Heegner Points and Rankin L-Series
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Correspondence

BRYAN BIRCH AND BENEDICT GROSS

Gross to Birch: March 1, 1982

Dear Birch,

I recently found an amusing method to study Heegner points on $J_0(N)$. Let E be an elliptic curve over $\mathbb{Q}$ of level N, together with a parametrization $J_0(N) \to_\pi E$. Let K be a quadratic field of discriminant d_K prime to N; let χ be the associated quadratic Dirichlet character and E^χ the twisted curve.

Let F be an imaginary quadratic field in which all prime factors of N split and choose an integral ideal $\mathfrak{n}$ with $\mathfrak{n}\bar{\mathfrak{n}} = N$ and $(\mathfrak{n}, \bar{\mathfrak{n}}) = 1$. Assume further that d_χ divides d_F, so K is contained in H, the Hilbert class field of F. The modular data $x = (\mathbb{C}/\mathscr{O}_F, \ker \mathfrak{n})$ defines a point of $X_0(N)$ rational over H and the divisor $e_f = \pi\big(\sum_{\mathrm{Aut}(H)} \chi(\sigma)\sigma x\big)$ gives a point of $E(K)^-$, or equivalently, a rational point on E^χ. One can check that e_F is killed by 2 whenever the sign in the functional equation for E^χ is $+1$. Do your computations support the following?

Conjecture. *e_F has infinite order iff* $\operatorname{rank} E^\chi(\mathbb{Q}) = 1$. *If this is the case and π is a strong Weil parametrization, let M denote the subgroup generated by the points e_F. Then* $(E^\chi(\mathbb{Q}) : M)^2 = \mathrm{Card}(\text{Ш}(E^\chi/\mathbb{Q}))$.

I think I can prove that the point e_F has infinite order whenever the image of the cuspidal group on E has order divisible by $p \geq 3$ *and* certain p-class groups are trivial. In all these cases, the rank is 1.

Here is a simple case which illustrates the method. Let $E = J_0(11)$ and let K be a *real* quadratic field in which the prime 11 is inert. Choose F as above, and let K' denote the other imaginary quadratic field contained in FK.

Proposition. *If* $h_K h_{K'} \not\equiv 0 \pmod 5$ *then* $E^\chi(\mathbb{Q}) \simeq \mathbb{Z}$, $e_F \neq 0$ *in* $E^\chi(\mathbb{Q})/5E^\chi(\mathbb{Q})$, *and* $\text{Ш}(E^\chi)_5 = (0)$.

Proof. A 5-descent, combined with the fact that $h_\chi \not\equiv 0 \pmod 5$, gives an exact sequence

$$0 \to E^\chi(\mathbb{Q})/5E^\chi(\mathbb{Q}) \to^{\delta} \mathscr{O}_K^*/(\mathscr{O}_K^*)^5 \to \text{Ш}(E^\chi/\mathbb{Q})_5 \to 0$$

with $\mathcal{O}_K^*/(\mathcal{O}_K^*)^5 \simeq \mathbb{Z}/5$. It will suffice to show $\delta(e_F) \neq 0$ in $\mathcal{O}_K^*/(\mathcal{O}_K^*)^5$, as $E^\chi(\mathbb{Q})_5 = (1)$.

But the map δ arises from the cohomology of the covering $\tilde{E} \to E$ with fibre μ_5 which is obtained by taking a 5^{th} root of the modular unit $f(z) = \Delta(z)/\Delta(11z)$. Some calculation then gives $\delta(e_F) = \delta\big(\sum_{\mathrm{Aut}(H)} \chi(\sigma)\sigma x\big) \equiv \big(\prod_{\mathfrak{a}} \Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})^{\chi(\mathfrak{a})}\big)^2$ mod $(\mathcal{O}_K^*)^5$, where the product is taken over ideals representing the classes of $\mathcal{O}_F$. Kronecker's limit formula can be used to show that the "elliptic unit" $\prod \Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})^{\chi(\mathfrak{a})}$ is equal to the $24h_K h_{K'}$ power of a fundamental unit for K. By our assumptions on h_K and $h_{K'}$, we see $\delta(e_F) \neq 0$.

Could you send me a preprint of your height computations, and anything else you might have on the subject?

Best wishes,

Dick Gross

Birch to Gross: May 6, 1982

Dear Dick,

I received your letter of March 1. I thought it was beautiful and decided to reply when I'd sorted it all out and had time to spare — which never happens, so I must apologize for not replying sooner.

First, the conjecture on your first page is insufficiently elaborate — in your notation there are three fields involved K, F and K', with corresponding characters χ, $\chi\chi'$ and χ'. For each of them, there corresponds a twisted elliptic curve defined over $\mathbb{Q}$. The correct conjecture appears to be

$$\begin{matrix}\text{canonical}\\ \text{height}\end{matrix}\begin{pmatrix}\text{point of } E^{(\chi)}(\mathbb{Q})\\ \text{given by } e_F\end{pmatrix} \doteq \frac{L'(E^{(\chi)},1)L'(E^{(\chi')},1)}{\begin{matrix}\text{real}\\ \text{period}\end{matrix}(E^{(\chi)})\ \begin{matrix}\text{real}\\ \text{period}\end{matrix}(E^{(\chi')})}$$

where $\doteq$ means "equal except for one or two stray factors like 2 or 3 that come in because I presumably haven't got quite the correct model."

Accordingly, your index

$$\begin{aligned}(E^{(\chi)}(\mathbb{Q}) : M)^2 \doteq{}& \mathrm{Card}(\text{Ш}(E^{(\chi)}/\mathbb{Q})) \cdot \mathrm{Card}(\text{Ш}(E^{(\chi')}/\mathbb{Q}))\\ &\cdot \begin{pmatrix}\text{factors coming from the}\\ \text{bad primes of } E^{(\chi)}\end{pmatrix} \cdot \begin{pmatrix}\text{factors coming from the}\\ \text{bad primes of } E^{(\chi')}\end{pmatrix};\end{aligned}$$

and I guess the torsion comes into it too. The significant point is that one needs the contribution from the "other" twist $E^{(\chi')}$; the less significant point is, of course, that there are more junk factors than you acknowledged. The conspicuous "corollary of the conjecture" is that e_F should have infinite order precisely when $E^{(\chi)}(\mathbb{Q})$ has rank 1 and $E^{(\chi')}(\mathbb{Q})$ has rank 0; if $\operatorname{rank} E^{(\chi)}(\mathbb{Q}) \geq 3$ or $\operatorname{rank} E^{(\chi')}(\mathbb{Q}) \geq 2$ the thing, as a matter of experience, is trivial.

I found your second page very nice indeed, and the more I think about it the more I like it; there seem to be an awful lot of doors it may open, and I'm almost reluctant to push them in case there is a block that I can't see! It's so much more down to earth than Barry's ideas, and seems to give a lot more information. I feel foolish not to have thought of something like it years ago — but then if one is set on points of order 2, the fact that the image of δ in the sequence

$$0 \to E^{(\chi)}(\mathbb{Q})/2E^{(\chi)}(\mathbb{Q}) \xrightarrow{\delta} \mathscr{O}_K^\times/\mathscr{O}_K^{\times\,2}$$

is automatically trivial is a very thick hedge!

Yours,

Bryan Birch

Gross to Birch: May 14, 1982

Dear Bryan,

Thanks for your note. In the interim between my letter and yours, I had noticed the point that I was neglecting the field K', as you kindly pointed out. Then I noticed some really *amazing* things — like the following:

1) The product $L'(E^{(\chi)}, 1)L(E^{(\chi')}, 1)$ is just the derivative at $s = 1$ of the L-series $L(E \otimes \mathrm{Ind}_F^Q \chi, s)$, where χ is viewed as a character of $\mathrm{Gal}(\overline{F}/F)$ and $\mathrm{Ind}\,\chi$ is the 2-dimensional induced representation to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
2) The L-series $L(E \otimes \mathrm{Ind}\,\chi, s)$ has a beautiful integral expression — by Rankin's method — as $L(\mathrm{Ind}\,\chi, s)$ is the L-series of a modular form of wt 1 !
3) The product of periods: *real period*$(E^{(\chi)}) \cdot$ *real period*$(E^{(\chi')})$ is *equal* to the integral $\int_{E(\mathbb{C})} \omega \wedge \overline{\omega}/\sqrt{d_F}$, as the characters χ and χ' have opposite parity. This integral can also be expressed as a Petersson inner product when E is a Weil curve (ω is a Néron differential on E).

Anyhow, this led me to drop the restriction of considering only *elliptic curves over* $\mathbb{Q}$, and I've arrived at the following crazy business.

Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a newform of wt 2 on $\Gamma_0(N)$ with coefficients in a subfield of $\mathbb{R}$; let $\omega_f = 2\pi i f(z)\,dz$ be the corresponding holomorphic 1-form on $X = X_0(N)$.

Let $J = J_0(N)$; there is a canonical symmetric pairing $\langle\ ,\ \rangle$ on $J(\overline{\mathbb{Q}}) \times J(\overline{\mathbb{Q}})$ with values in $\mathbb{R}$ which is obtained by composing the Poincaré height on $J \times J^\vee$ with the standard isomorphism $J \simeq J^\vee$. We can use the action of the Hecke algebra on J to *refine* this to a pairing $\langle\ ,\ \rangle_f : J(\overline{\mathbb{Q}}) \times J(\overline{\mathbb{Q}}) \to \mathbb{R}$ which satisfies $\langle T_l x, y\rangle_f = \langle x, T_l y\rangle_f = a_l \langle x, y\rangle_f$ for all $l \not| N$. If f has rational coefficients, so corresponds to the Weil curve $X \to_\pi E$, then $\langle x, y\rangle_f = \frac{1}{\deg \pi}\langle \pi x, \pi y\rangle_E$.

Now let F be an imaginary quadratic field in which all primes dividing N *split*, and fix a factorisation $N = \mathfrak{n} \cdot \overline{\mathfrak{n}}$ with $(\mathfrak{n}, \overline{\mathfrak{n}}) = 1$. The modular data $x = (\mathbb{C}/\mathscr{O}_F, \ker \mathfrak{n})$ defines a point of X which is rational over H, the Hilbert class field of F. Let y be the class of the divisor $\frac{1}{\delta}\{(x) - (\infty)\}$ in $J(H) \otimes \mathbb{Q}$, where $\delta = 1$ unless $F = \mathbb{Q}(i), \mathbb{Q}(\rho)$ in which case $\delta = 2, 3$. If χ is any character of $\mathrm{Gal}(H/K)$ we let $y_\chi = \sum \chi^{-1}(\sigma) y^\sigma$ in $(J(H) \otimes \mathbb{C})^\chi$.

By studying the functional equation of the L-series $L(f \otimes \mathrm{Ind}\,\chi, s)$ one can show that $L(f \otimes \mathrm{Ind}\,\chi, 1) = 0$. The "crude" conjecture then becomes

The following are equivalent: a) $\pi(y_\chi) \neq 0$ in $(A_f(H) \otimes \mathbb{C})^\chi$

b) $\dim_{\mathbb{C}}(A_f(H) \otimes \mathbb{C})^\chi = \dim A_f$

c) $L'(f \otimes \operatorname{Ind}\chi, 1) \neq 0$

Here A_f is the quotient of J determined by the newform f: $\pi : J \to A_f$.

Right now, all my descent evidence only helps to show b) $\Longrightarrow$ a). But I think that the equivalence a) $\Longleftrightarrow$ c) may be tractable. "All one has to do" is to prove the formula:

$$L'(f \otimes \operatorname{Ind}\chi, 1) = \frac{1}{\sqrt{d_F}} \int_X \omega_f \wedge \overline{\omega_f} \cdot \langle y_\chi, y_{\chi^{-1}} \rangle_f. \tag{$*$}$$

I hope I've got the fudge factors right — it seems to check against some hand computation, but I'd like to put it up against your tables. I should remark that there is a similar conjecture for $L'(f \otimes \operatorname{Ind}\chi, 1)$ when χ is a ring class character of F of conductor prime to N. Here one uses the Heegner point x constructed from the corresponding order.

Formula $(*)$ for all characters χ is equivalent to

$$L'(f, \sigma, 1) = \frac{1}{\sqrt{d_F}} \int_X \omega_f \wedge \overline{\omega_f} \cdot h_F \langle y, y^\sigma \rangle_f \tag{$**$}$$

for all elements $\sigma \in \operatorname{Gal}(H/F)$. Here $L(f, \sigma, s) = \frac{1}{h_F} \sum_\chi L(f \otimes \operatorname{Ind}\chi, s)\chi^{-1}(\sigma)$ is a "partial L-function", which has an even *nicer* analytic expression via Rankin's method. Namely, let

$$g_\sigma = \frac{1}{2\delta}\left(1 + \sum_{m,n}{}' q^{B(m,n)}\right),$$

where $B(x, y) = ax^2 + bxy + cy^2$ is a binary quadratic form in the class of σ. Then g_σ is a modular form of wt 1 and character ε on $\Gamma_0(d_F)$, where ε is the quadratic character corresponding to the extension $F/\mathbb{Q}$. Put $M = N d_F$ and define the Eisenstein series

$$E(z, s) = \sum_{c,d}{}' \frac{\varepsilon(d) y^s}{|cMz + d|^{2s}(cMz + d)};$$

this has wt 1 and character ε on $\Gamma_0(M)$ (but isn't holomorphic when $s \neq 0$.) Modulo powers of π and simple Γ-factors, one has the equality

$$L(f, \sigma, s) = \int_{\mathcal{H}/\Gamma_0(M)} \overline{f(z)} g_\sigma(z) E(z, s-1)\, dx\, dy.$$

At this point things become even more interesting, as I believe the integral breaks into h pieces (after dealing with some imprimitivity factors) which one can hope correspond to the *local* heights of the points y and y^σ at the archimedean places of H. Anyhow, it begins to look more like the sort of statement an analytic number theorist can deal with, I may even dare a crack at it myself!

Best wishes,

Dick

P.S. the relation b) $\iff$ c) is pretty much your conjecture with Swinnerton-Dyer, as $L(A_f/H, s) = \prod_\alpha \bigl(\prod_\chi L(f^\alpha \otimes \operatorname{Ind}\chi, s)\bigr)$, where the f^α are the conjugates of the modular form f. Note that the crude conjecture implies

$$L'(f \otimes \operatorname{Ind}\chi, 1) \neq 0 \iff L'(f^\alpha \otimes \operatorname{Ind}\chi, 1) \neq 0.$$

P.P.S. Don't despair at 2, just don't project to the ε-component of $E(F)$! For example, let $X = X_0(17)$ and F where $(17) = \wp.\overline{\wp}$ is split. The modular unit $h(z) = \{\Delta(z)/\Delta(17z)\}^{1/4}$ has divisor $4\{(0) - (\infty)\}$ on X and the map $J(H) \to^\delta H^*/(H^*)^4$ defined on divisors prime to $0, \infty$ by $\delta(\mathfrak{a}) = h(\mathfrak{a})$ mod $(H^*)^4$ is a group homomorphism. The image of the Heegner point y is equal to $\{\Delta(\mathscr{O})/\Delta(\wp)\}^{1/4}$, provided $F \neq \mathbb{Q}(i)$, this modular unit and all its conjugates generate the ideal $\wp^3$. Hence $\delta(y_1) = \prod_\sigma \delta(y^\sigma)$ is an element of $F^*/(F^*)^4$ which generates the ideal $\wp^{3h}$. If $4 \not| h$ this is *not* a fourth power. And — again assuming $F \neq \mathbb{Q}(i)$ — it is not the image of a torsion point on $J(F)$. Therefore, y_1 has infinite order.

Birch to Gross: around September 6, 1982

Dear Dick,

Thank you for your stuff — comments at end of letter, at least preliminary ones.* But no doubt you would prefer to know the present state of play as regards computations.

Nelson (Stephens) has been calculating Heegner points wholesale, dealing with all discriminants $-D$ with $D < 1000$ for various curves. So far, I have good data for the curves 11B, 17C, 19B, 26D, 37A, 43A, 57E, 67A, 76A; plus a few others I haven't had time to sort out yet (the data are fairly bulky) 79A, 109A.

Let us set up notation: suppose our curve is E with conductor N, so that E is associated with a normalized (i.e. $a_1 = 1$) differential ω on $X_0(N)$; $-D$ is the discriminant of your ring $\mathscr{O}$ so that $-D = efm^2$ where e, f are 1 or discriminants of quadratic fields, $(e, f)m$ is your e, and the game gives rise to a point of $E(\mathbb{Q}(\sqrt{e}))$ which you have been denoting by e_χ (at any rate for $\chi \neq \chi_0$). In case χ is not principal, we are led to a rational point of the twist $E^{(e)}$, which Nelson denotes by $P(e, f, m, \omega, N)$; in the principal case we are liable to have to take $e_\chi - e_\chi^\tau$ like you do, and are led to a point $P^*(e, f, m, \omega, N)$ "which is twice as big as it ought to be."

For the purpose of this letter, let us restrict ourselves to the cases

$$(em, fm, N) = 1$$

in order to avoid nonsense and complication (but in case N is not square free, this seems to be an undesirable restriction — e.g., for $N = 76$ we certainly need to allow $m = 2$). At this stage, there is no need to take the "imprimitivity index" m as 1. We throw away the "rubbish" cases that are automatically trivial by your Lemma 11.1 — and the conjectures that follow do not apply to such rubbish.

What I've actually tabulated is a near integer M. In case the point

$$P(e, f, m, \omega, N)$$

is torsion, $M = 0$; otherwise, it has always happened that rank $E^{(e)}(\mathbb{Q}) = 1$ (as predicted by conjectures made on the basis of this evidence!), we *fix* a generator $Q(e, E)$ of $E^{(e)}(\mathbb{Q})$, and we define M by

$$P(e, f, m, \omega, N) = \frac{w}{2} M(e, f, m, \omega, N) Q(e, E), \text{ in case } \chi \text{ is not principal}$$

$$P^*(e, f, m, \omega, N) = 2\frac{w}{2} M(e, f, m, \omega, N) Q(e, E), \text{ in case } \chi \text{ is principal}$$

*Gross had sent Birch an early version of his manuscript "Heegner points on $X_0(N)$".

where of course $\frac{w}{2}$ is the usual factor

$$\frac{w}{2} = \begin{cases} 2 & \text{if } \; efm^2 = -4, \\ 3 & \text{if } \; efm^2 = -3, \\ 1 & \text{otherwise.} \end{cases}$$

So apart from cases with e or f equal to 1, $M \in \mathbb{Z}$; and though the sign of M is meaningless, the sign of a ratio $M(e, f_1, m_1, \omega, N)/M(e, f_2, m_2, \omega, N)$ is well-determined. Unfortunately, Nelson's present tables only seem to give the sign reliably when $e > 0$.

Our main conjecture is of course that

$$\text{canonical height}\,(P(e, f, 1, \omega, N)) = L^{*\prime}(E^{(e)}, 1) L^*(E^{(f)}, 1)$$

where the L-functions on the right are suitably normalized — one needs to divide by the real period, allow for the torsion, and do a little fudging (e.g. by $\frac{w}{2}$). So far, we have very few computations of $L^{*\prime}$, so we can't be absolutely specific — the [burden] of the conjecture at present is that the amount of fudging will be slight and predictable! In particular, Ш should *not* come into the formula. Particular cases, more easily verified experimentally, are

(1) $P(e, f, 1, \omega, N)$ is torsion if $L^{*\prime}(E^{(e)}, 1) = 0$ (i.e. rank $E^{(e)}(\mathbb{Q}) \geq 3$) or $L^*(E^{(f)}, 1) = 0$ (i.e. rank $E^{(f)}(\mathbb{Q}) \geq 2$).

(2) For fixed e and variable f,

$$\frac{h(P(e, f_1, 1, \omega, N))}{h(P(e, f_2, 1, \omega, N))} = \frac{L^*(E^{(f_1)}, 1)}{L^*(E^{(f_2)}, 1)}$$

For the curve $y^2 = x^3 - 1728$, the evidence is about 10 years old by now!

Experimental facts gleaned from our present computations are as follows:

(1) If $m > 1$, $P(e, f, m, \omega, N)$ is a predictable multiple of $P(e, f, 1, \omega, N)$; and this can be proved, I think. End of story! (Use Hecke operators, cf. your § 6).

So it is nearly enough to consider the case $m = 1$ — at any rate for square-free N.

(2) For fixed ω, N and any e_1, e_2, f_1, f_2 for which the relevant Heegner points all exist

$$M(e_1, f_1, 1, \omega, N) M(e_2, f_2, 1, \omega, N) = \pm M(e_1, f_2, \cdots) M(e_2, f_1, \cdots)$$

and the sign is + (whenever e_1, e_2 are positive).

(3) $M^2(e_1, f_1, 1, \omega, N) L^*(E^{(f_2)}, 1) = M^2(e_2, f_2, 1, \omega, N) L^*(E^{(f_1)}, 1)$.

(4) For certain curves (maybe all curves without 2-torsion?) the parity of $M(e, f, m, \omega, N)$ depends *only* on $-D = efm^2$, ω, N, and *not* on e, f, m separately. Nelson can prove this in some cases, and I can in others. Combined with (1) above, it is quite a strong criterion for proving Heegner points non-trivial, *different* from those already known.

(Corollary: Subject to various conditions, for square free e

$$L^*(E^{(e)},1) \equiv a_e \bmod 2,$$

where of course a_n is eigenvalue of Hecke T_n. Is this well known?)

My guess is that (1), (4), and probably (2) are accessible with present techniques — but not (3).

Our evidence for (1) and (2) is all the curves I've mentioned, for all $-D \geq -1000$; the evidence for the "+" in (2) is only for the curves 17, 19, 37, 67, only in the cases $e > 0$. We've verified (3) in all cases, except that $L^{(f)}(1)$ has usually only been calculated for f up to 200 or so.

Comments on your M/S* — nothing very useful to say, but you asked!

p. 1. It still isn't clear to me that your method yields all the known results when E has (cuspidal) 2-torsion.

Chapter I. Doesn't claim to contain anything new, but it's a very nice exposition. I like the statement of Lemma 11.1 — it is straightforward junk-dunking, but more elegant than my own explicit statement in terms of quadratic residues and the like. §§4-6 are nice and clear too.

A very small point — don't call c the *level*, it would be waste of a good word if it were spare, and it isn't — the level is N! After all, c has a perfectly good name (the conductor of $\mathcal{O}$) if it needs one.

More material points:

(i) $(c, N) = 1$ is just a bit too restrictive in general, e.g. one needs to allow $c = 2$ when $N = 76$.

(ii) I guess 11.2 is correct, but it is hard to produce any evidence for cases with $\chi^2 \neq 1$. I like my conjectures experimentally verified as well as theoretically hyperplausible!

Chapter II. This seems rather more technical than I would have expected, but I have to admit that I've only played with the cases where χ is quadratic, when the L-series are very easy to write down.

p. 16 Conjecture 17.1. Do you *really* think this is accessible? After all, one doesn't even know how to prove $L'(1) = 0$.

"all evidence". Is there any evidence other than Stephens–Birch?

Chapter III (Very nice too.)

§20. I note that this is Barry's proof with the mappings made explicit!

§22. The case N prime is pegged rather closely to the Eisenstein component theory, and the case N composite is rather vague. Being a very down-to-earth sort of person (who isn't?) I prefer to start at the other end. Just at the moment, the most general formulation seems to be that *you have a theorem whenever you*

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*See note on page 17.

have a nice function supported by the cusps — for instance, for $N = 26$ one writes down

$$\left(\frac{\eta(2z)\,\eta(26z)}{\eta(z)\,\eta(13z)}\right)^{12}$$

and the rest is automatic, whether N is prime or composite matters little. There is a prehistoric paper of Morris Newman that deals with explicit functions of shape $\prod \eta(dz)^{r_d}$; but I guess that the torsion doesn't all come from functions of this particular shape, and of course (e.g. in § 23) you don't need quite such special functions to apply the Kronecker formula.

I'm not clear that your argument includes the $n = 2$ case.

Incidentally, you will be amused to hear that in the cases e.g. for $N = 11$ where you predict that e_χ is trivial in $E^{(\chi)}/5E^{(\chi)}$ the computations, sure enough, give M divisible by 5 — but *not* (usually) trivial.

Have sent a copy of your draft to Nelson Stephens — hope this is O.K. Will try to keep you posted, despite my well known reluctance to set pen to page.

Bryan

Gross to Birch: September 17, 1982

Dear Bryan,

Thanks for your letter. Your comments were helpful, and I'll incorporate them into the next version. Some questions.

1) I can't see how to handle the case where $gcd(c, N) \neq 1$, as I can't find a nice formula for the sign in the functional equation of $L(f \otimes g_\chi, s)$ when χ is primitive. Can you?

2) When I proposed conjecture 17.1, I didn't know about your wholesale data. It was motivated by the theoretical evidence in chapter III and some retail computational checks (about 1,000 in number) I made with Joe Buhler. We seem to get:

$$2^r \cdot \delta^2 \cdot D^{1/2} L'(f \otimes g_\chi, 1) = (\omega_f, \omega_f)\langle \nu_{\chi,f}, \nu_{\chi,f}\rangle$$

where $r = \text{Card}\{p \mid \gcd(N, D)\}$ and $\delta = w/2$. I hope this is in agreement with your tables.

Of course, I never would have considered the possibility of such an identity if I hadn't once seen an old paper of yours on $y^2 = x^3 - 1728$. I seem to remember some spurious powers of 2 and 3 in your formulae there; perhaps that's because you're on a curve isogenous to $X_0(36)$. Could you send me a copy of this manuscript if it still exists? Also, I'd appreciate a summary of Stephens' data on $X_0(N)$ for $N = 11, 17, 19$. (if the tables can be brought down to size).

3) You're right about chapter III — once you find the right modular unit you're in like Flynn. The functions $\prod_{dN} \eta(dz)^{m(d)}$ give the *rational* cuspidal group; the entire group of modular units can be quite a mess to determine. To preserve my sanity, I restricted to those cases where the Galois eigenfunctions of the cuspidal group were obviously cyclic.

4) My method works at $p = 2$ only when $\chi = 1$. But yours is nicer anyhow. Could you send me a write-up of your recent parity result — that looks neat.

Best,

Dick

Gross to Birch: December 1, 1982

Dear Bryan,

Working with Don Zagier, I think I've assembled a proof of the identity following conjecture 17.1 in my paper on Heegner points. Up to now we've been assuming that both N and D are prime, but I'd be surprised if the techniques didn't work in the general case. The method is more or less as I suggested in my letter of May 14 ; one uses Rankin's method to obtain explicit formulae for the derivatives of the L-series and stare at these long enough until one begins to see the local heights of Heegner points emerging. *Something* should actually be written down by the late Spring, and you'll get the first copy.

Two requests: would you mind if we referred to the identity and the resulting 17.1 in the next write-up as the conjecture of Birch (or of Birch/Stephens). I know you only make conjectures with *lots* of evidence, and only really *believed* it when $\chi^2 = 1$ and f came from an elliptic curve, but you were the one who discovered this amazing phenomenon, and without the security blanket of your evidence, I would never have dared a proof.

Second: could you send us some of your computations on $X_0(11)$, $X_0(17)$, and $X_0(19)$? The fun of the subject seems to me to be in the *examples*.

Best wishes,

Dick

Birch to Gross: December 27, 1982

Dear Dick,

Wonderful news. Does this mean that in particular you can show $L' = 0$ when it ought to (thus fulfilling Dorian Goldfeld's requirements?).

Will send O/P when Xmas recedes — at the moment all offices, not to mention the mail system, are inert.

Yours,

Bryan

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