = \frac{\begin{vmatrix} z_1 & z_2 \\ dz_1 & dz_2 \end{vmatrix}}{\partial P/\partial z_0} z_0^{n-3} = z_0^{n-3}\eta,$$

with $n = \deg P$, is a well-defined (again by (2.8)) rational 1-form on X. Moreover, as η is regular on X, the 1-form ω is also regular if $n \geq 3$. If $n=3$ then $\omega = \eta$ and if $n \geq 4$ then ω is regular and vanishes along the ample divisor $[z_0^{n-3}{=}0] \cap X$. Thus for any homogeneous polynomial $Q = Q(z_0, z_1, z_2)$ of degree $n-3$, the 1-form

$$\frac{Q}{z_0^{n-3}}\,\omega = Q\eta$$

is regular on C and vanishes along $[Q{=}0]$. Note that the dimension of the vector space of homogeneous polynomials of degree $n-3$ (a basis is given by all possible monomials) is

$$\tfrac{1}{2}(n-1)(n-2) = \text{ genus of } X.$$

We summarize these observations:

Proposition 2.20. *Let* $X = \{[z_0, z_1, z_2] \in \mathbb{P}^2(\mathbb{C}) \mid P(z_0, z_1, z_2) = 0\}$ *be a nonsingular curve of degree* $d \geq 3$. *If* $d=3$ *then the space of regular* 1*-forms on* X *is* $\{c\eta \mid c \in \mathbb{C}\}$, *where* η *is defined by* (2.2). *If* $d \geq 4$ *take the set*

$$\{Q_i \mid Q_i \textit{ is a monomial of degree } d-3 \textit{ for } 1 \leq i \leq \tfrac{1}{2}(d-1)(d-2)\}$$

as an ordered basis of homogeneous polynomials of degree $d-3$. *Then*

$$\{\omega_i = Q_i\eta \mid 1 \leq i \leq \tfrac{1}{2}(d-1)(d-2)\}$$

is a basis of the space of regular 1*-forms on* X.

Using the preceding we may write down explicitly a basis for $H^0(\mathcal{J}_k^m X)$. We demonstrate via examples. For $d=4$, $h^0(\mathcal{J}_2^2 X) = h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X) = 6+3=9$ and, since the genus is 3, there are 3 linearly independent 1-forms $\omega_1, \omega_2, \omega_3$ which, as shown above, may be taken as

$$\omega_1 = \frac{z_0(z_0\,dz_1 - z_1\,dz_0)}{\partial P/\partial z_2}, \quad \omega_2 = \frac{z_1(z_0\,dz_1 - z_1\,dz_0)}{\partial P/\partial z_2}, \quad \omega_3 = \frac{z_2(z_0\,dz_1 - z_1\,dz_0)}{\partial P/\partial z_3}.$$

A basis for $H^0(\mathcal{J}_2^2 X)$ is given by

$$\omega_1^{\otimes 2},\ \omega_2^{\otimes 2},\ \omega_3^{\otimes 2},\ \omega_1 \otimes \omega_2,\ \omega_1 \otimes \omega_3,\ \omega_2 \otimes \omega_3,\ \delta\omega_1,\ \delta\omega_2,\ \delta\omega_3,$$

where δ is the derivation defined in (1.9). The first six of these provide a basis of $H^0(\mathcal{K}_X^2)$ and the last three may be identified with a basis of $H^0(\mathcal{K}_X)$. For

$\mathcal{J}_2^3 X$ we have

$$\begin{aligned} h^0(\mathcal{J}_2^3 X) &= h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X^3) \\ &= h^0\big(\mathcal{O}_X(2(d-3))\big) + h^0\big(\mathcal{O}_X(3(d-3))\big) \\ &= C_2^{2d-4} - C_2^{d-4} + C_2^{3d-7} - C_2^{2d-7}. \end{aligned}$$

In particular, for $d = 4$, $h^0(\mathcal{J}_2^3 X) = h^0(\mathcal{K}_X^2) + h^0(\mathcal{K}_X^3) = 6 + 10 = 16$. A basis for $H^0(\mathcal{J}_2^3 X)$ is given by the six elements (identified with a basis of $H^0(\mathcal{K}_X^2)$)

$$\delta\omega_1^{\otimes 2},\ \delta\omega_2^{\otimes 2},\ \delta\omega_3^{\otimes 2},\ \delta(\omega_1 \otimes \omega_2),\ \delta(\omega_1 \otimes \omega_3),\ \delta(\omega_2 \otimes \omega_3)$$

and the 10 elements (a basis of $H^0(\mathcal{K}_X^3)$):

$$\begin{aligned} &\omega_1^{\otimes 3},\ \omega_2^{\otimes 3},\ \omega_3^{\otimes 3},\ \omega_1 \otimes \omega_2 \otimes \omega_3, \\ &\omega_1^{\otimes 2} \otimes \omega_2,\ \omega_1^{\otimes 2} \otimes \omega_3,\ \omega_2^{\otimes 2} \otimes \omega_1,\ \omega_2^{\otimes 2} \otimes \omega_3,\ \omega_3^{\otimes 2} \otimes \omega_1,\ \omega_3^{\otimes 2} \otimes \omega_2. \end{aligned}$$

3. Computation of Chern Classes
The Case of Surfaces

SUMMARY. *We exhibit here the explicit formulas due to* [Stoll and Wong 2002] (*see also* [Green and Griffiths 1980]) *for the Chern numbers of the projectivized parametrized jet bundles of a compact complex surface. The most important is the index formula given in Theorem* 3.9:

$$\iota(\mathcal{J}_k^m X) = (\alpha_k c_1^2 - \beta_k c_2) m^{2k+1} + O(m^{2k})$$

where $c_i = c_i(X), \alpha_k = \beta_k + \gamma_k$ *and*

$$\beta_k = \frac{2}{(k!)^2(2k+1)!} \sum_{i=1}^{k} \frac{1}{i^2}, \quad \gamma_k = \frac{2}{(k!)^2(2k+1)!} \sum_{i=1}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}.$$

This implies that $\alpha_k/\beta_k \to \infty$ *hence* $\alpha_k/\beta_k > c_2/c_1^2$ *for* k *sufficiently large provided that* $c_1^2 > 0$. *For example,* $c_2/c_1^2 = 11$ *for a smooth hypersurface of degree* $d = 5$ *and the explicit formula shows that* $\alpha_k/\beta_k > 11$ *for all* $k \geq 199$. (*See the table at the end of this section*). *The explicit formulas for* α_k *and* β_k *are crucial in the proof of the Kobayashi conjecture in Section* 7.

We now treat the case of a complex surface (complex dimension 2). The computations here are more complicated than those of Section 2 as we must deal with the second Chern number. The computation of the first Chern class is relatively easy since the *Whitney formula* is linear in this case; that is, if $0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}'' \to 0$ is exact, then $c_1(\mathcal{S}) = c_1(\mathcal{S}') + c_1(\mathcal{S}'')$. The Whitney formula for the second Chern classes on the other hand is nonlinear: $c_2(\mathcal{S}) = c_2(\mathcal{S}') + c_2(\mathcal{S}'') + c_1(\mathcal{S}')c_1(\mathcal{S}'')$. The (minor) nonlinearity may seem harmless at first but for filtrations the nonlinearity carries over at each step and the complexity increases rapidly. Thus the correct way to deal with the problem is not to calculate the second Chern class directly but to calculate the index $\iota(\mathcal{J}_k^m X) = c_1^2(\mathcal{J}_k^m X) - 2c_2(\mathcal{J}_k^m X)$ which does behave linearly, that is, $\iota(\mathcal{S}) = \iota(\mathcal{S}') + \iota(\mathcal{S}'')$. We then recover the second Chern class from the formula $c_2(\mathcal{J}_k^m X) = (c_1^2(\mathcal{J}_k^m X) - \iota(\mathcal{J}_k^m X))/2$. In order to compute the jet differentials we must first calculate the Chern classes and indices of the sheaves $\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$ where $I = (i_1, \ldots, i_k)$. For details of the computations see [Stoll and Wong 2002].

By results from [Tsuji 1987; 1988; Maruyama 1981], the exterior, symmetric and tensor products of the cotangent sheaf of a manifold of general type are semistable in the sense of Mumford–Takemoto. For a coherent sheaf $\mathcal{S}$ on a variety of dimension n the *index of stability* relative to the canonical class is defined to be

$$\mu(\mathcal{S}) = \frac{c_1^{n-1}(\mathcal{S})\, c_1(T^*X)}{(\operatorname{rk} \mathcal{S}) c_1^n(T^*X)}.$$

A sheaf $\mathcal{S}$ is said to be *semistable* in the sense of Mumford–Takemoto (relative to the canonical class) if $\mu(\mathcal{S}') \leq \mu(\mathcal{S})$ for all coherent subsheaves $\mathcal{S}'$ of $\mathcal{S}$. For a nonsemistable sheaf a subsheaf $\mathcal{S}'$ satisfying $\mu(\mathcal{S}') > \mu(\mathcal{S})$ is said to be a *destabilizing subsheaf*. In view of Tsuji's result it would seem reasonable to expect that the sheaves of jet differentials are also semistable. However using the explicit formulas for the Chern classes computed below we shall see that this is not the case. Tsuji's result is used in [Lu and Yau 1990] (see also [Lu 1991]) to show that a projective surface X satisfying the conditions that $\mathcal{K}_X$ is nef and $c_1^2(T^*X) - 2c_2(T^*X) > 0$ contains no rational nor elliptic curves. The instability of the jet differentials implies that the analogous result of Lu–Yau requires a different argument.

We list below some basic but very useful formulas (see [Wong 1999; Stoll and Wong 2002]):

Lemma 3.1. *Let X be a nonsingular complex surface and E be a vector bundle of rank 2 over X. Then* $\operatorname{rk}(\bigodot^m E) = m+1$ *and*

$$c_1(\textstyle\bigodot^m E) = \frac{1}{2}m(m+1)c_1(E),$$
$$c_2(\textstyle\bigodot^m E) = \frac{1}{24}m(m^2-1)(3m+2)c_1^2(E) + \frac{1}{6}m(m+1)(m+2)c_2(E).$$

Consequently the index is given by the formula:

$$\iota(\textstyle\bigodot^m E) = \frac{1}{6}m(m+1)(2m+1)c_1^2(E) - \frac{1}{3}m(m+1)(m+2)c_2(E).$$

Moreover, *if* $c_1^2(E) \neq 0$ *then*

$$\delta_\infty(E) \overset{\text{def}}{=} \lim_{m\to\infty} \frac{c_2(\bigodot^m E)}{c_1^2(\bigodot^m E)} = \frac{1}{2}.$$

Note that $\delta_\infty(E)$ is independent of $c_2(E)/c_1^2(E)$. The next formula gives the *Chern numbers for tensor products of different bundles*.

Lemma 3.2. *Let E_i, $i = 1, \ldots, k$, be holomorphic vector bundles*, *of respective rank r_i*, *over a nonsingular complex surface X. Let $R = \prod_{l=1}^k r_l$. Then*:

(i) $c_1\left(\bigotimes_{i=1}^{k} E_i\right) = \sum_{i=1}^{k}(r_1 \dots r_{i-1}r_{i+1}\dots r_k)c_1(E_i) = Rr_i\sum_{i=1}^{k}\frac{c_1(E_i)}{r_i}.$

(ii) $c_2\left(\bigotimes_{i=1}^{k} E_i\right) = R\left(\sum_{i=1}^{k}\frac{c_2(E_i)}{r_i} + (R-1)\sum_{1\le i<j\le k}\frac{c_1(E_i)c_1(E_j)}{r_ir_j}\right) + \sum_{i=1}^{k}\binom{\prod_{l=1,l\neq i}^{k} r_j}{2}c_1^2(E_i).$

In particular:

(iii) $c_2(E_1\otimes E_2) = \binom{r_2}{2}c_1^2(E_1) + (r_1r_2-1)c_1(E_1)c_1(E_2) + \binom{r_1}{2}c_1^2(E_2) + r_2c_2(E_1) + r_1c_2(E_2);$

(iv) $c_2\left(\bigotimes_{i=1}^{3} E_i\right) = r_1r_2r_3\left(\sum_{i=1}^{3}\frac{c_2(E_i)}{r_i} + (r_1r_2r_3-1)\sum_{1\le i<j\le 3}\frac{c_1(E_i)c_1(E_j)}{r_ir_j}\right) + \sum_{i=1}^{3}\binom{r_1r_2r_3/r_i}{2}c_1^2(E_i);$

and the index $\iota(E_1\otimes E_2) = c_1^2(E_1\otimes E_2) - 2c_2(E_1\otimes E_2)$ *is given by*

(v) $\iota(E_1\otimes E_2) = r_2c_1^2(E_1) + r_1c_1^2(E_2) + 2c_1(E_1)c_1(E_2) - 2r_2c_2(E_1) - 2r_1c_2(E_2).$

With the preceding formulas the computation of the *Chern numbers for* $\mathcal{J}_k^m X$ can now be carried out by using the filtration given in Theorem 2.3, reducing the calculation to the Chern numbers of bundles of the form

$$\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes\cdots\otimes \bigodot^{i_k} T^*X,$$

where the indices $I = (i_1,\dots,i_k)$ satisfy the condition $i_1+2i_2+\cdots+ki_k = m$. More precisely, take

$$\mathcal{I}_{km} = \big\{I = (i_1,\dots,i_k) \mid i_j \in \mathbb{N}\cup\{0\},\ i_1+2i_2+\cdots+ki_k = m\big\}$$

together with a fixed ordering of $\mathcal{I}_{km}$ (say, the lexicographical ordering). Then a brute force computation, applying Lemma 3.2 and Lemma 3.3 repeatedly yields the following formulas:

Theorem 3.3. *Let X be a nonsingular complex surface and let* $\mathcal{S}_{m-2i,i} = \bigodot^{m-2i} T^*X \otimes \bigodot^{i} T^*X$. *Denote by* $c_1 = c_1(T^*X), c_2 = c_2(T^*X)$. *Then*

$$\begin{aligned}
\mathrm{rk}\,(\mathcal{S}_{m-2i,i}) &= (m-2i+1)(i+1) = (m+1)+(m-1)i-2i^2,\\
c_1(\mathcal{S}_{m-2i,i}) &= \tfrac{1}{2}(m-i)(m-2i+1)(i+1)c_1\\
&= \tfrac{1}{2}m(m+1)+(m^2-2m-1)i-(3m-1)i^2+2i^3c_1,\\
c_2(\mathcal{S}_{m-2i,i}) &= \tfrac{1}{24}\{m(m^2-1)(3m+2)+2(3m^4-5m^3-3m^2+4m+1)i\\
&\quad+(3m^4-30m^3+12m^2+6m-7)i^2-2(9m^3-27m^2+5m+1)i^3\\
&\quad+(39m^2-42m+7)i^4-4(3m-1)i^5+4i^6\}c_1^2+\tfrac{1}{6}b_{m-2i,i}c_2,
\end{aligned}$$

where $c_i = c_i(T^*X)$. *The index is given by*

$$\iota(\mathcal{S}_{m-2i,i}) = \tfrac{1}{6}(a_{m-2i,i}c_1^2 - 2b_{m-2i,i}c_2),$$

where $a_{m-2i,i}$ *and* $b_{m-2i,i}$ *are polynomials given by*

$$\begin{aligned} a_{m-2i,i} &= m(m{+}1)(2m{+}1) + (2m^3 - 6m^2 - 7m - 1)i - (9m^2 - 6m - 5)i^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (14m-2)i^3 - 8i^4, \\ b_{m-2i,i} &= m(m{+}1)(m{+}2) + (m^3 - 3m^2 - 8m - 2)i - (6m^2 - 3m - 7)i^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (13m-1)i^3 - 10i^4. \end{aligned}$$

The rank rk $\mathcal{J}_2^m X$ *of the sheaf of* $\mathcal{J}_2^m X$ *is given by*

$$\begin{aligned} \frac{1}{24}(m+1)(m+3)(m+5) &= \frac{1}{2^3 3}(m^3 + 9m^2 + 23m + 15), &&\textit{if } m \textit{ is odd,} \\ \frac{1}{24}(m+2)(m+3)(m+4) &= \frac{1}{2^3 3}(m^3 + 9m^2 + 26m + 24), &&\textit{if } m \textit{ is even;} \end{aligned}$$

and the first Chern class of same sheaf, $c_1(\mathcal{J}_2^m X)$, *is*

$$\begin{aligned} \frac{(m{+}1)(m{+}3)(m{+}5)(3m{+}1)}{192}c_1 &= \frac{3m^4{+}28m^3{+}78m^2{+}68m{+}15}{2^6 3}c_1, &&m \textit{ odd,} \\ \frac{m(m{+}2)(m{+}4)(3m{+}10)}{192}c_1 &= \frac{3m^4{+}28m^3{+}84m^2{+}80m}{2^6 3}c_1, &&m \textit{ even.} \end{aligned}$$

The index of $\mathcal{J}_2^m X$ *is given by*

$$\iota(\mathcal{J}_2^m X) = c_1^2(\mathcal{J}_2^m X) - 2c_2(\mathcal{J}_2^m X) = a_m c_1^2 - b_m c_2,$$

where the coefficients a_m *and* b_m *are polynomials in* m *given by*

$$a_m = \begin{cases} \frac{1}{2^6 15}(7m^5 + 75m^4 + 270m^3 + 390m^2 + 203m + 15) & \textit{if } m \textit{ is odd,} \\ \frac{1}{2^6 15}(7m^5 + 75m^4 + 280m^3 + 420m^2 + 208m) & \textit{if } m \textit{ is even;} \end{cases}$$

$$b_m = \begin{cases} \frac{1}{2^6 15}(5m^5 + 75m^4 + 390m^3 + 810m^2 + 565m + 75) & \textit{if } m \textit{ is odd,} \\ \frac{1}{2^6 15}(5m^5 + 75m^4 + 400m^3 + 900m^2 + 720m) & \textit{if } m \textit{ is even.} \end{cases}$$

The formula for the index also yields the formula for $c_2(\mathcal{J}_2^m X)$:

$$c_2(\mathcal{J}_2^m X) = \frac{1}{2}\{c_1^2(\mathcal{J}_2^m X) - (a_m c_1^2 - b_m c_2)\} = \frac{1}{2}\{\lambda_m c_1^2 + b_m c_2\} \tag{3.1}$$

where the coefficients a_m and b_m are given by Theorem 3.3, and the coefficient λ_m is given by

$$\lambda_m = \begin{cases} \left(\frac{1}{192}(m+1)(m+3)(3m^2+16m+5)\right)^2 - a_m, & m \text{ odd,} \\ \left(\frac{1}{192}m(m+2)(m+4)(3m+10)\right)^2 - a_m, & m \text{ even.} \end{cases}$$

In particular:

COROLLARY 3.4. *Let X be a nonsingular complex surface and assume that $c_1^2(T^*X) > 0$. Then*

$$\delta(\mathcal{J}_2^m X) = \lim_{m\to\infty} \frac{c_2(\mathcal{J}_2^m X)}{c_1^2(\mathcal{J}_2^m X)} = \frac{1}{2}.$$

For simplicity, set $c_1 = c_1(T^*X)$, $c_2 = c_2(T^*X)$. For any sheaf $\mathcal{S}$, define

$$\iota(\mathcal{S}) = c_1^2(\mathcal{S}) - 2c_2(\mathcal{S}), \quad \mu(\mathcal{S}) = \frac{c_1(\mathcal{S})\, c_1}{(\operatorname{rk}\mathcal{S})c_1^2}, \quad \delta(\mathcal{S}) = \frac{c_2(\mathcal{S})}{c_1^2(\mathcal{S})}, \tag{3.2}$$

provided that the denominators are not zero. Denote for simplicity $\delta = \delta(TX) = c_2/c_1^2$. It is well known that $c_1^2 \le 3c_2$ and $c_2 \le 5c_1^2 + 36$ for a surface of general type with $c_1^2 > 0$ [Barth et al. 1984, p. 228]. Thus, for such surfaces, δ satisfies the estimate

$$\frac{1}{3} \le \delta \le 5 + \frac{36}{c_1^2} \le 41. \tag{3.3}$$

We give the precise numbers for a few special cases:

- $\mathcal{J}_2^2 X$. In this case $k = 2$, $m = 2$ and there are two weighted partitions (i_1, i_2) corresponding to the two solutions of $i_1 + 2i_2 = 2$ (Example 2.9), namely $I_1 = (2,0)$ and $I_2 = (0,1)$. The corresponding sheaves are $\mathcal{S}_{I_1} = \bigodot^2 T^*X$, $\mathcal{S}_{I_2} = T^*X$. The various invariants of these sheaves are as follows:

I	$\mathcal{S}$	rank	$c_1(\mathcal{S})$	$c_2(\mathcal{S})$	$\iota(\mathcal{S})$	$\mu(\mathcal{S})$	$\delta(\mathcal{S})$
$(2,0)$	$\bigodot^2 T^*X$	3	$3c_1$	$2c_1^2 + 4c_2$	$5c_1^2 - 8c_2$	1	$\frac{1}{9}(2+4\delta)$
$(0,1)$	T^*X	2	c_1	c_2	$c_1^2 - 2c_2$	$\frac{1}{2}$	δ
	$\mathcal{J}_2^2 X$	5	$4c_1$	$5c_1^2 + 5c_2$	$6c_1^2 - 10c_2$	$\frac{4}{5}$	$\frac{5}{16}(1+\delta)$

The Chern numbers are calculated using Lemma 3.1 and Lemma 3.2. Note that $\bigodot^2 T^*X$ is a subsheaf of $\mathcal{J}_2^2 X$ (by Example 2.5, $0 \to \bigodot^2 T^*X \to \mathcal{J}_2^2 X \to T^*X \to 0$ is an exact sequence) with $\mu(\bigodot^2 T^*X) > \mu(\mathcal{J}_2^2 X)$. A subsheaf with such a property is said to be a *destabilizing subsheaf.* On the other hand T^*X is a quotient sheaf of $\mathcal{J}_2^2 X$ with $\mu(T^*X) < \mu(\mathcal{J}_2^2 X)$. A quotient sheaf with such a property is said to be a *destabilizing quotient sheaf.*

- $\mathcal{J}_2^3 X$. In this case $k = 2$, $m = 3$ and there are two weighted partitions $I_1 = (3,0)$ and $I_2 = (1,1)$ corresponding to the two solutions of $i_1 + 2i_2 = 3$.

I	$\mathcal{S}$	rk	$c_1(\mathcal{S})$	$c_2(\mathcal{S})$	$\iota(\mathcal{S})$	$\mu(\mathcal{S})$	$\delta(\mathcal{S})$
$(3,0)$	$\bigodot^3 T^*X$	4	$6c_1$	$11c_1^2 + 10c_2$	$14c_1^2 - 20c_2$	$\frac{3}{2}$	$\frac{1}{36}(11+10\delta)$
$(1,1)$	$\bigotimes^2 T^*X$	4	$4c_1$	$5c_1^2 + 4c_2$	$6c_1^2 - 8c_2$	1	$\frac{1}{16}(5+\delta)$
	$\mathcal{J}_2^3 X$	8	$10c_1$	$40c_1^2 + 14c_2$	$20c_1^2 - 28c_2$	$\frac{5}{4}$	$\frac{1}{50}(20+7\delta)$

The sheaves $\bigodot^3 T^*X$ and $\bigotimes^2 T^*X$ are respectively a destabilizing subsheaf and a destabilizing quotient sheaf of $\mathcal{J}_2^3 X$. The sequence $0 \to \bigodot^3 T^*X \to \mathcal{J}_2^3 X \to \bigotimes^2 T^*X \to 0$ is exact, by Example 2.5.

- $\mathcal{J}_2^4 X$. In this case $k = 2$, $m = 4$ and there are 3 weighted partitions $I_1 = (4, 0)$, $I_2 = (2, 1)$ and $I_3 = (0, 2)$ corresponding to the 3 solutions of $i_1 + 2i_2 = 4$.

I	$\mathcal{S}$	rk	$c_1(\mathcal{S})$	$c_2(\mathcal{S})$	$\iota(\mathcal{S})$	$\mu(\mathcal{S})$	$\delta(\mathcal{S})$
$(4,0)$	$\bigodot^4 T^*X$	5	$10c_1$	$35c_1^2+20c_2$	$30c_1^2-40c_2$	2	$\frac{7}{20}+\frac{1}{5}\delta$
$(2,1)$	$\bigodot^2 T^*X\otimes T^*X$	6	$9c_1$	$31c_1^2+11c_2$	$19c_1^2-22c_2$	$\frac{3}{2}$	$\frac{31}{81}+\frac{11}{81}\delta$
$(0,2)$	$\bigodot^2 T^*X$	3	$3c_1$	$2c_1^2+4c_2$	$5c_1^2-8c_2$	1	$\frac{2}{9}+\frac{4}{9}\delta$
	$\mathcal{J}_2^4 X$	14	$22c_1$	$215c_1^2+35c_2$	$54c_1^2-70c_2$	$\frac{11}{7}$	$\frac{215}{484}+\frac{35}{484}\delta$

The sheaves $\bigodot^4 T^*X$ and $\bigodot^2 T^*X$ are respectively a destabilizing subsheaf and a destabilizing quotient sheaf of $\mathcal{J}_2^4 X$. Note that $\bigodot^2 T^*X\otimes T^*X$ is neither a subsheaf nor a quotient sheaf of $\mathcal{J}_2^4 X$. We have two exact sequences:

$$0\to\mathcal{F}_2^1\to\mathcal{J}_2^4X\to\bigodot^2 T^*X\to 0,$$
$$0\to\bigodot^4 T^*X\to\mathcal{F}_2^1\to\bigodot^2 T^*X\otimes T^*X\to 0.$$

- $\mathcal{J}_2^5 X$. In this case $k = 2$, $m = 5$ and there are 3 weighted partitions $I_1 = (5, 0)$, $I_2 = (3, 1)$ and $I_3 = (1, 2)$ corresponding to the 3 solutions of $i_1 + 2i_2 5$.

I	$\mathcal{S}$	rk	$c_1(\mathcal{S})$	$c_2(\mathcal{S})$	$\iota(\mathcal{S})$	$\mu(\mathcal{S})$	$\delta(\mathcal{S})$
$(5,0)$	$\bigodot^5 T^*X$	6	$15c_1$	$85c_1^2+35c_2$	$55c_1^2-70c_2$	$\frac{5}{2}$	$\frac{17}{45}+\frac{7}{45}\delta$
$(3,1)$	$\bigodot^3 T^*X\otimes T^*X$	8	$16c_1$	$106c_1^2+24c_2$	$44c_1^2-48c_2$	2	$\frac{53}{128}+\frac{3}{32}\delta$
$(1,2)$	$T^*X\otimes\bigodot^2 T^*X$	6	$9c_1$	$31c_1^2+11c_2$	$19c_1^2-22c_2$	$\frac{3}{2}$	$\frac{31}{81}+\frac{11}{81}\delta$
	$\mathcal{J}_2^5 X$	20	$40c_1$	$741c_1^2+70c_2$	$118c_1^2-140c_2$	2	$\frac{741}{1600}+\frac{7}{160}\delta$

The sheaves $\bigodot^5 T^*X$ and $\bigodot^2 T^*X\otimes T^*X$ are respectively a destabilizing subsheaf and a destabilizing quotient sheaf of $\mathcal{J}_2^5 X$. Note that $\bigodot^3 T^*X\otimes T^*X$ is neither a subsheaf nor a quotient sheaf of $\mathcal{J}_2^5 X$. We have two exact sequences:

$$0\to\mathcal{F}_2^1\to\mathcal{J}_2^5X\to T^*X\otimes\bigodot^2 T^*X\to 0,$$
$$0\to\bigodot^5 T^*X\to\mathcal{F}_2^1\to\bigodot^3 T^*X\otimes T^*X\to 0.$$

- $\mathcal{J}_2^6 X$. In this case $k = 2$, $m = 6$ and there are 4 weighted partitions $I_1 = (6, 0)$, $I_2 = (4, 1)$, $I_3 = (2, 1)$ and $I_4 = (0, 3)$ corresponding to the 3 solutions of $i_1 + 2i_2 = 6$.

I	$\mathcal{S}$	rk	$c_1(\mathcal{S})$	$c_2(\mathcal{S})$	$\iota(\mathcal{S})$	$\mu(\mathcal{S})$	$\delta(\mathcal{S})$
$(6,0)$	$\bigodot^6 T^*X$	7	$21c_1$	$175c_1^2+56c_2$	$91c_1^2-112c_2$	3	$\frac{25}{63}+\frac{8}{63}\delta$
$(4,1)$	$\bigodot^4 T^*X\otimes T^*X$	10	$25c_1$	$270c_1^2+45c_2$	$85c_1^2-90c_2$	$\frac{5}{2}$	$\frac{54}{125}+\frac{9}{125}\delta$
$(2,2)$	$\bigodot^2 T^*X\otimes\bigodot^2 T^*X$	9	$18c_1$	$138c_1^2+24c_2$	$48c_1^2-48c_2$	2	$\frac{23}{54}+\frac{2}{27}\delta$
$(0,3)$	$\bigodot^3 T^*X$	4	$6c_1$	$11c_1^2+10c_2$	$14c_1^2-20c_2$	$\frac{3}{2}$	$\frac{11}{36}+\frac{5}{18}\delta$
	$\mathcal{J}_2^6 X$	30	$70c_1$	$2331c_1^2+135c_2$	$238c_1^2-270c_2$	$\frac{7}{3}$	$\frac{333}{700}+\frac{27}{980}\delta$

We have three exact sequences:

$$0 \to \mathcal{F}_2^2 \to \mathcal{J}_2^6 X \to \bigodot^3 T^*X \to 0,$$
$$0 \to \mathcal{F}_2^1 \to \mathcal{F}_2^2 \to \bigodot^2 T^*X \otimes \bigodot^2 T^*X \to 0,$$
$$0 \to \bigodot^6 T^*X \to \mathcal{F}_2^1 \to \bigodot^4 T^*X \otimes \bigodot^2 T^*X \to 0.$$

The sheaves $\bigodot^6 T^*X$ and $\bigodot^3 T^*X$ are respectively a destabilizing subsheaf and a destabilizing quotient sheaf of $\mathcal{J}_2^6 X$.

REMARK 3.5. For each partition $I = (i_1, i_2)$ satisfying $i_1 + 2i_2 = m$ we associate the (nonweighted) sum $|I| = i_1 + i_2$. Let $I_{\max} = \max_I\{|I|\}$ and $I_{\min} = \max_I\{|I|\}$. Then the sheaf $\mathcal{S}_{I_{\max}}$ is a destabilizing subsheaf and the sheaf $\mathcal{S}_{I_{\min}}$ is a destabilizing quotient sheaf.

We now deal with the case of general k. We shall be content with asymptotic formulas as the general formulas become complicated since the general formula for sums of powers can only be given recursively. However the highest order term is quite simple; indeed, we have

$$\sum_{i=1}^{m} i^d = \frac{m^{d+1}}{d+1} + O(m^d). \tag{3.4}$$

Before dealing with the jet bundles $\mathcal{J}_k^m X$ we must first find the formulas for the sheaves $\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$. This is easier due to the symmetry of the sheaves and we know, a priori, that the formulas can be expressed in terms of the symmetric functions in the exponents $i_1, \ldots, i_k$. For general k we introduce some notation for the j-th symmetric functions on k indices:

$$s_{0;k} = 1, \quad s_{1;k} = \sum_{p=1}^{k} i_p, \quad s_{2;k} = \sum_{1 \le p < q \le k} i_p i_q, \quad \ldots, \quad s_{k;k} = \prod_{p=1}^{k} i_p. \tag{3.5}$$

We have

$$\mu_k = \prod_{p=1}^{k} (i_p + 1) = \sum_{p=0}^{k} s_{p;k}. \tag{3.6}$$

Let $I = (i_1, \ldots, i_k)$ and $I' = (i_1, \ldots, i_{k-1})$, so that

$$\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_{k-1}} T^*X \otimes \bigodot^{i_k} T^*X = \mathcal{S}_{I'} \otimes \bigodot^{i_k} T^*X.$$

By Lemma 3.1, Lemma 3.2 and induction we obtain the following result, where we abbreviate $c_i = c_i(T^*X)$:

LEMMA 3.6. *Let X be a nonsingular complex surface and $\mathcal{S}_I = \mathcal{S}_{i_1, i_2, \ldots, i_k} = \bigodot^{i_1} T^*X \otimes \bigodot^{i_2} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$ where $i_1, i_2, \ldots, i_k$ are nonnegative integers.*

Then $\operatorname{rk} \mathcal{S}_I = \mu_k$,

$$\begin{aligned} c_1(\mathcal{S}_I) &= \frac{1}{2}\sum_{j=1}^{k} i_j \prod_{j=1}^{k}(i_j+1)\,c_1 = \tfrac{1}{2}s_{1;k}\mu_k c_1(T^*X) = \tfrac{1}{2}s_{1;k}\sum_{p=0}^{k} s_{p;k}, \\ \iota(\mathcal{S}_I) &= \tfrac{1}{6}\mu_k\big((2s_{1;k}^2+s_{1;k}-s_{2;k})c_1^2-2(s_{1;k}^2+2s_{1;k}-2s_{2;k})c_2\big), \\ c_2(\mathcal{S}_I) &= \tfrac{1}{24}\mu_k s_{j;k}\big((3s_{1;k}^2\mu_k-4s_{1;k}^2-2s_{1;k}+2s_{2;k})c_1^2+4(s_{1;k}^2+2s_{1;k}-2s_{2;k})c_2\big), \end{aligned}$$

where $s_{j;k}, 1 \le j \le k$ *are the symmetric functions in* $i_1, \dots, i_k$ *as defined in* (3.5) *and* $\mu_k = \sum_{j=0}^{k} s_{j;k}$.

These formulas, together with the filtrations of Green–Griffiths, are now used to get the formulas for $\mathcal{J}_k^m X$. First we have the formula for the rank (the proof is similar to that of Theorem 2.15 though somewhat more complicated):

THEOREM 3.7. *For any positive integer* $k \ge 2$ *we have*

$$\operatorname{rk} \mathcal{J}_k^m X = \sum_{(i_1,\dots,i_k)\in\mathcal{I}_{k,m}} \prod_{j=1}^{k}(i_j+1) = A_k m^{2k-1} + O(m^{2k-2})$$

where the coefficient is given by

$$A_k = \frac{1}{\prod_{l=2}^{k} l^2(2l-2)(2l-1)} = \frac{1}{(k!)^2(2k-1)!}.$$

Next we derive the formulas for $c_1(\mathcal{J}_k^m X)$ from the formulas for $c_1(\mathcal{S}_I)$, for $I \in \mathcal{I}_{k,k}$. By Whitney's formula, we see that $c_1(\mathcal{J}_k^m X)$ is given by

$$c_1(\mathcal{J}_k^m X) = \sum_{i_k=0}^{[m/k]} \sum_{I'\in\mathcal{I}_{k-1,m-ki_k}} \Big(c_1(\mathcal{S}_{I'}) \operatorname{rk} \bigodot^{i_k} T^*X + c_1\big(\bigodot^{i_k} T^*X\big) \operatorname{rk} \mathcal{S}_{I'}\Big), \tag{3.7}$$

where $i_1+\dots+ki_k = m$ and $\mathcal{I}_{k-1,m-ki_k}$ consists of all indices $I' = (i_1, \dots, i_{k-1})$ satisfying $i_1 + 2i_2 + \dots + (k-1)i_{k-1} = m - ki_k$. We have already seen that

$$\begin{aligned} c_1(\mathcal{J}_1^m X) &= \big(\tfrac{1}{2}m^2 + O(m)\big)c_1, \\ c_1(\mathcal{J}_2^m X) &= \big(\tfrac{1}{2^6}m^4 + O(m^3)\big)c_1, \end{aligned}$$

where $c_1 = c_1(T^*X)$. For general k we have (using (3.7) and along the lines of the proof of Theorem 2.16):

THEOREM 3.8. *Let* X *be a nonsingular complex surface. Then, for any positive integer* $k \ge 2$,

$$c_1(\mathcal{J}_k^m X) = \big(B_k m^{2k} + O(m^{2k-1})\big)c_1,$$

where the coefficient B_k *is given by*

$$B_k = \frac{1}{(k!)^2(2k)!}\sum_{i=1}^{k}\frac{1}{i} = \frac{A_k}{2k}\sum_{i=1}^{k}\frac{1}{i}.$$

We now compute the *index of* $\mathcal{J}_k^m X$ for general k. As in the case of the first Chern number, the filtration theorem implies that

$$\iota(\mathcal{J}_k^m X) = \sum_{I \in \mathcal{I}_{k,m}} \iota(\mathcal{S}_I).$$

Since $\iota(\mathcal{S}_I) = (\mathrm{rk} \bigodot^{i_k} T^*X)\, \iota(\mathcal{S}_{I'}) + (\mathrm{rk}\, \mathcal{S}_{I'})\, \iota(\bigodot^{i_k} T^*X) + 2c_1(\mathcal{S}_{I'})\, c_1(\bigodot^{i_k} T^*X)$, where $I = (i_1, \ldots, i_k)$ and $I' = (i_1, \ldots, i_{k-1})$, we get

$$\begin{aligned}\iota(\mathcal{J}_k^m X) = \sum_{i_k=0}^{[m/k]} \Big((i_k+1) \textstyle\sum_{I'} \iota(\mathcal{S}_{I'}) + \iota(\bigodot^{i_k} T^*X) \sum_{I'} \mathrm{rk}\,(\mathcal{S}_{I'}) \\ + i_k(i_k+1) \textstyle\sum_{I'} c_1(\mathcal{S}_{I'}) \Big),\end{aligned}$$

where we abbreviate $\sum_{I' \in \mathcal{I}_{k-1, m-ki_k}}$ by $\sum_{I'}$. Using the formulas for $\iota(\mathcal{S}_{I'})$ and $\mathrm{rk}\,(\mathcal{S}_{I'})$ obtained previously (Lemma 3.6) and induction we get:

Theorem 3.9. *Let X be a nonsingular complex surface. For any positive integer $k \geq 2$,*

$$\iota(\mathcal{J}_k^m X) = (\alpha_k c_1^2 - \beta_k c_2) m^{2k+1} + O(m^{2k}),$$

where the coefficients α_k and β_k satisfy the respective recursive relations:

$$\begin{aligned}\alpha_k &= \frac{\alpha_{k-1}}{2k^3(2k+1)} + \frac{B_{k-1}}{k^4(4k^2-1)} + \frac{A_{k-1}}{2k^5(k-1)(4k^2-1)},\\ \beta_k &= \frac{\beta_{k-1}}{2k^3(2k+1)} + \frac{A_{k-1}}{2k^5(k-1)(4k^2-1)}\end{aligned}$$

with $\alpha_1 = \beta_1 = \frac{1}{3}$ and A_i, B_i are the numbers given in Theorems 3.7 *and* 3.8 *respectively. The coefficients are given explicitly as $\alpha_k = \beta_k + \gamma_k$, where $\gamma_1 = 0$ and for $k \geq 2$*

$$\beta_k = \frac{2}{(k!)^2(2k+1)!} \sum_{i=1}^{k} \frac{1}{i^2}, \qquad \gamma_k = \frac{2}{(k!)^2(2k+1)!} \sum_{i=1}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}.$$

Corollary 3.10. *With the assumptions and notations of Theorem* 3.9,

$$\lim_{k\to\infty} \frac{\alpha_k}{\beta_k} = \lim_{k\to\infty} \frac{\gamma_k}{\beta_k} = \infty.$$

Consequently if $c_1^2 > 0$ then $\iota(\mathcal{J}_k^m X) = c m^{2k+1} c_1^2 + O(m^{2k})$ for some positive constant c.

The asymptotic expansion for $c_2(\mathcal{J}_k^m X)$ now follows readily from Corollary 3.10 along with Theorems 3.8 and 3.9:

Theorem 3.11. *Let X be a nonsingular complex surface. For any positive integer k,*

$$c_2(\mathcal{J}_k^m X) = \tfrac{1}{2}(c_1^2(\mathcal{J}_k^m X) - \iota(\mathcal{J}_k^m X)) = \tfrac{1}{2} c_1^2(\mathcal{J}_k^m X) = \tfrac{1}{2} A_k^2 c_1^2 m^{4k} + O(m^{4k-1}).$$

We tabulate the ratios α_k/β_k on the next page (they can be checked readily using Mathematica or Maple):

k	α_k	β_k	α_k/β_k
2	$\frac{7}{2^6 15}$	$\frac{5}{2^6 15}$	1.40000
3	$\frac{17}{2^7 3^6 7}$	$\frac{7}{2^7 3^6 5}$	1.73469
4	$\frac{83}{2^{16} 3^8 7}$	$\frac{41}{2^{16} 3^8 7}$	2.02439
5	$\frac{1717}{2^{17} 3^8 5^6 11}$	$\frac{479}{2^{17} 3^8 5^6 7}$	2.28108
6	$\frac{1927}{2^{21} 3^{11} 5^6 11\, 13}$	$\frac{59}{2^{21} 3^{11} 5^6 11}$	2.51239
7	$\frac{726301}{2^{22} 3^{12} 5^7 7^6 11\, 13}$	$\frac{266681}{2^{22} 3^{12} 5^7 7^6 11\, 13}$	2.72348
8	$\frac{3144919}{2^{34} 3^{12} 5^7 7^6 11\, 13\, 17}$	$\frac{63397}{2^{34} 3^{12} 5^7 7^6 11\, 13}$	2.91804
9	$\frac{2754581}{2^{35} 3^{20} 5^7 7^6 13\, 17\, 19}$	$\frac{514639}{2^{35} 3^{20} 5^7 7^6 11\, 13\, 17}$	3.09879
10	$\frac{2923673}{2^{39} 3^{21} 5^{10} 7^7 13\, 17\, 19}$	$\frac{178939}{2^{39} 3^{21} 5^9 7^7\, 13\, 17\, 19}$	3.26779
11	$\frac{315566191}{2^{40} 3^{21} 5^{10} 7^7 11^6 17\, 19\, 23}$	$\frac{10410343}{2^{40} 3^{21} 5^9 7^7 11^6\, 13\, 17\, 19}$	3.42666
12	$\frac{330851461}{2^{47} 3^{24} 5^{12} 7^7 11^6 17\, 19\, 23}$	$\frac{18500393}{2^{47} 3^{24} 5^{11} 7^7 11^6\, 17\, 19\, 23}$	3.57670
197			10.9808
198			10.9987
199			11.0165
200			11.0345

4. Finsler Geometry of Projectivized Vector Bundles

Summary. *Our use of projectivized jet bundles is initiated by the recognition that, for projectivized vector bundles, the algebraic geometric concept of ampleness is equivalent to the existence of a Finsler (not hermitian in general) metric with negative mixed holomorphic bisectional curvature. It is known, at least in the case of the tangent bundle that, even for Finsler metrics, negative holomorphic bisectional curvature implies hyperbolicity. We provide in this section some of the basic notions from Finsler geometry. For more details see* [Cao and Wong 2003; Chandler and Wong 2004] *and the references there.*

Many questions concerning a complex vector bundle E of rank greater than 1 may be reduced to problems about the tautological *line* bundle (or its dual) over the projectivization $\mathbb{P}(E)$. For example the algebraic geometric concept of ampleness (and the numerical effectiveness) of a holomorphic vector bundle

E may be interpreted in terms of Finsler geometry (see [Cao and Wong 2003], and also [Aikou 1995; 1998]; for general theory on Finsler geometry we refer to [Bao and Chern 1991; Bao et al. 1996; Abate and Patrizio 1994]). For the relationship with the *Monge–Ampère equation* see [Wong 1982]. We also provide some implications of this reformulation. For applications of the formulation using projectivized bundles to complex analysis see [Dethloff et al 1995a, b]. The dual of a vector bundle E will be denoted by E^*. For any positive integer k, denote by $\bigodot^k E$ the k-fold symmetric product. The dual vector bundle E^* is said to be *ample* if and only if the line bundle $\mathcal{L}_{\mathbb{P}(E)}$ is ample.

By a *Finsler metric* along the fibers of E we mean a function $h : E \to \mathbb{R}_{\geq 0}$ with the following properties:

(FM1) h is of class $\mathcal{C}^0$ on E and is of class $\mathcal{C}^\infty$ on $E \setminus \{\text{zero section}\}$.
(FM2) $h(z, \lambda v) = |\lambda| h(z, v)$ for all $\lambda \in \mathbb{C}$.
(FM3) $h(z, v) > 0$ on $E \setminus \{\text{zero section}\}$.
(FM4) For z and v fixed, the function $\eta_{z,v}(\lambda) = h^2(z, \lambda v)$ is smooth even at $\lambda = 0$.
(FM5) $h|_{E_z}$ is a *strictly pseudoconvex* function on $E_z \setminus \{0\}$ for all $z \in M$.

Denote by $\pi : TE \to E$ the projection and $\mathcal{V} = \ker \pi \subset TE$ the vertical subbundle. A Finsler metric F defines naturally a hermitian inner product on the vertical bundle $\mathcal{V} \subset TE$ by

$$\langle V, W \rangle_{\mathcal{V}} = \sum_{i,j=1}^{r} g_{i\bar{j}}(z, v) V^i \overline{W}^j, \quad g_{i\bar{j}}(z, v) = \frac{\partial^2 F^2(z, v)}{\partial v^i \partial \bar{v}^j} \tag{4.1}$$

for horizontal vector fields $V = \sum_i V^i \partial/\partial v^i, W = \sum_i W^i \partial/\partial v^i \in \mathcal{V}$ on E where $v_1, \dots, v_r$ are the fiber coordinates. (The difference between a Finsler metric and a hermitian metric is that, for a hermitian metric, the components $(g_{i\bar{j}})$ of the hermitian inner product on the vertical bundle are independent of the fiber coordinates). The hermitian inner product defines uniquely a hermitian connection (known as the *Chern connection*) $\theta = (\theta_i^k)$ and the associate hermitian curvature $\Theta = (\Theta_i^k)$. If $(g_{i\bar{j}})$ comes from a hermitian metric then the curvature forms depend only on the base coordinates; however if it comes from a general Finsler metric then the curvature forms will have horizontal, vertical and mixed components:

$$\Theta_i^k = \sum_{\alpha,\beta=1}^{n} K_{i\alpha\bar{\beta}}^k \, dz^\alpha \wedge d\bar{z}^\beta + \sum_{j,l=1}^{r} \kappa_{ij\bar{l}}^k \, dv^j \wedge d\bar{v}^l + \sum_{\alpha=1}^{n} \sum_{l=1}^{r} \mu_{i\alpha\bar{l}}^k \, dz^\alpha \wedge d\bar{v}^l + \sum_{j=1}^{r} \sum_{\beta=1}^{n} \nu_{ij\bar{\beta}}^k \, dv^j \wedge d\bar{z}^\beta.$$

Denote by $P = \sum_{i=1}^r v^i \partial/\partial v^i$ the position vector field on E. The *mixed holomorphic bisectional curvature* of the Finsler metric is defined, for any nonzero vector field $X \in \Gamma(M, TM)$, to be

$$\langle \Theta(X,X)P,\, P\rangle_{\mathcal{V}} = \sum_{i,j,k=1}^{r} \sum_{\alpha,\beta=1}^{n} g_{k\bar{j}} K^{k}_{i\alpha\bar{\beta}} v^i \bar{v}^j X^\alpha \overline{X}^\beta, \tag{4.2}$$

where the inner product is defined by (4.1). The following result can be found in [Cao and Wong 2003].

THEOREM 4.1. *Let E be a rank $r \geq 2$ holomorphic vector bundle over a compact complex manifold X. The following statements are equivalent*:

(1) *E^* is ample* (*resp. nef*).
(2) *$\bigodot^k E^*$ is ample* (*resp. nef*) *for some positive integer k.*
(3) *The dual $\mathcal{L}_{\mathbb{P}(\bigodot^k E)}$ of the tautological line bundle over the projectivized bundle $\mathbb{P}(\bigodot^k E)$ is ample* (*resp. nef*) *for some positive integer k.*
(4) *There exists a Finsler metric along the fibers of E with negative* (*resp. nonpositive*) *mixed holomorphic bisectional curvature.*
(5) *For some positive integer k there exists a Finsler metric along the fibers of $\bigodot^k E$ with negative* (*resp. nonpositive*) *mixed holomorphic bisectional curvature.*

From the algebraic geometric point of view the key relationship between a vector bundle and its projectivization is the *Fundamental Theorem of Grothendieck* [Grothendieck 1958]:

THEOREM 4.2. *Let $p : E \to X$ be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold X of dimension n. Then for any analytic sheaf $\mathcal{S}$ on X and any $m \geq 1$,*

$$p^i_* \mathcal{L}^m_{\mathbb{P}(E)} \cong \begin{cases} \bigodot^m E^*, & \text{if } i = 0,\\ 0, & \text{if } i > 0,\end{cases}$$

where $p^i_* \mathcal{L}^m_{\mathbb{P}(E)}$ is the i-th direct image of $\mathcal{L}^m_{\mathbb{P}(E)}$. Consequently,

$$H^i(X, \textstyle\bigodot^m E^* \otimes \mathcal{S}) \cong H^i(\mathbb{P}(E), \mathcal{L}^m_{\mathbb{P}(E)} \otimes p^*\mathcal{S}) \quad \text{for all } i \geq 0.$$

The theorem implies that the cohomology groups vanish beyond the dimension n of X although the dimension of $\mathbb{P}(E)$ is $n+r-1 > n$; moreover, $\chi(\bigodot^m E^* \otimes \mathcal{S}) = \chi(\mathcal{L}^m_{\mathbb{P}(E)} \otimes p^*\mathcal{S})$. For a vector bundle F over a smooth surface X, the *Chern character* and the *Todd class* are defined by

$$\begin{aligned} \mathrm{ch}(F) &= \mathrm{rk}(F) + c_1(F) + \tfrac{1}{2}(c_1^2(F) - 2c_2(F)),\\ \mathrm{td}(F) &= 1 + \tfrac{1}{2}c_1(F) + \tfrac{1}{12}(c_1^2(F) + c_2(F)). \end{aligned} \tag{4.3}$$

The *Riemann–Roch formula* is

$$\chi(F) = \mathrm{ch}(F)\cdot \mathrm{td}(TX)[X] = \big(\tfrac{1}{2}(\iota(F) - c_1(F)\,c_1) + \tfrac{1}{12}\mathrm{rk}(F)(c_1^2 + c_2)\big)[X], \tag{4.4}$$

where $c_i = c_i(T^*X) = -c_i(TX)$. The notation $\omega[X]$ indicates the evaluation of a form of top degree on the *fundamental cycle* $[X]$, that is,

$$\omega[X] = \int_X \omega.$$

Assume rank $F = 2$ over a nonsingular complex surface X. Then, by Lemma 3.1,

$$\begin{aligned}
\iota(\textstyle\bigodot^m F) &= \tfrac{1}{6}m(m+1)(2m+1)c_1^2(F) - \tfrac{1}{3}m(m+1)(m+2)c_2(F),\\
\operatorname{ch}(\textstyle\bigodot^m F) &= m+1+\tfrac{1}{2}m(m+1)c_1(F) + \tfrac{1}{12}m(m+1)\big((2m+1)c_1^2(F)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 2(m+2)c_2(F)\big),\\
\chi(\textstyle\bigodot^m F) &= \tfrac{1}{12}m(m+1)\big((2m+1)c_1^2(F) - 2(m+2)c_2(F)\big)\\
&\qquad\qquad - \tfrac{1}{4}m(m+1)c_1(F)\,c_1 + \tfrac{1}{12}(m+1)(c_1^2+c_2).
\end{aligned}$$

For example, taking $F = T^*X$,

$$\chi(\textstyle\bigodot^m T^*X) = \tfrac{1}{12}(m+1)\big((2m^2-2m+1)c_1^2 - (2m^2+4m-1)c_2\big);$$

in particular,

$$\chi(T^*X) = \tfrac{1}{6}(c_1^2 - 5c_2), \quad \chi(\textstyle\bigodot^2 T^*X) = \tfrac{1}{4}(5c_1^2 - 15c_2).$$

In any case we have:

THEOREM 4.3. *Let $p : E \to X$ be a holomorphic vector bundle of rank $r = 2$ over a complex surface X. Then $\dim \mathbb{P}(E) = 3$ and for any positive integer m,*

$$\chi(\textstyle\bigodot^m E^*) = \chi(\mathcal{L}^m_{\mathbb{P}(E)}) = \dfrac{m^3}{3!}(c_1^2(E^*) - c_2(E^*)) + O(m^2) = \dfrac{m^3}{3!}c_1^3(\mathcal{L}^m_{\mathbb{P}(E)}) + O(m^2).$$

Suppose that $h^2(\mathcal{L}^m_{\mathbb{P}(E)})$ $(= h^2(\bigodot^m E)) = O(m^2)$ and that $c_1^3(\mathcal{L}^m_{\mathbb{P}(E)}) > 0$ (equivalently, $c_1^2(E) - c_2(E) > 0$). The preceding theorem implies that E (or equivalently $\mathcal{L}_{\mathbb{P}(E)}$) is big, that is,

$$h^0(\mathcal{L}^m_{\mathbb{P}(E)}) = h^0(\textstyle\bigodot^m E) \geq Cm^3$$

for some constant $C > 0$. Recall the following fact (from [Cao and Wong 2003] or [Kobayashi and Ochiai 1970], for example):

THEOREM 4.4. *Let E be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold X. Then the canonical bundles of X and $\mathbb{P}(E)$ are related by the formula*

$$\mathcal{K}_{\mathbb{P}(E)} \cong [p_E]^*(\mathcal{K}_X \otimes \det E^*) \otimes \mathcal{L}^{-r}_{\mathbb{P}(E)}$$

where $\mathcal{L}^{-r}_{\mathbb{P}(E)}$ is the dual of the r-fold tensor product of $\mathcal{L}_{\mathbb{P}(E)}$. In particular, we have

$$\mathcal{K}_{\mathbb{P}(TX)} \cong [p_{TX}]^*\mathcal{K}_X^2 \otimes \mathcal{L}^{-n}_{\mathbb{P}(TX)} \text{ and } \mathcal{K}_{\mathbb{P}(T^*X)} \cong \mathcal{L}^{-n}_{\mathbb{P}(T^*X)}$$

where $n = \dim X$.

Corollary 4.5. *Let X be a complex manifold of dimension n.*

(i) *TX is ample (resp. nef) if and only if $\mathcal{K}^{-1}_{\mathbb{P}(T^*X)}$ is ample (resp. nef).*
(ii) *If $\mathcal{K}_X$ is nef then $\mathcal{K}_{\mathbb{P}(TX)} \otimes \mathcal{L}^n_{\mathbb{P}(TX)}$ is nef.*
(iii) *If T^*X is ample then $\mathcal{K}_{\mathbb{P}(TX)} \otimes \mathcal{L}^n_{\mathbb{P}(TX)}$ is nef and $\mathcal{K}_{\mathbb{P}(TX)} \otimes \mathcal{L}^{n+1}_{\mathbb{P}(TX)}$ is ample.*

We have the following *vanishing theorem* [Cao and Wong 2003] (for variants see [Chandler and Wong 2004]):

Corollary 4.6. *Let E be a nef holomorphic vector bundle of rank $r \geq 2$ over a compact complex manifold M of dimension n. Then*

$$H^i\big(X, \textstyle\bigodot^m E \otimes \det E \otimes \mathcal{K}_X\big) = 0,$$
$$H^i\big(X, \textstyle\bigodot^m (\bigotimes^k E) \otimes \det(\bigotimes^k E) \otimes \mathcal{K}_X\big) = 0,$$

for all $i, m, k \geq 1$. Consequently, if $E = TX$ then $H^i(X, \bigodot^m TX) = 0$ for all $i, m \geq 1$.

For a holomorphic line bundle $\mathcal{L}$ over a compact complex manifold Y with $h^0(\mathcal{L}^m) > 0, m$ a positive integer, define a meromorphic map

$$\Phi_m = [\sigma_0, \dots, \sigma_N] : Y \to \mathbb{P}^N$$

where $\sigma_0, \dots, \sigma_N$ is a basis of $H^0(\mathcal{L}^m)$. The *Kodaira–Iitaka dimension* of $\mathcal{L}$ is defined to be

$$\kappa(\mathcal{L}) = \begin{cases} -\infty, & \text{if } \ h^0(\mathcal{L}^m) = 0 \ \text{ for all } \ \text{m}, \\ \max\{\dim \Phi_m(X) \mid h^0(\mathcal{L}^m) > 0\}, & \text{otherwise.} \end{cases}$$

The line bundle $\mathcal{L}$ is said to be *big* if $k(\mathcal{L}) = \dim Y$. This is equivalent to saying that, for $m \gg 0$

$$h^0(\mathcal{L}^m) \geq C m^{\dim Y}$$

for some positive constant C; in other words, the dimension of the space of sections $h^0(\mathcal{L}^m)$ has maximum possible growth rate. See [Chandler and Wong 2004] for a discussion of the differential geometric meaning of big bundles. Riemann–Roch asserts that if $c_1^{\dim Y}(\mathcal{L}) > 0$ the Euler characteristic is big:

$$\chi(\mathcal{L}^m) = \frac{c_1^{\dim Y}(\mathcal{L})}{(\dim Y)!} m^{\dim Y} + O(m^{\dim Y - 1}).$$

This, in general, is not enough to conclude that $\mathcal{L}$ is big. However, Corollary 4.6 implies that if T^*X is nef then the cohomology groups $H^i(X, T^*X) = 0$ for all $i \geq 1$. Hence T^*X is big if the Euler characteristic is big. In fact, for surfaces the weaker condition that $\mathcal{K}_X$ is nef suffices:

Corollary 4.7. *Suppose that the canonical bundle $\mathcal{K}_X$ of a nonsingular surface is nef and that $c_1^{\dim \mathbb{P}(TX)}(\mathcal{L}_{\mathbb{P}(TX)}) > 0$. Then $\mathcal{L}_{\mathbb{P}(TX)}$ is big.*

A vector bundle E of rank > 1 is said to be big if the line bundle $\mathcal{L}_{\mathbb{P}(E)}$ is big. By Theorem 4.3, for a surface X the condition $c_1^{\dim \mathbb{P}(TX)}(\mathcal{L}_{\mathbb{P}(TX)}) > 0$ is equivalent to the condition that $c_1^2(TX) - c_2(TX) = c_1^2(T^*X) - c_2(T^*X)$ is positive. Thus we may restate Corollary 4.7 as follows:

COROLLARY 4.8. *Let X be a nonsingular compact surface such that $c_1^2(T^*X) - c_2(T^*X) > 0$ and $\mathcal{K}_X$ is nef. Then T^*X is big.*

The preceding corollary implies the following theorem which may be viewed as an analogue, for surfaces, of the classical theorem that a curve of positive genus is hyperbolic [Lu and Yau 1990; Lu 1991; Dethloff et al. 1995a; 1995b]:

THEOREM 4.9. *Let X be a nonsingular surface such that $c_1^2(T^*X) - 2c_2(T^*X) > 0$ and $\mathcal{K}_X$ is nef. Then X is hyperbolic.*

We refer the readers to [Dethloff et al. 1995a; 1995b] for further information and refinements of the preceding theorem. The condition $c_1^2(T^*X) - c_2(T^*X) > 0$ is not satisfied by hypersurfaces in $\mathbb{P}^3$ which is the main reason that jet differentials are introduced. The computations in the previous section will provide conditions (on the Chern numbers $c_1^2(T^*X)$ and $c_2(T^*X)$) under which the sheaves of jet differentials $\mathcal{J}_k^m X$ must be big.

5. Weighted Projective Spaces and Projectivized Jet Bundles

SUMMARY. *The fibers of the k-jet bundles $\mathbb{P}(J^kX)$ are special types of weighted projective spaces. We collect some of the known facts of these spaces in this section. The main point is that these spaces are, in general, not smooth but with very mild singularities and we show that the usual theory of fiber integration for smooth manifolds extends to $\mathbb{P}(J^kX)$. This will be used in later sections.*

We follow the approach of the previous section by reducing questions concerning k-jet differentials to questions about the line bundle over the projectivization $\mathbb{P}(J^kX)$. Since J^kX is only a $\mathbb{C}^*$-bundle rather than a vector bundle the fibers of the projectivized bundle $\mathbb{P}(J^kX)$ is not the usual projective space but a special type of weighted projective space. We give below a brief account concerning these spaces; see [Beltrametti and Robbiano 1986; Dolgachev 1982; Dimca 1992] for more detailed discussions and further references. The general theory of the projectivization of coherent sheaves can be found in [Banica and Stanasila 1976].

Consider $\mathbb{C}^{r+1}$ together with a vector $Q = (q_0, \dots, q_r)$ of positive integers. The space $\mathbb{C}^{r+1}$ is then denoted $(\mathbb{C}^{r+1}, Q)$ and we say that each coordinate z_i, $0 \le i \le r$, has *weight* (or *degree*) q_i. A $\mathbb{C}^*$-action is defined on $(\mathbb{C}^{r+1}, Q)$ by

$$\lambda.(z_0, \dots, z_r) = (\lambda^{q_0} z_0, \dots, \lambda^{q_r} z_r) \quad \text{for } \lambda \in \mathbb{C}^*.$$

The quotient space $\mathbb{P}(Q) = (\mathbb{C}^{r+1}, Q)/\mathbb{C}^*$ is the *weighted projective space* of type Q. The equivalence class of an element $(z_0, \dots, z_r)$ is denoted by $[z_0, \dots, z_r]_Q$.

For $Q = (1, \dots, 1) = \mathbf{1}$, $\mathbb{P}(Q) = \mathbb{P}_r$ is the usual complex projective space of dimension r and an element of $\mathbb{P}_r$ is denoted simply by $[z_0, \dots, z_r]$. The case $r = 1$ is special as it can be shown that $\mathbb{P}(q_0, q_1) \cong \mathbb{P}^1$ for any tuple (q_0, q_1). This is not so if $r \geq 2$. For a tuple Q define a map $\psi_Q : (\mathbb{C}^{r+1}, \mathbf{1}) \to (\mathbb{C}^{r+1}, Q)$ by

$$\psi_Q(z_0, \dots, z_r) = (z_0^{q_0}, \dots, z_r^{q_r}).$$

It is easily seen that ρ_Q is compatible with the respective $\mathbb{C}^*$-actions and hence descends to a well-defined morphism:

$$[\psi_Q] : \mathbb{P}_r \to \mathbb{P}(Q), \quad [\psi_Q]([z_0, \dots, z_r]) = [z_0^{q_0}, \dots, z_r^{q_r}]_Q. \tag{5.1}$$

The weighted projective space can also be described as follows. Denote by Θ_{q_i} the group consisting of all q_i-th roots of unity. The group $\Theta_Q = \bigoplus_{i=0}^r \Theta_{q_i}$ acts on $\mathbb{P}_r$ by coordinatewise multiplication:

$$(\theta_0, \dots, \theta_r).[z_0, \dots, z_r] = [\theta_0 z_0, \dots, \theta_r z_r], \quad \theta_i \in \Theta_{q_i},$$

and the quotient space is denoted by $\mathbb{P}_r/\Theta_Q$. The next result is easily verified [Dimca 1992]:

THEOREM 5.1. *The weighted projective space $\mathbb{P}(Q)$ is isomorphic to the quotient $\mathbb{P}_r/\Theta_Q$. In particular*, $\mathbb{P}(Q)$ *is irreducible and normal* (*the singularities are cyclic quotients and hence rational*).

Given a tuple Q we assign the degree (or weight) q_i to the variable z_i $(i = 1, \dots, q)$ and denote by $S_Q(m)$ the space of homogeneous polynomials of degree m. In other words, a polynomial P is in $S(Q)(m)$ if and only if $P(\lambda \cdot (z_0, \dots, z_r)) = \lambda^m P(z_0, \dots, z_r)$. We may express such a polynomial explicitly as

$$P = \sum_{(i_0, \dots, i_r) \in \mathcal{I}_{Q,m}} a_{i_0 \dots i_r} z_0^{i_0} \dots z_r^{i_r},$$

where the index set $\mathcal{I}_{Q,m}$ is defined by

$$\mathcal{I}_{Q,m} = \bigl\{ (i_0, \dots, i_r) \bigm| \textstyle\sum_{j=0}^r q_j i_j = m,\ i_l \in \mathbb{N} \cup \{0\} \bigr\}.$$

The sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)$, $m \in \mathbb{N}$, is by definition the sheaf over $\mathbb{P}(Q)$ whose global regular sections are precisely the elements of $S_Q(m)$, i.e., $H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(m)) = S_Q(m)$. For a negative integer $-m$ the sheaf $\mathcal{O}_{\mathbb{P}(Q)}(-m)$ is defined to be the dual of $\mathcal{O}_{\mathbb{P}(Q)}(m)$ and $\mathcal{O}_{\mathbb{P}(Q)}(0)$ is the structure sheaf $\mathcal{O}_{\mathbb{P}(Q)}$ of $\mathbb{P}(Q)$. Here are some basic properties of these sheaves (see [Beltrametti and Robbiano 1986]):

THEOREM 5.2. *Let $Q = (q_0, \dots, q_r)$ be an $r+1$-tuple of positive integers.*

(i) *For any for any $m \in \mathbb{Z}$, the line sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is a reflexive coherent sheaf.*
(ii) $\mathcal{O}_{\mathbb{P}(Q)}(m)$ *is locally free if m is divisible by each q_i* (*hence by the least common multiple*).
(iii) *Let m_Q be the least common multiple of $\{q_0, \dots, q_r\}$. Then $\mathcal{O}_{\mathbb{P}(Q)}(m_Q)$ is ample.*

(iv) *There exists an integer* n_0 *depending only on* Q *such that* $\mathcal{O}_{\mathbb{P}(Q)}(nm_Q)$ *is very ample for all* $n \geq n_0$.

(v) $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m_Q) \otimes \mathcal{O}_{\mathbb{P}(Q)}(\beta) \cong \mathcal{O}_{\mathbb{P}(Q)}(\alpha m_Q + \beta)$ *for any* $\alpha, \beta \in \mathbb{Z}$.

For $Q = \mathbf{1}$ the assertions of the preceding theorem reduce to well-known properties of the usual twisted structure sheaves of the projective space. For any subset $J \subset \{0, 1, \dots, r\}$ denote by m_J the least common multiple of $\{q_j, j \in J\}$ and define

$$m(Q) = -\sum_{i=0}^{q} q_i + \frac{1}{r} \sum_{i=2}^{r+1} \frac{\sum_{\#J=i} m_J}{\binom{r-1}{i-2}},$$

where $\#J$ is the number of elements in the set J. It is known that we may take $n_0 = m(Q)+1$ in assertion (iv) above. In general the line sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is not invertible if m is not an integral multiple of m_Q. It can be shown that for $Q = (1,1,2)$ the sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1)$ is not invertible and hence, neither is $\mathcal{O}_{\mathbb{P}(Q)}(1) \otimes \mathcal{O}_{\mathbb{P}(Q)}(1)$. On the other hand, by part (ii) of the preceding theorem we know that $\mathcal{O}_{\mathbb{P}(Q)}(2)$ is invertible, thus $\mathcal{O}_{\mathbb{P}(Q)}(1) \otimes \mathcal{O}_{\mathbb{P}(Q)}(1) \ncong \mathcal{O}_{\mathbb{P}(Q)}(2)$. The following theorem on the cohomologies of the sheaf $\mathcal{O}_{\mathbb{P}}(Q)(p)$ is similar to the case of standard projective space (see [Beltrametti and Robbiano 1986] or [Dolgachev 1982]):

THEOREM 5.3. *If* $Q = (q_0, \dots, q_r)$ *is an* $(r+1)$*-tuple of positive integers then for* $p \in \mathbb{Z}$,

$$H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}}(Q)(p)) = \begin{cases} \{0\}, & i \neq 0, r \\ S_Q(p), & i = 0, \\ S(Q)(-p-|Q|), & i = r, \end{cases}$$

where $|Q| = q_0 + \cdots + q_r$.

The cohomology group $H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(p))$ vanishes provided that $i \neq 0, r$. Let $Q = (q_0, \dots, q_r)$ be a $(r+1)$-tuple of positive integers and define, for $k = 1, \dots, r$,

$$l_{Q,k} = \operatorname{lcm}\left\{ \frac{q_{i_0} \cdots q_{i_k}}{\gcd\,(q_0, \dots, q_{i_k})} \;\middle|\; 0 \leq i_0 < \cdots < i_k \leq r \right\}.$$

For integral cohomology we have:

THEOREM 5.4. *Let* Q *be an* $(r+1)$*-tuple of positive integers. Then*

$$H^i(\mathbb{P}(Q); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } i \text{ is even}, \\ 0, & \text{if } i \text{ is odd}. \end{cases}$$

Further, take $[\psi_Q] : \mathbb{P}^r \to \mathbb{P}(Q)$ *as the quotient map defined by* (5.1). *Then the diagram*

$$\begin{array}{ccc} H^{2k}(\mathbb{P}(Q);\mathbb{Z}) & \xrightarrow{[\psi_Q]^*} & H^{2k}(\mathbb{P}^r;\mathbb{Z}) \\ \cong \big\downarrow & & \big\downarrow \cong \\ \mathbb{Z} & \xrightarrow{\quad l_{Q,k} \quad} & \mathbb{Z} \end{array}$$

commutes, where the lower map is given by multiplication by the number $l_{Q,k}$.

Note that the number $l_{Q,r}$ is precisely the number of preimages of a point in $\mathbb{P}(Q)$ under the quotient map $[\psi_Q]$ (see (5.1)). The proof of the preceding Theorem for $k=r$ is easy; the reader is referred to [Kawasaki 1973] for the general case.

Let $Q=(q_0,q_1,\dots,q_r)$, $r\ge 1$, be an $(r+1)$-tuple of positive integers. The tuple Q is said to be *reduced* if the greatest common divisor (gcd) of $(q_0,q_1,\dots,q_r)$ is 1. In general, if the gcd is d, the tuple

$$Q_{\mathrm{red}}=Q/d=(q_0/d,\dots,q_r/d)$$

is called the *reduction* of Q. Let

$$\begin{aligned} d_0 &= \gcd(q_1,\dots,q_r),\\ d_i &= \gcd(q_0,q_1,\dots,q_{i-1},q_{i+1},\dots,q_r), \quad 1\le i\le r-1,\\ d_r &= \gcd(q_0,\dots,q_{r-1}) \end{aligned}$$

and define

$$\begin{aligned} a_0 &= \mathrm{lcm}(d_1,\dots,d_r),\\ a_i &= \mathrm{lcm}(d_0,d_1,\dots,d_{i-1},d_{i+1},\dots,d_r), \quad 1\le i\le r-1,\\ a_r &= \mathrm{lcm}(d_0,\dots,d_{r-1}), \end{aligned}$$

where "lcm" is short for "least common multiple". Define the normalization of Q by

$$Q_{\mathrm{norm}}=(q_0/a_0,\dots,q_r/a_r).$$

A tuple Q is said to be *normalized* if $Q=Q_{\mathrm{norm}}$.

THEOREM 5.5. *Let Q be a normalized $(r+1)$-tuple of positive integers. Then the Picard group* $\mathrm{Pic}(\mathbb{P}(Q))$ *and the divisor class group* $\mathrm{Cl}(\mathbb{P}(Q))$ *are both isomorphic to* $\mathbb{Z}$, *and are generated, respectively, by*

$$\bigl[\mathcal{L}_{\mathbb{P}(Q)}^{m_Q}=\mathcal{O}_{\mathbb{P}(Q)}(m_Q)\bigr] \quad \textit{and} \quad \bigl[\mathcal{L}_{\mathbb{P}(Q)}=\mathcal{O}_{\mathbb{P}(Q)}(1)\bigr].$$

Note that the generators of the two groups are different in general. For the standard projective space we have $m_Q=1$ and so the generators are the same. For the k-jet bundles the fibers of their projectivization are weighted projective spaces with $m_Q=k!$, so we shall only be concerned with the case where $n,k\ge 1$ are positive integers and

$$Q=\bigl((\underbrace{1,\dots,1}_{n}),(\underbrace{2,\dots,2}_{n}),\dots,(\underbrace{k,\dots,k}_{n})\bigr),$$

which is normalized. In this case we shall write $\mathbb{P}_{n,k}$ for $\mathbb{P}(Q)$. Note that $r=\dim\mathbb{P}_{n,k}=nk-1$; the least common multiple of Q is

$$m_Q=k! \quad \text{and} \quad l_{Q,r}=(k!)^n. \tag{5.2}$$

Define a positive function

$$\rho_Q(z_0,\dots,z_r)=\sum_{i=0}^{r}|z_i|^{2/q_i} \tag{5.3}$$

on $(\mathbb{C}^{r+1},Q)\setminus\{0\}$. Then

$$\rho_Q(\lambda^{q_0}z_0,\dots,\lambda^{q_r}z_r)=|\lambda|^2\sum_{i=0}^{r}|z_i|^{2/q_i}=|\lambda|^2\rho_Q(z_0,\dots,z_r)$$

and

$$\psi^*(\rho_Q)(z_0,\dots,z_r)=\sum_{i=0}^{r}|z_i^{q_i}|^{2/q_i}=\sum_{i=0}^{r}|z_i|^2=\rho_{\mathbf{1}}(z_0,\dots,z_r)$$

is the standard Euclidean norm function on $(\mathbb{C}^{r+1},\mathbf{1})$. The function ρ_Q is not differentiable along $Z=\bigcup\{[z_{q_i}=0],\, q_i\neq 1\}$. However, on $\mathbb{C}^{r+1}\setminus Z$, we deduce from the above that

$$\partial\bar\partial\log\rho_Q(\lambda^{q_0}z_0,\dots,\lambda^{q_r}z_r)=\partial\bar\partial\log\rho_Q(z_0,\dots,z_r)$$

and that

$$\psi_Q^*(\partial\bar\partial\log\rho_Q)=\partial\bar\partial\log\rho_{\mathbf{1}}.$$

The first identity shows that $\partial\bar\partial\log\rho_Q$ is invariant under the $\mathbb{C}^*$-action hence descends to a well-defined (1, 1)-form ω_Q on $\mathbb{P}(Q)\setminus\pi_Q(Z)$. The second identity says that $\psi_Q^*(\omega_Q)$ is the Fubini–Study metric ω_{FS} on the standard projective space $\mathbb{P}^r\setminus\pi(Z)$ (hence actually extends smoothly across $\pi(Z)$). The Fubini–Study metric $[\omega_{FS}]$ is the first Chern form of $\mathcal{O}_{\mathbb{P}^r}(1)$ which is the (positive) generator of $\operatorname{Pic}\mathbb{P}^r=\operatorname{Cl}\mathbb{P}^r$. Hence $[\omega_{FS}]$ is the positive generator of $H^2(\mathbb{P}^r,\mathbb{Z})$. Theorem 5.4 implies that $[l_{Q,1}\omega_Q]$ is the generator of $H^2(\mathbb{P}(Q),\mathbb{Z})$.

Consider the function $\big(\sum_{i=0}^r|z_i|^{2\kappa}\big)^{1/\kappa}$, for κ a positive integer, defined on $\mathbb{C}^{r+1}$. It clearly satisfies

$$\big(\textstyle\sum_{i=0}^r|\lambda z_i|^{2\kappa}\big)^{1/\kappa}=|\lambda|^2(\sum_{i=0}^r|z_i|^{2\kappa})^{1/\kappa};$$

hence is a metric along the fibers of the tautological line bundle over $\mathbb{P}^r$. Moreover, the form

$$\partial\bar\partial\log\big(\textstyle\sum_{i=0}^r|z_i|^{2\kappa}\big)^{1/\kappa}$$

descends to a well-defined form on the standard projective space $\mathbb{P}^r$, indeed a Chern form, denoted by η_κ, for the hyperplane bundle of $\mathbb{P}^r$; moreover it is cohomologous to the Fubini–Study form. With this we may define an alternative to ρ_Q,

$$\tau_Q(z_0,\dots,z_r)=\left(\sum_{i=0}^{r}|z_i|^{2\kappa/q_i}\right)^{1/\kappa},\quad \kappa=\prod_{i=0}^{r}q_i. \tag{5.4}$$

It is of class $\mathcal{C}^\infty$ on $\mathbb{C}^{r+1}\setminus\{0\}$. Just like ρ_Q, the function τ_Q satisfies

$$\tau_Q(\lambda^{q_0}z_0,\dots,\lambda^{q_r}z_r)=\left(\sum_{i=0}^r|\lambda|^{2\kappa}|z_i|^{2\kappa/q_i}\right)^{1/\kappa}=|\lambda|^2\left(\sum_{i=0}^r|z_i|^{2\kappa/q_i}\right)^{1/\kappa}$$

and

$$(\psi^*\tau_Q)(z_0,\dots,z_r)=\tau_Q(z_0^{q_0},\dots,z_r^{q_r})=\left(\sum_{i=0}^r|z_i|^{2\kappa}\right)^{1/\kappa}.$$

These equalities imply that $\partial\bar\partial\log\tau_Q$ descends to a well-defined form γ_Q on $\mathbb{P}(Q)$ with the property that $\psi_Q^*\gamma_Q=\eta_k$, and consequently is cohomologous to ω_Q.

Let $\pi: J^kX\to X$ be the parametrized k-jet bundle of a complex manifold X and denote by $p:\mathbb{P}(J^kX)\to X$ and $pr: p^*J^kX\to\mathbb{P}(J^kX)$ the corresponding projection maps. The following diagram is commutative:

$$\begin{tikzcd} p^*J^kX \arrow[r, "p_*"] \arrow[d, "pr"'] & J^kX \arrow[d, "\pi"] \\ \mathbb{P}(J^kX) \arrow[r, "p"] & X \end{tikzcd}$$

and the tautological subsheaf of p^*J^kX is the line sheaf defined by

$$\{([\xi],\eta)\in p^*J^kX\mid[\xi]\in\mathbb{P}(J^kX),\,p([\xi])=\pi(\eta)=x,\,[\eta]=[\xi]\}$$

where, for ξ (resp. η) in J^kX, its equivalence class in $\mathbb{P}(J^kX)$ is denoted by $[\xi]$ (resp. $[\eta]$). The "hyperplane sheaf", denoted $\mathcal{L}=\mathcal{L}_k$, is defined to be the dual of the tautological line sheaf. The fiber $\mathbb{P}(J^k_xX)$ over a point $x\in X$ is the weighted projective space of type $Q=((1,\dots,1);\dots;(k,\dots,k))$ and the restriction of $\mathcal{L}_k$ to $\mathbb{P}(J^k_xX)$ is the line sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1)$ as defined in Theorem 5.2. The next result follows readily from Theorem 5.2:

THEOREM 5.6. *Let X be a complex manifold.*

(i) *For any $m\in\mathbb{Z},\mathcal{L}^m_{\mathbb{P}(J^kX)}$ is a reflexive coherent sheaf.*
(ii) *$\mathcal{L}^{k!}_{\mathbb{P}(J^kX)}$ is the generator of* $\mathrm{Pic}(\mathbb{P}(J^kX))$, *that is, $\mathcal{L}^m_{\mathbb{P}(J^kX)}$ is locally free if m is divisible by $k!$.*
(iii) *For any $\alpha,\beta\in\mathbb{Z}$, $\mathcal{L}^{k!\alpha}_{\mathbb{P}(J^kX)}\otimes\mathcal{L}^{\beta}_{\mathbb{P}(J^kX)}\cong\mathcal{L}^{k!\alpha+\beta}_{\mathbb{P}(J^kX)}$.*

The Chern class of the bundle $\mathcal{L}^{k!}_{\mathbb{P}(J^kX)}$ is $k!\,\omega_Q=l_{Q,1}\omega_Q$, where ω_Q is constructed after Theorem 5.5. By (5.3) the function $\rho_Q^{k!}$ is a Finsler metric along the fibers of $\mathcal{L}^{k!}_{\mathbb{P}(J^kX)}$. The same is of course also true if we use γ_Q and τ_Q instead. Just as in the case of projectivized vector bundles we still have the identification of the spaces $\mathcal{L}^{-1}_{\mathbb{P}(J^kX)}\setminus\{0\}$ with $J^kX\setminus\{0\}$, which is compatible with the respective $\mathbb{C}^*$ action. Thus, as in the case of vector bundles, we conclude that a metric along the fibers of $\mathcal{L}^{-1}_{\mathbb{P}(J^kX)}$ is identified with a Finsler metric along the fibers of

J^kX. As J^kX, in general, is only a $\mathbb{C}^*$-bundle and not a vector bundle, we see that Finsler geometry is indispensable.

Note that, for $k \geq 2$, the sheaf $\mathcal{L}_{\mathbb{P}(J^kX)}$ is not locally free and, in general,

$$\mathcal{L}^a_{\mathbb{P}(J^kX)} \otimes \mathcal{L}^b_{\mathbb{P}(J^kX)}(a) \ncong \mathcal{L}^{a+b}_{\mathbb{P}(J^kX)}.$$

Hence some of the proofs of the results that are valid for projectivized vector bundles require modifications. Basically, things work well if we use integer multiples of $k!$ (that is, $\mathcal{J}^{mk!}_k X$); for example, Grothendieck's Theorem (Theorem 4.2) remains valid:

THEOREM 5.7. *Let X be a complex manifold and $p : \mathbb{P}(J^kX) \to X$ the k-th parametrized jet bundle and let $\mathcal{S}$ be an analytic sheaf on X. For any $m \geq 1$,*

$$p^i_* \mathcal{L}^{mk!}_{\mathbb{P}(J^kX)} \cong \begin{cases} \mathcal{J}^{mk!}_k X, & \text{if } i = 0, \\ 0, & \text{if } i > 0, \end{cases}$$

where $p^i_* \mathcal{L}^{mk!}_{\mathbb{P}(J^kX)}$ is the i-th direct image of $\mathcal{L}^{mk!}_{\mathbb{P}(J^kX)}$. Consequently, we have

$$H^i(X, \mathcal{J}^{mk!}_k X \otimes \mathcal{S}) \cong H^i(\mathbb{P}(J^kX), \mathcal{L}^{mk!}_{\mathbb{P}(J^kX)} \otimes p^*\mathcal{S})$$

for all i.

In the case of vector bundles, Theorem 4.3 provides a relation between the Chern numbers of the bundle and that of the line bundle over the projectivization. Theorem 4.3 may be proved directly via fiber integration. Although the projectivized k-jet bundles are not smooth for $k \geq 2$ this correspondence is still valid. These technicalities are needed when we deal with problem of degeneration; as we shall see in Sections 6 and 7, under the condition that $\mathcal{L}^{k!}_k$ is big, k-jets of holomorphic maps into X are *algebraically degenerate*, that is, the images are contained in some (special type of) subvarieties of $\mathbb{P}(J^kX)$ which may be very singular. In order to calculate the Euler characteristic of $\mathcal{L}^{k!}_k X$ of these subvarieties it is necessary to compute the intersection numbers, as usual, via Chern classes and this is best handled by going down, via fiber integration, to the base variety X which is nonsingular. We take this opportunity to formulate a criterion for certain type of singular spaces on which fiber integration works well. The purpose here is not to exhibit the most general results but results general enough for our purpose. First we recall some basic facts concerning fiber integration. Let P and X be complex manifolds and $p : P \to X$ be a holomorphic surjection. The map p is said to be *regular* at a point $y \in P$ if the Jacobian of p at y is of maximal rank. The set of regular points is an open subset of P and p is said to be *regular* if every point of P is a regular point. The following statements concerning fiber integration are well-known (see [Stoll 1965], for example):

THEOREM 5.8. *Let P and X be connected complex manifolds of dimension N and n respectively. Let $p : P \to X$ be a regular holomorphic surjection. Let r, s be integers with $r, s \geq N - n = q$. Then for any (r,s)-form ω of class $\mathcal{C}^k$ on*

P that is integrable along the fibers of p, the fiber integral $p_*\omega$ is a well defined $(r-q, s-q)$ form of class $\mathcal{C}^k$ on X. Moreover:

(i) *For any $(N-r,\, N-s)$-form on X such that $\omega \wedge p^*\eta$ is integrable on P, we have*
$$\int_P \omega \wedge p^*\eta = \int_X p_*\omega \wedge \eta.$$
(ii) *If ω is of class $\mathcal{C}^1$ and if ω and $d\omega$ are integrable along the fibers p then $dp_*\omega = p_*d\omega, \partial p_*\omega = p_*\partial\omega$ and $\overline{\partial} p_*\omega = p_*\overline{\partial}\omega$.*

(iii) *If ω is nonnegative and integrable along the fibers of p then $p_*\omega$ is also nonnegative.*

(iv) *Suppose that Y is another connected complex manifold of dimension n' with a regular holomorphic surjection $\pi : X \to Y$. Assume that ω is a (r, s)-form such that $r, s \geq q+q'$ where $q = N-n$ and $q' = n-n'$. If ω is integrable along the fibers of p and $p_*\omega$ is integrable along the fibers of π then $\pi_* p_*\omega = (\pi\circ p)_*\omega$.*

If ω is a form of bidegree (r, s) so that either $r < q$ or $s < q$, where q is the fiber dimension, then we set $p_*\omega = 0$. If $p : P \to X$ is a holomorphic fiber bundle with smooth fiber S, then p is a regular surjection and the preceding Theorem is applicable. Consider now P, an irreducible complex space of complex dimension N, with a holomorphic *surjection* $p : P \to X$ where X is *nonsingular* and of complex dimension n. The map p is said to be *regular* if there exists a connected complex manifold $\tilde{P}$ of the same dimension as P and a surjective morphism $\tau : \tilde{P} \to P$ such that the composite map $\tilde{p} = p\circ\tau : \tilde{P} \to X$ is regular. Let $U \subset P$ be an open set and $\iota : U \to V \subset \mathbb{C}^{N'}$ a local embedding, where V is an open set of $\mathbb{C}^{N'}$ for some N'. If η is a differential form on V then $\iota^*\eta$ is a differential form on U. Conversely, a differential form ω on U is of the form $\iota^*\eta$ for some embedding $\iota : U \to V$ and some differential form η on V. Suppose that ω is a differential form on P of bidegree (r, s); hence $\tau^*\omega$ is a differential form on $\tilde{P}$ of bidegree (r, s). If either r or s is less than the fiber dimension $q = N-n$ then $p_*\omega$ is defined to be zero. For the case $r, s \geq q$ and assuming that $\tau^*\omega$ is integrable along the fibers of $\tilde{p}$ (for example, this is the case if ω is integrable along the fibers of p), we are in the nonsingular situation; hence $\tilde{p}_*\tau^*\omega$ is defined. The pushforward $p_*\omega$ is naturally defined by
$$p_*\omega \stackrel{\text{def}}{=} \tilde{p}_*\tau^*\omega. \tag{5.5}$$
From this definition it is clear (since $\tilde{p} = p\circ\tau$) that the basic properties of fiber integrals remain valid in the more general situation:

THEOREM 5.9. *Let $\tilde{\tau} : \tilde{P} \to P, p : P \to X$ and $\tilde{p} : \tilde{P} \to X$ be as above and let ω be a form of bidegree (r, s) on P with $r, s \geq N-n$ where $n = \dim X, N = \dim P = \dim \tilde{P}$. Then*:

(i) *If $\tau^*\omega$ is integrable along the fibers $\tilde{P}_y = \tilde{p}^{-1}(x)$ for almost all $x \in X$ then for any $(N-r, N-s)$-form on X such that $\omega \wedge p^*\eta$ is integrable on P and*

$\tau^*\omega \wedge \tilde{p}^*\eta$ is integrable on $\tilde{P}$,

$$\int_P \omega \wedge p^*\eta = \int_X p_*\omega \wedge \eta = \int_{\tilde{P}} \tilde{p}_*\tau^*\omega \wedge \tau^*\eta.$$

(ii) *If ω is of class $\mathcal{C}^1$ and if $\tau^*\omega$ and $\tau^*d\omega$ are integrable along the fibers $\tilde{P}_x$ for all $x \in X$ then $dp_*\omega = p_*d\omega, \partial p_*\omega = p_*\partial\omega$ and $\overline{\partial} p_*\omega = p_*\overline{\partial}\omega$.*

(iii) *If $\tau^*\omega$ is integrable along $\tilde{M}_x$ for all $x \in X$ then $p_*\omega$ is a form of type $(p-N+n, q-N+n)$ on X.*

(iv) *If ω is a continuous nonnegative form and $\tau^*\omega$ is integrable along $\tilde{P}_x$ for all $x \in X$ then $p_*\omega$ is also nonnegative.*

The converse of part (iv) is not true in general.

The next theorem shows that the preceding theorem is applicable to the projectivized k-jet bundles (we refer the readers to [Stoll and Wong 2002] for details).

Theorem 5.10. *Let X be a complex manifold of complex dimension n and let $p : P = \mathbb{P}(J^kX) \to X$ be the projectivized k-jet bundle of X. Then there exists a complex manifold $\tilde{P}$ of the same dimension as P and a surjective finite morphism $\tilde{\tau} : \tilde{P} \to P$ such that $\tilde{p} = p \circ \tau : \tilde{P} \to X$ is a regular holomorphic surjection. Moreover, $\tilde{P}$ can be chosen so that each of the fibers of $\tilde{p}$ is the complex projective space $\mathbb{P}^q$ where $q = nk - 1$.*

A similar argument (see [Stoll and Wong 2002]) shows that in general we have:

Theorem 5.11. *Let X be a connected complex manifold of complex dimension n. Suppose that P is an irreducible complex space for which there exists a holomorphic surjective morphism $p : P \to X$ that is locally trivial; that is, for any $x \in X$ there exists an open neighborhood V of X, a complex space Y and a biholomorphic map $\alpha_V : p^{-1}V \to V \times Y$ such that the diagram*

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "\alpha_V", "\cong"'] \arrow[d, "p"'] & V \times Y \arrow[d, "p_V"] \\ V \arrow[r, equal] & V \end{tikzcd}$$

commutes, where p_V is the projection onto the first factor. Then there exists a complex manifold $\tilde{P}$ of the same dimension as P and a surjective morphism $\tau : \tilde{P} \to P$ such that $\tilde{p} = p \circ \tau : \tilde{P} \to X$ is a regular holomorphic surjection.

Next we extend the definition of *pushforward* of forms to subvarieties of a complex space P with a projection map $p : P \to X$ satisfying the local triviality condition of the preceding theorem. In general the pushforwards exist only as currents. Suppose that $Y \subset P$ is an irreducible subvariety of dimension ν of P and assume that $p|_Y : Y \to X$ is surjective. Let $\Sigma \subset Y$ be the set of singular points of Y; so the set $S_1 = \{z \in X \mid (p|_Y)^{-1}(z) \subset \Sigma\}$ is a subvariety of codimension at least one in X. Note that

$$p|_{Y\setminus\Sigma} : Y \setminus \Sigma \to X \setminus S_1$$

is surjective, hence generically regular; that is, there exists a subvariety $S_2 \subset X$ of codimension at least 1 such that

$$p|_{Y\setminus(\Sigma\cup(p|_Y)^{-1}(S_2))} : Y_1 = Y \setminus (\Sigma \cup (p|_Y)^{-1}(S_2)) \to X_1 = X \setminus (S_1 \cup S_2)$$

is a regular surjection. Let ω be a smooth (r,s)-form on Y, $r,s \geq N-n =$ generic fiber dimension of $p|_{Y_1}$, which is integrable along the fibers of $p|_{Y_1}$. Then $(p|_{Y_1})_*\omega$ is a $(p-N+n, q-N+n)$-form on X_1. Meanwhile, the pushforward $(p|_Y)_*\omega$ exists as a current on X, that is,

$$(p|_Y)_*\omega((p|_Y)^*\phi) \overset{\text{def}}{=} \omega(\phi) = \int_Y (p|_Y)^*\phi \wedge \omega \tag{5.6}$$

for any $(N-r, N-s)$-form ϕ with compact support on X. Clearly, $(p|_Y)_*\omega|_{X_1} = (p|_Y)_*\omega$. Note that as a current the pushforward commutes with exterior differentiation, that is, $d(p|_Y)_*\omega = p_*d\omega, \partial(p|_Y)_*\omega = (p|_Y)_*\partial\omega$ and $\overline{\partial}(p|_Y)_*\omega = (p|_Y)_*\overline{\partial}\omega$. Also, by definition, the pushforward preserves nonnegativity.

The *Riemann–Roch formulas for jet differentials* follow from those of the bundles $\bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$, given below (see [Stoll and Wong 2002] for details):

THEOREM 5.12. *Let X be a smooth compact complex surface. Set $I = (i_1, \ldots, i_k)$ and $\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$, where each i_j is a nonnegative integer. Then*

$$\begin{aligned}\chi(X;\mathcal{S}_I) = \tfrac{1}{12}\mu_k(2s_{1;k}^2 - 2s_{1;k} - s_{2;k} + 1)c_1^2(T^*X) \\ - \tfrac{1}{12}\mu_k(2s_{1;k}^2 + 4s_{1;k} - 4s_{2;k} - 1)c_2(T^*X),\end{aligned}$$

where $s_{j;k}$, for $1 \leq j \leq k$, is the degree-j symmetric function in $i_1, \ldots, i_k$ and $\mu_k = \sum_{j=0}^k s_{j;k}$ is as in (3.6).

Given an exact sequence of coherent sheaves $0 \to E_1 \to E_2 \to E_3 \to 0$ the ranks, the first Chern classes, the Chern characters, the indices and the Euler characteristics are additive in the sense that $\operatorname{rk} E_2 = \operatorname{rk} E_1 + \operatorname{rk} E_3, c_1(E_2) = c_1(E_1) + c_1(E_3), \iota(E_2) = \iota(E_1) + \iota(E_3), ch(E_2) = ch(E_1) + ch(E_3)$ and $\chi(X;E_2) = \chi(X;E_1) + \chi(X;E_3)$. The Euler characteristic of $\mathcal{J}_k^{k!}X$ is given thus:

THEOREM 5.13. *Let X be a nonsingular surface. We have, for $m \gg k$,*

$$\chi(\mathcal{J}_k^{k!m}X) = \tfrac{1}{2}\iota(\mathcal{J}_k^{k!m}X) + O(m^{2k}) = \tfrac{1}{2}(k!)^{2k+1}(\alpha_k c_1^2 - \beta_k c_2)\, m^{2k+1} + O(m^{2k}),$$

where α_k and β_k are constants given in Theorem 3.9.

Example 5.14. We record below explicit formulas for the sheaves that occur in the preceding computations:

$\mathcal{S}$	$\mathrm{ch}(\mathcal{S})$	$\chi(\mathcal{S})$
T^*X	$2+\ c_1+\ \frac{1}{2}(c_1^2-2c_2)$	$\frac{1}{6}(c_1^2-5c_2)$
$\bigodot^2 T^*X$	$3+\ 3c_1+\ \frac{1}{2}(5c_1^2-8c_2)$	$\frac{1}{4}(5c_1^2-15c_2)$
$\bigodot^3 T^*X$	$4+\ 6c_1+\ 7c_1^2-10c_2$	$\frac{1}{3}(13c_1^2-29c_2)$
$\bigodot^4 T^*X$	$5+\ 10c_1+\ 15c_1^2-20c_2$	$\frac{1}{12}(125c_1^2-235c_2)$
$\bigodot^5 T^*X$	$6+\ 15c_1+\frac{1}{2}(55c_1^2-70c_2)$	$\frac{1}{2}(41c_1^2-84c_2)$
$\bigodot^6 T^*X$	$7+\ 21c_1+\frac{1}{2}(91c_1^2-112c_2)$	$\frac{1}{12}(427c_1^2-665c_2)$
$\bigodot^7 T^*X$	$7+\ 21c_1+\frac{1}{2}(91c_1^2-112c_2)$	$\frac{1}{3}(170c_1^2-250c_2)$
$T^*X\otimes T^*X$	$4+\ 4c_1+\ 3c_1^2-4c_2$	$\frac{1}{3}(4c_1^2-11c_2)$
$(\bigodot^2 T^*X)\otimes T^*X$	$6+\ 9c_1+\frac{1}{2}(19c_1^2-22c_2)$	$\frac{1}{2}(11c_1^2-21c_2)$
$(\bigodot^3 T^*X)\otimes T^*X$	$8+\ 16c_1+\ 22c_1^2-24c_2$	$\frac{1}{3}(44c_1^2-70c_2)$
$(\bigodot^2 T^*X)\otimes(\bigodot^2 T^*X)$	$9+\ 18c_1+\ 24c_1^2-24c_2$	$\frac{1}{4}(63c_1^2-93c_2)$
$(\bigodot^4 T^*X)\otimes T^*X$	$10+\ 25c_1+\frac{1}{2}(85c_1^2-90c_2)$	$\frac{1}{6}(185c_1^2-265c_2)$
$(\bigodot^3 T^*X)\otimes(\bigodot^2 T^*X)$	$12+\ 30c_1+\ 49c_1^2-46c_2$	$35c_1^2-45c_2$
$(\bigodot^5 T^*X)\otimes T^*X$	$12+\ 36c_1+\ 73c_1^2-76c_2$	$56c_1^2-75c_2$
$\mathcal{J}_2^2X$	$5+\ 4c_1+\ 3c_1^2-5c_2$	$\frac{1}{12}(17c_1^2-55c_2)$
$\mathcal{J}_2^3X$	$8+\ 10c_1+\ 10c_1^2-14c_2$	$\frac{1}{4}(23c_1^2-53c_2)$
$\mathcal{J}_2^4X$	$14+\ 22c_1+\ 27c_1^2-35c_2$	$\frac{1}{6}(103c_1^2-207c_2)$
$\mathcal{J}_2^5X$	$20+\ 40c_1+\ 59c_1^2-70c_2$	$\frac{1}{3}(122c_1^2-205c_2)$
$\mathcal{J}_2^6X$	$30+\ 70c_1+\ 119c_1^2-135c_2$	$\frac{1}{2}(173c_1^2-265c_2)$
$\mathcal{J}_2^7X$	$40+110c_1+\ 214c_1^2-200c_2$	$\frac{1}{3}(487c_1^2-590c_2)$

Although the space $\mathbb{P}(J^kX)$ is not smooth, the following Riemann–Roch Theorem is still valid, by Theorems 5.7 and 5.13:

Theorem 5.15. *Let X be a nonsingular surface and $p:\mathbb{P}(J^kX)\to X$ the k-jet bundle. Then*

$$\begin{aligned}\chi(\mathcal{L}^m_{\mathbb{P}(J^kX)}) &= \mathrm{ch}(\mathcal{L}^m)\,.\,\mathrm{td}(\mathbb{P}(J^kX))[\mathbb{P}(J^kX)]\\ &= \mathrm{ch}(\mathcal{L}^m)\,.\,\mathrm{td}(T_p)\,.\,p^*\mathrm{td}(X)[\mathbb{P}(J^kX)]\\ &= p_*\big(\mathrm{ch}(\mathcal{L}^m)\,.\,\mathrm{td}(T_p)\big).\mathrm{td}(X)[X],\end{aligned}$$

where T_p is the relative tangent sheaf of the projection $p:\mathbb{P}(J^kX)\to X$, that is, the restriction of T_p to each fiber of p is the tangent sheaf of the weighted projective space $\mathbb{P}(Q)$.

On $\mathbb{P}(J^k X)$ we have

$$\mathrm{ch}(\mathcal{L}^m) = \sum_{i=0}^{2k+1} \frac{c_1^i(\mathcal{L}^m)}{i!} = \sum_{i=0}^{2k+1} \frac{c_1^i(\mathcal{L})}{i!} m^i$$

which implies that

$$\chi(\mathcal{L}^m) = \frac{c_1^{2k+1}(\mathcal{L})}{(2k+1)!} m^{2k+1} + O(m^{2k}).$$

Theorems 5.13 and 5.15 imply:

Corollary 5.16. *Let X be a nonsingular surface and $p : \mathbb{P}(J^k X) \to X$ the k-jet bundle. Then*

$$p_*\left(\frac{c_1^{2k+1}(\mathcal{L})}{(2k+1)!}\right) = \frac{1}{2}\iota(\mathcal{J}_k X) = \frac{1}{2}(\alpha_k c_1^2 - \beta_k c_2)$$

where α_k and β_k are constants given in Theorem 3.9.

6. The Lemma of Logarithmic Derivatives and the Schwarz Lemma

Summary. *In this section we use Nevanlinna Theory to show* (*Corollary* 6.2) *that if ω is a holomorphic k-jet differential of weight m vanishing on an effective divisor of a projective manifold X then $f^*\omega \equiv 0$ for any holomorphic map $f : \mathbb{C} \to X$.* (*For our application in Section* 7 *it is enough to assume that the divisor is a hyperplane section.*) *This implies* (*see Theorem* 6.4) *that if $f : \mathbb{C} \to X$ is an algebraically nondegenerate holomorphic map then the irreducible component of the base locus containing $[j^k f]$ is of codimension at most $(n-1)k$; equivalently the dimension is at least $n+k-1, n = \dim X$. This result is crucial in the proof of our main result in Section* 7. *We must point out that the method of this section works only for the parametrized jet bundles but not the full jet bundles.* (*Otherwise we could have avoided the complicated computations of the Chern numbers of the parametrized jet bundles; computing the Chern numbers of the full jet bundles, as honest vector bundles, is much simpler!*) *The idea of the proof is relatively standard from the point of view of Nevanlinna Theory. The main step is to construct, using a standard algebraic geometric argument, a Finsler metric of logarithmic type, reducing the problem to a situation in which the Lemma of Logarithmic Derivatives is applicable. If $f^*\omega \not\equiv 0$ this lemma implies that the integral of $\log|f^*\omega|$ is small. On the other hand, the first Crofton formula in Nevanlinna Theory asserts that the integral of $\log|f^*\omega|$* (*as the counting function of the zeros of $f^*\omega$ by the Poincaré–Lelong formula*) *is not small. This contradiction establishes Theorem* 6.1 *and Corollary* 6.2. *Theorem* 6.4 *and Corollary* 6.5 *then follow from the Schwarz Lemma via a reparametrization argument often used in Nevanlinna Theory. The main point is that a reparametrization does*

not change, as a set, the (algebraic closure of the) image of the map f but does change the image of $f^*\omega$ if ω is a k-jet differential and $k \geq 2$.

The *classical Schwarz Lemma* in one complex variable asserts that a holomorphic map $f : \mathbb{C} \to X$ is constant for a compact Riemann surface X of genus ≥ 2; that is, there are at least 2 independent regular 1-forms on X. Further, there is a noncompact version; namely, let X be a compact Riemann surface and let D be a finite number of points in X. Then a holomorphic map $f : \mathbb{C} \to X \setminus D$ is constant if the logarithmic genus is at least 2, that is, there are at least 2 independent 1-forms regular on $X \setminus D$ and with no worse than logarithmic singularity at each of the points in D. There are of course many proofs of this classical result, one of which is to find a nontrivial holomorphic 1-form (or, logarithmic 1-form) on X such that $f^*\omega \equiv 0$ for any entire holomorphic map $f : \mathbb{C} \to X$. This is not so difficult to do because $g \geq 2$ implies that T^*X is ample (and a priori, spanned). The main difficulty of proving the preceding comes from the fact that a big bundle is not necessary spanned. A coherent sheaf $\mathcal{S}$ is said to be *spanned* (by global regular sections) if, for every $v \in \mathcal{S}_x$ there is a global regular section $\sigma \in H^0(\mathcal{S})$ such that $\sigma(x) = v$. However, it is easily seen that a coherent sheaf $\mathcal{S}$ is spanned by global rational sections. For example, the complex projective space has no global regular 1-form. Hence it cannot span any of the fibers of $T^*\mathbb{P}^n$. However take any point $x \in \mathbb{P}^n$, assuming without loss of generality that $x = [x_0, \ldots, x_n]$ with $x_0 \neq 0$, then $T_x^*\mathbb{P}^n$ is spanned by dt_i, $i = 1, \ldots, n$, where $t_i = x_i/x_0$. Now dt_i is a global rational one-form since t_i is a global rational function; in fact

$$dt_i = \frac{x_0 dx_i - x_i dx_0}{x_0^2}$$

has a pole of order 2 along the "hyperplane at infinity", $[x_0 = 0]$. This shows that $T^*\mathbb{P}^n$ is spanned by global rational one-forms. In fact we can do better, namely, we may replace dt_i by $d \log t_i$

$$d \log t_i = \frac{dt_i}{t_i} = \frac{dx_i}{x_i} - \frac{dx_0}{x_0}.$$

A simple argument shows that there is a finite set of logarithmic one-forms $\{dL_i/L_i\}$ where each L_i is a rational function which span $T^*\mathbb{P}^n$ at every point. The mild singularity can be dealt with using the classical Lemma of Logarithmic Derivatives in Nevanlinna Theory and a weak form of the analytic Bézout Theorem known as Crofton's Formula.

It is not hard to see that the preceding procedure can be extended to deal with jet differentials. The details are given in the next theorem. The most convenient way to get to the Schwarz Lemma is via Nevanlinna Theory. First we recall some standard terminology. The characteristic function of a map $f : \mathbb{C} \to X$ is

$$T_f(r) = \int_0^r \frac{1}{t} \int_{\Delta_t} f^* c_1(H),$$

where H is a hyperplane section in X and the characteristic function of a non-trivial holomorphic function $F:\mathbb{C}\to\mathbb{C}$ is

$$T_F(r)=\int_0^{2\pi}\log^+|F(re^{\sqrt{-1}\theta})|\frac{d\theta}{2\pi}.$$

Note that $\omega(j^kf)$ is a holomorphic function if f is a holomorphic map and ω is a k-jet differential of weight m.

THEOREM 6.1 (LEMMA OF LOGARITHMIC DERIVATIVES). *Let X be a projective variety and let* (*i*) *D be an effective divisor with simple normal crossings*, *or* (*ii*) *D be the trivial divisor in X* (*that is*, *the support of D is empty or equivalently*, *the line bundle associated to D is trivial*). *Let $f:\mathbb{C}\to X$ be an algebraically nondegenerate holomorphic map and $\omega\in H^0(X,\mathcal{J}_k^mX(\log D))$* (*resp. $H^0(X,\mathcal{J}_k^mX)$ in case* (*ii*)) *a jet differential such that $\omega\circ j^kf$ is not identically zero*. *Then*

$$T_{\omega\circ j^kf}(r)=\int_0^{2\pi}\log^+\bigl|\omega(j^kf(re^{\sqrt{-1}\theta}))\bigr|\,\frac{d\theta}{2\pi}\;\le\;O(\log T_f(r))+O(\log r).$$

PROOF. We claim that there exist a finite number of rational functions $t_1,\dots,t_q$ on X such that

(†) *the logarithmic jet differentials $\{(d^{(j)}t_i/t_i)^{m/j}\mid 1\le i\le q,1\le j\le k\}$ span the fibers of $\mathcal{J}_k^mX(\log D)$* (*resp.$\mathcal{J}_k^mX$*) *over every point of X*.

Note that rational jet differentials span the fibers of $\mathcal{J}_k^mX(\log D)$ (resp.$\mathcal{J}_k^mX$); the claim here is that this can be achieved by those of logarithmic type. Without loss of generality we may assume that D is ample; otherwise we may replace D by $D+D'$ so that $D+D'$ is ample. (This is so because a section of $\mathcal{J}_k^mX(\log D)$ is a priori a section of $\mathcal{J}_k^mX(\log(D+D'))$.) Observe that if s is a function that is holomorphic on a neighborhood U such that $[s=0]=D\cap U$ then $[s^\tau=0]=\tau D\cap U$ for any rational number τ. Thus $\delta^{(j)}(\log s^\tau)=\tau\delta^{(j)}(\log s)$ is still a jet differential with logarithmic singularity along $D\cap U$ so the multiplicity causes no problem. This implies that we may assume without loss of generality that D is very ample (after perhaps replacing D with τD for some τ for which τD is very ample).

Let $u\in H^0(X,[D])$ be a section such that $D=[u=0]$. At a point $x\in D$ choose a section $v_1\in H^0(X,[D])$ so that $E_1=[v_1=0]$ is smooth, $D+E_1$ is of *simple normal crossings* and v_1 is nonvanishing at x. (This is possible because the line bundle $[D]$ is very ample.) The rational function $t_1=u_1/v_1$ is regular on the affine open neighborhood $X\setminus E_1$ of x and $(X\setminus E_1)\cap[t_1=0]=(X\setminus E_1)\cap D$. Choose rational functions $t_2=u_2/v_2,\dots,t_n=u_n/v_n$ where u_i and v_i are sections of a very ample bundle $\mathcal{L}$ so that $t_2,\dots,t_n$ are regular at x, the divisors $D_i=[u_i=0], E_i=[v_i=0]$ are smooth and the divisor $D+D_2+\dots+D_n+E_1+\dots+E_n$ is of simple normal crossings. Further, since the bundles involved are very ample the sections can be chosen so that $dt_1\wedge\dots\wedge dt_n$ is nonvanishing at x; the complete system of sections provides an embedding. Hence at each point there are $n+1$ sections with the property that n of the quotients of these $n+1$ sections form a local coordinate system on some open neighborhood U_x of x. This implies that (†) is satisfied over U_x. Since D is compact it is covered by a finite number of such open neighborhoods, say $U_1, \dots, U_p$, and a finite number of rational functions (constructed as above for each U_i) on X so that (†) is satisfied on $\bigcup_{1\le i\le p} U_i$. Moreover, there exist relatively compact open subsets U_i' of U_i $(1 \le i \le p)$ such that $\bigcup_{1\le i\le p} U_i'$ still covers D.

Next we consider a point x in the compact set $X \setminus \bigcap_{1\le i\le p} U_i'$. Repeating the procedure as above we may find rational functions $s_1 = a_1/b_1, \dots, s_n = a_n/b_n$ where a_i and b_i are sections of some very ample line bundle so that $s_1, \dots, s_n$ form a holomorphic local coordinate system on some open neighborhood V_x of x. Thus (†) is satisfied on V_x by the rational functions $s_1, \dots, s_n$. Note that we must also choose these sections so that the divisor $H = [s_1 \ \dots \ s_n = 0]$ together with those divisors (finite in number), which have been already constructed above, is still a divisor with simple normal crossings (this is possible by the very ampleness of the line bundle $\mathcal{L}$.) Since $X \setminus \bigcap_{1\le i\le p} U_i'$ is compact, it is covered by a finite number of such coordinate neighborhoods. The coordinates are rational functions and finite in number and by construction it is clear that the condition (†) is satisfied on $X \setminus \bigcap_{1\le i\le p} U_i'$. Since $\bigcup_{1\le i\le p} U_i$ together with $X \setminus \bigcap_{1\le i\le p} U_i'$ covers X, the condition (†) is satisfied on X. If D is the trivial divisor, then it is enough to use only the second part of the construction above and again (†) is verified with $\mathcal{J}_k^m X(\log D) = \mathcal{J}_k^m X$. To obtain the estimate of the theorem observe that the function $\rho : J^k X(-\log D) \to [0, \infty]$ defined by

$$\rho(\xi) = \sum_{i=1}^{q} \sum_{j=1}^{k} \left| (d^{(j)} t_i/t_i)^{m/j}(\xi) \right|^2, \quad \xi \in J^k X(-\log D), \tag{6.1}$$

$\{t_i\}$ being the family of rational functions satisfying condition (†), is continuous in the extended sense; it is continuous in the usual sense outside the fibers over the divisor E (the sum of the divisors associated to the rational functions $\{t_i\}$; note that E contains D). Over the fiber of each point $x \in X - E$, $\left| (d^{(j)} t_i/t_i)^{m/j}(\xi) \right|^2$ is finite for $\xi \in J^k X(-\log D)_x$, thus ρ is not identically infinite. Moreover, since

$$\left\{ (d^{(j)} t_i/t_i)^{m/j} \mid 1 \le i \le q, 1 \le j \le k \right\}$$

span the fiber of $\mathcal{J}_k^m X(\log D)$ over every point of X, ρ is strictly positive (possibly $+\infty$) outside the zero section of $J^k X(-\log D)$. The quotient

$$|\omega|^2/\rho : J^k X(-\log D) \to [0, \infty]$$

does not take on the extended value ∞ when restricted to $J^k X(-\log D) \setminus$ {zero section} because, as we have just observed, ρ is nonvanishing (although it does blow up along the fibers over E so that the reciprocal $1/\rho$ is zero there) and the singularity of $|\omega|$ is no worse than that of ρ since the singularity of ω occurs only along D (which is contained in E) and is of log type. Thus the restriction to $J_k X(-\log D)\setminus\{\text{zero section}\}$,

$$|\omega|^2/\rho : J^k X(-\log D)\setminus\{\text{zero section}\}\to[0,\infty),$$

is a continuous nonnegative function. Moreover, $|\omega|$ and ρ have the same homogeneity,

$$|\omega(\lambda.\xi)|^2=|\lambda|^{2m}|\omega(\lambda.\xi)|^2 \quad\text{and}\quad \rho(\lambda.\xi)=|\lambda|^{2m}\rho(\xi),$$

for all $\lambda\in\mathbb{C}^*$ and $\xi\in J^kX(-\log D)$; therefore $|\omega|^2/\rho$ descends to a well-defined function on $\mathbb{P}(E_{k,D})=(J^kX(-\log D)\setminus\{\text{zero section}\})/\mathbb{C}^*$, that is,

$$|\omega|^2/\rho : \mathbb{P}(E_{k,D})\to[0,\infty)$$

is a well-defined continuous function and so, by compactness, there exists a constant c with the property that $|\omega|^2\le c\rho$. This implies that

$$\begin{aligned} T_{\omega\circ j^k f}(r) &= \int_0^{2\pi}\log^+\bigl|\omega(j^k f(re^{\sqrt{-1}\theta}))\bigr|\,\frac{d\theta}{2\pi}\\ &\le \int_0^{2\pi}\log^+\bigl|\rho(j^k f(re^{\sqrt{-1}\theta}))\bigr|\,\frac{d\theta}{2\pi}+O(1).\end{aligned}$$

Since t_i is a rational function on X, the function

$$(d^{(j)}t_i/t_i)^{m/j}(j^kf)=\bigl((t_i\circ f)^{(j)}/t_i\circ f\bigr)^{m/j}$$

(m is divisible by $k!$) is meromorphic on $\mathbb{C}$ and so, by the definition of ρ,

$$\log^+|\rho(j^kf)|\le O\bigl(\max_{1\le i\le q,\,1\le j\le k}\log^+|(t_i\circ f)^{(j)}/t_i\circ f|\bigr)+O(1).$$

Now by the classical lemma of logarithmic derivatives for meromorphic functions,

$$\int_0^{2\pi}\log^+\bigl|(t_i\circ f)^{(j)}/t_i\circ f\bigr|\,\frac{d\theta}{2\pi}\ \cdot\!\le\!\cdot\ O(\log r)+O(\log T_{t_i\circ f}(r)),$$

where $\cdot\!\le\!\cdot$ indicates that the estimate holds outside a set of finite Lebesgue measure in $\mathbb{R}_+$. Since t_i is a rational function,

$$\log T_{t_i\circ f}(r)\le O(\log T_f(r))+O(\log r)$$

and we arrive at the estimate

$$\begin{aligned}\int_0^{2\pi}\log^+\bigl|\rho(j^kf(re^{\sqrt{-1}\theta}))\bigr|\,\frac{\theta}{2\pi}\ &\le\ O\biggl(\int_0^{2\pi}\log^+|(t_i\circ f)^{(j)}/t_i\circ f|\frac{d\theta}{2\pi}\biggr)+O(1)\\ &\cdot\!\le\!\cdot\ O(\log T_f(r))+O(\log r).\end{aligned}$$

This implies that $T_{\omega\circ j^kf}(r)\ \cdot\!\le\!\cdot\ O(\log T_f(r))+O(\log r)$, as claimed. □

We obtain as a consequence the following Schwarz type lemma for logarithmic jet differentials.

Corollary 6.2. *Let X be a projective variety and D be an effective divisor (possibly the trivial divisor) with simple normal crossings. Let $f:\mathbb{C}\to X\setminus D$ be a holomorphic map. Then*

$$\omega(j^kf)\equiv 0 \quad \textit{for all } \omega\in H^0(X,\mathcal{J}_k^mX(\log D)\otimes[-H]),$$

where H is a generic hyperplane section (and hence any hyperplane section).

Proof. If f is constant the corollary holds trivially, so we may assume that f is nonconstant. Now suppose that $\omega\circ j^kf\not\equiv 0$. Since f is nonconstant, we may assume without loss of generality that $\log r = o(T_f(r))$ (after perhaps replacing f with $f\circ\phi$, where ϕ is a transcendental function on $\mathbb{C}$). By Theorem 6.1, we have

$$\int_0^{2\pi}\log^+|\omega\circ j^kf|\,\frac{d\theta}{2\pi} = T_{\omega\circ j^kf}(r) \;\cdot\!\leq\!\cdot\; O\big(\log(rT_f(r))\big).$$

On the other hand, since ω vanishes on H and H is generic, we obtain via Jensen's Formula ,

$$\begin{aligned} T_f(r) &\leq N_f(H;r)+O\big(\log(rT_f(r))\big)\\ &= \int_0^{2\pi}\log|\omega\circ j^kf|\,\frac{d\theta}{2\pi}+O(\log\big(rT_f(r))\big),\end{aligned}$$

which, together with the preceding estimate, implies that

$$T_f(r)\leq O\big(\log(rT_f(r))\big).$$

This is impossible; hence we must have $\omega\circ j^kf\equiv 0$. If $H_1=[s_1{=}0]$ is any hyperplane section then it is linearly equivalent to a generic hyperplane section $H=[s{=}0]$. If ω vanishes along H' then $(s/s_1)\omega$ vanishes along H. The preceding discussion implies that $(s/s_1)\omega(j^kf)\equiv 0$. Further, this implies that $\omega(j^kf)\equiv 0$ as we may choose a generic section H so that the image of f is not entirely contained in H. $\square$

Interpreting this corollary via Grothendieck's isomorphism we may restate the result in terms of sections of $\mathcal{L}^m_{\mathbb{P}(J^kX)}|_Y\otimes p|_Y^*[-D]$ on the projectivized bundle:

Corollary 6.3. *Let $Y\subset\mathbb{P}(J^kX)$ be a subvariety and suppose that there exists a nontrivial section*

$$\sigma\in H^0\big(Y,\,\mathcal{L}^m_{\mathbb{P}(J^kX)}|_Y\otimes p|_Y^*[-D]\big),$$

where D is an ample divisor in X and $p:\mathbb{P}(J^kX)\to X$ is the projection map. If the image of the lifting $[j^kf]:\mathbb{C}\to\mathbb{P}(J^kX)$ of a holomorphic curve $f:\mathbb{C}\to X$ is contained in Y, then $\sigma([j^kf])\equiv 0$.

Theorem 6.1 and Corollaries 6.2 and 6.3 tell us about the *base locus* $B_k^m(D)$ of the line sheaves $\mathcal{L}_k^m\otimes p^*[-D]$, where we write for simplicity $\mathcal{L}_k^m=\mathcal{L}^m_{\mathbb{P}(J^kX)}$ and

D is an ample divisor in X; by that we mean the (geometric) intersection of all possible sections of powers of $\mathcal{L}_k$:

$$B_k^m(D) = \bigcap_{\sigma \in H^0(\mathbb{P}(J^k X), \mathcal{L}_k^m)} [\sigma = 0]. \tag{6.2}$$

Indeed, Corollary 6.3 implies that the image of the (projectivized k-jet) $[j^k f] : \mathbb{C} \to \mathbb{P}(J^k X)$ of a nonconstant holomorphic map $f : \mathbb{C} \to X$ must be contained in $B_k^m(D)$ for all $m \in \mathbb{N}$ and $D \in \mathcal{A} =$ the cone of all ample divisors; that is, the image $[j^k f](\mathbb{C})$ is contained in

$$B_k(\mathcal{L}_k) = \bigcap_{m \in \mathbb{N}} \bigcap_{D \in \mathcal{A}} B_k^m(D), \tag{6.3}$$

which is a subvariety of $\mathbb{P}(J^k X)$. Moreover, the image $[j^k f](\mathbb{C})$, being a connected set, must be contained in an irreducible component of $B_k(\mathcal{L})$. If f is algebraically nondegenerate then $\dim \overline{f(\mathbb{C})} = \dim X = n$. Since $p_* \overline{[j^k f(\mathbb{C})]} = \overline{f(\mathbb{C})}$ and $\overline{[j^k f(\mathbb{C})]} \subset B_k(\mathcal{L})$ (where $p : \mathbb{P}(J^k X) \to X$ is the projection) we conclude that *the dimension of the base locus is at least* $n = \dim X$ *if* f *is algebraically nondegenerate*. We shall show that the dimension is actually higher, for $k \geq 2$, by considering a reparametrization of the curve f.

Define

$$\begin{aligned} \mathcal{A} &= \{\phi \mid \phi : \mathbb{C} \to \mathbb{C} \text{ is a nonconstant holomorphic map}\}, \\ \mathcal{A}_{\zeta_0} &= \{\phi \in \mathcal{A} \mid \phi(\zeta_0) = \zeta_0, \phi'(\zeta_0) \neq 0\}, \\ \mathcal{A}_{\zeta_0,\zeta_1} &= \{\phi \in \mathcal{A} \mid \phi(\zeta_0) = \zeta_1, \phi'(\zeta_0) \neq 0\}. \end{aligned}$$

By a *reparametrization* of f we mean the composite map $f \circ \phi : \mathbb{C} \to X$, where $\phi \in \mathcal{A}$. It is clear that, as a set, the algebraic closure of the image of f is invariant by reparametrization. Moreover, since a reparametrization is again a curve in X, the Schwarz Lemma implies that its k-jet is contained in the base locus $B_k(\mathcal{L})$. As remarked earlier, if f is algebraically nondegenerate the dimension of the base locus is at least n.

The first order jet of a reparametrization is given by

$$j^1(f \circ \phi) = (f(\zeta),\, f'(\phi)\phi').$$

Thus, if $\phi \in \mathcal{A}_0$ (that is, $\phi(0) = 0$), then

$$j^1(f \circ \phi)(0) = \big(f(\phi(0)),\, f'(\phi(0))\phi'(0)\big) = \big(f(0),\, f'(0)\phi'(0)\big),$$

which implies that the projectivization satisfies

$$[j^1(f \circ \phi)(0)] = [f'(0)\phi'(0)] = [f'(0)] = [j^1 f(0)];$$

that is, the fiber $\mathbb{P}_{f(\zeta_0)}(J^1 X)$ is invariant by $\phi \in \mathcal{A}_{\zeta_0}$.

Assume from here on that the map f is algebraically nondegenerate. For $k \geq 2$ we have

$$j^2(f(\phi)) = \big(f(\phi),\, f'(\phi)\phi',\, f'(\phi)\phi'' + f''(\phi)(\phi')^2\big).$$

Moreover, $\phi \in \mathcal{A}_0$ implies

$$\begin{aligned} j^2(f\circ\phi)(0) &= \big((f\circ\phi)(0),\, (f\circ\phi)'(0),\, (f\circ\phi)''(0)\big) \\ &= \big(f(0),\, s_\phi f'(0),\, t_\phi f'(0) + s_\phi^2 f''(0)\big), \end{aligned}$$

where $s_\phi = \phi'(0)$ and $t_\phi = \phi''(0)$. We are free to prescribe the complex numbers s_ϕ and t_ϕ. The bundle $\mathbb{P}(J^2X)$ is algebraic and locally trivial, hence locally algebraically trivial (as a $\mathbb{C}^*$-bundle). In particular, we have a $\mathbb{C}^*$-isomorphism $J^2_{f(0)}X \cong \mathbb{C}^n \oplus \mathbb{C}^n$, where $\lambda(\boldsymbol{z},\boldsymbol{w}) = (\lambda\boldsymbol{z}, \lambda^2\boldsymbol{w})$ for $(\boldsymbol{z},\boldsymbol{w}) \in \mathbb{C}^n \oplus \mathbb{C}^n$. The Jacobian matrix of the map

$$(s_\phi, t_\phi) \mapsto (s_\phi f'(0), t_\phi f'(0) + s_\phi^2 f''(0))$$

is given by

$$\begin{pmatrix} \dfrac{\partial(f\circ\phi)'(0)}{\partial s_\phi} & \dfrac{\partial(f\circ\phi)''(0)}{\partial s_\phi} \\ \dfrac{\partial(f\circ\phi)'(0)}{\partial t_\phi} & \dfrac{\partial(f\circ\phi)''(0)}{\partial t_\phi} \end{pmatrix} = \begin{pmatrix} f'(0) & 2s_\phi f''(0) \\ 0 & f'(0) \end{pmatrix}.$$

It is clear that the rank is 2 if $f'(0) \neq 0$ (which we may assume without loss of generality because $f' \not\equiv 0$ so $f'(\zeta) \neq 0$ for generic ζ). Thus, as ϕ varies through the space $\mathcal{A}_0$, $j^2(f\circ\phi)(0)$ sweeps out a complex 2-dimensional set in the fiber $J^2_{f(0)}X$ over the point $f(0) \in X$, and the projectivization is a set of dimension at least 1 in $\mathbb{P}(J^2_{f(0)}X)$. If f is algebraically nondegenerate, the algebraic closure of $[j^2 f(\mathbb{C})]$ is of dimension $n = \dim X$, as remarked earlier. The preceding argument shows that

$$\bigcup_{\phi\in\mathcal{A}} \overline{[j^2(f\circ\phi)(\mathbb{C})]}$$

is of dimension at least $n{+}1$. By Schwarz's Lemma the set $\bigcup_{\phi\in\mathcal{A}} \overline{[j^2(f\circ\phi)(\mathbb{C})]}$ is contained in the base locus $B_2(\mathcal{L})$ thus $\dim B_2(\mathcal{L}) \geq n+1$. Since $\dim \mathbb{P}(J^2X) = n(2+1)-1$, the codimension of $B_2(\mathcal{L})$ in $\mathbb{P}(J^2X)$ is at most $3n-1-(n+1) = 2(n-1)$.

For $k = 3$ we get

$$j^3(f(\phi)) = \\ \big(f(\phi),\, f'(\phi)\phi',\, f'(\phi)\phi'' + f''(\phi)(\phi')^2,\, f'(\phi)\phi''' + 3f''(\phi)\phi'\phi'' + f'''(\phi)(\phi')^3\big).$$

Hence, for $\phi \in \mathcal{A}_0$,

$$j^3(f\circ\phi)(0) = \big(f(0),\, s_\phi f'(0),\, t_\phi f'(0) + s_\phi^2 f''(0),\, u_\phi f'(0) + 3s_\phi t_\phi f''(0) + s_\phi^3 f'''(0)\big),$$

where $s_\phi = \phi'(0)$, $t_\phi = \phi''(0)$, and $u_\phi = \phi'''(0)$. The Jacobian matrix of the map

$$(s_\phi, t_\phi, u_\phi) \mapsto \big(s_\phi f'(0),\, t_\phi f'(0) + s_\phi^2 f''(0),\, u_\phi f'(0) + 3s_\phi t_\phi f''(0) + s_\phi^3 f'''(0)\big)$$

is

$$\begin{pmatrix} \dfrac{\partial(f\circ\phi)'(0)}{\partial s_\phi} & \dfrac{\partial(f\circ\phi)''(0)}{\partial s_\phi} & \dfrac{\partial(f\circ\phi)'''(0)}{\partial s_\phi} \\ \dfrac{\partial(f\circ\phi)'(0)}{\partial t_\phi} & \dfrac{\partial(f\circ\phi)''(0)}{\partial t_\phi} & \dfrac{\partial(f\circ\phi)'''(0)}{\partial t_\phi} \\ \dfrac{\partial(f\circ\phi)'(0)}{\partial u_\phi} & \dfrac{\partial(f\circ\phi)''(0)}{\partial u_\phi} & \dfrac{\partial(f\circ\phi)'''(0)}{\partial u_\phi} \end{pmatrix}$$

$$= \begin{pmatrix} f'(0) & 2s_\phi f''(0) & 3t_\phi f''(0) + 3s_\phi^2 f'''(0) \\ 0 & f'(0) & 3t_\phi f''(0) \\ 0 & 0 & f'(0) \end{pmatrix}.$$

It is clear that the rank is 3 if $f'(0) \neq 0$ (which we may assume without loss of generality). Thus, as ϕ varies through the space $\mathcal{A}_0, j^3(f\circ\phi)(0)$ sweeps out a complex 3-dimensional set in the fiber $J^3_{f(0)}X$ over the point $f(0) \in X$ and the projectivization is a set of dimension at least 2 in $\mathbb{P}(J^3_{f(0)}X)$. If f is algebraically nondegenerate then the set $\bigcup_{\phi\in\mathcal{A}} \overline{[j^3(f\circ\phi)(\mathbb{C})]}$ is of dimension at least $n+2$ in $\mathbb{P}(J^3X)$. By Schwarz's Lemma this same set is contained in the base locus $B_3(\mathcal{L})$ thus $\dim B_3(\mathcal{L}) \geq n+2$. Since $\dim \mathbb{P}(J^3X) = n(3+1)-1$, the codimension of $B_3(\mathcal{L})$ in $\mathbb{P}(J^3X)$ is at most $4n-1-(n+2) = 3(n-1)$.

The case for general k is argued in a similar fashion. Define polynomials P_{ij}, $1 \leq j \leq i$, by setting $P_{1,1} = \phi'$, $P_{2,1} = \phi''$, $P_{2,2} = (\phi')^2$ and, for $i \geq 3$,

$$\begin{aligned} P_{i,1} &= \phi^{(i)}, \\ P_{i,2} &= P_{i-1,1} + P'_{i-1,2}, \dots \\ P_{i-1,i-1} &= P_{i-1,i-2} + P'_{i-1,i-1}, \\ P_{i,i} &= (\phi')^i. \end{aligned}$$

In particular, $P_{i,1}$ is the only polynomial involving $\phi^{(i)}$; each $P_{i,j}$, for $j \geq 2$, involves only derivatives of ϕ of order less than i. We get, by induction:

$$(f\circ\phi)^{(i)} = \sum_{j=1}^{i} f^{(j)}(\phi)P_{i,j} = f'(\phi)\phi^{(i)} + \sum_{j=2}^{i} f^{(j)}(\phi)P_{i,j}.$$

Thus the k-th jet $j^k(f\circ\phi)$ is given by

$$\Bigl(f'(\phi)\phi', \cdots, f'(\phi)\phi^{(i)} + \sum_{j=2}^{i} f^{(j)}(\phi)P_{i,j}, \cdots, f^{(k)}(\phi)\phi^{(k)} + \sum_{j=2}^{k} f^{(j)}(\phi)P_{k,j}\Bigr),$$

and we have k parameters $s_{\phi,i} = \phi^{(i)}(0)$, $i = 1, \dots, k$. The Jacobian matrix of the map (with $\phi(0) = 0$)

$$(s_{\phi,1}, \dots, s_{\phi,k}) \mapsto \Bigl(s_{\phi,1}f'(0), s_{\phi,2}f'(0) + s_{\phi,1}^2 f''(0), \dots, s_{\phi,k}f'(0) + \sum_{j=2}^{k} f^{(j)}(0)P_{k,j}\Bigr)$$

is given by the $k \times nk$ matrix

$$\begin{pmatrix} f'(0) & 2s_\phi f''(0) & 3t_\phi f''(0)+3s_\phi^2 f'''(0) & \dots & . & . \\ 0 & f'(0) & 3t_\phi f''(0) & \dots & . & . \\ 0 & 0 & f'(0) & \dots & . & . \\ . & . & 0 & \dots & . & . \\ . & . & . & \dots & f'(0) & . \\ 0 & 0 & . & \dots & 0 & f'(0) \end{pmatrix}.$$

It is clear that the rank is k if $f'(0) \neq 0$ (which we may assume without loss of generality). Thus, as ϕ varies through the space $\mathcal{A}_0$, $j^k(f\circ\phi)(0)$ sweeps out a complex k-dimensional set in the fiber $J^k_{f(0)}X$ over the point $f(0) \in X$, and the projectivization is a set of dimension at least $k-1$ in $\mathbb{P}(J^k_{f(0)}X)$. If f is algebraically nondegenerate then the set $\bigcup_{\phi\in\mathcal{A}} \overline{[j^k(f\circ\phi)(\mathbb{C})]}$ is of dimension at least $n+k-1$ in $\mathbb{P}(J^kX)$. By Schwarz's Lemma this same set is contained in the base locus $B_k(\mathcal{L})$ thus $\dim B_k(\mathcal{L}) \geq n+k-1$. Since $\dim \mathbb{P}(J^kX) = n(k+1)-1$, the codimension of $B_k(\mathcal{L})$ in $\mathbb{P}(J^kX)$ is at most $(k+1)n-1-(n+k-1) = k(n-1)$. This completes the proof of the following Theorem:

Theorem 6.4. *Let X be a connected compact manifold of dimension n and let $\mathcal{L}_k$ be the dual of the tautological line bundle over $\mathbb{P}(J^kX), k \geq 2$. Suppose that $f:\mathbb{C}\to X$ is an algebraically nondegenerate holomorphic map. Then the irreducible component of the base locus containing $[j^kf]$ is of codimension at most $(n-1)k$; equivalently the dimension is at least $n+k-1$.*

Corollary 6.5. *Let X be a connected projective manifold of complex dimension n and let $\mathcal{L}_k$ be the dual of the tautological line bundle over $\mathbb{P}(J^kX)$. If the dimension of the base locus $B_k(\mathcal{L}_k) \leq n+k-2$ then every holomorphic map $f:\mathbb{C}\to X$ is algebraically degenerate.*

7. Surfaces of General Type

Summary. *In this section we shall show that every holomorphic map $f:\mathbb{C}\to X$ is algebraically degenerate, where X is a minimal surface of general type such that $p_g(X) > 0$ and* $\operatorname{Pic} X \cong \mathbb{Z}$. *These conditions, together with the explicit calculations in Section* 3, *imply that $\mathcal{J}_kX$ is big (equivalently, the line bundle $\mathcal{L}_k$ over $\mathbb{P}(J^kX)$ is big) for $k \gg 0$. The Schwarz Lemma of the preceding section implies that the image of the lifting $[j^kf]:\mathbb{C}\to\mathbb{P}(J^kX)$ is contained in the base locus $B_k(\mathcal{L}_k)$ (see* (6.3)). *(Note that the dimension of* $\mathbb{P}(J^kX)$ *is* $2k+1$.) *Moreover, if f is algebraically nondegenerate,* $\dim B_k(\mathcal{L}_k) \geq k+1$.

On the other hand, we show (Theorem 7.20) *that the base locus is at most of dimension k. This contradiction establishes the theorem. The result in Theorem* 7.20 *is obtained by a cutting procedure (each cut lowers the dimension of the base locus by one) pioneered by Lu and Yau and extended by Dethloff–Schumacher–Wong (in which the condition* $\operatorname{Pic} X \cong \mathbb{Z}$ *was first introduced).*

The starting point in the process is the explicit formulas obtained by Stoll and Wong in Theorem 3.9 *and Corollary* 3.10, *namely, the index* $\iota(\mathcal{J}_k^m X) = \chi(\mathcal{L}_k^m) + O(m^{2k}) = (\alpha_k c_1^2 - \beta_k c_2)m^{2k+1} + O(m^{2k})$ (*here* $c_i = c_i(X)$) *is very big*; *indeed we have*, $\lim_{k\to\infty} \alpha_k/\beta_k = \infty$. *Consequently if* $c_1^2 > 0$, *which is the case if* X *is minimal*, *then* $\chi(\mathcal{L}_k^m) = cm^{2k+1}c_1^2 + O(m^{2k})$ *for some positive constant* c (*as*, *eventually*, $\alpha_k/\beta_k > c_2/c_1^2$). *If the base locus* Y_1 *were of codimension one* (*which we show that there is no loss of generality in assuming that it is irreducible*) *then for* $k \gg 0, \chi(\mathcal{L}_k|_{Y_1})$ *is still big and Schwarz Lemma implies that the base locus must be of codimension* 2. *The computation is based on the intersection formulas obtained in Lemma* 7.15 (*requiring the assumption* $\operatorname{Pic} X \cong \mathbb{Z}$) *and Theorem* 7.16. *The cutting procedure can be repeated and*, *as to be expected*, *each time with a loss which can be explicitly estimated using the intersection formulas*. *These losses are compensated by taking a larger* k. *In the proof of Theorem* 7.20 *we show that*, *after* k *cuts*, *the Euler characteristic is bounded below by*

$$\mu_k\left(\delta_k c_1^2 - \left(\sum_{i=1}^k \frac{1}{i^2}\right)c_2\right),$$

where μ_k *is a positive integer and*

$$\delta_k = \left(\sum_{i=1}^k \frac{1}{i^2} + \sum_{i=2}^k \frac{1}{i}\sum_{j=1}^{i-1}\frac{1}{j}\right) - \frac{1}{4}\left(\sum_{i=1}^k \frac{1}{i}\right)^2 + \frac{(k+1)}{4(k!)^2}\left(\sum_{i=1}^k \frac{1}{i}\right)^2.$$

It remains to show that

$$\frac{\delta_k}{\sum_{i=1}^k \frac{1}{i^2}} > \frac{c_2}{c_1^2}$$

for k *sufficiently large*. *A little bit of combinatorics shows that*

$$\lim_{k\to\infty} \frac{\delta_k}{\sum_{i=1}^k \frac{1}{i^2}} = \infty$$

(*compare the proof of Corollary* 3.10). *This completes the proof of our main result*. *Indeed*, *for a hypersurface of degree* $d \geq 5$ *in* $\mathbb{P}^3$, *our colleague B*. *Hu checked*, *using Maple*, *that* $k \geq 2283$ *is sufficient*. *This*, *together with a result of Xu implies that a generic hypersurface of degree* $d \geq 5$ *in* $\mathbb{P}^3$ *is hyperbolic*.

We recall first some well-known results on manifolds of general type. The following result can be found in [Barth et al. 1984]:

THEOREM 7.1. *Let* X *be a minimal surface of general type*. *The following Chern-number inequalities hold*:

(i) $c_1^2(T^*X)[X] > 0$.
(ii) $c_2(T^*X)[X] > 0$.
(iii) $c_1^2(T^*X)[X] \leq 3c_2(T^*X)[X]$.
(iv) $5c_1^2(T^*X)[X] - c_2(T^*X[X]) + 36 \geq 0$ *if* $c_1^2(T^*X)[X]$ *is even.*
(v) $5c_1^2(T^*X)[X] - c_2(T^*X)[X] + 30 \geq 0$ *if* $c_1^2(T^*X)[X]$ *is odd.*

Let L_0 be a nef line bundle on a variety X of complex dimension n. A coherent sheaf E over X is said to be *semistable* (or *semistable in the sense of Mumford–Takemoto*) with respect to L_0 if $c_1(E)\,.\,c_1^{n-1}(L_0) \geq 0$ and if, for any coherent subsheaf $\mathcal{S}$ of E with $1 \leq \operatorname{rk}\mathcal{S} < \operatorname{rk} E$, we have $\mu_{\mathcal{S},L_0} \leq \mu_{E,L_0}$, where

$$\mu_{\mathcal{S},L_0} \overset{\text{def}}{=} \frac{c_1(\mathcal{S})\,.\,c_1^{n-1}(L_0)}{\operatorname{rk}\mathcal{S}}[X] \quad \text{and} \quad \mu_{E,L_0} \overset{\text{def}}{=} \frac{c_1(E)\,.\,c_1^{n-1}(L_0)}{\operatorname{rk} E}[X]. \tag{7.1}$$

It is said to be *stable* if the inequality is strict, that is, $\mu_{\mathcal{S},L_0} < \mu_{E,L_0}$.

The number $\mu_{\mathcal{S},L_0}$ shall be referred to as *the normalized degree relative to* L_0. We shall write $\mu_{\mathcal{S}}$ for $\mu_{\mathcal{S},L_0}$ if L_0 is the canonical bundle. If X is of general type then (see [Maruyama 1981] in the case of surfaces and [Tsuji 1987, 1988] for general dimensions):

THEOREM 7.2. *Let X be a smooth variety of general type. Then the bundles $\bigotimes^m T^*X$, $\bigodot^m T^*X$ are semistable with respect to the canonical bundle $\mathcal{K}_X$.*

Recall from Section 2 that for a vector bundle E of rank r,

$$\operatorname{rk} \bigodot^m E = \frac{(m+r-1)!}{(r-1)!\,m!}, \quad c_1(\bigodot^m E) = \frac{(m+r-1)!}{r!\,(m-1)!}\,c_1(E).$$

Thus, for surfaces of general type, we have

$$\mu_{\odot^m T^*X} = \tfrac{1}{2} m c_1^2(T^*X)[X]$$

with respect to the canonical bundle. More generally:

THEOREM 7.3. *Let X be a surface of general type. If D is a divisor in X such that $H^0(X, \mathcal{S}_I \otimes [-D]) \neq 0$ where $\mathcal{S}_I = (\bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X)$ and $I = (i_1, \ldots, i_k)$ is a k-tuple of positive integers satisfying $m = i_1 + 2i_2 + \cdots + k i_k$, then*

$$\mu_{[D]} \leq \mu_{\mathcal{S}_I} = \frac{\sum_{j=1}^k i_j}{2} c_1^2(T^*X)[X] \leq \tfrac{1}{2} m c_1^2(T^*X)[X],$$

where $[D]$ is the line bundle associated to the divisor D.

The examples at the end of Section 2 show that the sheaves of k-jet differentials are not semistable unless $k = 1$. However we do have (by Theorems 3.7 and 3.8):

THEOREM 7.4. *Let X be a surface of general type. Then*

$$\mu_{\mathcal{J}_k^m X} = \frac{\sum_I c_1(\mathcal{S}_I)\, c_1(T^*X)}{\sum_I \operatorname{rk}\mathcal{S}_I} = \sum_I \frac{\operatorname{rk}\mathcal{S}_I}{\sum_I \operatorname{rk}\mathcal{S}_I} \mu_{\mathcal{S}_I} \leq \frac{m}{2} c_1^2(T^*X),$$

and equality holds if and only if $k = 1$; moreover, asymptotically,

$$\mu_{\mathcal{J}_k^m X} = \left(\frac{\sum_{i=1}^k \frac{1}{i}}{2k} m + O(1) \right) c_1^2(T^*X).$$

A coherent sheaf E is said to be *Euler semistable* if for any coherent subsheaf $\mathcal{S}$ of E with $1 \le \operatorname{rk}\mathcal{S} < \operatorname{rk} E$, we have

$$\frac{\chi(\mathcal{S})}{\operatorname{rk}\mathcal{S}} \le \frac{\chi(E)}{\operatorname{rk} E} \tag{7.2}$$

It is said to be *Euler stable* if the inequality is strict.

There is a concept of semistability due to Gieseker–Maruyama (see [Okonek et al. 1980]) for coherent sheaves on $\mathbb{P}^n$ in terms of the Euler characteristic that differs from the concept introduced here.

EXAMPLE 7.5. From the exact sequence

$$0 \to \bigodot^2 T^*X \to \mathcal{J}_2^2 X \to T^*X \to 0,$$

we get, via the table on page 163,

$$\chi(\mathcal{J}_2^2 X) = \chi(T^*X) + \chi(\bigodot^2 T^*X) = \tfrac{1}{6}(c_1^2 - 5c_2) + \tfrac{1}{4}(5c_1^2 - 15c_2) = \tfrac{1}{12}(17c_1^2 - 55c_2).$$

Theorem 6.1 yields $c_1^2 - 3c_2 \le 0$, which implies that

$$\chi(T^*X) = \frac{c_1^2 - 5c_2}{6} < 0.$$

Thus $\chi(\mathcal{J}_2^2 X) < \chi(\bigodot^2 T^*X)$, that is, $\mathcal{J}_2^2 X$ is not semistable in the sense of (7.2).

Recall that the index of each of the sheaves $\mathcal{S}_I$ and $\mathcal{J}_k^m X$ of a surface X is of the form $ac_1^2(T^*X) + bc_2(T^*X)$. Thus the ratio $\gamma(X) = \gamma(T^*X) = c_2(T^*X)/c_1^2(T^*X)$ is an important invariant. More generally, we define

$$\gamma(\mathcal{S}) = \frac{c_2(\mathcal{S})}{c_1^2(\mathcal{S})}, \tag{7.3}$$

provided that $c_1^2(\mathcal{S}) \ne 0$.

Let X be a smooth hypersurface in $\mathbb{P}^3$. Then

$$c_1 = c_1(TX) = -c_1(T^*X) = d - 4,$$
$$c_2 = c_2(TX) = c_2(T^*X) = d^2 - 4d + 6.$$

Hence the ratio of $c_1^2(T^*X)$ and $c_2(T^*X)$ is given by

$$\gamma_d(\mathcal{J}_1 X) = \gamma_d(T^*X) = \frac{c_2(T^*X)}{c_1^2(T^*X)} = \frac{d^2 - 4d + 6}{(d-4)^2} = 1 + \frac{4d - 10}{(d-4)^2}, \tag{7.4}$$

provided that $d \ne 4$. Note that $\gamma_\infty(T^*X) = \lim_{d\to\infty} \gamma_d(T^*X) = 1$. Table A on the next page shows the first few values of $\gamma_g = \gamma_d(\mathcal{J}_1 X)$.

Recall from Theorem 5.12 that

$$\begin{aligned}\chi(X; \bigodot^m T^*X) &= \tfrac{1}{12}(m+1)\big((2m^2 - 2m + 1)c_1^2 - (2m^2 + 4m - 1)c_2\big)\\ &= \tfrac{1}{12}(m+1)\big((2m^2 - 2m + 1)(d-4)^2 - (2m^2 + 4m - 1)(d^2 - 4d + 6)\big)\\ &= \tfrac{1}{3}(5 - 2d)3m^3 + O(m^2).\end{aligned}$$

d	γ_d	d	γ_d	d	γ_d
5	11	12	$\frac{51}{32} \sim 1.5937$	19	$\frac{97}{75} \sim 1.2934$
6	$\frac{9}{2} = 4.5$	13	$\frac{14}{27} \sim 1.5175$	20	$\frac{163}{128} \sim 1.2735$
7	3	14	$\frac{73}{50} = 1.46$	21	$\frac{363}{289} \sim 1.2561$
8	$\frac{19}{8} = 2.375$	15	$\frac{171}{121} \sim 1.4132$	22	$\frac{67}{54} \sim 1.2408$
9	$\frac{51}{25} = 2.04$	16	$\frac{11}{8} = 1.375$	23	$\frac{443}{361} \sim 1.2244$
10	$\frac{11}{6} = 1.8\bar{3}$	17	$\frac{227}{169} \sim 1.3432$	24	$\frac{243}{200} = 1.215$
11	$\frac{83}{49} \sim 1.6939$	18	$\frac{129}{98} \sim 1.3164$	25	$\frac{59}{49} \sim 1.2041$

Table A. Values of $\gamma_d(\mathcal{J}_1X)$ as a function of d.

It is clear that $\chi(X; \bigodot^m T^*X) < 0$ for all $m \geq 1$ if $d \geq 3$. If $d \geq 5$ it is well-known that $H^0(X, \bigodot^m T^*X) = 0$, whence the following nonvanishing theorem:

Theorem 7.6. *Let X be a smooth hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. Then*

$$\begin{aligned}\dim H^1(X, \textstyle\bigodot^m T^*X) &\geq \dim H^1(X, \textstyle\bigodot^m T^*X) - \dim H^2(X, \textstyle\bigodot^m T^*X)\\ &= \tfrac{1}{6}(m+1)\big(2(2d-5)m^2 - (3d^2-16d+28)m - (d^2-6d+11)\big)\\ &= \tfrac{1}{3}(2d-5)3m^3 + O(m^2)\end{aligned}$$

for all $m \gg 0$.

Next we consider the case of 2-jets. We have, by Riemann–Roch:

$$\chi(\mathcal{J}_2^m X) = \frac{1}{2}\big(\iota(\mathcal{J}_2^m X) - c_1(\mathcal{J}_2^m X) \cdot c_1\big) + \frac{1}{12}(\operatorname{rk} \mathcal{J}_2^m X)(c_1^2 + c_2).$$

(Here $c_1 = c_1(T^*X), c_2 = c_2(T^*X)$ and, using the formulas for $c_1(\mathcal{J}_2^m X)$, $\operatorname{rk} \mathcal{J}_2^m X$ and $\iota(\mathcal{J}_2^m X)$ in Theorem 3.3 we get:

$$\chi(\mathcal{J}_2^m X) = \frac{1}{2^7 3^2 5}(p_m c_1^2 - q_m c_2)$$

with

$$p_m = \begin{cases} 21m^5 + 180m^4 + 410m^3 + 180m^2 + 49m + 120, & \text{if } m \text{ is odd,}\\ 21m^5 + 180m^4 + 420m^3 + 180m^2 - 56m + 480, & \text{if } m \text{ is even;}\end{cases}$$

$$q_m = \begin{cases} 15m^5 + 225m^4 + 1150m^3 + 2250m^2 + 1235m - 75, & \text{if } m \text{ is odd,}\\ 15m^5 + 225m^4 + 1180m^3 + 2520m^2 + 1640m - 480, & \text{if } m \text{ is even.}\end{cases}$$

The index $\chi(\mathcal{J}_2^m X)$ is positive if and only if $p_m/q_m > c_2/c_1^2$, and taking the limit as $m \to \infty$ yields the inequality $c_2/c_1^2 \leq \frac{7}{5}$. For a smooth hypersurface of degree d in $\mathbb{P}^3$ the ratio $c_2/c_1^2 = 1 + \big((4d-10)/(d-4)^2\big)$ and we arrive at the inequality

$$\frac{4d-10}{(d-4)^2} \leq \frac{2}{5},$$

which is equivalent to the inequality $0 \leq d^2 - 18d + 41 = (d-9)^2 - 40$. We deduce:

THEOREM 7.7. *Let X be a smooth hypersurface in $\mathbb{P}^3$. Then $\chi(\mathcal{J}_2^m X)$ is big if and only if $d = \deg X \geq 16$.*

We use the terminology that the Euler characteristic is *big* if and only if there is a constant $c > 0$ such that

$$\chi(\mathcal{J}_2^m X) \geq cm^5 + O(m^4)$$

for all $m \gg 0$. In order to lower the degree in the preceding theorem we must use jet differentials of higher order. We see from Table A on page 177 that the ratio c_2/c_1^2 of a hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$ is bounded above by 11. By Theorem 3.7,

$$\iota(\mathcal{J}_k^m X) = (\alpha_k c_1^2 - \beta_k c_2) m^{2k+1} + O(m^{2k})$$

thus the index is positive if and only if

$$\frac{\alpha_k}{\beta_k} > \frac{c_2}{c_1^2}.$$

In the table on page 148 we see that the ratio α_k/β_k crosses the threshold 11 as k increases from 198 to 199. Putting this together with Theorem 5.13, we get:

THEOREM 7.8. *Let X be a generic smooth hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. For each $k \geq 199$,*

$$\chi(\mathcal{J}_k^m X) \geq cm^5 + O(m^4)$$

for all $m \gg k$.

For a minimal surface of general type, Theorem 7.1 implies that

$$\frac{1}{3} \leq \gamma(X) = \frac{c_2(X)}{c_1^2(X)} \leq \begin{cases} 5 + 36c_1^{-2} \leq 41 & \text{if } c_1^2 \text{ is even,} \\ 5 + 30c_1^{-2} \leq 34 & \text{if } c_1^2 \text{ is odd.} \end{cases}$$

The ratio α_k/β_k was shown to tend to ∞ as $k \to \infty$. Thus Theorem 7.8 extends to any minimal surface of general type:

THEOREM 7.9. *Let X be a smooth minimal surface of general type. Then*

$$\chi(\mathcal{J}_k^m X) \geq cm^5 + O(m^4) \quad \textit{for all } m \gg k \gg 0.$$

In [Green and Griffiths 1980] we find the following result:

THEOREM 7.10. *Let X be a smooth surface of general type. If $i_1 + \cdots + i_k$ is even then a nontrivial section of the bundle $\bigodot^{i_1} TX \otimes \cdots \otimes \bigodot^{i_k} TX \otimes \mathcal{K}^{(i_1+\cdots+i_k)/2}$ is nonvanishing.*

Using this, Green and Griffiths deduced the following vanishing Theorem. We include their argument here, with minor modifications.

THEOREM 7.11. *Let X be a smooth surface of general type. Assume that the canonical bundle $\mathcal{K}_X$ admits a nontrivial section. Then $H^2(X, \mathcal{J}_k^m X) = 0$ for all $k \geq 1$ and $m > 2k$.*

Proof. Let σ be a nontrivial section of $\mathcal{K}_X$, so that we have an exact sequence:

$$0 \to \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2-1} \xrightarrow{\otimes\sigma} \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2} \to \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2}|_D \to 0$$

where $D = [\sigma{=}0]$, $\mathcal{S}_I = \bigodot^{i_1} TX \otimes \cdots \otimes \bigodot^{i_k} TX$ and $i_1+\cdots+i_k$ is even. Hence,

$$0 \to H^0(X, \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2-1}) \xrightarrow{\otimes\sigma} H^0(X, \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2})$$

is exact. By Theorem 7.10 the image of the map $\otimes\sigma$ is 0; hence

$$H^0(X, \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2-1}) = 0.$$

The argument applies also to the exact sequence:

$$0 \to \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2-l} \xrightarrow{\otimes\sigma} \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2=l+1} \to \mathcal{S}_I \otimes \mathcal{K}^{(/i_1+\cdots+i_k)2}|_D \to 0$$

for any $l \geq 1$ and we conclude via induction that

$$H^0(X, \mathcal{S}_I \otimes \mathcal{K}^q) = 0$$

for all $q < (i_1+\cdots+i_k)/2$. If $i_1+\cdots+i_k$ is odd then taking $i_{k+1} = 1$ we have

$$H^0(X, \mathcal{S}_{i_1,\ldots,i_k,i_{k+1}} \otimes \mathcal{K}^{q+1}) = 0$$

provided that $q+1 < (i_1+\cdots+i_k+1)/2$ (equivalently $q < (i_1+\cdots+i_k-1)/2$). Suppose that $H^0(X, \mathcal{S}_I \otimes \mathcal{K}^q) \neq 0$. Then there exists a nontrivial section ρ of $H^0(X, \mathcal{S}_I \otimes \mathcal{K}^q)$ and we obtain a nontrivial section $\rho \otimes \sigma$ of $\mathcal{S}_{i_1,\ldots,i_k,i_{k+1}} \otimes \mathcal{K}^{q+1}$. This shows that:

$$H^0(X,\, \mathcal{S}_I \otimes \mathcal{K}^q) = \begin{cases} 0, & \text{for all } q < \frac{1}{2}(i_1+\cdots+i_k-1) \text{ if } i_1+\cdots+i_k \text{ is odd,} \\ 0 & \text{for all } q < \frac{1}{2}(i_1+\cdots+i_k) \text{ if } i_1+\cdots+i_k \text{ is even.} \end{cases}$$

By Serre duality,

$$H^2(X, \mathcal{S}_I \otimes \mathcal{K}^{1-q}) = \begin{cases} 0, & \text{for all } q < \frac{1}{2}(i_1+\cdots+i_k-1) \text{ if } i_1+\cdots+i_k \text{ is odd,} \\ 0 & \text{for all } q < \frac{1}{2}(i_1+\cdots+i_k) \text{ if } i_1+\cdots+i_k \text{ is even,} \end{cases}$$

where $\mathcal{S}_I = \bigodot^{i_1} T^*X \otimes \cdots \otimes \bigodot^{i_k} T^*X$. If $|I| = i_1+\cdots+i_k \geq 3$ then we may take $q = 1$ in the formulas above. Thus we have: $H^2(X, \mathcal{S}_I) = 0$, if $|I| \geq 3$. Note $\mathcal{J}_k^m X$ admits a composition series by $\mathcal{S}_I$ satisfying the condition $\sum_{j=1}^k j i_j = m$. Thus $H^2(X, \mathcal{J}_k^m X) = 0$ if each of these $\mathcal{S}_I$ satisfies the condition $|I| \geq 3$. If $k = 2$ we have:

$$i_1 + 2i_2 = m \iff i_2 = (m-i_1)/2 \iff i_1 + i_2 = (m+i_1)/2.$$

Thus $i_1 + i_2 \geq 3$ if and only if $m \geq 6 - i_1$. Since $i_1 \geq 0$ we conclude that $m \geq 6$ implies $i_1 + i_2 \geq 3$. If $k = 3$ then

$$\begin{aligned} i_1{+}2i_2{+}3i_3 = m &\iff (i_1{+}i_3) + 2(i_2{+}i_3) = m \iff i_2{+}i_3 = \tfrac{1}{2}(m{-}i_1{-}i_3) \\ &\iff i_1{+}i_2{+}i_3 = \tfrac{1}{2}(m{+}i_1{-}i_3). \end{aligned}$$

Thus $i_1+i_2+i_3\geq 3$ if and only if $m\geq 6-i_1+i_3\geq 6+i_3$. Since i_3 is at most $[m/3]$ we conclude that $i_1+i_2+i_3\geq 3$ if $m\geq 9$. The case of general k can be established by an induction argument. □

For our purpose only the following weaker result is needed:

THEOREM 7.12. *Let E be a holomorphic vector bundle of rank $r\geq 2$ over a nonsingular projective surface X. Assume that*

(i) $\mathcal{K}_X$ *is nef and not the trivial bundle*;
(ii) $\operatorname{Pic} X\cong\mathbb{Z}$;
(iii) $\det E^*$ *is nef*;
(iv) *there exists a positive integer s with the property that there is a nontrivial global regular section ρ of $(\mathcal{K}_X\otimes\det E^*)^s$ such that the zero divisor $[\rho=0]$ is smooth.*

Then $H^i(X,\bigodot^m E^*)=0$ for all $i\geq 2$ and for m sufficiently large.

The canonical bundle $\mathcal{K}_X$ of a minimal surface X of general type is nef. If $\operatorname{Pic} X\cong\mathbb{Z}$ then $\mathcal{K}_X$ is ample, so $\mathcal{K}_X\otimes\det\bigl(\bigodot^{i_1}T^*X\otimes\cdots\otimes\bigodot^{i_k}T^*X\bigr)$ is ample for any nonnegative integers $i_1,\dots,i_k$. Hence:

COROLLARY 7.13. *Let X be a nonsingular minimal surface of general type. Assume that* $\operatorname{Pic} X\cong\mathbb{Z}$ *and $p_g(X)>0$. Let $I=(i_1,\dots,i_k)$ be a k-tuple of nonnegative integers. Then $H^2\bigl(X,\bigodot^{i_1}T^*X\otimes\cdots\otimes\bigodot^{i_k}T^*X\bigr)=0$ if $i_1+\cdots+i_k$ is sufficiently large*; *consequently, $H^2(X,\mathcal{J}_k^m X)=0$ if $m\gg k$.*

COROLLARY 7.14. *Let X be a nonsingular minimal surface of general type with* $\operatorname{Pic} X\cong\mathbb{Z}$. *Then*

$$h^0(X,\mathcal{J}_k^{k!m}X)\geq cm^{2k+1}+O(m^{2k})$$

for some positive constant c; *that is, $\mathcal{J}_k^{k!}X$ is big.*

A good source for the general theory of vanishing theorems is [Esnault and Viehweg 1992].

Next we deal with the question of algebraic degeneration of holomorphic maps and hyperbolicity of surfaces of general type. The condition that $\mathcal{J}_k^{k!}X$ is big implies that $\mathcal{J}_k^{k!}X\otimes[-D]$ is big for any ample divisor D on X. We may write $D=a_0D_0$ for $a_0>0$, with D_0 as the positive generator of $\operatorname{Pic} X$. The Schwarz Lemma for jet differentials implies that the image of $[j^kf]$ is contained in the zero set of all k-jet differentials vanishing along an ample divisor. Thus we may assume that $[j^kf](\mathbb{C})$ is contained in an effective irreducible divisor in $\mathbb{P}(J^kX)$ and is the zero set of a section

$$\sigma\in H^0\bigl(\mathbb{P}(J^kX),\,\mathcal{L}_k^{k!m_k}\otimes p^*[-\nu_kD_0]\bigr),\tag{7.5}$$

where we abbreviate $\mathcal{L}_k=\mathcal{L}_{\mathbb{P}(J^kX)}$ (note that $\operatorname{Pic}\mathbb{P}(J^kX)\cong\mathbb{Z}\langle\mathcal{L}_k^{k!}\rangle\oplus\operatorname{Pic} X$). Our aim is to show that the restriction $\mathcal{L}_k^{k!}|_{[\sigma=0]}$ is big. First we need a lemma:

LEMMA 7.15. *Let X be a nonsingular minimal surface of general type with* $\operatorname{Pic} X \cong \mathbb{Z}$. *Suppose that $H^0(\mathcal{J}_k^{k!m}X \otimes [-D]) \neq 0$ for some divisor D in X. Then for all $m \gg 0$,*

$$c_1([D]) \leq \frac{B_k}{A_k} k!\, m c_1(T^*X) + O(1)$$

where A_k and B_k are the constants defined in Theorems 3.7 *and* 3.8.

PROOF. The assumption that $\operatorname{Pic} X \cong \mathbb{Z}$ implies that $c_1(\mathcal{J}_k^{k!}X \otimes [-D]) = qc_1$, where $c_1 = c_1(T^*X)$ and $q \in \mathbf{Q}$. Let σ be a nontrivial section of $\mathcal{J}_k^{k!}X \otimes [-D]$. The Poincaré–Lelong formula implies that

$$\begin{aligned} 0 = \int_X dd^c \log ||\sigma||^2 \wedge c_1 &\geq \int_{[\sigma=0]} c_1 - \int_X c_1(\mathcal{J}_k^{k!m}X \otimes [-D]) \wedge c_1 \\ &= \int_{[\sigma=0]} c_1 - q\int_X c_1^2, \end{aligned}$$

implying $q > 0$. On the other hand, the usual formula for Chern classes yields

$$0 < c_1(\mathcal{J}_k^{k!m}X \otimes [-D]) = c_1(\mathcal{J}_k^{k!m}X) - (\mathrm{rk}\mathcal{J}_k^{k!m}X)c_1([D]).$$

By the asymptotic formula in Section 3, we have

$$c_1(\mathcal{J}_k^{k!m}X) = B_k(k!m)^{2k}c_1 + O((k!m)^{2k-1});$$

hence the preceding inequality may be written as

$$A_k c_1([D])(k!m)^{2k-1} = (\mathrm{rk}\mathcal{J}_k^{k!m}X)c_1([D]) < B_k(k!m)^{2k}c_1 + O((k!m)^{2k-1}),$$

where A_k and B_k are the constants defined in Theorems 3.7 and 3.8. Thus we get the estimate

$$c_1([D]) \leq \frac{B_k}{A_k} k!\, m c_1 + O(1). \qquad \square$$

THEOREM 7.16. *Let X be a smooth surface and $\mathcal{L}_k$ be the "hyperplane line sheaf" over $\mathbb{P}(J^kX)$. Then*

$$\begin{aligned} p_* c_1^{2k+1}(\mathcal{L}_k^{k!}) &= (2k+1)!\chi(\mathcal{L}_k^{k!}) = \tfrac{1}{2}(2k+1)!(k!)^{2k+1}(\alpha_k c_1^2 - \beta_k c_2) \\ &= (k!)^{2k-1}\left(\sum_{i=1}^{k}\frac{1}{i^2} + \sum_{i=2}^{k}\frac{1}{i}\sum_{j=1}^{i-1}\frac{1}{j}\right)c_1^2 - \left(\sum_{i=1}^{k}\frac{1}{i^2}\right)c_2, \end{aligned}$$

$$p_*\big(c_1^{2k}(\mathcal{L}_k^{k!})p^*c_1\big) = \tfrac{1}{2}(k!)^{2k}(2k!)B_k c_1^2 = \left(\frac{(k!)^{2k-2}}{2}\sum_{i=1}^{k}\frac{1}{i}\right)c_1^2,$$

$$p_*\big(c_1^{2k-1}(\mathcal{L}_k^{k!})p^*c_1^2\big) = (k!)^{2k-1}(2k-1)!A_k c_1^2 = (k!)^{2k-3}c_1^2.$$

PROOF. Let E be a coherent sheaf of rank r and L be a line bundle. Then

$$c_k(E \otimes L) = \sum_{i=0}^{k} \frac{(r-i)!}{(k-i)!(r-k)!} c_i(E)c_1(L)^{k-i}.$$

For a surface we have only two Chern classes, $c_1(E\otimes L)=rc_1(L)+c_1(E)$ and $c_2(E\otimes L)=\frac{1}{2}r(r-1)c_1^2(L)+(r-1)c_1(E)c_1(L)+c_2(E)$. From this we get

$$\begin{aligned}\iota(E\otimes L)&=r^2c_1^2(L)+2rc_1(L)c_1(E)+c_1^2(E)-r(r-1)c_1^2(L)\\&\qquad\qquad-2(r-1)c_1(L)c_1(E)-2c_2(E)\\&=c_1^2(E)-2c_2(E)+rc_1^2(L)+2c_1(L)c_1(E)\\&=\iota(E)+rc_1^2(L)+2c_1(L)c_1(E)\end{aligned}$$

and the Euler characteristic (with $c_i=c_i(T^*X)$):

$$\begin{aligned}\chi(E\otimes L)&=\tfrac{1}{2}\big(\iota(E\otimes L)-c_1(E\otimes L)c_1\big)+\frac{1}{12}\mathrm{rk}(E\otimes L)(c_1^2+c_2)\\&=\tfrac{1}{2}\big(\iota(E)+rc_1^2(L)+2c_1(L)c_1(E)-(rc_1(L)+c_1(E))c_1\big)+\tfrac{1}{12}r(c_1^2+c_2)\\&=\chi(E)+\tfrac{1}{2}\big(rc_1^2(L)+2c_1(L)c_1(E)-rc_1(L)\,c_1\big).\end{aligned}$$

For the sheaf of jet differentials we have the asymptotic expansions

$$\begin{aligned}c_1(\mathcal{J}_k^mX)&=B_km^{2k}c_1+O(m^{2k-1}),\\\mathrm{rk}\ \mathcal{J}_k^mX&=A_km^{2k-1}+O(m^{2k-2})\\\chi(\mathcal{J}_k^mX)&=\chi(\mathcal{L}^m)=\tfrac{1}{2}(\alpha_kc_1^2-\beta_kc_2)m^{2k+1}+O(m^{2k})';\end{aligned}$$

hence

$$\begin{aligned}c_1(\mathcal{J}_k^{k!m}X)&=(k!)^{2k}B_km^{2k}c_1+O(m^{2k-1}),\\\mathrm{rk}\ \mathcal{J}_k^{k!m}X&=(k!)^{2k-1}A_km^{2k-1}+O(m^{2k-2})\\\chi(\mathcal{J}_k^{k!m}X)&=\chi(\mathcal{L}^{k!m})=\tfrac{1}{2}(k!)^{2k+1}(\alpha_kc_1^2-\beta_kc_2)m^{2k+1}+O(m^{2k}).\end{aligned}$$

We get from these the asymptotic expansion for $\chi(\mathcal{J}_k^{k!m}X\otimes L^m)$:

$$\begin{aligned}\chi(\mathcal{J}_k^{k!m}X\otimes L^m)&=\chi(\mathcal{J}_k^{k!m}X)+\tfrac{1}{2}m\big(m(\mathrm{rk}\,\mathcal{J}_k^{k!m}X)c_1^2(L)+c_1(L)c_1(\mathcal{J}_k^{k!m}X)\\&\qquad\qquad-(\mathrm{rk}\,\mathcal{J}_k^{k!m}X)c_1(L)c_1\big)\\&=\tfrac{1}{2}\big((k!)^{2k+1}(\alpha_kc_1^2-\beta_kc_2)\\&\qquad+(k!)^{2k-1}A_kc_1^2(L)+(k!)^{2k}B_kc_1(L)c_1\big)m^{2k+1}+O(m^{2k}).\end{aligned}$$

If $c_1(L)=\lambda c_1$ then

$$\begin{aligned}&\chi((\mathcal{L}_k^{k!}\otimes p^*L)^m)\\&\quad=\chi((\mathcal{J}_k^{k!m}X\otimes L^m)\\&\quad=\tfrac{1}{2}\big((k!)^{2k+1}(\alpha_kc_1^2-\beta_kc_2)+(\lambda^2(k!)^{2k-1}A_k+\lambda(k!)^{2k}B_k)c_1^2\big)m^{2k+1}+O(m^{2k})\\&\quad=\chi(\mathcal{J}_k^{k!m}X)+\tfrac{1}{2}\lambda^2(k!)^{2k-1}A_kc_1^2m^{2k+1}+\tfrac{1}{2}\lambda(k!)^{2k}B_kc_1^2m^{2k+1}+O(m^{2k}).\end{aligned}$$

Since $c_1^i(p^*L)=0$ for all $i\geq 3$, we have

$$\begin{aligned}&c_1^{2k+1}(\mathcal{L}_k^{k!}\otimes p^*L)\\&\qquad=(c_1(\mathcal{L}_k^{k!})+c_1(p^*L))^{2k+1}\\&\qquad=c_1^{2k+1}(\mathcal{L}_k^{k!})+(2k+1)c_1^{2k}(\mathcal{L}_k^{k!})c_1(p^*L)+k(2k+1)c_1^{2k-1}(\mathcal{L}_k^{k!})c_1^2(p^*L),\end{aligned}$$

and we get, *up to* $O(m^{2k})$,

$$
\begin{aligned}
&\chi((\mathcal{L}_k^{k!}\otimes p^*L)^m)\\
&\quad= \frac{c_1^{2k+1}(\mathcal{L}_k^{k!}\otimes p^*L)}{(2k+1)!}m^{2k+1}\\
&\quad= \frac{c_1^{2k+1}(\mathcal{L}_k^{k!})+(2k+1)c_1^{2k}(\mathcal{L}_k^{k!})c_1(p^*L)+k(2k+1)c_1^{2k-1}(\mathcal{L}_k^{k!})c_1^2(p^*L)}{(2k+1)!}m^{2k+1}\\
&\quad= \frac{c_1^{2k+1}(\mathcal{L}_k^{k!})}{(2k+1)!}m^{2k+1}+\frac{c_1^{2k}(\mathcal{L}_k^{k!})c_1(p^*L)}{(2k)!}m^{2k+1}+\frac{1}{2}\frac{c_1^{2k-1}(\mathcal{L}_k^{k!})c_1^2(p^*L)}{(2k-1)!}m^{2k+1}\\
&\quad= \frac{c_1^{2k+1}(\mathcal{L}_k^{k!})}{(2k+1)!}m^{2k+1}+\lambda\frac{c_1^{2k}(\mathcal{L}_k^{k!})p^*c_1}{(2k)!}m^{2k+1}+\lambda^2\frac{1}{2}\frac{c_1^{2k-1}(\mathcal{L}_k^{k!})p^*c_1^2}{(2k-1)!}m^{2k+1}.
\end{aligned}
$$

Comparing the two expressions for $\chi((\mathcal{L}_k^{k!}\otimes p^*L)^m)$ we deduce that

$$
\begin{aligned}
p_*c_1^{2k+1}(\mathcal{L}_k^{k!}) &= (2k+1)!\chi(\mathcal{L}_k^{k!}),\\
p_*c_1^{2k}(\mathcal{L}_k^{k!})p^*c_1 &= \tfrac{1}{2}(k!)^{2k}(2k!)B_kc_1^2,\\
p_*c_1^{2k-1}(\mathcal{L}_k^{k!})p^*c_1^2 &= (k!)^{2k-1}(2k-1)!A_kc_1^2.
\end{aligned}
$$

The theorem follows from these by substituting the asymptotic expansions for $\chi(\mathcal{L}_k^{k!})$, A_k and B_k into the expressions above. □

As a means toward understanding the general case we treat the special case of 2-jets and 3-jets (for the case of $\mathbb{P}(TX)$, that is, 1-jets, see [Miyaoka 1977; Lu and Yau 1990; Lu 1991; Dethloff et al. 1995b]). For 2-jets the intersection formulas in Lemma 7.15 and Theorem 7.16 read as:

$$
\begin{aligned}
c_1([D]) &\le \tfrac{3}{4}mc_1,\\
p_*c_1^5(\mathcal{L}_2^2) &= 14c_1^2-10c_2,\\
p_*c_1^4(\mathcal{L}_2^2)p^*c_1 &= 3c_1^2,\\
p_*c_1^3(\mathcal{L}_2^2)p^*c_1^2 &= 2c_1^2.
\end{aligned}
\tag{7.6}
$$

We shall use these formulas to deal with holomorphic maps from the complex plane into a minimal surface X of general type satisfying the conditions that $\operatorname{Pic}X\cong\mathbb{Z}$ and $\mathcal{K}_X$ is effective and nontrivial (for example X is a hypersurface in $\mathbb{P}^3$ of degree $d\ge 5$). *The condition was first introduced in* [Dethloff et al. 1995b] *and is crucial in the rest of this article.* We shall use the following terminology. An irreducible subvariety Y in $\mathbb{P}(J^kX)$ is said to be *horizontal* if $p(Y)=X$, where $p:\mathbb{P}(J^kX)\to X$ is the projection; otherwise it is said to be *vertical*. A variety is said to be horizontal (resp. vertical) if every irreducible component is horizontal (resp. vertical). A subvariety Y may be decomposed as $Y=Y^{\text{hor}}+Y^{\text{ver}}$, where Y^{hor} and Y^{ver} consist respectively of the horizontal and vertical components. Note that $Y^{\text{ver}}=(p^{-1}C)\cap Y$, where C is a subvariety of X; indeed $C=p(Y^{\text{ver}})$. We shall need a lemma:

LEMMA 7.17. *Let X be a surface such that $p_g(X) > 0$ and $\operatorname{Pic} X \cong \mathbb{Z}$ with ample generator $[D_0]$. There exist positive integers m and a and a nontrivial section $\sigma \in H^0\big(\mathbb{P}(J^k X),\, \mathcal{L}_k^{k!m} \otimes p^*[-aD_0]\big)$ such that $[\sigma = 0]^{\mathrm{hor}}$ is reduced and irreducible, that is, there exists exactly one horizontal component with multiplicity* 1.

For the proof of the case $k = 1$, see [Dethloff et al. 1995b, Lemmas 3.5 and 3.6]. The proof depends only on the assumption $\operatorname{Pic} X \cong \mathbb{Z}$, which implies that $\operatorname{Pic} \mathbb{P}(J^1 X) \cong \mathbb{Z} \oplus \mathbb{Z}$. This is of course also valid for $\operatorname{Pic} \mathbb{P}(J^k X)$ for any k. Indeed the proof (with $J^1 X$ replaced by $J^k X$) is word for word the same.

THEOREM 7.18. *Let X be a minimal surface of general type with effective ample canonical bundle such that $\operatorname{Pic} X \cong \mathbb{Z}$, $p_g(X) > 0$, and*

$$17c_1^2(T^*X) - 16c_2(T^*X) > 0.$$

(*This is satisfied if X is a hypersurface of degree $d \geq 70$.*) *Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate.*

PROOF. We start with the weaker assumption $7c_1^2(T^*X) - 5c_2(T^*X) > 0$. (By Theorem 7.7, this is satisfied for smooth hypersurfaces in $\mathbb{P}^3$ if and only if $\deg X \geq 16$.) Under this assumption the sheaf $\mathcal{J}_2^2 X$ is big. This implies that, for any ample divisor D in X there is a section $0 \not\equiv \sigma_1 \in H^0(\mathcal{L}_2^{2m} \otimes p^*[-aK])$ provided that $m \gg 0$ where $a > 0$ and K is the canonical divisor. By Schwarz Lemma (Corollary 6.3) the image of $[j^2 f]$ (as f is algebraically nondegenerate) is contained in the horizontal component of $[\sigma_1 = 0]$. By Lemma 7.17 we may assume that the horizontal component of $[\sigma_1 = 0]$ is irreducible. The vertical component of $[\sigma_1 = 0]$ must be of the form $p^*(bK)$ for some $b \geq 0$ which admits a section s^b. Replacing σ_1 with $\sigma_1 \otimes s^{-b} \in H^0(\mathcal{L}_2^{2m_1} \otimes p^*[-(a-b)K])$, $Y_1 = [\sigma_1 \otimes s^{-b} = 0]$ is horizontal, irreducible and contains the image of $[j^2 f]$. Since $\dim \mathbb{P}(J^2 X) = 5$ the dimension of Y_1 is 4. As remarked earlier we may assume that $a_1 = a - b \geq 0$. We get from the first and third intersection formulas of (7.6):

$$\begin{aligned} c_1^4(\mathcal{L}_2^2|_{Y_1}) &= c_1^4(\mathcal{L}_2^2) \,.\, (c_1(\mathcal{L}_2^{2m_1}) - a_1 p^* c_1) \geq m_1\big(c_1^5(\mathcal{L}_2^2) - a_1 c_1^4(\mathcal{L}_2^2) \,.\, p^* c_1\big) \\ &\geq m_1\big((14c_1^2 - 10c_2) - \tfrac{9}{4}c_1^2\big) = \frac{m_1}{2^2}(47c_1^2 - 40c_2) > 0. \end{aligned}$$

(For a hypersurface of degree d in $\mathbb{P}^n$ we have $c_1^2 = (d-4)^2$, $c_2 = d^2 - 4d + 6$. Thus, for $d = 16$, $47c_1^2 = 6768$ and $40c_2 = 7920$, so $\chi(\mathcal{L}_2^2|_Y) < 0$; however,

$$47c_1^2 - 40c_2 = 47(d^2 - 8d + 16) - 40(d^2 - 4d + 6) = 7d(d - 30) - 2(3d - 256)$$

is positive if and only if $d \geq 40$.) We claim that $\mathcal{L}_2^2|_{Y_1}$ is big. It suffices to show that $H^2(\mathcal{L}_2^{2m_1} \otimes [-Y_1]) = 0$ for $m \gg 0$. To see this consider the exact sequence

$$0 \to \mathcal{L}_2^{2m_1} \otimes [-Y_1] \xrightarrow{\otimes\sigma} \mathcal{L}_2^{2m_1} \to \mathcal{L}_2^{2m_1}|_{Y_1} \to 0$$

and the induced exact sequence

$$\cdots \to H^2(\mathcal{L}_2^{2m_1} \otimes [-Y_1]) \xrightarrow{\otimes\sigma} H^2(\mathcal{L}_2^{2m_1}) \to H^2(\mathcal{L}_2^{2m_1}|_{Y_1}) \to 0.$$

The vanishing of $H^2(\mathcal{L}_2^{2m_1}|_{Y_1})$ for $m_1 \gg 0$ follows from the vanishing of $H^2(\mathcal{L}_2^{2m})$. By Schwarz's Lemma, the image of $[j^k f]$ is contained in the zero set of any nontrivial section $\sigma_2 \in H^0(Y_1, \mathcal{L}_2^{2m_1}|_{Y_1} \otimes p^*[-a_2 K])$, $a_2 > 0$ and $m_2 \gg 0$. Since Y_1 is irreducible $Y_2 = [\sigma_2 = 0] \cap Y_1$ is of codimension 2 (so $\dim Y_2 = 3$) in $\mathbb{P}(J^2 X)$ where $\sigma_2 \in H^0(\mathcal{L}_2^{2m_2}|_{Y_1} \otimes [-a_2 D])$. By Schwarz's Lemma the reparametrized k-jets $\{[j^k(f \circ \phi)]\}$ is contained in Y_2.

We may assume that Y_2 is irreducible. Otherwise $Y_2 = \sum_{i=1}^n Y_{2,i}$, where $n \geq 2$ and each $Y_{2,i}$, is irreducible and hence effective. We have $\bigotimes_{i=1}^n [Y_{2,i}] = [Y_2] = \mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_1}$ (we use the notation $[Z]$ to denote the line bundle associated to a divisor Z). The image $[j^k f](\mathbb{P}^n)$ is contained in Y_{2,i_0} for some $1 \leq i_0 \leq n$. Let s_i be the (regular) section such that $[s_i = 0] = Y_{2,i}$ (an effective divisor in Y_1); then we have an exact sequence

$$0 \to [Y_{2,i_0}] \overset{\rho_{i_0}}{\to} \mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_1} \to \mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_{2,i_0}} \to 0.$$

In particular, we have an injection

$$0 \to [Y_{2,i_0}] \overset{\rho_{i_0}}{\to} \mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_1},$$

where the map ρ_{i_0} is defined by multiplication with the section $\bigotimes_{i=1, i \neq i_0}^n s_i$. In other words we may consider each $[Y_{2,i_0}]$ as a subsheaf of $\mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_1}$ hence a section of $[Y_{2,i_0}]$ is identified also as a section of $\mathcal{L}_k^{k! m_2} \otimes p^*[-a_2 K]|_{Y_1}$. The Schwarz Lemma applies and we conclude that $s_{i_0}([j^k f]) \equiv 0$ for each i. Thus we may assume that Y_2 is irreducible by replacing Y_2 with Y_{2,i_0}.

We now repeat the previous calculation for Y_1 to Y_2 using again the intersection formulas listed above; we get

$$\begin{aligned}
c_1^3(\mathcal{L}_2^2|_{Y_2}) &= c_1^3(\mathcal{L}_2^2) \,.\, (c_1(\mathcal{L}_2^{2m_1}) - a_1 p^* c_1) \,.\, (c_1(\mathcal{L}_2^{2m_2}) - a_2 p^* c_1) \\
&\geq \big(m_1 m_2 c_1^5(\mathcal{L}_2^2) - (a_1 m_2 + m_1 a_2) c_1^4(\mathcal{L}_2^2) \,.\, p^* c_1 + a_1 a_2 c_1^3(\mathcal{L}_2^2) \,.\, p^* c_1^2\big) \\
&= m_1 m_2 \big(c_1^5(\mathcal{L}_2^2) - (l_1 + l_2) c_1^4(\mathcal{L}_2^2) \,.\, p^* c_1 + l_1 l_2 c_1^3(\mathcal{L}_2^2) \,.\, p^* c_1^2\big) \\
&= m_1 m_2 \big((14 c_1^2 - 10 c_2) - 3(l_1 + l_2) c_1^2 + 2 l_1 l_2 c_1^2\big),
\end{aligned}$$

where $0 \leq l_i = a_i / m_i \leq \frac{3}{4}$, for $i = 1, 2$. Elementary calculus shows that the function $14 - 3(l_1 + l_2) + 2 l_1 l_2$ achieves its minimum value $14 - 3(\frac{3}{4} + \frac{3}{4}) + 2(\frac{3}{4})^2 = \frac{85}{8}$ at $l_1 = l_2 = \frac{3}{4}$; thus we get

$$c_1^3(\mathcal{L}_2^2|_{Y_2}) \geq m_1 m_2 (\tfrac{85}{8} c_1^2 - 10 c_2) = \frac{5 m_1 m_2}{2^3} (17 c_1^2 - 16 c_2) > 0.$$

This shows that $\mathcal{L}_2^2|_{Y_2}$ is big and the image of $[j^2 f]$ is contained in

$$Y_3 = Y_2 \cap [\sigma_3 = 0]$$

(where the intersection is taken over all global sections σ_3 of $\mathcal{L}_2^{2m}|_{Y_2}$ vanishing on an ample divisor), which is of dimension 2. By Corollary 6.5 the dimension of the base locus is at least 3 if f is algebraically nondegenerate. Thus f must be algebraically degenerate (and if X contains no rational or elliptic curve then X is hyperbolic). $\square$

Note that the intersection procedure was applied twice. For a smooth hypersurface X in $\mathbb{P}^3$, condition (iii) is satisfied if and only degree of $X \geq 70$ (this is easily checked from the formulas $c_1^2 = (d-4)^2$, $c_2 = d^2 - 4d + 6$). This can be improved if we use 3-jets. For 3-jets the intersection formulas of Lemma 7.15 and Theorem 7.16 are given explicitly as follows:

$$\begin{aligned}
c_1([D]) &\leq \frac{11m}{3!} c_1,\\
p_* c_1^7(\mathcal{L}_3^{3!}) &= \frac{7!(3!)^7}{2}\Big(\frac{17}{2^7 3^6 7} c_1^2 - \frac{7}{2^7 3^6 5} c_2\Big) = (3!)^3(85c_1^2 - 49c_2),\\
p_* c_1^6(\mathcal{L}_3^{3!}) p^* c_1 &= \frac{(3!)^3 11}{2} c_1^2,\\
p_* c_1^5(\mathcal{L}_3^{3!}) p^* c_1^2 &= (3!)^3 c_1^2.
\end{aligned}$$

THEOREM 7.19. *Let X be a minimal surface with* $\operatorname{Pic} X \cong \mathbb{Z}$, $p_g(X) > 0$, *and*

$$389c_1^2(T^*X) - 294c_2(T^*X) > 0. \qquad (*)$$

Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate.

PROOF. The sheaf $\mathcal{J}_3 X$ is big if and only if degree $d \geq 11$. As in the case of 2-jets we know that the image of $[j^3 f]$ is contained in $Y_1 = [\sigma_1 = 0]$ for some $\sigma_1 \in H^0(\mathcal{L}_2^{2m_1} \otimes p^*[-a_1 K])$. Since $\dim \mathbb{P}(J^3 X) = 7$ the dimension of Y_1 is 6. From the intersection formulas listed above we get

$$\begin{aligned}
c_1^6(\mathcal{L}_2^2|_{Y_1}) &= c_1^6(\mathcal{L}_2^{3!}) \,.\, (c_1(\mathcal{L}_2^{2m_1}) - a_1 p^* c_1) \geq m_1\big(c_1^7(\mathcal{L}_2^{3!}) - a_1 c_1^6(\mathcal{L}_2^{3!}) \,.\, p^* c_1\big)\\
&\geq (3!)^3 m_1\big((85c_1^2 - 49c_2) - \tfrac{1}{12} 11^2 c_1^2\big) = (3!)^3 \tfrac{1}{12} m_1 (899c_1^2 - 588c_2) > 0.
\end{aligned}$$

For a smooth hypersurface in $\mathbb{P}^3$, $899c_1^2 - 588c_2 > 0$ if and only if $d \geq 13$.

Continuing as in the case of 2-jets, we see that the image of $[j^3 f]$ is contained in the zero set of any nontrivial section $\sigma_2 \in H^0\big(Y_1, \mathcal{L}_2^{(3!)m_1}|_{Y_1} \otimes p^*[-a_2 K]\big)$, $a_2 > 0$ and $m_2 \gg 0$. The dimension of $Y_2 = [\sigma_2 = 0] \cap Y_1$ is 5. By Schwarz's Lemma the reparametrized 3-jet $\{[j^3(f \circ \phi)]\}$ is contained in Y_2. As in the case of 2-jets we may assume that Y_2 is irreducible. We now repeat the previous calculation using the intersection formulas above:

$$\begin{aligned}
c_1^5(\mathcal{L}_3^{3!}|_{Y_2}) &= c_1^5(\mathcal{L}_3^{3!}) \,.\, (c_1(\mathcal{L}_3^{(3!)m_1}) - a_1 p^* c_1) \,.\, (c_1(\mathcal{L}_3^{(3!)m_2}) - a_2 p^* c_1)\\
&\geq \big(m_1 m_2 c_1^7(\mathcal{L}_3^{3!}) - (a_1 m_2 + m_1 a_2) c_1^6(\mathcal{L}_3^{3!}) \,.\, p^* c_1 + a_1 a_2 c_1^5(\mathcal{L}_3^{3!}) \,.\, p^* c_1^2\big)\\
&= m_1 m_2 \big(c_1^7(\mathcal{L}_3^{3!}) - (l_1 + l_2) c_1^6(\mathcal{L}_3^{3!}) \,.\, p^* c_1 + l_1 l_2 c_1^5(\mathcal{L}_3^{3!}) \,.\, p^* c_1^2\big)\\
&= (3!)^3 m_1 m_2 \big((85c_1^2 - 49c_2) - \tfrac{11}{2}(l_1 + l_2) c_1^2 + l_1 l_2 c_1^2\big)
\end{aligned}$$

where $0 \leq l_i = a_i/m_i \leq \frac{11}{6}$ for $i = 1, 2$. Elementary calculus shows that the function $85 - \frac{11}{2}(l_1 + l_2) + l_1 l_2$ achieves its minimum value at $l_1 = l_2 = \frac{11}{6}$; thus we have

$$c_1^5(\mathcal{L}_3^{3!}|_{Y_2}) \geq (3!)^3 m_1 m_2 \big(\tfrac{2455}{36} c_1^2 - 49 c_2\big) > 0.$$

For a smooth hypersurface in $\mathbb{P}^3$ this occurs if and only if $d \geq 18$.

Now the image of $[j^3 f]$ is contained in $Y_3 = [\sigma_3 = 0] \cap Y_2$ and has dimension 4. Moreover, an argument identical to the case of Y_2 shows that we may assume Y_3 irreducible. Continuing with the procedure we get

$$c_1^4(\mathcal{L}_3^{3!}|_{Y_3}) = c_1^4(\mathcal{L}_3^{3!}) \prod_{i=1}^{3} \big(c_1(\mathcal{L}_3^{(3!)m_i}) - a_i p^* c_1\big).$$

Expanding the right-hand side above yields (note that $p^* c_1^3 \equiv 0$ because the dimension of the base space is 2 hence $c_1^3 = c_1^3(X) \equiv 0$)

$$\begin{aligned} m_1 m_2 m_3 c_1^7(\mathcal{L}_3^{3!}) - (a_1 m_2 m_3 + m_1 a_2 m_3 + m_1 m_2 a_3) c_1^6(\mathcal{L}_3^{3!}) \,.\, p^* c_1 \\ + (a_1 a_2 m_3 + a_1 m_2 a_3 + m_1 a_2 a_3) c_1^5(\mathcal{L}_3^{3!}) \,.\, p^* c_1^2, \end{aligned}$$

so we have

$$\begin{aligned} c_1^4(\mathcal{L}_3^{3!}|_{Y_3}) &\geq \big(m_1 m_2 m_3 c_1^7(\mathcal{L}_3^{3!}) - (a_1 m_2 m_3 + m_1 a_2 m_3 + m_1 m_2 a_3) c_1^6(\mathcal{L}_3^{3!}) \,.\, p^* c_1 \\ &\qquad + (a_1 a_2 m_3 + a_1 m_2 a_3 + m_1 a_2 a_3) c_1^5(\mathcal{L}_3^{3!}) \,.\, p^* c_1^2\big) \\ &= m_1 m_2 m_3 \big(c_1^7(\mathcal{L}_3^{3!}) - (l_1 + l_2 + l_3) c_1^6(\mathcal{L}_3^{3!}) \,.\, p^* c_1 \\ &\qquad + (l_1 l_2 + l_2 l_3 + l_3 l_1) c_1^5(\mathcal{L}_3^{3!}) \,.\, p^* c_1^2\big) \\ &= (3!)^3 m_1 m_2 m_3 \big((85 c_1^2 - 49 c_2) - \tfrac{11}{2}(l_1 + l_2 + l_3) c_1^2 \\ &\qquad + (l_1 l_2 + l_2 l_3 + l_3 l_1) c_1^2\big), \end{aligned}$$

where $0 \leq l_i = a_i / m_i \leq \frac{11}{6}$ for $i = 1, 2, 3$. Elementary calculus shows that the function $85 - \frac{11}{2}(l_1 + l_2 + l_3) + (l_1 l_2 + l_2 l_3 + l_3 l_1)$ achieves its minimum value at $l_1 = l_2 = l_3 = \frac{11}{6}$; thus we get

$$c_1^4(\mathcal{L}_3^{3!}|_{Y_3}) \geq \frac{(3!)^3}{6} m_1 m_2 m_3 (389 c_1^2 - 294 c_2) > 0.$$

For hypersurfaces in $\mathbb{P}^3$ this happens if and only if $d \geq 20$. Thus the image of $[j^3 f]$ is contained in a subvariety $Y_4 = Y_3 \cap [\sigma_4 = 0]$ which is of dimension 3. By Corollary 6.5 the map f must be algebraically degenerate. □

Note that the intersection procedure was applied three times. In order to remove condition $(*)$ in Theorem 7.19 we must use very high order jets, and if we use k-jets then it is necessary to carry out the intersection procedure k times. The preceding proof underscores the importance of the explicit formulas obtained in Section 3.

THEOREM 7.20. *Let X be a smooth minimal surface of general type with $p_g(X) > 0$ and $\operatorname{Pic} X \cong \mathbb{Z}$. Then every holomorphic map $f : \mathbb{C} \to X$ is algebraically degenerate. If, in addition, the surface X contains no rational nor elliptic curve then X is hyperbolic.*

PROOF. As remarked earlier, we have to work with the k-jet bundles for k sufficiently large. In the case of 2-jets the cutting procedure was applied twice and for 3-jets, 3 times. Now we have to do this k-times, each time making sure (by using the explicit formulas of section 3) that the bundle is still big.

The assumption implies that $\mathcal{L}_k^{k!}$ is big for $k \gg 0$ hence there exists $m_1 \gg k$ and $a_1 > 0$ such that $h^0(\mathbb{P}(J^kX), \mathcal{L}_k^{k!m_1} \otimes p^*[-a_1K]) > 0$ where K is the canonical divisor. As in the proof of Theorem 7.18 (and Theorem 7.19) we may, by Lemma 7.17, assume that there exists $\sigma_1 \in H^0(\mathbb{P}(J^kX), \mathcal{L}_k^{k!m} \otimes p^*[-a_1K])$ such that $Y_1 = [\sigma_1 = 0]$ is horizontal and irreducible. This implies that codim $Y_1 = 1$ (equivalently, $\dim Y_i = \dim \mathbb{P}(J^kX) - 1 = 2k+1-1 = 2k$).

By the Schwarz Lemma of the preceding section, we conclude that the image of $[j^kf]$ is contained in Y_1. The proof of Theorem 7.19 shows that $\mathcal{L}_{\mathbb{P}(J^kX)}|_{Y_1}$ is still big and so there exists $\sigma_2 \in H^0(Y_1, \mathcal{L}_k^{k!m_2} \otimes p^*[-a_2K]), m_2, a_2 > 0$ and (because Y_1 is irreducible) that $Y_2 = [\sigma_2 = 0]$ is of codimension 2 in $\mathbb{P}(J^kX)$. Schwarz's Lemma implies that the image of $[j^kf]$ is contained in Y_2. As was shown earlier, we may assume that Y_2 is irreducible. A calculation similar to that of Theorem 7.19 shows that $\mathcal{L}_k^{k!}|_{Y_2}$ is still big (see the calculation below).

The process can be continued k times, resulting in a sequence of reduced and irreducible horizontal subvarieties,

$$\mathbb{P}(J^kX) = Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_k \supset [j^kf](\mathbb{C}),$$

where codim $Y_i = i$ (equivalently, $\dim Y_i = 2k+1-i$ as $\dim \mathbb{P}(J^kX) = 2k+1$), and each of the subvarieties is the zero set of a section σ_i:

$$Y_i = [\sigma_i = 0],\ \sigma_i \in H^0\big(Y_{i-1},\, \mathcal{L}_k^{k!m_i} \otimes p^*[-a_iK]\big) \quad \text{for } 1 \le i \le k.$$

We claim that $\mathcal{L}_k^{k!}|_{Y_i}$ *is big for* $1 \le i \le k$, *by a calculation* (*to be carried out below*) *analogous to that in Theorems* 7.19 *and* 7.20.

Assuming this for the moment, we see that there exists a nontrivial section $\sigma_{k+1} \in H^0\big(Y_k,\, \mathcal{L}_k^{k!m_k} \otimes p^*[-a_{k+1}K]\big)$ and $[j^kf](\mathbb{C})$ is contained in an irreducible component of $[\sigma_{k+1} = 0] \cap Y_k$. Since Y_k is irreducible this component, denoted Y_{k+1}, is of codimension $k+1$ (equivalently, $\dim Y_{k+1} = 2k+1-(k+1) = k$). This however contradicts Corollary 6.5 that the component containing all the reparametrization $[j^k(f \circ \phi)](\mathbb{C})$ must be of codimension at most k (equivalently, dimension at least $k+1$) if f is algebraically nondegenerate. Thus the map f must be algebraically degenerate. Since the image of an algebraically degenerate map must be contained in a rational or an elliptic curve in X, we conclude readily that X is hyperbolic if it contains no rational nor elliptic curve.

It remains to verify the claim by carrying out the computations for k-jets—more precisely, computations for the Chern numbers $c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}|_{Y_\lambda})$, for $1 \le \lambda \le k$—using Theorem 7.16 the intersection formulas obtained in Lemma 7.15:

$c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}|_{Y_\lambda})$

$$\begin{aligned}
&= c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}) \prod_{i=1}^{\lambda} \bigl(m_i c_1(\mathcal{L}_k^{k!}) - a_i c_1\bigr) \\
&= \Bigl(\prod_{i=1}^{\lambda} m_i\Bigr) c_1^{2k+1}(\mathcal{L}_k^{k!}) - \Bigl(\sum_{i=1}^{\lambda} a_i \prod_{1\le j\ne i\le\lambda} m_j\Bigr) c_1^{2k}(\mathcal{L}_k^{k!}X) c_1 \\
&\qquad\qquad + \Bigl(\sum_{1\le i<j\le l} a_i a_j \prod_{1\le q\ne i,j\le\lambda} m_q c_1^{2k-1}(\mathcal{L}_k^{k!})\Bigr) c_1^2 \\
&= (k!)^{2k-3} \Bigl(\prod_{i=1}^{\lambda} m_i\Bigr) \Biggl((k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}\Bigr) c_1^2 - (k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2}\Bigr) c_2 \\
&\qquad\qquad - \Bigl(\sum_{i=1}^{\lambda} l_i\Bigr)\Bigl(\frac{k!}{2} \sum_{i=1}^{k} \frac{1}{i}\Bigr) c_1^2 + \Bigl(\sum_{1\le i<j\le\lambda} l_i l_j\Bigr) c_1^2\Biggr)
\end{aligned}$$

for $1 \le l_j = a_j/m_j \le (B_k/A_k)k!$ and $1 \le \lambda \le k$. The coefficient of c_1^2 is

$$D_{k,\lambda} = (k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}\Bigr) - \Bigl(\sum_{i=1}^{\lambda} l_i\Bigr)\Bigl(\frac{k!}{2} \sum_{i=1}^{k} \frac{1}{i}\Bigr) + \sum_{1\le i<j\le\lambda} l_i l_j.$$

The minimum occurs at

$$l_j = \frac{B_k}{A_k} k! = \frac{(k-1)!}{2} \sum_{i=1}^{k} \frac{1}{i}$$

for all $1 \le j \le \lambda \le k$. By the intersection formulas in Lemma 7.15 and Theorem 7.16, we have:

$$D_{k,\lambda} \ge (k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}\Bigr) - \lambda \frac{(k-1)!\,k!}{4} \Bigl(\sum_{i=1}^{k} \frac{1}{i}\Bigr)^2 + \frac{\lambda(\lambda+1)}{4k} \Bigl(\sum_{i=1}^{k} \frac{1}{i}\Bigr)^2.$$

It is clear that the worst case occurs for $\lambda = k$, namely, $D_{k,\lambda} \ge D_{k,k}$, and that

$$D_{k,k} \ge (k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j}\Bigr) - \frac{(k!)^2}{4} \Bigl(\sum_{i=1}^{k} \frac{1}{i}\Bigr)^2 + \frac{(k!)^2(k+1)}{4(k!)^2} \Bigl(\sum_{i=1}^{k} \frac{1}{i}\Bigr)^2.$$

In other words, denoting the expression on the right-hand side above by $(k!)^2\delta_k$, we have

$$\begin{aligned}
c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}|_{Y_\lambda}) &= c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}) \prod_{i=1}^{\lambda} \bigl(m_i c_1(\mathcal{L}_k^{k!}X) - a_i c_1\bigr) \\
&\ge (k!)^{2k-3} \Bigl(\prod_{i=1}^{\lambda} m_i\Bigr) \Bigl((k!)^2 \delta_k c_1^2 - (k!)^2 \Bigl(\sum_{i=1}^{k} \frac{1}{i^2}\Bigr) c_2\Bigr) \\
&= (k!)^{2k-1} \Bigl(\prod_{i=1}^{\lambda} m_i\Bigr) \Bigl(\delta_k c_1^2 - \Bigl(\sum_{i=1}^{k} \frac{1}{i^2}\Bigr) c_2\Bigr)
\end{aligned}$$

for $1 \leq \lambda \leq k$. It remains to show that

$$\delta_k c_1^2 - \left(\sum_{i=1}^{k} \frac{1}{i^2} \right) c_2 > 0$$

or, equivalently,

$$\frac{\delta_k}{\sum_{i=1}^{k} 1/i^2} > \frac{c_2}{c_1^2} \tag{7.7}$$

for k sufficiently large. We claim that

$$\lim_{k\to\infty} \frac{\delta_k}{\sum_{i=1}^{k} 1/i^2} = \infty, \tag{7.8}$$

where

$$\delta_k = \left(\sum_{i=1}^{k} \frac{1}{i^2} + \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j} \right) - \frac{1}{4} \left(\sum_{i=1}^{k} \frac{1}{i} \right)^2 + \frac{(k+1)}{4(k!)^2} \left(\sum_{i=1}^{k} \frac{1}{i} \right)^2.$$

Observe that

$$\sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j} = \sum_{1\leq i<j\leq k} \frac{1}{ij} \quad \text{and} \quad \left(\sum_{i=1}^{k} \frac{1}{i} \right)^2 = \sum_{i=1}^{k} \frac{1}{i^2} + 2 \sum_{1\leq i<j\leq k} \frac{1}{ij};$$

hence

$$\delta_k \geq \frac{1}{2} \sum_{1\leq i<j\leq k} \frac{1}{ij} - \frac{3}{4} \sum_{i=1}^{k} \frac{1}{i^2},$$

and the ratio satisfies

$$\frac{\delta_k}{\sum_{i=1}^{k} 1/i^2} \geq \frac{1}{2} \frac{\sum_{1\leq i<j\leq k} 1/(ij)}{\sum_{i=1}^{k} 1/i^2} - \frac{3}{4}.$$

Since

$$\sum_{1\leq i<j\leq k} \frac{1}{ij} = \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j},$$

we must show that

$$\lim_{k\to\infty} \frac{\sum_{i=2}^{k} 1/i \sum_{j=1}^{i-1} 1/j}{\sum_{i=1}^{k} 1/i^2} = \infty,$$

just as in the limit in Corollary 3.10. But this is clear, because

$$\lim_{k\to\infty} \sum_{i=2}^{k} \frac{1}{i} \sum_{j=1}^{i-1} \frac{1}{j} \geq \lim_{k\to\infty} \sum_{i=2}^{k} (i-1) \frac{1}{i^2} = \infty,$$

whereas $\lim_{k\to\infty} \sum_{i=1}^{k} 1/i^2 < \infty$. Thus (7.7) is verified.

We remark that $c_2/c_1^2 = 11$ for a smooth hypersurface of degree $d = 5$ in $\mathbb{P}^3$. Thus, by (7.7), it is enough to choose k so that

$$\frac{\delta_k}{\sum_{i=1}^{k} 1/i^2} > 11.$$

With the aid of a computer, we found that this occurs at $k = 2283$ (for $k = 2282$ the ratio on the left above is approximately 10.9998). By Theorem 7.1,

$$\begin{cases} 5c_1^2 - c_2 + 36 \geq 0, & \text{if } c_1^2 \text{ is even,} \\ 5c_1^2 - c_2 + 30 \geq 0, & \text{if } c_1^2 \text{ is odd,} \end{cases}$$

which implies that

$$\begin{cases} 23 \geq 5 + (36/c_1^2) \geq c_2/c_1^2, & \text{if } c_1^2 \text{ is even,} \\ 35 \geq 5 + (30/c_1^2) \geq c_2/c_1^2, & \text{if } c_1^2 \text{ is odd.} \end{cases}$$

Thus, by (7.7), we need k so that the ratio $\delta_k / \sum_{i=1}^k 1/i^2$ is > 23 if c_1^2 is even and > 35 if it is odd. We did not find the explicit k satisfying these conditions; this would take a lot of time, even for the computer. However we do know from (7.8) that k exists. This shows that $c_1^{2k+1-\lambda}(\mathcal{L}_k^{k!}|_{Y_\lambda}) > 0$ hence

$$\sum_{i=0}^{2} (-1)^i H^i(Y_\lambda, \mathcal{L}_k^{k!}|_{Y_\lambda}) = \chi(\mathcal{L}_k^{k!}|_{Y_\lambda}) > 0.$$

To show that $\mathcal{L}_k^{k!}|_{Y_\lambda}$ is big it is sufficient to show that $H^2(Y_\lambda, \mathcal{L}_k^{k!}|_{Y_\lambda}) = 0$ for $0 \leq \lambda \leq k$. This is done as in Theorems 7.18 and 7.19 by considering the exact sequences

$$0 \to \mathcal{L}_k^{k!m_\lambda}|_{Y_{\lambda-1}} \otimes [-Y_\lambda] \xrightarrow{\otimes \sigma_\lambda} \mathcal{L}_k^{k!m_\lambda}|_{Y_{\lambda-1}} \to \mathcal{L}_k^{k!m_\lambda}|_{Y_\lambda} \to 0$$

and the induced exact sequence

$$\cdots \to H^2\big(\mathcal{L}_k^{k!m_\lambda}|_{Y_{\lambda-1}} \otimes [-Y_\lambda]\big) \to H^2(\mathcal{L}_k^{k!m_\lambda}|_{Y_{\lambda-1}}) \to H^2(\mathcal{L}_k^{k!m_\lambda}|_{Y_\lambda}) \to 0.$$

By induction $H^2(\mathcal{L}_k^{k!m_\lambda}|_{Y_{\lambda-1}}) = 0$ for $m_\lambda \gg 0$ and the exact sequence above implies the vanishing of $H^2(\mathcal{L}_k^{k!m_\lambda}|_{Y_\lambda})$. This completes the proof of the theorem. □

COROLLARY 7.21. *A generic hypersurface surface of degree $d \geq 5$ in $\mathbb{P}^3$ is hyperbolic.*

PROOF. The assumptions of Theorem 7.19 are satisfied by a generic hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$. Thus the image of a holomorphic map $f : \mathbb{C} \to X$ is contained in a curve, necessarily rational or elliptic curve. By a theorem of Xu [1994] a generic hypersurface surface of degree $d \geq 5$ in $\mathbb{P}^3$ contains no rational nor elliptic curve. Hence f must be a constant. □

The generic condition in Xu means that the statement holds for all curves outside a countable union of Zariski closed sets. A variety X satisfying the condition that every holomorphic curve $f : \mathbb{C} \to X$ is constant is usually referred to as Brody hyperbolic. In general *Kobayashi hyperbolic* implies *Brody hyperbolic*. For compact varieties Brody hyperbolic is equivalent to Kobayashi hyperbolic but for open varieties this is not the case. As a consequence of Corollary 7.21 we have:

Corollary 7.22. *There exists a curve C of degree $d = 5$ in $\mathbb{P}^2$ such that $\mathbb{P}^2 \setminus C$ is Kobayashi hyperbolic.*

Proof. It is well-known that the complement of 5 lines, in general position, in $\mathbb{P}^2$ is Kobayashi hyperbolic. By a Theorem of Zaidenberg [1989] any sufficiently small (in the sense of the classical topology, rather than the Zariski topology) deformation of a Brody hyperbolic manifold is Brody hyperbolic. Thus, for any curve C of degree 5 in a sufficiently small open (in the classical topology) neighborhood U, of 5 lines in general position, the complement $\mathbb{P}^2 \setminus C$ is Brody hyperbolic. Let $\bigcup Z_i$ be a countable union of Zariski closed sets in the space of surfaces of degree 5 in $\mathbb{P}^3$ such that any surface $S \notin \cup Z_i$ is hyperbolic. Embed $\mathbb{P}^2$ in $\mathbb{P}^3$ as a linear subspace. Any curve $C \in \mathcal{C} = \{S \cap \mathbb{P}^2 \mid S \notin \cup Z_i\}$ is a curve of degree 5 and is hyperbolic. It is clear that $\mathcal{C} \cap U$ is nonempty. Thus there exists a hyperbolic curve C of degree 5 in $\mathbb{P}^2$ such that $\mathbb{P}^2 \setminus C$ is Brody hyperbolic. It is well-known that this implies that $\mathbb{P}^2 \setminus C$ is Kobayashi hyperbolic. □

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