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Anisotropic and Crystalline Mean Curvature Flow

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Contents

1. Introduction

Motion by mean curvature of an embedded smooth hypersurface without boundary has been the subject of several recent papers, because of its geometric interest and of its application to different areas, see for instance the pioneering book [Brakke 1978], or the papers [Allen and Cahn 1979], [Huisken 1984], [Osher and Sethian 1988], [Evans and Spruck 1991], [Almgren et al. 1993]. A smooth boundary ∂E of an open set $E = E(0) \subset \mathbb{R}^n$ flows by mean curvature if there exists a time-dependent family $(\partial E(t))_{t\in[0,T]}$ of smooth boundaries satisfying the following property: the normal velocity of any point $x \in \partial E(t)$ is equal to the sum of the principal curvatures of $\partial E(t)$ at x. One can show that, at each time t, such an evolution process reduces the area of $\partial E(t)$ as fast as possible. Mean curvature flow has therefore a variational character, since it can be interpreted as the gradient flow associated with the area functional $\partial E \to \mathcal{H}^{n-1}(\partial E)$, where $\mathcal{H}^{n-1}$ indicates the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

In several physical processes (for instance in certain models of dendritic growth and crystal growth, see [Cahn et al. 1992], or in statistical physics (see for example [Spohn 1993]) it turns out, however, that the evolution of the surface is not simply by mean curvature, but is an anisotropic evolution. From the energy point of view, this means that the functional of which we are taking the gradient flow is not the area of ∂E anymore, but is a weighted area, which can be derived by looking at $\mathbb{R}^n$ as a normed space. Let $\phi : \mathbb{R}^n \to [0, +\infty[$ be a norm on $\mathbb{R}^n$. One of the most common area measures of ∂E in the normed space $(\mathbb{R}^n, \phi)$ is the so-called Minkowski content $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ of ∂E induced by ϕ. Denoting by $d_\phi(x,y) := \phi(y-x)$ the distance on $\mathbb{R}^n$ induced by ϕ and by $\mathcal{H}^n_{d_\phi}$ the n-dimensional Hausdorff measure with respect to d_ϕ, $\mathcal{M}^{n-1}_{d_\phi}$ is defined as

$$\mathcal{M}^{n-1}_{d_\phi}(\partial E) := \lim_{\rho\to 0^+} \frac{1}{2\rho}\mathcal{H}^n_{d_\phi}\big(\{z \in \mathbb{R}^n : \inf_{x\in\partial E} d_\phi(z,x) < \rho\}\big). \tag{1–1}$$

Since it is possible to prove that $\mathcal{H}^n_{d_\phi}$ coincides with the Lebesgue measure $|\cdot|$ multiplied by the factor

$$c_{n,\phi} := \frac{\omega_n}{|\{\xi \in \mathbb{R}^n : \phi(\xi) \le 1\}|},$$

ω_n being a normalizing constant, we have

$$\mathcal{M}^{n-1}_{d_\phi}(\partial E) = c_{n,\phi} \lim_{\rho\to 0^+} \frac{1}{2\rho}\big|\{z \in \mathbb{R}^n : \inf_{x\in\partial E} d_\phi(z,x) < \rho\}\big|. \tag{1–2}$$

Therefore, $\mathcal{M}^{n-1}_{\phi}(\partial E)$ measures (for small $\rho > 0$) the ratio between the volume of a ρ-tubular neighborhood of ∂E and ρ. Definition (1–1) can be made more explicit, since it turns out that

$$\mathcal{M}^{n-1}_{d_\phi}(\partial E) = c_{n,\phi} \int_{\partial E} \phi^o(\nu^E)\ d\mathcal{H}^{n-1}. \tag{1–3}$$

Here ν^E is the Euclidean unit normal to ∂E pointing outside of E, and the function $\phi^o : \mathbb{R}^n \to [0, +\infty[$ is the dual norm of ϕ. Physically, $\phi^o(\nu)$ plays the rôle of a surface tension of a flat surface whose normal is ν, and can be considered as the anisotropy. The functional (1–3) is the above mentioned weighted area, whose gradient flow gives raise to the so-called anisotropic motion by mean curvature. In the *regular* case, that is, when ϕ^2 is smooth and strictly convex, the relevant quantity is the so-called Cahn–Hoffman vector field n^E_ϕ on ∂E, which is the image of $\nu^E_\phi := \nu^E/\phi^o(\nu^E)$ through the map $\frac{1}{2}\nabla((\phi^o)^2)$, and whose divergence is the anisotropic mean curvature of ∂E (denoted by κ_ϕ^E). The anisotropic mean curvature is derived (again by a variational principle) in the computation of the first variation of (1–3). Therefore, for any time t, anisotropic mean curvature flow is defined in such a way to decrease $\mathcal{M}_{d_\phi}^{n-1}(\partial E(t))$ as fast as possible.

Besides the regular case, other anisotropies can be considered; we are particularly interested in the *crystalline case*, when the function ϕ is piecewise linear (or equivalently when the unit ball $B_\phi := \{\phi \leq 1\}$ is a polytope). From the mathematical point of view, this field of research was initiated by the work of J. Taylor [Taylor 1978], [Taylor 1986], [Taylor 1991], [Cahn et al. 1992], [Taylor 1992], [Taylor 1993], [Almgren and Taylor 1995]. See also the papers [Hoffman and Cahn 1972], [Cahn and Hoffman 1974], [Cahn et al. 1993]. Recently, several authors contributed to the subject: see for instance

- [Girao and Kohn 1994], [Girao 1995], [Rybka 1997], [Ishii and Soner 1999] and [Giga and Giga 2000] for general properties of the crystalline flow in two dimensions and for the convergence of a crystalline algorithm;
- [Stancu 1996] for self-similar solutions of the crystalline flow in two dimensions;
- [Fukui and Giga 1996], [Giga and Giga 1997], [Giga and Giga 1998b], [Giga and Giga 1999] for the crystalline evolution of graphs in two dimensions;
- [Giga and Gurtin 1996] for a comparison theorem for crystalline evolutions in two dimensions;
- [Roosen and Taylor 1994] for the crystalline evolution in a diffusion field, and [Giga and Giga 1998a] for the crystalline flow with a driving force in two dimensions;
- [Ambrosio et al. 2002] for some regularity properties of solutions to crystalline variational problems in two dimensions;
- [Yunger 1998] and [Paolini and Pasquarelli 2000] for some properties of the crystalline flow in three dimensions.

In two dimensions (that is, for crystalline curvature evolution of curves) the situation is essentially understood, since the notion of crystalline curvature is clear, as well as the corresponding geometric evolution law. For instance, for polygonal initial curves (whose geometry is compatible with the geometry of ∂B_ϕ) a comparison principle is available and the flow admits local existence and uniqueness. It turns out that each edge of the curve translates in normal direction during the flow, and the evolution can be described by a system of ordinary differential equations.

However, in three space dimensions the situation is not so clear. As in the two-dimensional case, it is necessary to redefine what is a smooth boundary ∂E, in order to assign to our interface some notion of ϕ-mean curvature. To this purpose we recall a (rather strong) notion of smoothness, the Lipschitz ϕ-regularity. Even with this notion at our disposal, the definition of the crystalline mean curvature is quite involved. In addition, once the crystalline mean curvature κ_ϕ^E is defined on ∂E, one realizes that, in general, it is not constant on two-dimensional facets F of ∂E. This fact is a source of difficulties, since (being κ_ϕ^E identified with the normal velocity of ∂E at the initial time) facets can split in several pieces, or can even bend (forming curved regions) during the subsequent evolutionary process. These phenomena (which probably should not be considered as singularities of the flow) partially explain why when $n = 3$, a short time existence theorem for crystalline mean curvature flow is still missing (even for convex initial data E). Concerning this kind of behavior, we refer also to the work [Yunger 1998].

Before illustrating the plan of the paper, we observe that other choices of area measures are possible in $(\mathbb{R}^n, \phi)$, which are as natural as the ϕ-Minkowski content. For example, one could consider the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}_{d_\phi}(\partial E)$ of ∂E with respect to d_ϕ. Also for this notion of area, an integral representation theorem is available, which shows in particular that $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ and $\mathcal{H}^{n-1}_{d_\phi}(\partial E)$ may differ. Therefore, taking the first variation of $\mathcal{H}^{n-1}_{d_\phi}$ would give a notion of mean curvature (see [Shen 1998]) which is different from κ_ϕ^E. This, in turn, implies that the gradient flow of the d_ϕ-Hausdorff measure functional is a different geometric evolution process. Our viewpoint will be to work with the ϕ-Minkowski content of ∂E. We remark that all the theory that we develop can be similarly constructed for $\mathcal{H}^{n-1}_{d_\phi}$: indeed, what is really relevant is that $\mathcal{H}^{n-1}_{d_\phi}(\partial E)$ can also be represented as an integral on ∂E, by means of an integrand (weighting ν^E) which is *convex*. Finally, we recall that the mean curvature obtained from the first variation of the volume form for the Minkowski content on regular hypersurfaces in a Finsler manifold is considered in [Shen 2001].

The content of the paper is the following. In Section 2 we give some notation. In particular, in Subsection 2.1 we introduce the norm ϕ, its unit ball B_ϕ and the induced distance d_ϕ. In Subsection 2.2 we recall the main properties of the dual ϕ^o of ϕ and of the duality maps T_ϕ and T_{ϕ^o}. In particular, we discuss the geometric properties of such maps, also in the crystalline case. In Subsection 3.1 we discuss the integral representation of $\mathcal{H}^{n-1}_{d_\phi}$ (Theorem 3.3). In Subsection 3.2 we discuss the integral representation of $\mathcal{M}^{n-1}_{d_\phi}$ (Theorem 3.7). The relations between $\mathcal{H}^{n-1}_{d_\phi}$ and $\mathcal{M}^{n-1}_{d_\phi}$ are considered in Subsection 3.3. Section 3 is based on the results proved in [Bellettini et al. 1996]. In Section 4 we recall the first variation of area in the Euclidean case, and the main definitions and properties of Euclidean mean curvature flow. We rely heavily on the notion of oriented distance function from ∂E. In Subsection 4.2 we focus attention on the regular case and on the first variation of the weighted area. The definition of ϕ-mean curvature is given in (4–9). Also here the oriented ϕ-distance function plays a crucial rôle. In Subsection 4.3 we state some generalizations of the previous results when the norm is space-dependent. Subsections 4.2 and 4.3 are based on the results proved in [Bellettini and Paolini 1996]. The crystalline case is deepened in Section 5. Lipschitz ϕ-regularity is introduced in Definition 5.2, and illustrated with examples. The geometry of a facet $F \subset \partial E$ is studied in Section

6, which is preliminary to the definition of crystalline mean curvature on F (Definition 7.2). We will restrict for simplicity to polyhedral Lipschitz ϕ-regular sets. After some examples, in Subsection 7.1 we illustrate some properties of those facets having constant crystalline mean curvature. Sections 5, 6 and 7 are based on the results originally proved in [Bellettini et al. 1999], [Bellettini et al. 2001a], [Bellettini et al. 2001b], [Bellettini et al. 2001c]. We conclude the paper with Section 8, where we briefly summarize the main ideas and motivations behind our approach.

2. Notation

Given two vectors $v, w \in \mathbb{R}^n$, $n \geq 2$, we denote by $\langle v, w \rangle$ the scalar product between v and w. We also set $|v| := \sqrt{\langle v, v \rangle}$ and $S^{n-1} := \{v \in \mathbb{R}^n : |v| = 1\}$. Given an integer $k \in [0, n]$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure in $\mathbb{R}^n$ (see [Federer 1969] and [Ambrosio et al. 2000]). If $B \subset \mathbb{R}^n$ is a Borel set, we let $|B|$ to be the Lebesgue measure of B (which equals $\mathcal{H}^n(B)$) and $\mathrm{dist}(x, B)$ to be the distance of the point $x \in \mathbb{R}^n$ from B, defined as $\inf\{|y-x| : y \in B\}$. Even if B is a smooth hypersurface, it is not difficult to realize that the distance function $\mathrm{dist}(\,\cdot\,, B)$ is not differentiable *on* B. We let $\omega_m := \pi^{m/2} \big/ \int_0^{+\infty} s^{m/2} e^{-s}\, ds$, which turns out to be the Lebesgue measure of $\{x \in \mathbb{R}^m : |x| \leq 1\}$, for an integer $m \in [0, n]$.

We say that the set $M \subset \mathbb{R}^n$ is an $(n-1)$-dimensional Lipschitz manifold if M can be written, locally, as the graph of a Lipschitz function (with respect to a suitable orthogonal system of coordinates) defined on an open subset of $\mathbb{R}^{n-1}$. We say that the open set $E \subset \mathbb{R}^n$ is Lipschitz (or that its topological boundary ∂E is Lipschitz) if ∂E can be written, locally, as the graph of a Lipschitz function of $(n-1)$ variables (with respect to a suitable orthogonal system of coordinates) and E is locally the subgraph. We recall (see [Federer 1969]) that, if E has Lipschitz boundary, then at $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, the unit (Euclidean) normal vector to ∂E pointing toward $\mathbb{R}^n \setminus E$ is well defined and, in the sequel, will be denoted by $\nu^E(x)$.

If f is a smooth function defined on an open subset of $\mathbb{R}^n$, we denote by $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ the gradient of f and by Δf the Laplacian of f.

2.1. The norm ϕ and the distance d_ϕ.

In what follows we indicate by $\phi : \mathbb{R}^n \to [0, +\infty[$ a *convex* function which satisfies the properties

$$\phi(\xi) \geq \lambda |\xi|, \quad \xi \in \mathbb{R}^n, \tag{2–1}$$

for a suitable constant $\lambda \in \,]0, +\infty[$, and

$$\phi(a\xi) = |a|\, \phi(\xi), \quad \xi \in \mathbb{R}^n,\ a \in \mathbb{R}. \tag{2–2}$$

Notice that our assumptions ensure that there exists a constant $\Lambda \geq \lambda$ such that $\phi(\xi) \leq \Lambda |\xi|$ for any $\xi \in \mathbb{R}^n$. The function ϕ is a norm on $\mathbb{R}^n$, called a Minkowski norm (or Minkowski metric). The vector space $\mathbb{R}^n$ endowed with ϕ is a normed space called Minkowski space and is probably the simplest example of a Finsler manifold: in this case, the manifold is $\mathbb{R}^n$, and ϕ is a norm (independent of the position) on its tangent space which obviously is a copy of $\mathbb{R}^n$. We set

$$B_\phi := \{\xi \in \mathbb{R}^n : \phi(\xi) \leq 1\}$$

(the unit ball of ϕ), a bounded convex set containing the origin in its interior and centrally symmetric (a so-called symmetric convex body). B_ϕ is usually called the indicatrix (sometimes also Wulff shape). The function ϕ can be identified with B_ϕ, since given a symmetric convex body K, the function $\xi \to \inf\{\alpha > 0 : \xi \in \alpha K\}$ is a convex function satisfying (2–1), (2–2) and having K as unit ball. Clearly ϕ is uniquely determined by its values on the unit sphere S^{n-1}.

Remark 2.1. If we weaken assumption (2–2) into $\phi(a\xi) = a\phi(\xi)$ for any $\xi \in \mathbb{R}^n$ and any $a \geq 0$, we have that B_ϕ is not centrally symmetric anymore. Some of the results in the next sections can be generalized to nonsymmetric functions ϕ; in the sequel, for simplicity we will restrict to the symmetric case.

It is always useful to keep in mind the Euclidean case, which corresponds to the choice $\phi(\xi) = |\xi|$. The Riemannian case corresponds to a norm ϕ whose B_ϕ is an ellipsoid: $\phi(\xi) := \sqrt{\langle A\xi, \xi\rangle}$ for a real positive definite symmetric matrix A.

Definition 2.2. We say that ϕ is regular if B_ϕ has boundary of class $\mathcal{C}^\infty$ and each principal curvature of ∂B_ϕ is strictly positive at each point of ∂B_ϕ.

Example 2.1. Let $p \in [1, +\infty[$, $p \neq 2$ and set $\phi(\xi) := (\sum_{i=1}^n |\xi_i|^p)^{1/p}$. Then ϕ is not regular: indeed, if $p > 2$ then B_ϕ is of class $\mathcal{C}^2$ but the requirement on principal curvatures in Definition 2.2 is not fulfilled. On the other hand, if $p < 2$ then ∂B_ϕ is not of class $\mathcal{C}^2$.

Definition 2.3. We say that ϕ is crystalline if B_ϕ is a polytope.

Example 2.2. The norms $\phi(\xi) := \sum_{i=1}^n |\xi_n|$ and $\phi(\xi) := \max\{|\xi_1|, \ldots, |\xi_n|\}$ are crystalline.

It is well known that, given the norm ϕ, we can measure distances in $\mathbb{R}^n$ by "integrating" ϕ as follows: the ϕ-distance $d_\phi(x, y)$ between two points $x, y \in \mathbb{R}^n$ is given by

$$\begin{aligned} d_\phi(x, y) &= \inf\Big\{\int_0^1 \phi(\dot\gamma)\, dt : \gamma \in AC([0,1]; \mathbb{R}^n), \gamma(0) = x, \gamma(1) = y\Big\} \\ &= \phi(y - x), \end{aligned} \tag{2–3}$$

where $AC([0,1]; \mathbb{R}^n)$ is the class of all absolutely continuous curves from $[0,1]$ to $\mathbb{R}^n$. The last equality in (2–3) is, for instance, a consequence of Jensen's inequality. The function d_ϕ is nonnegative, symmetric, vanishes only if $x = y$, and satisfies the triangular property. To be consistent with the beginning of this section, when $\phi(\xi) = |\xi|$ (Euclidean case) we omit the subscript ϕ in the notation of the distance function.

REMARK 2.4. We recall the following interesting fact. Given a distance $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ on $\mathbb{R}^n$, by differentiation we can construct a new function $\psi_d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ which, under suitable assumptions, turns out to be a Finsler metric (which, in this case, depends on the position x):

$$\psi_d(x, \xi) := \limsup_{t \to 0} \frac{d(x, x + t\xi)}{t}, \quad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n.$$

Some of the properties of the functions ψ_d, d_{ψ_d} have been investigated for instance in the papers [De Giorgi 1989], [De Giorgi 1990], [Venturini 1992], [Bellettini et al. 1996], [Amar et al. 1998] (see also [De Cecco and Palmieri 1993], [De Cecco and Palmieri 1995] for related results on Lipschitz manifolds).

2.2. The dual norm ϕ^o. The duality maps. Given the norm ϕ acting on vectors, we define the dual norm $\phi^o : \mathbb{R}^n \to [0, +\infty[$ of ϕ (acting on covectors) as

$$\phi^o(\xi^o) := \sup\{\langle \xi, \xi^o \rangle : \xi \in B_\phi\}, \quad \xi^o \in \mathbb{R}^n. \tag{2–4}$$

It is not difficult to verify that ϕ^o is a norm on (the dual of) $\mathbb{R}^n$, and that $(\phi^o)^o = \phi$. It is possible to prove that if ϕ is regular then also ϕ^o is regular, and that if ϕ is crystalline then ϕ^o is crystalline. The function ϕ^o is strictly related to the Legendre–Fenchel transform ϕ^* of ϕ, defined as $\phi^*(\xi^o) := \sup\{\langle \xi, \xi^o \rangle - \phi(\xi) : \xi \in \mathbb{R}^n\}$. Indeed $\phi^*(\xi^o) = +\infty$ if $\xi \notin B_{\phi^o}$, and $\phi^*(\xi^o) = 0$ if $\xi \in B_{\phi^o}$.

EXAMPLE 2.3. Figure 1 describes how to construct B_{ϕ^o} starting from ϕ. Assume we have been given a smooth symmetric convex body $\{\phi \le 1\}$ as (the ellipse) in Figure 1. Let $\nu \in S^{n-1}$ be a unit vector. By definition and using the homogeneity, computing $\phi^o(\nu)$ is equivalent to solve the maximum problem $\max\{\langle \nu, z \rangle : z \in \partial B_\phi\}$.

The vector $\xi = \xi(\nu)$ in Figure 1 is the solution, hence $\phi^o(\nu) = \langle \nu, \xi \rangle$. Observe that the strict convexity of B_ϕ ensures uniqueness of the solution of the maximum problem (2–4); it is clear that if ∂B_ϕ contains some flat region, problem (2–4) has in general more than one solution.

REMARK 2.5. Notice that

$$\phi^o(\nu) = \frac{1}{2} \mathcal{H}^1(\mathrm{pr}_\nu(B_\phi)), \tag{2–5}$$

where $\mathrm{pr}_\nu(B_\phi)$ denotes the orthogonal projection of B_ϕ onto the line $\mathbb{R}\nu$; see Figure 1.

The map described in Figure 1 associating to the vector $\xi \in \partial B_\phi$ the vector $\nu / \phi^o(\nu) \in \partial B_{\phi^o}$ (extended in a one-homogeneous way on the whole of $\mathbb{R}^n$) is called the duality map, and can also be defined in the crystalline case.

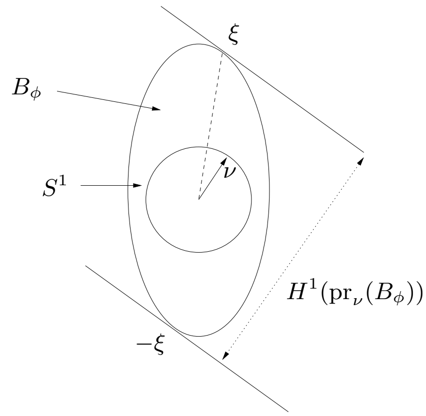

Figure 1. ν is a unit vector. The vector $\xi \in \partial B_\phi$ (dotted line) is such that $\phi^o(\nu) = \langle \nu, \xi \rangle$. Observe that the line tangent to ∂B_ϕ at ξ is orthogonal to ν. The distance between this line and its parallel line tangent to ∂B_ϕ at $-\xi$ is the length of the orthogonal projection of B_ϕ onto the one-dimensional subspace $\mathbb{R}\nu$, and equals $2\phi^o(\nu)$.

DEFINITION 2.6. By T_ϕ and T_{ϕ^o} we denote the (possibly multivalued) duality maps defined as

$$\begin{aligned} T_\phi(\xi) &:= \{\xi^o \in \mathbb{R}^n : \langle \xi^o, \xi \rangle = \phi(\xi)^2 = (\phi^o(\xi^o))^2\}, \quad \xi \in \mathbb{R}^n, \\ T_{\phi^o}(\xi^o) &:= \{\xi \in \mathbb{R}^n : \langle \xi, \xi^o \rangle = (\phi^o(\xi^o))^2 = \phi(\xi)^2\}, \quad \xi^o \in \mathbb{R}^n. \end{aligned} \tag{2–6}$$

Possibly adopting the conventions on multivalued mappings (see for instance [Brezis 1973]) one can check that $T_\phi(a\xi) = |a| T_\phi(\xi)$ for any $\xi \in \mathbb{R}^n$ and any $a \in \mathbb{R}$, and similarly for T_{ϕ^o}. Moreover T_ϕ takes ∂B_ϕ onto ∂B_{ϕ^o}, T_{ϕ^o} takes ∂B_{ϕ^o} onto ∂B_ϕ, and $T_\phi T_{\phi^o} = T_{\phi^o} T_\phi = \mathrm{Id}$.

REMARK 2.7. Let ϕ be regular. Then T_ϕ and T_{ϕ^o} are single valued. Moreover

$$T_\phi = \frac{1}{2}\nabla((\phi)^2) = \phi\nabla\phi, \quad T_{\phi^o} = \frac{1}{2}\nabla((\phi^o)^2) = \phi^o\nabla\phi^o. \tag{2–7}$$

EXAMPLE 2.4. When $\phi(\xi) = |\xi|$ (Euclidean case), then $T_\phi = \mathrm{Id}$. When $\phi(\xi) = \sqrt{\langle A\xi, \xi \rangle}$ is Riemannian, then $T_\phi(\xi) = A\xi$.

Figure 2 illustrates how to construct T_ϕ in a regular case. First of all, since T_ϕ is one-homogeneous, it is enough to evaluate T_ϕ on ∂B_ϕ.

The point ξ belongs to ∂B_ϕ; since $T_\phi(\xi) = \nabla\phi(\xi)$ and ∂B_ϕ is a level set of ϕ, it is clear that $T_\phi(\xi)$ is orthogonal to ξ. In addition, $\phi^o(\nabla\phi(\xi)) = 1 = \langle \xi, \nabla\phi(\xi) \rangle$, which implies $|T_\phi(\xi)| = \langle \xi, \nabla\phi(\xi)/|\nabla\phi(\xi)| \rangle^{-1}$.

In what follows, it is important to keep in mind that the duality maps in (2–6) are still well defined under our assumptions on a Minkowski norm, in particular in the crystalline case. If ϕ is not regular, the equalities in (2–7) become

$$T_\phi = \tfrac{1}{2}\nabla^-((\phi)^2) = \phi\nabla^-\phi, \quad T_{\phi^o} = \tfrac{1}{2}\nabla^-((\phi^o)^2) = \phi^o\nabla^-\phi^o, \tag{2–8}$$

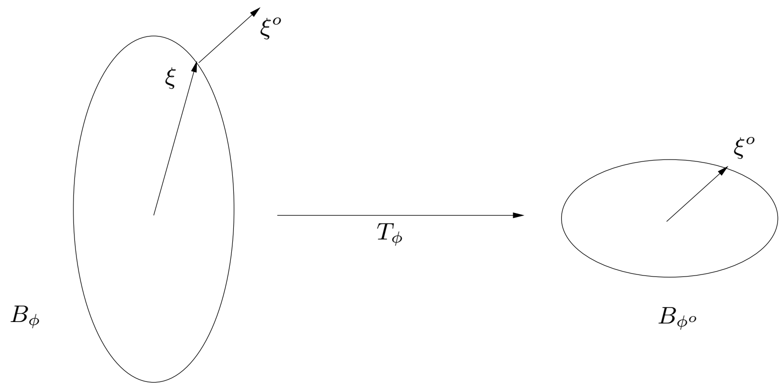

Figure 2. The point ξ belongs to ∂B_ϕ. The point $T_\phi(\xi) := \xi^o \in \partial B_{\phi^o}$ is the unit normal $\nu^{B_\phi}(\xi)$ to ∂B_ϕ at ξ, multiplied by the factor $\langle \xi, \nu^{B_\phi}(\xi)\rangle^{-1}$.

where ∇^- denotes the usual subdifferential in convex analysis (see for instance [Rockafellar 1972]); the main feature of the maps T_ϕ and T_{ϕ^o} is that they are no longer one-to-one.

Geometrically, if $\xi \in \partial B_\phi$, then $T_\phi(\xi)$ is the intersection of the closed outward normal cone to ∂B_ϕ with ∂B_{ϕ^o}. In Figure 3 we show an example of B_ϕ and of its dual body B_{ϕ^o}: B_ϕ is the Cartesian product of a planar regular hexagon with the interval $[-1, 1]$. If $\xi \in \partial B_\phi$ is a point in the relative interior of a facet, then the normal cone $T_\phi(\xi)$ to ∂B_ϕ at ξ is a singleton (a vertex in ∂B_{ϕ^o}); if $\xi \in \partial B_\phi$ is a point in the relative interior of an edge, then $T_\phi(\xi)$ is a one-dimensional closed segment (a closed edge in ∂B_{ϕ^o}); if $\xi \in \partial B_\phi$ is a vertex, then $T_\phi(\xi)$ is a closed triangle (a closed facet in ∂B_{ϕ^o}).

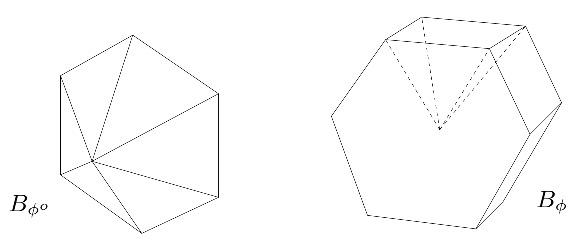

Figure 3. Dual polytopes. Duality maps take vertices into closed facets, points in the relative interior of a facet into vertices, and points in the relative interior of an edge into closed edges. The B_{ϕ^o} depicted here is supposed to look like an umbrella viewed from above.

3. Area Measures in $(\mathbb{R}^n, \phi)$

3.1. The $(n-1)$-dimensional Hausdorff measure. We recall the definition of Hausdorff measure in the metric space $(\mathbb{R}^n, d_\phi)$; see [Federer 1969].

DEFINITION 3.1. If $A \subseteq \mathbb{R}^n$ is a Borel subset of $\mathbb{R}^n$ and $m \in \{n-1, n\}$ we set

$$\mathcal{H}^m_{d_\phi}(A) := \frac{\omega_m}{2^m} \lim_{\delta \to 0^+} \inf \left\{ \sum_{i=1}^{+\infty} (\mathrm{diam}_{d_\phi}(S_i))^m : A \subseteq \bigcup_{i=1}^{+\infty} S_i, \ \mathrm{diam}_{d_\phi}(S_i) < \delta \right\}, \tag{3–1}$$

where $\mathrm{diam}_{d_\phi}(S_i) := \sup\{d_\phi(s, \sigma) : (s, \sigma) \in S_i \times S_i\}$ is the diameter of the set S_i with respect to d_ϕ.

It is not difficult to prove that the limit in (3–1) exists. Consistent with the notation in Section 2, when ϕ is the Euclidean norm we omit the subscript d_ϕ in the notation of the Hausdorff measure. The following integral representation result provides an explicit formula for computing $\mathcal{H}^n_{d_\phi}$ and $\mathcal{H}^{n-1}_{d_\phi}$.

THEOREM 3.2. *Let ϕ be a norm on $\mathbb{R}^n$. Then*

$$\mathcal{H}^n_{d_\phi}(A) = \int_A \frac{\omega_n}{|B_\phi|} \, dx = \frac{\omega_n}{|B_\phi|} |A|,$$

for any Borel set $A \subseteq \mathbb{R}^n$.

Given $\nu \in S^{n-1}$, denote by

$$S_\nu(B_\phi) := \{\xi \in B_\phi : \langle \nu, \xi \rangle = 0\}$$

the section of B_ϕ with the hyperplane orthogonal to ν passing through the origin; moreover, set

$$I\phi(\nu) := \frac{1}{\mathcal{H}^{n-1}(S_\nu(B_\phi))}. \tag{3–2}$$

THEOREM 3.3. *Let ϕ be a norm on $\mathbb{R}^n$. Let M be a $(n-1)$-dimensional Lipschitz manifold in $\mathbb{R}^n$. Then*

$$\mathcal{H}^{n-1}_{d_\phi}(A \cap M) = \omega_{n-1} \int_{A \cap M} I\phi(\nu^M) \, d\mathcal{H}^{n-1}, \tag{3–3}$$

where $\nu^M(x)$ is a unit vector normal to M at x and $A \subseteq \mathbb{R}^n$ is a Borel set.

The measure $\mathcal{H}^{n-1}_{d_\phi}$ is also called Busemann surface measure, see the books [Thompson 1996], [Schneider 1993] for detailed information on this topic and for complete references.

We will still denote by $I\phi$ the one-homogeneous extension of the function in (3–2) on the whole of $\mathbb{R}^n$. We then have that the unit ball of $I\phi$ can be written as

$$\{\xi \in \mathbb{R}^n : I\phi(\xi) \le 1\} = \left\{\xi \in \mathbb{R}^n : |\xi| \le \mathcal{H}^{n-1}(S_{\xi/|\xi|}(B_\phi))\right\},$$

that is, $\{I\phi \le 1\}$ is the so-called intersection body $I(B_\phi)$ of B_ϕ. An interesting result of Busemann ensures that $I(B_\phi)$ is convex (see [Busemann 1947], [Busemann 1949], [Thompson 1996], [Schneider 1993] and references therein). This, in turn, is essentially equivalent to the following semicontinuity property of $\mathcal{H}^{n-1}_{d_\phi}$:

if $\{E_k\}_k$ is a sequence of finite perimeter sets whose characteristic functions converge in $L^1(\mathbb{R}^n)$ to the characteristic function of a finite perimeter set E, then $\mathcal{H}^{n-1}_{d_\phi}(\partial E) \leq \liminf_{k\to+\infty} \mathcal{H}^{n-1}_{d_\phi}(\partial E_k)$.

3.2. The $(n-1)$-dimensional Minkowski content.

Definition 3.4. Let M be a $(n-1)$-dimensional Lipschitz manifold. We define the $(n-1)$-dimensional Minkowski content $\mathcal{M}^{n-1}_{d_\phi}(M)$ of M with respect to d_ϕ as

$$\mathcal{M}^{n-1}_{d_\phi}(M) := \lim_{\rho\to 0^+} \frac{\mathcal{H}^n_{d_\phi}\left(\{z \in \mathbb{R}^n : \mathrm{dist}_\phi(z, M) < \rho\}\right)}{2\rho}. \tag{3–4}$$

Under our regularity assumption on ∂E it is possible to prove that the limit in (3–4) exists (see [Federer 1969], [Ambrosio et al. 2000]).

In the Euclidean case $\phi(\xi) = |\xi|$, the $(n-1)$-dimensional Minkowski content coincides with the $(n-1)$-dimensional Hausdorff measure.

Observe that $\mathcal{H}^n_{d_\phi}$ and $\mathcal{M}^{n-1}_{\phi}$ are invariant under isometries between the normed ambient spaces, while $c_{n,\phi}$ is not invariant.

Remark 3.5. The Minkowski content $\mathcal{M}^{n-1}_{d_\phi}$ provides a notion of surface measure which is constructed by means of the orthogonal projections of B_ϕ onto the (one-dimensional) normal spaces to the manifold M; on the other hand the Hausdorff measure $\mathcal{H}^{n-1}_{d_\phi}$ is constructed by means of the intersections of B_ϕ with the ($(n-1)$-dimensional) tangent spaces to M. Other notions of surface measure, different in general from these two notions, can be considered, such as the Holmes–Thompson measure, see [Thompson 1996], or the definitions introduced in [De Giorgi 1995] (see [Ambrosio and Kirchheim 2000]).

Remark 3.6. It can be proved that, for a Lipschitz set E, $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ coincides with the perimeter of the set E with respect to ϕ, whose definition is given in a distributional way.

The following representation result provides an explicit integral formula for computing the Minkowski content of a sufficiently smooth set.

Theorem 3.7. *Let M be a $(n-1)$-dimensional Lipschitz manifold. Then*

$$\mathcal{M}^{n-1}_{d_\phi}(M) = c_{n,\phi} \int_M \phi^o(\nu^M)\ d\mathcal{H}^{n-1}. \tag{3–5}$$

The validity of Theorem 3.7 can be explained as follows: the measure $\mathcal{H}^n_{d_\phi}$ is $c_{n,\phi}$ times the Lebesgue measure. Moreover the Lebesgue measure of the ρ-tubular neighborhood (in the distance d_ϕ) in (3–4) is approximately $\mathcal{H}^{n-1}(M)$ multiplied by the 1-dimensional length of the orthogonal projection of $\{\phi \leq \rho\}$ in the direction ν, see Figure 4. This length equals $2\rho\phi^o(\nu)$, hence (3–5) follows.

Since ϕ^o is convex, the following semicontinuity property of $\mathcal{M}^{n-1}_{d_\phi}$ holds: if $\{E_k\}_k$ is a sequence of finite perimeter sets whose characteristic functions converge in $L^1(\mathbb{R}^n)$ to the characteristic function of a finite perimeter set E, then $\mathcal{M}^{n-1}_{d_\phi}(\partial E) \leq \liminf_{k\to+\infty} \mathcal{M}^{n-1}_{d_\phi}(\partial E_k)$.

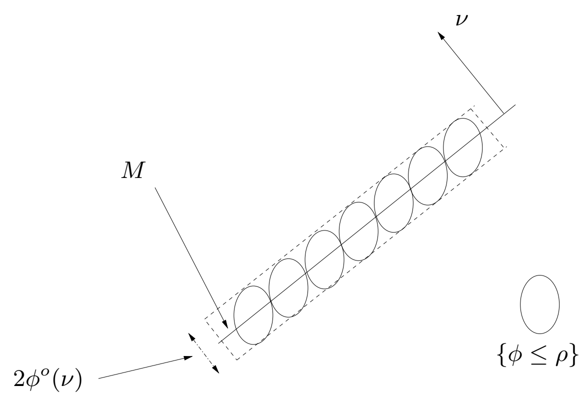

Figure 4. We assume for simplicity that M is flat, and that $\nu \in S^{n-1}$ is orthogonal to M. The ρ-tubular neighborhood (in the distance d_ϕ) is obtained by centering rescaled sets $\{\phi \leq \rho\}$ of $\{\phi \leq 1\}$ at points of M.

3.3. From the Hausdorff measure to the Minkowski content. As already remarked in the Introduction, in what follows we will work with $\mathcal{M}^{n-1}_{d_\phi}$ and this will affect the value of the ϕ-mean curvature. However, one could consider $\mathcal{H}^{n-1}_{d_\phi}$ as well, and change all subsequent definitions by replacing ϕ^o with $I\phi$. We refer to [Shen 1998] for a definition of mean curvature obtained by considering the intrinsic Hausdorff measure.

The following result follows from Theorems 3.3 and 3.7.

Proposition 3.8. *Let ϕ be a norm on $\mathbb{R}^n$. Then*

$$\mathcal{H}^{n-1}_{d_\phi} = \mathcal{M}^{n-1}_{d_\psi}, \quad \psi := \left(\frac{\omega_{n-1}|B_{(I\phi)^o}|}{\omega_n}\right)^{1/(n-1)} (I\phi)^o. \tag{3–6}$$

4. First Variation of Area and Mean Curvature Flow. Regular Case

In this section we define the anisotropic ϕ-mean curvature. We first recall some facts concerning the Euclidean case.

4.1. Preliminaries on the Euclidean case. Let $E \subset \mathbb{R}^n$ be an open set with smooth compact boundary. It is known (see for instance [Giusti 1984], [Ambrosio 1999]) that, under these assumptions, there exists a tubular neighborhood U of ∂E such that the oriented distance function d^E from ∂E negative inside E, defined as

$$d^E(z) := \text{dist}(z, E) - \text{dist}(z, \mathbb{R}^n \setminus E), \quad z \in \mathbb{R}^n,$$

is smooth on U, and $|\nabla d^E(z)| = 1$ for any $z \in U$ (eikonal equation). Hence, given any $x \in \partial E$, $\nabla d^E(x)$ is the outer unit normal $\nu^E(x)$ to ∂E at x. In addition $\Delta d^E(x)$ is the sum of the principal curvatures (the mean curvature) of ∂E at x. Therefore $-\Delta d^E(x)\nabla d^E(x)$ is the mean curvature vector of ∂E at x.

In order to compute the first variation of area, we need to introduce a class of admissible variations. We define a family Ψ_λ of compactly supported diffeomorphisms of the ambient space $\mathbb{R}^n$ as follows. Denote by $\Psi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ a smooth vector field. Given any $\lambda \in \mathbb{R}$, define $\Psi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ as $\Psi_\lambda(x) := \Psi(x, \lambda)$. Assume that $\Psi_0 = \mathrm{Id}$ and that $\Psi_\lambda - \mathrm{Id}$ has compact support for any $\lambda \in \mathbb{R} \setminus \{0\}$. The following theorem is the classical result on the first variation of area, see for instance [Giusti 1984].

THEOREM 4.1. *For any* $\lambda \in \mathbb{R}$ *define* $E_\lambda := \Psi_\lambda(E)$. *Then*

$$\frac{d}{d\lambda}\mathcal{H}^{n-1}(\partial E_\lambda)_{|\lambda=0} = \int_{\partial E} \Delta d^E \ \langle X, \nabla d^E \rangle \ d\mathcal{H}^{n-1}, \tag{4–1}$$

where $X := (\partial \Psi_\lambda / \partial \lambda)_{|\lambda=0}$.

Observe that only the normal component of X enters in formula (4–1).

We are now in a position to define the smooth mean curvature flow starting from a given ∂E.

DEFINITION 4.2. Let $E \subset \mathbb{R}^n$ be an open set with smooth compact boundary. Let $T > 0$ and, for any $t \in [0, T]$, let $E(t)$ be a set with compact boundary. We say that $(E(t))_{t\in[0,T]}$ is a smooth mean curvature flow in $[0, T]$ starting from $E = E(0)$ if the following conditions hold:

(i) there exists an open set $A \subset \mathbb{R}^n$ containing $\partial E(t)$ for any $t \in [0, T]$ such that, if we set

$$\bar{d}(z, t) := \mathrm{dist}(z, E(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus E(t)), \quad z \in \mathbb{R}^n, \ t \in [0, T],$$

we have $\bar{d} \in \mathcal{C}^\infty\big(A \times [0, T]\big)$;

(ii)
$$\frac{\partial}{\partial t}\bar{d}(x, t) = \Delta \bar{d}(x, t), \quad x \in \partial E(t), \ t \in [0, T]. \tag{4–2}$$

Condition (i) implies that each $\partial E(t)$ is a smooth boundary, smoothly evolving in time. The vector $-(\partial \bar{d}/\partial t)\nabla \bar{d}(x, t)$ is the projection of the velocity of the point x on the normal space to $\partial E(t)$ at x (see for instance [Ambrosio 1999]).

EXAMPLE 4.1. The main explicit example of mean curvature flow is the one of the sphere $E = \{|z| < R_0\}$, which shrinks self-similarly. Indeed in this case $\bar{d}(z, t) = |z| - R(t)$, and the equation (4–2) becomes $\dot{R} = -(n-1)/R$. Its solution represents the evolving sphere of radius $R(t) = \sqrt{R_0^2 - 2(n-1)t}$ for $t \in [0, \frac{1}{2}R_0^2/(n-1)]$, which disappears for times larger than $\frac{1}{2}R_0^2/(n-1)$.

It is customary to say that the evolution law (4–2) is the gradient flow of the area functional $\mathcal{H}^{n-1}(\partial E)$; the idea is that, at each time, the set $E(t)$ evolves in such a way to make $\mathcal{H}^{n-1}(\partial E(t))$ as small as possible. This assertion has been made rigorous in the paper [Almgren et al. 1993], where it is shown that the correct (nonsymmetric) distance between sets to use in order to obtain the mean curvature flow is

$$(E, F) \subset \mathbb{R}^n \times \mathbb{R}^n \to \int_{(E \setminus F) \cup (F \setminus E)} \operatorname{dist}(x, \partial F)\, dx. \tag{4–3}$$

Mean curvature flow has been the subject of several recent papers, of which we list some. We refer to:

- [Gage and Hamilton 1986] and [Evans and Spruck 1992a] for a local in time existence and uniqueness theorem of a smooth solution;
- [Ecker and Huisken 1989] and [Ecker and Huisken 1991] for the evolution of graphs;
- [Huisken 1984] for the evolution of convex sets;
- [Grayson 1987] and [Angenent 1991a] for local and global properties of the flow of curves;
- [Barles et al. 1993], [Evans and Spruck 1992b], [Evans and Spruck 1995], [White 1995], [White 2000], [White 2003] for some qualitative properties of the flow.
- Concerning global in time solutions, defined after the onset of singularities, we refer to:
- [Brakke 1978], where a geometric measure theory approach is introduced;
- [Evans and Spruck 1991] and [Chen et al. 1991] for the level set method and viscosity solutions;
- [Ilmanen 1992], [Ilmanen 1993a], [De Giorgi 1993], [De Giorgi 1994] and [Bellettini and Paolini 1995] for the barriers method (see also [Soner 1993]);
- [Almgren et al. 1993] for a variational approach based on time discretization;
- [Ilmanen 1993b] for the approximation of mean curvature flow by means of a sequence of reaction-diffusion equations;
- [Ilmanen 1994] for the elliptic regularization method;
- [Angenent 1991a], [Soner and Souganidis 1993], [Altschuler et al. 1995] and [Bellettini and Paolini 1994] for the analysis of some kind of singularity of the flow;
- [Fierro and Paolini 1996], [Paolini and Verdi 1992], [Angenent et al. 1995] for numerical simulations of certain singularities.

4.2. The anisotropic regular case. We assume in this subsection that ϕ is a regular norm. Let E be a set with smooth compact boundary. Also in this case it is possible to prove that there exists a tubular neighborhood U of ∂E such that the oriented ϕ-distance function d^E_ϕ from ∂E negative inside E, defined as

$$d^E_\phi(z) := \operatorname{dist}_\phi(z, E) - \operatorname{dist}_\phi(z, \mathbb{R}^n \setminus E), \quad z \in \mathbb{R}^n,$$

is smooth on U, and $\phi^o(\nabla d^E_\phi(z)) = 1$ for any $z \in U$ (anisotropic eikonal equation). Therefore

$$T_{\phi^o}(\nabla d^E_\phi(z)) = \phi^o(\nabla d^E_\phi(z)) \nabla \phi^o(\nabla d^E_\phi(z)) = \nabla \phi^o(\nabla d^E_\phi(z)), \quad z \in U. \tag{4–4}$$

In particular, given any $x \in \partial E$, we have

$$\nabla d^E_\phi(x) = \frac{\nu^E(x)}{\phi^o(\nu^E(x))} =: \nu^E_\phi(x). \tag{4–5}$$

The following result is a generalization of Theorem 4.1, and shows also the rôle of the duality map T_{ϕ^o}. Let Ψ_λ and E_λ be as in Subsection 4.1.

THEOREM 4.3. *For any $\lambda \in \mathbb{R}$ define $E_\lambda := \Psi_\lambda(E)$. Then*

$$\frac{d}{d\lambda}\mathcal{M}^{n-1}_{d_\phi}(\partial E_\lambda)_{|\lambda=0} = c_{n,\phi}\int_{\partial E} \mathrm{div}\ \mathrm{n}^E_\phi\ \langle X, \nabla d^E_\phi\rangle\ \phi^o(\nu^E) d\mathcal{H}^{n-1}, \tag{4–6}$$

where $X := \frac{\partial}{\partial\lambda}\Psi_{\lambda_{|\lambda=0}}$, *and*

$$\mathrm{n}^E_\phi(z) := T_{\phi^o}(\nabla d^E_\phi(z)), \quad z \in U. \tag{4–7}$$

The vector field n_ϕ is sometimes called the Cahn–Hoffman vector field, and satisfies

$$\phi(n^E_\phi) = 1 = \langle \nabla d^E_\phi, n^E_\phi\rangle \quad \text{on } U. \tag{4–8}$$

In the Euclidean case $\phi(\xi) = |\xi|$ we have $n^E_\phi = \nu^E_\phi = \nu^E$ on ∂E.

REMARK 4.4. The left hand side of (4–6) depends on the values of ϕ^o only on S^{n-1} (recall (3–5)). Hence also the right hand side of (4–6), written in terms of the one-homogeneous extension of ϕ^o on the whole of $\mathbb{R}^n$, must depend only on the values of ϕ^o on S^{n-1}.

Observe that

$$\frac{d}{d\lambda}\mathcal{M}^{n-1}_{d_\phi}(\partial E_\lambda)_{|\lambda=0} = \int_{\partial E} \mathrm{div}\ \mathrm{n}^E_\phi\ \langle X, \nabla d^E_\phi\rangle\ d\mathcal{P}^{n-1}_\phi,$$

where $d\mathcal{P}^{n-1}_\phi$ is the measure on ∂E having $c_{n,\phi}\phi^o(\nu^E)$ as density with respect to $\mathcal{H}^{n-1}$.

We are in a position to give the following definition.

DEFINITION 4.5. Let E be an open set with smooth compact boundary. We define the ϕ-mean curvature κ^E_ϕ of ∂E as

$$\kappa^E_\phi(x) := \mathrm{div}\ \mathrm{n}^E_\phi(x), \quad x \in \partial E. \tag{4–9}$$

It is possible to prove that $\kappa^E_\phi(x)$ is also the tangential divergence of n^E_ϕ evaluated at $x \in \partial E$. Indeed, define $f(z) := \langle \nu^E(x), \mathrm{n}^E_\phi(z)\rangle$ for any $z \in U$. Thanks to (4–8), f has a maximum at x (with value $\phi^o(\nu^E(x))$). Therefore ∇f vanishes at x, that is, $\nu^E(x)\nabla \mathrm{n}^E_\phi(x) = 0$. This implies that the tangential divergence of n^E_ϕ at x equals div $\mathrm{n}^E_\phi(x)$.

EXAMPLE 4.2. The ϕ-mean curvature of ∂B_ϕ is constantly equal to $n-1$. Indeed $\nabla d^{B_\phi}_\phi(z) = \nabla\phi(z)$ and $T_{\phi^o}(\nabla d^{B_\phi}_\phi(z)) = z/\phi(z)$ for any z in $\mathbb{R}^n \setminus \{0\}$. Then a computation gives $\mathrm{div}(z/\phi(z)) = n - 1$ on ∂B_ϕ.

EXAMPLE 4.3. Let $n = 2$, and write $\phi^o(\xi^o) = |\xi^o|\phi^o(\xi^o/|\xi^o|) =: \rho\psi(\theta)$, where $\xi_o = (\xi^o_1, \xi^o_2) = (\rho\cos\theta, \rho\sin\theta)$. Then $\kappa^E_\phi = \kappa^E(\psi + \psi'')$, where κ^E is the Euclidean curvature of ∂E.

EXAMPLE 4.4. Observe that from (4–4) we derive $\kappa^E_\phi = \mathrm{tr}(\nabla^2\phi^o(\nabla d^E_\phi)\nabla^2 d^E_\phi)$ on ∂E, where ∇^2 denotes the Hessian matrix and tr is the trace operator.

REMARK 4.6. In the paper [Bellettini and Fragalà 2002] the second variation of $\mathcal{M}^{n-1}_{d_\phi}$ is computed. What replaces the squared length of the second fundamental form of ∂E is the term $\mathrm{tr}(\nabla \mathrm{n}^E_\phi \nabla \mathrm{n}^E_\phi)$. In the same paper, a sort of Laplace–Beltrami operator is introduced, see also [Bao et al. 2000] (for references therein), [Shen 2001], [Mugnai 2003].

We are now in a position to define what is a smooth anisotropic mean curvature flow, for a regular anisotropy ϕ.

DEFINITION 4.7. Let $E \subset \mathbb{R}^n$ be an open set with smooth compact boundary. Let $T > 0$ and, for any $t \in [0,T]$, let $E(t)$ be a set with compact boundary. We say that $(E(t))_{t\in[0,T]}$ is a smooth ϕ-mean curvature flow in $[0,T]$ starting from $E = E(0)$ if the following conditions hold:

(i) there exists an open set $A \subset \mathbb{R}^n$ containing $\partial E(t)$ for any $t \in [0,T]$ such that, if we set

$$\bar{d}_\phi(z,t) := \mathrm{dist}_\phi(z, E(t)) - \mathrm{dist}_\phi(z, \mathbb{R}^n \setminus E(t)), \quad z \in \mathbb{R}^n,\ t \in [0,T],$$

we have $d_\phi \in \mathcal{C}^\infty\big(A \times [0,T]\big)$;

(ii)
$$\frac{\partial}{\partial t} d_\phi(x,t) = \mathrm{div}\mathrm{n}^{E(t)}_\phi(x), \quad x \in \partial E(t),\ t \in [0,T]. \tag{4–10}$$

EXAMPLE 4.5. We show that $\{\xi \in \mathbb{R}^n : \phi(\xi) < R_0\}$ shrinks self-similarly under the flow (4–10). We have in this case $\bar{d}_\phi(z,t) = \phi(z) - R(t)$, and (4–10) (thanks to Example 4.2) becomes $\dot{R} = -(n-1)/R$. Its solution represents the evolving set $\{\xi \in \mathbb{R}^n : \phi(\xi) < R(t)\}$, where $R(t) = \sqrt{R_0^2 - 2(n-1)t}$ for $t \in [0, \frac{1}{2}R_0^2/(n-1)]$, which disappears for times larger than $\frac{1}{2}R_0^2/(n-1)$.

The evolution law (4–10) is the gradient flow of $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$. This can be seen, for instance, by using the (nonsymmetric) distance between sets in (4–3), where $\mathrm{dist}(x, \partial F)$ is replaced by $\mathrm{dist}_\phi(x, \partial F)$.

For what concerns anisotropic mean curvature flows, we refer to the following (largely incomplete) list of papers: [Hoffman and Cahn 1972], [Angenent 1991b], [Spohn 1993], [Cahn et al. 1993], [Gage 1994], [Giga and Goto 1992] (see also [Giga et al. 1991] for weak solutions to a large class of anisotropic equations).

4.3. On space dependent norms. In this subsection we list some generalization of the previous results. Assume that $\phi = \phi(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ depends on the position x, ϕ^2 is smooth, $\phi^2(x,\cdot)$ is strictly convex (in the sense of Definition 2.2) and ϕ satisfies (2–1), (2–2) for any $x \in \mathbb{R}^n$, with λ independent of

x. Define $\phi^o(x,\xi^o) := \sup\{\langle \xi, \xi^o\rangle : \xi \in B_\phi(x)\}$, $B_\phi(x) := \{\xi \in \mathbb{R}^n : \phi(x,\xi) \le 1\}$, and d_ϕ as in the first equality of (2–3), with $\phi(\gamma,\dot\gamma)$ in place of $\phi(\dot\gamma)$. T_ϕ and T_{ϕ^o} are defined as in (2–8) taking x fixed. Then (3–3) holds with

$$I\phi(x,\nu^M(x)) = \frac{1}{\mathcal{H}^{n-1}(S_\nu(B_\phi(x))}$$

in place of $I\phi(\nu^M(x))$. The n-dimensional d_ϕ-Hausdorff measure $\mathcal{H}^n_{d_\phi}$ in $\mathbb{R}^n$ has the integral representation

$$\mathcal{H}^n_{d_\phi}(A) = \omega_n \int_A \frac{1}{|B_\phi(x)|}\, dx,$$

for any Borel set $A \subseteq \mathbb{R}^n$. Then (3–5) becomes

$$\mathcal{M}^{n-1}_{d_\phi}(M) = \omega_n \int_M \phi^o(x,\nu^M(x)) \frac{1}{|B_\phi(x)|}\, d\mathcal{H}^{n-1}(x).$$

In addition, the function ψ in (3–6) becomes $\psi(x,\xi) = f(x)(I\phi)^o(x,\xi)$, where f depends only on x and has the expression

$$f(x) = \left(\frac{\omega_{n-1}|B_{(I\phi)^o}(x)|}{\omega_n}\right)^{1/(n-1)}.$$

Concerning the first variation of area and ϕ-mean curvature, (4–6) of Theorem 4.3 reads as

$$\frac{d}{d\lambda}\mathcal{M}^{n-1}_{d_\phi}(\partial E_\lambda)_{|\lambda=0} = \omega_n \int_{\partial E} \mathrm{div}_\phi\ \mathrm{n}^E_\phi\ \langle X, \nabla d^E_\phi\rangle \frac{1}{|B_\phi(x)|}\ \phi^o(x,\nu^E) d\mathcal{H}^{n-1},$$

where the operator div_ϕ acts on a smooth vector field v as follows:

$$\mathrm{div}_\phi\ v := \mathrm{div}\ v + \left\langle v,\ \nabla\Big(\log \frac{1}{|B_\phi(x)|}\Big)\right\rangle,$$

and

$$\mathrm{n}^E_\phi(z) := \nabla\phi^o(z, \nabla d^E_\phi(z)),$$

$\nabla\phi^o$ being the gradient of ϕ^o with respect to the ξ variable.

5. The Crystalline Case: Lipschitz ϕ-Regular Sets

In this section we assume that ϕ is a crystalline norm. The main difficulties when trying to generalize the notion of ϕ-curvature given in (4.5) to the crystalline case are due to the loss of regularity, both of ∂E and of the norm ϕ^o. Observe that the explicit computation of κ^E_ϕ in (4.5) requires the computation of the hessian of ϕ^o which, in the crystalline case, is just a (nonnegative) measure. Recall also that, in the crystalline case, the duality maps are not single valued anymore. This will force us to consider inclusions in place of equalities, and suitable selection principles will be required. Finally, we have to keep in mind that, whichever definition of smoothness we choose, the set ∂B_ϕ must be smooth.

Unlike the regular case, in the crystalline case we have to redefine what is a smooth boundary. The idea is to define smoothness of ∂E by requiring the existence of at least one Lipschitz selection of a normal (in a suitable sense) vector field.

Before giving formal definitions, we recall some notation. At points $x \in \partial E$ where $\nu^E(x)$ exists we set $\nu_\phi^E(x) := \nu^E(x)/\phi^o(\nu^E(x))$. We indicate by $\mathrm{Lip}(\partial E;\mathbb{R}^n)$ the class of all Lipschitz vector fields defined on the Lipschitz boundary ∂E.

Definition 5.1. If $E \subset \mathbb{R}^n$ is Lipschitz we define

$$\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n) := \{N \in L^\infty(\partial E;\mathbb{R}^n) : N(x) \in T^o(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1}\text{a.e. } x \in \partial E\},$$
$$\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n) := \mathrm{Lip}(\partial E;\mathbb{R}^n) \cap \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n).$$

As we shall see, in general smooth sets E in the usual sense do not admit even one element in the class $\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. The best of smoothness that we can hope is described by the following definition [Bellettini and Novaga 1998]:

Definition 5.2. Let $E \subset \mathbb{R}^n$ be an open set with compact boundary. We say that E is Lipschitz ϕ-regular if ∂E is Lipschitz continuous and there exists a vector field $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. With a small abuse of notation, the pair (E,η) will also denote a Lipschitz ϕ-regular set.

Observe that, unlike ν_ϕ^E, the vector field η is defined everywhere on ∂E.

To better understand the meaning of Definition 5.2, we consider examples.

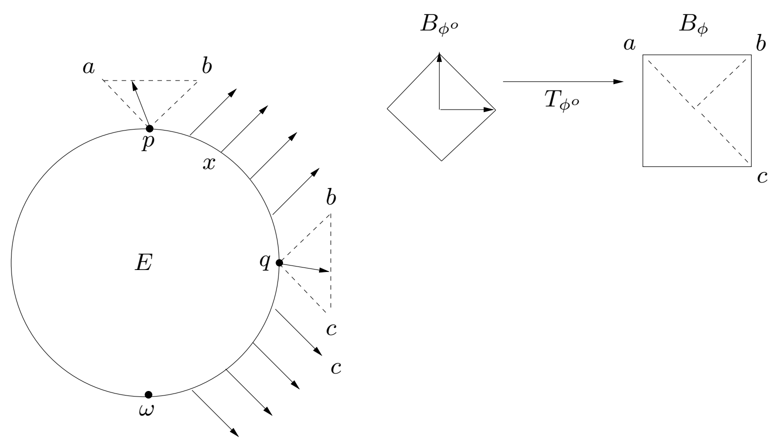

Figure 5. The open ball E is *not* Lipschitz ϕ-regular when we endow $\mathbb{R}^2$ with the norm $\phi(\xi) = \max\{|\xi_1|, |\xi_2|\}$.

Example 5.1. Let $n = 2$ and $\phi(\xi) := \max\{|\xi_1|, |\xi_2|\}$, in such a way that B_ϕ is the square of side 2, see Figure 5. Let $E := \{z \in \mathbb{R}^2 : |z| < 1\}$ be the open unit disk. Then E is *not* Lipschitz ϕ-regular. Indeed, to see that E is Lipschitz ϕ-regular, we have to compute $T^o(\nu_\phi^E(x))$, for $x \in \partial E$, and to show that we can produce a vector field η on ∂E which is a Lipschitz selection for the multivalued map $x \in \partial E \to T_{\phi^o}(\nu_\phi^E(x))$. Observe now that $T^o(\nu_\phi^E(p))$ is the upper horizontal segment $[a, b]$ of ∂B_ϕ; we depict therefore a corresponding dotted triangle at p. Similarly, $T_{\phi^o}(\nu_\phi^E(q))$ is the right vertical segment $[b, c]$ of ∂B_ϕ, and again we depict the corresponding dotted triangle at q. On the other hand, any point x on ∂E lying in the (relatively) open arc A between p and q is such that $T_{\phi^o}(\nu_\phi^E(x)) = b$. We deduce that $\eta \equiv b$ on A, and $\eta \equiv c$ on the open arc on ∂E between q and ω. Hence, *any* vector we choose inside the dotted triangles (for instance, the triangle at q) will produce a *discontinuity* in the vector field η (at q). We can conclude that the circle, considered in $(\mathbb{R}^2, \phi)$, is not Lipschitz ϕ-regular, and that it takes the rôle of the square in the usual Euclidean plane.

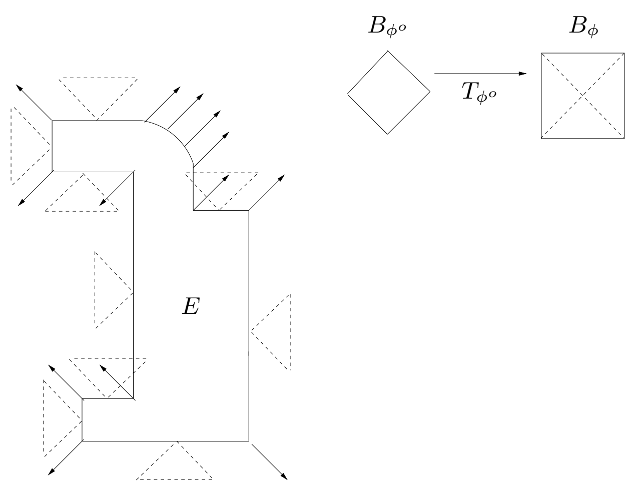

Figure 6. Example of a Lipschitz ϕ-regular set E when $\phi(\xi) = \max\{|\xi_1|, |\xi_2|\}$. The values of η are uniquely determined at the vertices and on the curved arc of ∂E. Any Lipschitz extension of these values on the interior of the edges, which lies in the dotted triangles, produces a Lipschitz vector field satisfying the required properties (that is, making E Lipschitz ϕ-regular). Examples are depicted in Figures 10 and 13.

Example 5.2. Let $\phi(\xi) := \max\{|\xi_1|, |\xi_2|\}$. In Figure 6 we show an example of a Lipschitz ϕ-regular set E.

At the vertices of ∂E the vector ν_ϕ^E is not defined. Let v be a vertex of ∂E, and let F_1 and F_2 be the two arcs of ∂E having v as a vertex (arcs can also be flat, i.e., segments). For any x in the relative interior of F_i, the closed convex set $T_{\phi^o}(\nu_\phi^E(x))$ is either a segment or a singleton; in both cases is independent of x and depends only on F_i. Denote it by K_i. The crucial property that makes E Lipschitz ϕ-regular is that the intersection $\cap_{i=1}^2 K_i$ is a singleton, see Figure 6. This produces a unique vector at each vertex of ∂E; at this point, it is easy to realize that we can construct infinitely many vector fields $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^2)$ lying inside the dotted triangles with the assigned values at the vertices (see for instance Figures 10 and 13).

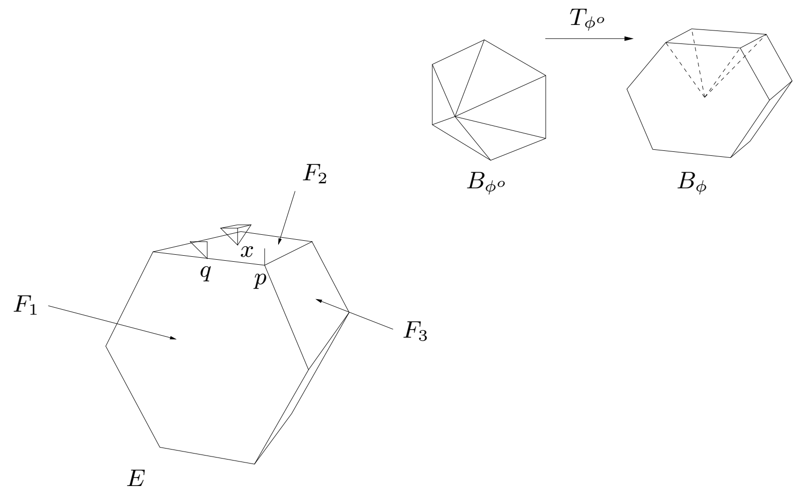

Figure 7. An example of Lipschitz ϕ-regular set E for the norm ϕ whose unit ball is the product of an hexagon with $[-1,1]$.

Example 5.3. In Figure 7 it is shown an example of Lipschitz ϕ-regular set in $\mathbb{R}^3$ when the unit ball of ϕ is prism with hexagonal basis. Observe that the vector ν_ϕ^E is not defined on the vertices and on the edges of ∂E. Let $p \in \partial E$ be a vertex of ∂E and let F_1, F_2, F_3 be the three (relatively) closed facets of ∂E having p as a common vertex. For any x in the relative interior of F_i, the convex set $T_{\phi^o}(\nu_\phi^E(x))$ is a closed facet of ∂B_ϕ which is independent of x and depends only on F_i. Denote it by K_i. The intersection $\cap_{i=1}^3 K_i$ is a singleton (and is the corresponding vertex of ∂B_ϕ). In Figure 7 we have depicted such an intersection as a vector at the point p. On the other hand, if F_1 and F_2 have in common the segment S, and q is a point in the relative interior of S, then $\cap_{i=1}^2 K_i$ is the corresponding edge of ∂B_ϕ. We have depicted this set as a triangle. Finally, if x is a point in the relative interior of a facet (for instance, the top facet F), then $\nu_\phi^E(x)$ coincides with the top vertex of the ∂B_{ϕ^o}, and therefore $T_{\phi^o}(\nu_\phi(F))$ is the top facet of ∂B_ϕ, and we have depicted this set on the interior of F as a pyramid. Showing that E is Lipschitz ϕ-regular means to exhibit a Lipschitz vector field $\eta : \partial E \to \mathbb{R}^3$ which on the vertices of ∂E is fixed (to be the corresponding vertices of ∂B_ϕ), on the relative interior of an edge of ∂E is constrained to lie in the corresponding segment of ∂B_ϕ, and in the relative interior of a facet of ∂E is constrained to lie in the corresponding facet of ∂B_ϕ. It is at this point easy to realize that such a choice can be made (in infinitely many different ways) for the set E in Figure 7.

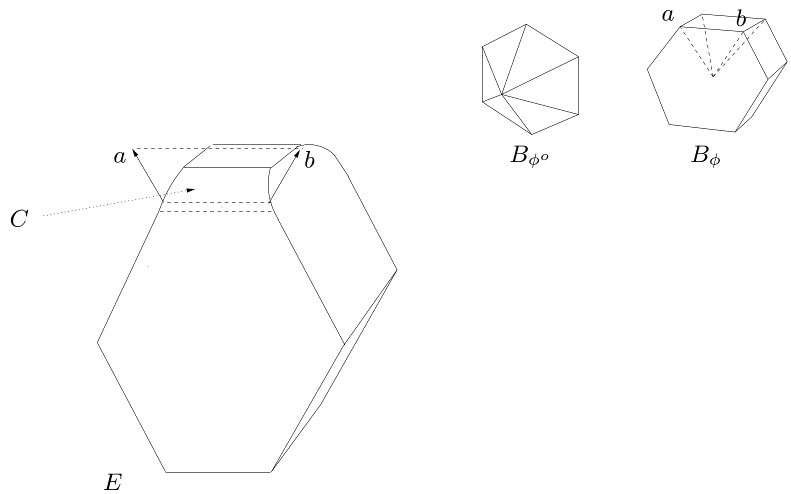

Figure 8. A non polyhedral Lipschitz ϕ-regular set E.

Example 5.4. In Figure 8 we show an example of Lipschitz ϕ-regular set E in $\mathbb{R}^3$ which is not polyhedral. The curved region C is *ruled*; if S is any horizontal segment in C (see the dotted lines), any vector field $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^3)$ must lie, on S, in the corresponding segment $[a, b]$ of ∂B_ϕ.

In two dimensions the structure of Lipschitz ϕ-regular sets E having a finite number of arcs (arcs can be also segments) can be described as follows. The arcs are located in a precise order consistent with ϕ-regularity, and are divided into two classes. In the first class there are edges which are parallel to some facet of B_ϕ and have the same exterior Euclidean normal vector (and we say that the edge *corresponds to* a facet of B_ϕ). The second class consists of the arcs (some of which can be flat) not belonging to the first class, where there is only one possible choice of the vector field η consistent with the ϕ-regularity. The arcs of the second class have therefore zero ϕ-curvature (see [Taylor 1993], [Giga and Gurtin 1996]).

6. On Facets of Polyhedral Lipschitz ϕ-Regular Sets

We have seen in Example 5.2 that, even in two dimensions, if (E, η) is a Lipschitz ϕ-regular set, there are in general infinitely many vector fields $\overline{\eta} \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^2)$ such that $(E, \overline{\eta})$ is Lipschitz ϕ-regular. Therefore the divergence of each of these $\overline{\eta}$ could be considered as a ϕ-mean curvature of ∂E, and in this way the Lipschitz boundary ∂E would have infinitely many different ϕ-mean curvatures. This approach could be pursued to some extent; however, we shall see that, among all vector fields satisfying the required constraints, there are some which are distinguished, have the *same* divergence, and such a uniquely defined divergence is what we can call the ϕ-mean curvature of ∂E, at least from the evolutionary point of view. Indeed, this notion of ϕ-mean curvature should be identified with the initial velocity of the interface under crystalline mean curvature flow.

To simplify notation, in this section we shall consider, when $n \geq 3$, only polyhedral Lipschitz ϕ-regular sets E with a finite number of facets, that will be understood as (relatively) closed connected $(n-1)$ dimensional flats with Lipschitz (polyhedral) boundary. The symbol F will always denote such a facet. This is surely a restriction, since in general facets can produce curved regions during the flow.

6.1. Some notation. If $F \subset \partial E$ is a facet, we denote by ∂F and $\mathrm{int}(F)$ the relative boundary and relative interior of F. An edge of ∂E with vertices p, q will be denoted by $[p, q]$, and its relative interior by $]p, q[$. We denote by Π_F the affine hyperplane spanned by the facet F.

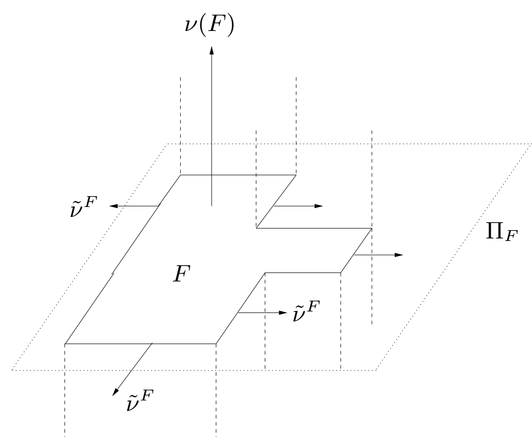

Figure 9. A facet F. The dotted lines delimit the solid set E having F as a facet.

We define $\nu(F)$ to be the unit normal to $\mathrm{int}(F)$ which points outside of E and we set $\nu_\phi(F) := \nu(F)/\phi^o(\nu_\phi(F))$. We indicate by $\tilde{\nu}^F$ the ($\mathcal{H}^{n-2}$-almost everywhere defined) unit normal to ∂F pointing outside of F; see Figure 9. Only facets F such that $T_{\phi^o}(\nu_\phi(F))$ is a facet of B_ϕ (that is., facets of ∂E corresponding to some facet of ∂B_ϕ) will be considered.

Definition 6.1. Let (E, η) be a Lipschitz ϕ-regular set. We define the trace function $c_F \in L^\infty(\partial F)$ as

$$c_F := \langle \eta, \tilde{\nu}^F \rangle. \tag{6–1}$$

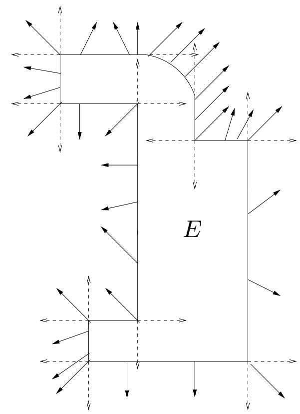

Figure 10. The normal trace of η on the boundary of each one-dimensional facet of ∂E is independent of η itself (among all vector fields making E Lipschitz ϕ-regular).

EXAMPLE 6.1. In Figure 10 we depict a vector field η which makes ∂E Lipschitz ϕ-regular (B_ϕ is the square as in Figure 6). Since the values of η are uniquely determined at the vertices of ∂E, the constants c_F do not depend on the particular choice of η. The dotted vectors at the vertices indicate the unit normals (in the line containing the facet F) pointing outward F (that is, $\tilde{\nu}^F$).

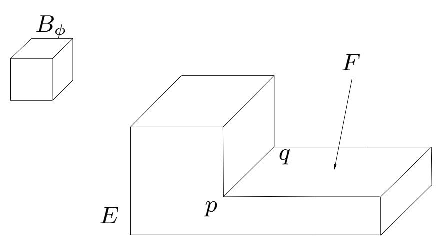

Figure 11. A Lipschitz ϕ-regular set when B_ϕ is the cube.

EXAMPLE 6.2. Consider the Lipschitz ϕ-regular set (E, η) of Figure 11 (B_ϕ is the unit cube). In Figure 12 the bold vectors at the vertices of ∂E are the unique possible values for η. The vector field $\tilde{\nu}^F$ points outside F, and on $]p, q[$ points inside E. The pyramids with vertex on the relative interior of the two facets having $[p, q]$ in common represent the corresponding facets of ∂B_ϕ (for instance, $T_{\phi^o}(\nu_\phi(F))$ for the facet F), that is, the range of admissibility of η. It follows that c_F is negative on $]p, q[$, while c_F is positive on the remaining relatively open edges of ∂F.

Given a Lipschitz ϕ-regular set (E, η), in general it is possible to prove that c_F does not depend on the choice of η in $\mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. More precisely, for

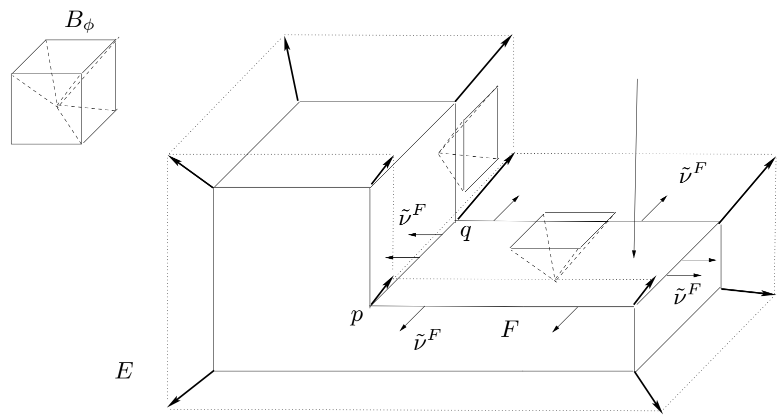

Figure 12. On the relative interior of $[p, q]$ the function c_F is negative (and constant).

$\mathcal{H}^{n-1}$-almost every $x \in \partial F$, we have

$$c_F(x) = \begin{cases} \max\{\langle \xi, \tilde{\nu}^F(x)\rangle : \xi \in T_{\phi^o}(\nu_\phi(F))\} & \text{if } \tilde{\nu}^F(x) \text{ points outside } E, \\ \min\{\langle \xi, \tilde{\nu}^F(x)\rangle : \xi \in T_{\phi^o}(\nu_\phi(F))\} & \text{if } \tilde{\nu}^F(x) \text{ points inside } E. \end{cases} \tag{6–2}$$

7. ϕ-Mean Curvature on a Facet

In this section we want to define (pointwise almost everywhere) the ϕ-mean curvature on a facet F of a polyhedral Lipschitz ϕ-regular set (E, η). We need some preliminaries. We let

$$\mathrm{Nor}_\phi(F; \Pi_F) := \big\{N \in L^\infty(\mathrm{int}(F); \Pi_F) : \\ N(x) \in T_{\phi^o}(\nu_\phi(F)) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{int}(F)\big\}.$$

It is possible to prove (see [Giga et al. 1998]) that any $N \in \mathrm{Nor}_\phi(F; \Pi_F)$ with $\mathrm{div} N \in L^2(\mathrm{int}(F))$ admits a normal trace $\langle N, \tilde{\nu}^F\rangle$ on ∂F, for which the Gauss–Green Theorem holds on F (see [Anzellotti 1983]). We set

$$H(F; \Pi_F) := \\ \big\{N \in \mathrm{Nor}_\phi(F; \Pi_F) : \mathrm{div} N \in L^2(\mathrm{int}(F)),\ \langle N, \tilde{\nu}^F\rangle = c_F \ \ \mathcal{H}^{n-2} \text{ a.e. on } \partial F\big\}.$$

Remark 7.1. Thanks to (6–2), the class $H(F; \Pi_F)$ does not depend on the choice of the vector field η making E Lipschitz ϕ-regular.

We define the functional $\mathcal{F}(\,\cdot\,, F) : H(F; \Pi_F) \to [0, +\infty[$ as

$$\mathcal{F}(N, F) := c_{n,\phi} \int_{\mathrm{int}(F)} (\mathrm{div} N)^2 \ \phi^o(\nu^E) d\mathcal{H}^{n-1}. \tag{7–1}$$

The right hand side of (7–1) equals $c_{n,\phi}\phi^o(\nu(F)) \int_F (\mathrm{div} N)^2 \ d\mathcal{H}^{n-1}$, since F is flat.

Our definition of ϕ-mean curvature is based on the following result: the minimum problem

$$\inf \{\mathcal{F}(N, F) : N \in H(F; \Pi_F)\} \tag{7–2}$$

admits a solution, and any two minimizers have the *same* divergence.

Denote by $N^F_{\min}$ a solution of problem (7–2); since $\mathrm{div} N^F_{\min}$ is independent of the choice of $N^F_{\min}$ among all minimizers of (7–2), we can give the definition of crystalline mean curvature.

DEFINITION 7.2. We define the ϕ-mean curvature κ^F_ϕ on the relative interior of F as

$$\kappa^F_\phi(x) := \mathrm{div} N^F_{\min}(x), \quad \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{int}(F).$$

Observe that κ^F_ϕ is only a function in $L^2(\mathrm{int}(F))$. We then set $\kappa^E_\phi := \kappa^F_\phi$ on each facet F of ∂E: it turns out that the orthogonal projection of minimizing vector fields on the orthogonal to ∂F is continuous on ∂F.

REMARK 7.3. The minimum problem (7–2), which is at the basis of Definition 7.2, arises when looking at the best way to decrease the $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ through deformations of the ambient space, precisely in the computation of the first variation

$$\liminf_{\lambda \to 0^+} \frac{\mathcal{M}^{n-1}_{d_\phi}(\partial E_\lambda) - \mathcal{M}^{n-1}_{d_\phi}(\partial E)}{\lambda}$$

of $\mathcal{M}^{n-1}_{d_\phi}$ at ∂E. Here, using the notation of Theorem 4.3, we have $E_\lambda = \Psi_\lambda(E)$ and $\Psi_\lambda(x) = x + \lambda X(x)$, where X is a suitable Lipschitz vector field.

The ϕ-mean curvature of ∂B_ϕ is constantly equal to $n-1$. Indeed, the vector field $x/\phi(x)$ has constant divergence on ∂B_ϕ, hence it solves the Euler–Lagrange inequality derived from (7–2). We now use the (strict) convexity in the divergence to show that $x/\phi(x)$ is actually a minimizer of $\mathcal{F}(\,\cdot\,, F)$ on any facet $F \subset \partial B_\phi$.

The following example concerning crystalline curvature of curves is enlightening.

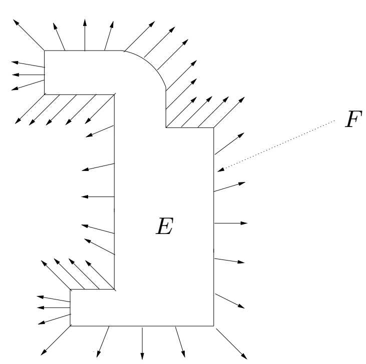

Figure 13. The vector field $\overline{N}^E_{\min} : \partial E \to \mathbb{R}^2$ is, on each facet F, the linear combination of the values of η at the vertices.

Example 7.1. Let $n = 2$. We compute explicitly the ϕ-curvature of a two-dimensional Lipschitz ϕ-regular set (E, η). Given a facet $F \subset \partial E$ (in this case F equals a segment $[z, w]$), the minimum problem (7–2) reads as

$$\inf\Big\{ \int_{]z,w[} (N'(s))^2 d\mathcal{H}^1(s) : N \in L^2(]z,w[;\Pi_{[z,w]}),\ N' \in L^2(]z,w[), \\ N(x) \in T_{\phi^o}(\nu_\phi^{[z,w]}(x)) \text{ for a.e. } x \in \,]z,w[,\ N(z) = c_z, N(w) = c_w \Big\},$$

where c_z and c_w are the orthogonal projections of $\eta(z)$ and $\eta(w)$ on the line $\Pi_{[z,w]}$, with the correct sign.

We now observe that the above minimum problem has a *unique* solution $\overline{N}^F_{\min}$, which is simply the linear function connecting c_z at z with c_w at w. Hence, when $n = 2$, not only the divergence of a minimizer is unique, but also the minimizer itself. If we now repeat this procedure for any facet, and on each facet we add to $\overline{N}^F_{\min}$ the proper (constant) normal component to F, we end up with the vector field $\overline{N}^E_{\min} : \partial E \to \mathbb{R}^2$ whose divergence is the ϕ-curvature of ∂E. An example of this vector field is depicted in Figure 13. Curved regions in ∂E have zero ϕ-curvature. On the other hand, if F is a facet of $\partial E \subset \mathbb{R}^2$ and $B_F \subset \partial B_\phi$ is the corresponding facet in ∂B_ϕ, κ^F_ϕ is *constant* on F and

$$\kappa^F_\phi = \delta_F \frac{|B_F|}{|F|} \quad \text{on } \mathrm{int}(F), \tag{7–3}$$

where $\delta_F \in \{0, \pm 1\}$ is a convexity factor: $\delta_F = 1$, -1 or 0 depending on whether E is locally convex at F, locally concave at F, or neither.

In two dimensions (7–3) is used to define the curvature flow of a Lipschitz ϕ-regular set (see the references quoted in the Introduction). If ∂E has a finite number of arcs, crystalline curvature flow can be described with a system of ordinary differential equations, since each arc (with nonzero ϕ-curvature) moves in normal direction in the evolution process: it cannot split or curve since, as dictated by (7–3), its normal velocity is constant. On the other hand, arcs or segments with zero ϕ-curvature stay still, and are progressively eaten by the other evolving arcs.

When the space dimension n is larger than or equal to 3, the computation of the ϕ-mean curvature on a facet is not, in general, an easy problem. As already mentioned in the Introduction, a short time existence theorem of a crystalline mean curvature flow is still missing. Concerning the comparison principle, only an indirect proof is available, in a certain class of crystalline evolutions, see [Bellettini et al. 2000].

Definition 7.4. We say that E is convex at F if E, locally around F, lies on one side of Π_F.

The following results show that the ϕ-curvature enjoys some additional regularity properties.

THEOREM 7.5. $\kappa_\phi^F \in L^\infty(\mathrm{int}(F))$. *Moreover*, κ_ϕ^F *has bounded variation on* $\mathrm{int}(F)$. *Finally*, *if* F *is convex and* E *is convex at* F, *then* κ_ϕ^F *is convex on* $\mathrm{int}(F)$.

Since the jump set of a function with bounded variation is well defined (see for instance [Ambrosio et al. 2000]), this theorem makes it possible to speak of the jump set of κ_ϕ^F on $\mathrm{int}(F)$, which should describe, at time zero, where the facet splits under crystalline mean curvature flow. For small times in the evolution problem, F is expected to translate parallel to itself if κ_ϕ^F is constant on $\mathrm{int}(F)$ or to bend if κ_ϕ^F is continuous but not constant on $\mathrm{int}(F)$. Facets with constant ϕ-mean curvature have been isolated and studied in [Bellettini et al. 1999], [Bellettini et al. 2001c], where the following notation was introduced.

DEFINITION 7.6. We say that F is ϕ-calibrable if κ_ϕ^F is constant on $\mathrm{int}(F)$.

More explicitly, F is ϕ-calibrable provided there exists a vector field $N : \mathrm{int}(F) \to \Pi_F$ which solves the following problem:

$$\begin{cases} N \in L^\infty(\mathrm{int}(F); \Pi_F), \\ N(x) \in T_{\phi^o}(\nu_\phi(F)) \text{ for } \mathcal{H}^{n-1} \text{ a.e. } x \in \mathrm{int}(F), \\ \langle N, \tilde{\nu}^F \rangle = c_F \quad \mathcal{H}^{n-2} \text{ a.e. on } \partial F, \\ \mathrm{div} N = \dfrac{1}{|F|} \displaystyle\int_{\partial F} c_F \; d\mathcal{H}^{n-1}. \end{cases} \tag{7–4}$$

Observe that the constant on the right hand side of the differential equation in (7–4) is determined by using the Gauss–Green theorem on F. The complete characterization of ϕ-calibrable facets F is not yet available. We conclude the paper by pointing out some known results in this direction.

7.1. Characterization of ϕ-calibrable facets in special cases. Assume that $n = 3$. Let $B_F \subset \partial B_\phi$ be the facet corresponding to F. If necessary, we identify B_F with its orthogonal projection on the plane parallel to Π_F passing through the origin of $\mathbb{R}^3$. We also assume that B_F contains the origin in its interior and that it is symmetric (this latter assumption can be weakened). Therefore B_F can be considered as the unit ball of a norm (in $\mathbb{R}^2$), which we denote by $\tilde{\phi}$. We assume that F is Lipschitz $\tilde{\phi}$-regular. Denote by $\kappa_{\tilde{\phi}}^F$ the $\tilde{\phi}$-curvature of ∂F and by $\tilde{\phi}^o$ the dual of $\tilde{\phi}$. The following result holds.

THEOREM 7.7. *Let* $n = 3$. *Assume that* F *is convex and that* E *is convex at* F. *Then* F *is* ϕ*-calibrable if and only if*

$$\sup_{\partial F} \kappa_{\tilde{\phi}}^F \leq \frac{1}{|F|} \int_{\partial F} \tilde{\phi}^o(\tilde{\nu}^F) \; d\mathcal{H}^1. \tag{7–5}$$

The sup in (7–5) is an essential supremum, since $\kappa_{\tilde{\phi}}^F$ is a function in $L^\infty(\partial F)$. Hence, under the assumptions of Theorem 7.7, problem (7–4) is solvable if and only if the $\tilde{\phi}$-curvature of ∂F is bounded above by the constant on the right hand side of (7–5); this means, roughly speaking, that the edges of ∂F cannot be too "short".

Finally, let us mention that examples of facets which are not ϕ-calibrable are given in [Bellettini et al. 1999], and that the problem of calibrability when B_ϕ is a portion of a cylinder (hence not in a crystalline setting) has been recently considered, under rather mild assumptions, in the papers [Bellettini et al. 2002] and [Bellettini et al. 2003].

8. Concluding Remarks

Fix a norm ϕ on $\mathbb{R}^n$ and denote by d_ϕ the distance induced by ϕ. As a starting point of our approach let us consider the $(n-1)$-dimensional measure $\mathcal{M}^{n-1}_{d_\phi}$, defined as in (1–1) on compact and sufficiently smooth boundaries ∂E of solid sets E. Such a notion is called the ϕ-Minkowski content of the manifold ∂E, and is a geometric invariant under isometries of the ambient space (because the n-dimensional Hausdorff measure $\mathcal{H}^n_{d_\phi}$ with respect to d_ϕ is invariant and the tubular neighborhoods are computed with respect to the distance d_ϕ). On the other hand $c_{n,\phi}$ is not invariant (recall that ω_n is a normalizing constant).

We recall that, in the generic (finite dimensional) normed space $(\mathbb{R}^n, \phi)$, there are other meaningful notions of surface measure, such as for instance the $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}_{d_\phi}$ with respect to d_ϕ, the Holmes–Thompson measure and the measure considered in [De Giorgi 1995]. Even in $n = 2$ dimensions, there are examples of norms ϕ on $\mathbb{R}^2$ for which $\mathcal{M}^1_{d_\phi}$ and $\mathcal{H}^1_{d_\phi}$ are different. Roughly speaking, this can be explained as follows. $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ is constructed by taking the projections of $B_\phi := \{\phi \le 1\}$ onto the (one-dimensional) normal spaces to ∂E. On the other hand, $\mathcal{H}^{n-1}_{d_\phi}(\partial E)$ is constructed by taking the intersections of B_ϕ with the ($(n-1)$-dimensional) tangent spaces to ∂E. We notice that, in any case, $\mathcal{H}^{n-1}_{d_\phi}$ can be seen as the Minkowski content with respect to another norm ψ.

Beside its geometric interest, our choice of working with $\mathcal{M}^{n-1}_{d_\phi}$ is motivated also by the physics of phase transitions, where it happens that some relevant phenomena are concentrated in a very thin tubular neighborhood of the interface (sometimes called diffuse interface), and lead in the limit to the Minkowski content.

The next step in our approach consists in the definition of the ϕ-mean curvature κ^E_ϕ of ∂E. This concept, which as usual depends also on the immersion of the manifold, is obtained by computing the first variation of $\mathcal{M}^{n-1}_{d_\phi}$ on ∂E, and is identified with the normal velocity of the initial datum $\partial E = \partial E(0)$ under the anisotropic mean curvature flow.

The computation of the first variation of $\mathcal{M}^{n-1}_{d_\phi}$ is rather direct when ∂B_ϕ is smooth and all its principal curvatures are strictly positive (regular case), but becomes more involved in the crystalline case (that is, when B_ϕ a polytope). In this latter situation, assuming that ∂E is polyhedral and has a geometry locally resembling the geometry of ∂B_ϕ, we compute κ_ϕ^E on its facets, by applying a suitable minimization principle. We then discuss the problem of characterizing ϕ-calibrable facets of ∂E in $n = 3$ dimensions, that is, those facets $F \subset \partial E$ for which κ_ϕ^E is constant on the relative interior $\mathrm{int}(F)$ of F.

We conclude by recalling that (a scalar multiple of) B_ϕ is a solution of the so-called isoperimetric problem, that is, the problem of minimizing $\mathcal{M}^{n-1}_{d_\phi}(\partial E)$ among all finite perimeter sets E with $\mathcal{H}^n_{d_\phi}(E)$ fixed. This is in agreement with the fact that ∂B_ϕ has constant ϕ-mean curvature, precisely equal to $n-1$.

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