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Inverse Problems for Time Harmonic Electrodynamics

PETRI OLA, LASSI PÄIVÄRINTA, AND ERKKI SOMERSALO

Abstract. We study the inverse boundary value and inverse scattering problems for time-harmonic Maxwell's equations. The goal is to recover electromagnetic material parameters (permittivity, conductivity and permeability) in an unaccessible region of space from field measurements outside this region. We review the known results concerning the isotropic material parameters. Maxwell's equations are formulated here using differential forms. This representation is found particularly useful when anisotropies are allowed.

Introduction

In his famous article *A Dynamical Theory of Electromagnetic Field* of 1864 James Clerk Maxwell wrote down differential equations that describe the laws of electromagnetism in full generality. The four equations of Maxwell,

$$\nabla \cdot \boldsymbol{D}(x,t) = \rho(x,t), \tag{0–1}$$

$$\nabla \cdot \boldsymbol{B}(x,t) = 0, \tag{0–2}$$

$$\frac{\partial \boldsymbol{B}(x,t)}{\partial t} + \nabla \times \boldsymbol{E}(x,t) = 0, \tag{0–3}$$

$$-\frac{\partial \boldsymbol{D}(x,t)}{\partial t} + \nabla \times \boldsymbol{H}(x,t) = \boldsymbol{J}(x,t), \tag{0–4}$$

describe the dynamics of the five vector fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{J}$. Here $\boldsymbol{E}(x,t)$ is the electric field, $\boldsymbol{D}(x,t)$ the *electric displacement*, $\boldsymbol{B}$ the *magnetic induction* or *magnetic flux density*, $\boldsymbol{H}(x,t)$ is the *magnetic field* and, finally, $\boldsymbol{J}(x,t)$ is the *electric current density*. Since modern vector calculus was unknown to Maxwell, he formulated these equations as twenty scalar equations. The present form of these equations originates from Oliver Heaviside from the 1880's.

Equation (0–1) is Gauss' law and it says that infinitesimally the total flux of the electric displacement is equal to the density of free charges. The scalar field ρ here is the *free charge density*. Equation (0–2) is the magnetic analogue of Gauss' law saying that there are no free magnetic charges. Equation (0–3), called Faraday's law, explains how a changing magnetic flux creates an electric current in a conductive loop, a law that is based on a series of experiments that Faraday performed during 1831 and 1832. At that time the phenomenon dual to Faraday's law of induction was known as Ampère's law. It explains how an electric current in a loop creates a magnetic field and in our notation reads

$$\nabla \times \boldsymbol{H}(x,t) = \boldsymbol{J}(x,t). \tag{0–5}$$

The asymmetry in the equations (0–3) and (0–5) worried Maxwell and he started to think about Faraday's idea of polarization. Under the influence of an electric field a medium starts to polarize. This results in a small change in the position of charges and hence an electric current. He added a new current term $\frac{\partial}{\partial t}\boldsymbol{D}(x,t)$ to Ampère's law and as a result of purely theoretical reasoning discovered, among other things, electromagnetic waves. The existence of these waves was later verified by the experiments of Herz.

We call equations (0–1) to (0–4) macroscopic, because they deal directly with observable physical quantities and explain how they are related to each other. In particular, the structure of the medium is of no consequence. Also, without any additional assumptions, these equations are not enough to determine the fields uniquely, as a moment's reflection reveals. In addition to his four differential equations, Maxwell described four so-called *structural* or *constitutive equations* that relate $\boldsymbol{E}$ with $\boldsymbol{D}$, $\boldsymbol{B}$ with $\boldsymbol{H}$ and $\boldsymbol{J}$ with $\boldsymbol{E}$:

$$\begin{aligned}
\boldsymbol{D}(x,t) &= \varepsilon(x)\boldsymbol{E}(x,t),\\
\boldsymbol{B}(x,t) &= \mu(x)\boldsymbol{H}(x,t),\\
\boldsymbol{J}(x,t) &= \boldsymbol{J}_0(x,t) + \sigma(x)\boldsymbol{E}(x,t).
\end{aligned}$$

Here $\varepsilon(x)$ is the *electric permittivity* or *dielectricity*, $\mu(x)$ the *magnetic permeability* and $\sigma(x)$ the *electric conductivity*. The current density is divided in two parts, $\boldsymbol{J}_0$ being the forced current density, while the second term is the ohmic (or volume) current density driven by the electric field. Roughly speaking, ε expresses the tendency of the material to form electric dipoles under the influence of an external electric field, while the conductivity is related to the mobility of free charges in the material. The permeability μ is analogous to ε, expressing the magnitude in which the material is forming magnetic dipoles in an external magnetic field.

The goal in electromagnetic inverse problems is to determine these parameters in an inaccessible region in a noninvasive way from field measurements outside this region. The application areas include geophysical prospecting, nondestructive testing and medical imaging. As an example, we mention here the problem of detecting leukemia by using electromagnetic waves. This is made possible by the fact that leukemia causes a change of electric permittivity in the bone marrow by a factor of up to two. For more details, we refer to [3] and Chapter 2 in [1].

In this article we review the uniqueness results and reconstruction algorithms for time-harmonic fixed frequency inverse problems. This means that the time dependence of all fields is assumed to be $e^{-i\omega t}$, the frequency $\omega > 0$ being fixed. Instead of describing the electromagnetic fields as vector fields in a Euclidean space, we have chosen to define them as differential forms on a Riemannian manifold. This not only gives additional generality but also clarifies the nature of different physical fields. As an example, the electric displacement and magnetic induction have a physically well defined flux through a surface, hence they are integrable over two dimensional surfaces and consequently they correspond naturally to 2-forms. The formulation using forms makes obvious the invariance properties of Maxwell's equations. At the same time some formulas, like the radiation condition, are considerably simplified.

The structure of this article is as follows. In the first two sections we describe the problems to be considered starting from Maxwell's equations, and set up the mathematical framework that is going to be used. We also offer references to aspects of the problem that are not covered in detail in these notes. In Section 3 we rescale Maxwell's equations and then complete them into a Dirac type elliptic system. For a similar time domain formulation for Maxwell's equations in a more general setting, see [9]. In Section 4 we introduce the exponentially growing fundamental solution, and use this to find a large enough family of solutions that we can test the media with. In Section 5 we introduce an integration by parts formula that connects the parameters in the interior to our boundary measurement. To keep the reconstruction algrorithm constructive (at least mathematically), the next step is to show that our boundary data makes it possible to determine the Cauchy data of these special solutions, and this is done in Section 6. In the final section we explain how the unique determination of the parameters is proved.

1. Time-Harmonic Maxwell Equations

We now assume that the time dependence of all fields in (0–1) to (0–4) is harmonic with frequency $\omega > 0$, i.e., all time dependent fields above are of the form $f(x,t) = e^{-i\omega t} f(x)$. Cancelling out the oscillatory exponential we end up with the system

$$i\omega\, \boldsymbol{D}(x) + \nabla \times \boldsymbol{H}(x) = \boldsymbol{J}(x), \tag{1–1}$$

$$\nabla \cdot \boldsymbol{D}(x) = \rho(x), \tag{1–2}$$

$$-\, i\omega\, \boldsymbol{B}(x) + \nabla \times \boldsymbol{E}(x) = 0, \tag{1–3}$$

$$\nabla \cdot \boldsymbol{B}(x) = 0. \tag{1–4}$$

We intepret the electric field and magnetic field as 1-forms by identifying a vector field $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\mathbf{k}$ with the 1-form (we are using the Einstein summation convention whenever convenient) $F = F_i dx^i$. To intepret the equations above in terms of forms we also have to identify vector fields with 2-forms as follows: The vector field $\boldsymbol{F} = F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}$ is identified with the 2-form $F_1\,(dx^2 \wedge dx^3) + F_2\,(dx^3 \wedge dx^1) + F_3(dx^1 \wedge dx^2)$. In terms of the Euclidean Hodge-star operator $*_e$, this 2-form can be expressed as as $*_e(F_i dx^i)$. Equations (1–1) to (1–4) now take the form

$$i\omega D(x) + d\,H(x) = J(x), \tag{1–5}$$

$$d\,D(x) = \tilde{\rho}(x), \tag{1–6}$$

$$-\,i\omega\,B(x) + d\,E(x) = 0, \tag{1–7}$$

$$d\,B(x) = 0. \tag{1–8}$$

where $D = *_e(D_i dx^i)$, $B = *_e(B_i dx^i)$, and $\tilde{\rho} = \rho\, dV_e$, with $dV_e = dx^1 \wedge dx^2 \wedge dx^3$ is the Euclidean volume element.

For the moment we consider the system above on an arbitrary smooth differentiable and orientable three-manifold M, with or without a boundary. The microscopic structure of the medium in the domain is modelled by introducing a Riemannian metric g on M, and postulating that the magnetic induction and the electric displacement (which are 2-forms) are related to the magnetic and electric fields via

$$D = \gamma(x) * E, \qquad B = \mu(x) * H. \tag{1–9}$$

Here $*$ is the Hodge-star operator with recpect to the metric g, and μ and γ are smooth scalar functions,

$$\gamma(x) = \varepsilon(x) + i\,\frac{\sigma(x)}{\omega}.$$

Furthermore, we assume that ε and μ are equal to constants ε_0 and μ_0, respectively, outside a compact set, both are bounded and strictly positive, and σ is a nonnegative compactly supported function. In this formulation, the ohmic part of the current density J is merged with the electric displacement D, and J in (1–5) represents the forced current density J_0. Note that Maxwell's equations (1–5) to (1–8) are purely topological, i.e., there is no reference to the underlying metric. The metric properties appear, as expected. in the constitutive equations (1–9). As Kepler wrote in his 1602 thesis, albeit in a different context, *Ubi materia, ibi geometria*: Where there is matter, there is geometry.

In the sequel, we shall assume throughout that on the manifold M,

$$\tilde{\rho} = 0 \quad \text{and} \quad J_0 = 0.$$

We arrive at Maxwell's equations for the so called *perfect media*,

$$d\,H(x) + i\omega\gamma(x) * E(x) = 0, \tag{1–10}$$

$$d\,\gamma * E(x) = 0, \tag{1–11}$$

$$d\,E(x) - i\omega\mu(x) * H(x) = 0, \tag{1–12}$$

$$d\,\mu * H(x) = 0. \tag{1–13}$$

We remark that in reality not all media obey the constitutive relations (1–9) used here. First of all, in some applications the functions $\mu(x)$ and $\gamma(x)$ also depend on the frequency via so called *dispersion relations.* In the time domain, the frequency dependency corresponds to the *memory* of the matter, i.e., the responses of the material, such as the polarization, are not instantaneous but depend on the past values of the fields. Mathematically, this means that in the time domain, the constitutive relations become causal time convolutions. Secondly, not all media are *isotropic.* The medium is isotropic if one can choose $\gamma(x)$ and $\mu(x)$ to be scalar functions. For example muscle tissue is anisotropic and these functions have to be allowed to be more general tensors. We refer to [9] for a discussion of Maxwell's equations for forms in anisotropic media. Finally, the constitutive relations might be more complicated. For example, both D and B can depend on a linear combination of E and H, which leads to *chiral media.* The dependence can also be nonlinear. Such materials are in abundance in nature. For example, several crystals are chiral and metals in strong magnetic fields behave in a nonlinear fashion. In this work we limit ourselves to perfect media.

2. Inverse Problems

In this section we formulate the inverse boundary value problem as well as the inverse scattering problems for Maxwell's equations.

We start by fixing certain notations. Assume first that M is a smooth compact oriented 3-manifold with $\partial M \neq \varnothing$. We denote by $\Omega^k M$, $0 \le k \le 3$ the vector bundle of smooth k-forms on M. Let $i : \partial M \to M$ denote the canonical imbedding. We define the *tangential trace* of k-forms as

$$\boldsymbol{t} : \Omega^k M \to \Omega^k \partial M,\ \boldsymbol{t}\omega = i^*\omega \quad \text{for } \omega \in \Omega^k M,\ 0 \le k \le 2.$$

where i^* is the pull-back of i. Similarly, we define the *normal trace* as

$$\boldsymbol{n} : \Omega^k M \to \Omega^{3-k} \partial M,\ \boldsymbol{n}\omega = i^*(*\omega) \quad \text{for } \omega \in \Omega^k M,\ \ 1 \le k \le 3.$$

Observe that for 1-forms, the tangential component corresponds to the tangential component of the vector field while for 2-forms, it corresponds to the transversal flux through the boundary. For the normal trace, the roles are interchanged. For more precise discussion, see e.g. [20]. (In fact, the definition of the normal trace here differs from that given in the cited reference.)

Stokes' formula can be written now as follows: Let $\delta : \Omega^k M \to \Omega^{k-1} M$ denote the codifferential for k-forms,

$$\delta = (-1)^{n(k+1)+1} * d* = (-1)^k * d * \quad \text{for dimension } n = 3.$$

We denote the inner product of k-forms over M as

$$(\omega,\eta)=\int_M \omega\wedge *\bar\eta,$$

while at the boundary we denote

$$\langle\omega,\eta\rangle=\int_{\partial M}\omega\wedge\bar\eta \quad \text{for } \omega\in\Omega^k\partial M,\ \eta\in\Omega^{2-k}\partial M.$$

We have the identity

$$(d\omega,\eta)-(\omega,\delta\eta)=\langle \boldsymbol{t}\omega,\boldsymbol{n}\eta\rangle \quad \text{for } \omega\in\Omega^k M,\ \eta\in\Omega^{k+1}M. \tag{2–1}$$

With these notations, we define the *admittance map* for Maxwell's equations at the boundary: Assume for simplicity that $\gamma-\varepsilon_0,\mu-\mu_0\in C_0^\infty(\mathrm{int}(M))$, i.e., the material parameters near the boundary ∂M are constants $\varepsilon_0>0$ and $\mu_0>0$, respectively. We define

$$\Lambda:\boldsymbol{t}(\varepsilon_0^{1/2}E)\mapsto \boldsymbol{t}(\mu_0^{1/2}H).$$

The inverse boundary value problem (IBP) we consider here can be stated as follows:

IBP. *From the knowledge of the admittance map* Λ *at the boundary, determine the material parameters* γ *and* μ *in* M.

One can also consider this problem for chiral media; see [11]. For anisotropic media, if γ and μ are conformally related to each other, then the linarization suggests that the nonuniqueness arises solely from boundary preserving diffeomorphisms of M to itself, see [22]. This result is proved in the time domain in [9].

Equally natural is the inverse scattering problem. For simplicity, we assume here that $M=\mathbb{R}^3$ endowed with the Euclidean metric $g=g_e$, and furthermore, $\varepsilon(x)=\varepsilon_0$ and $\mu(x)=\mu_0$ for $x\notin D$. Consider the following plane-wave solution of Maxwell's equations in vacuum,

$$E_{\mathrm{i}}(x)=e^{i\langle x,k\rangle}p,\qquad H_{\mathrm{i}}(x)=e^{i\langle x,k\rangle}q,$$

where k satisfies $|k|^2=\varepsilon_0\mu_0\omega^2$. To satisfy equations (1–10) and (1–12), we require that the polarization 1-forms p and q satisfy

$$k\wedge p=\omega\mu_0*q,\qquad k\wedge q=-\omega\varepsilon_0*p,$$

where we have identified the vector k with a 1-form through $k(v)=\langle k,v\rangle$. It follows then that

$$\mu_0\|q\|^2=\mu_0 q\wedge *q=\varepsilon_0\|p\|^2,\qquad p\wedge *q=0,$$

and furthermore, the equations (1–11) and (1–13) require that

$$k\wedge *p=0,\qquad k\wedge *q=0.$$

The total field is written as a sum of the incoming field above plus the scattered field,

$$E = E_{\rm i} + E_{\rm sc}, \qquad H = H_{\rm i} + H_{\rm sc}.$$

The scattered field needs to satisfy a radiation condition at infinity. To understand better the the radiation condition for differential forms, let us go back for a while to the physical time domain picture. In the discussion below, we write concisely $E_{\rm sc} = E_{\rm sc}(x,t)$ and $H_{\rm sc} = H_{\rm sc}(x,t)$ for the physical (real valued) scattered time domain fields. Consider a ball B_R of radius $R > 0$ containing the inhomogeneity D. The total energy of the scattered field in $B_R \setminus \overline{D}$ expressed as

$$\mathcal{E} = \tfrac{1}{2}\varepsilon_0 \|E_{\rm sc}\|_R^2 + \tfrac{1}{2}\mu_0 \|H_{\rm sc}\|_R^2 = \mathcal{E}_E + \mathcal{E}_H,$$

where we write

$$\|E_{\rm sc}\|_R^2 = \int_{B_R\setminus\overline{D}} E_{\rm sc} \wedge *E_{\rm sc} = (E_{\rm sc}, E_{\rm sc})_R,$$

and similarly for $H_{\rm sc}$. Consider the electric part of the energy. The time derivative of it gives us

$$\frac{\mathcal{E}_E}{\partial t} = \varepsilon_0 \Big(\frac{\partial E_{\rm sc}}{\partial t}, E_{\rm sc} \Big)_R = \frac{1}{\mu_0} (\delta B_{\rm sc}, E_{\rm sc})_R,$$

and further, by applying Stokes' law,

$$\frac{\mathcal{E}_E}{\partial t} = \frac{1}{\mu_0} (B_{\rm sc}, dE_{\rm sc})_R + \frac{1}{\mu_0} \big(\langle \boldsymbol{n} B_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial B_R} + \langle \boldsymbol{n} B_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial D} \big). \tag{2–2}$$

Again, from Maxwell's equations, we obtain

$$\frac{1}{\mu_0} (B_{\rm sc}, dE_{\rm sc})_R = -\frac{1}{\mu_0} \Big(B_{\rm sc}, \frac{\partial B_{\rm sc}}{\partial t} \Big) = -\frac{\partial \mathcal{E}_H}{\partial t}.$$

By substituting this identity into equation (2–2), we find that the total change of energy equals the flux through the boundaries ∂B_R and ∂D, i.e.,

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial t} = \frac{\partial \mathcal{E}_E}{\partial t} + \frac{\partial \mathcal{E}_H}{\partial t} &= \frac{1}{\mu_0} \big(\langle \boldsymbol{n} B_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial B_R} + \langle \boldsymbol{n} B_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial D} \big) \\ &= \langle \boldsymbol{t} H_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial B_R} + \langle \boldsymbol{t} H_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial D}. \end{aligned}$$

The radiation conditions are now defined in such a way that for large R, the energy flux through ∂B_R either becomes negative (outgoing waves) or positive (incoming wave). For the outgoing wave, we write

$$\langle \boldsymbol{t} H_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial B_R} = -\Big(\frac{\varepsilon_0}{\mu_0} \Big)^{1/2} \langle \boldsymbol{t} E_{\rm sc}, \boldsymbol{t} E_{\rm sc} \rangle_{\partial B_R} + \Big\langle \boldsymbol{t} H_{\rm sc} + \Big(\frac{\varepsilon_0}{\mu_0} \Big)^{1/2} \boldsymbol{t} E_{\rm sc}, \boldsymbol{t} E_{\rm sc} \Big\rangle_{\partial B_R}.$$

Hence, to assure that the last term has an asymptotically vanishing effect, we set the radiation condition

$$\boldsymbol{t}(\varepsilon_0^{1/2} E_{\rm sc} + \mu_0^{1/2} H_{\rm sc}) = o\Big(\frac{1}{|x|} \Big). \tag{2–3}$$

This is the outgoing *radiation condition* for differential forms that we impose for the scattered field in the frequency domain. Compared with the *Silver–Müller radiation condition* in the vector formalism, this appears strikingly simple.

By using the representations of the scattered fields in terms of Green's functions, it is possible to derive an asymptotic representation of the fields,

$$\boldsymbol{t}E_{\mathrm{sc}}(x) = E_\infty(\hat{x};k;p)\frac{e^{i|k||x|}}{|x|} + o(|x|^{-1}),$$
$$\boldsymbol{t}H_{\mathrm{sc}}(x) = H_\infty(\hat{x};k;p)\frac{e^{i|k||x|}}{|x|} + o(|x|^{-1}),$$

where $\hat{x} = x/|x|$, and the mutually orthogonal 1-forms (defined on the unit sphere) E_∞ and H_∞ are the electric and magnetic far-field patterns, respectively, corresponding to the polarization p and incidence direction k with $|k|^2 = \varepsilon_0\mu_0\omega^2$. Note that one only needs to specify one of these, the other one can then be immediately obtained from the radiation conditions. The inverse scattering problem (ISP) can now be formulated as follows.

ISP. *From the knowledge of $E_\infty(\hat{x};k;p)$ for all $\hat{x} \in S^2$, $k \in \mathbb{R}^3$ with $|k|$ fixed and for three linearly independent polarizations p determine the material parameters γ and μ.*

If one knows the admittance map on a smooth surface Γ enclosing the inhomogeneity, then the boundary value on Γ of the rescaled scattered field $e_{\mathrm{sc}} = \gamma^{1/2}E_{\mathrm{sc}}$ corresponding to the incoming plane wave with electric component $e_{\mathrm{i}} = \varepsilon_0^{1/2}pe^{i\langle x,k\rangle}$ can be solved for from the boundary integral equation

$$\tfrac{1}{2}\boldsymbol{t}e_{\mathrm{sc}} = \boldsymbol{t}e_{\mathrm{i}} + D_k\Lambda\boldsymbol{t}e_{\mathrm{sc}} - K_k\boldsymbol{t}e_{\mathrm{sc}}$$

on Γ. (The argument is similar the one used to derive Equation (6–2) below.) Here, the operators D_k and K_k are defined analogously to D and K introduced in Section 6 but using the standard outgoing fundamental solution $-e^{ik|x|}/4\pi|x|$ instead of Faddeev's Green's function. The mapping properties are unchanged in this replacement, and the unique solvability follows in a standard manner assuming that ω is not a resonance frequency. Hence we know the tangential boundary values of e and h on Γ, and thus also the far-fields are determined by the impedance map.

To be able to reduce the ISP to IBP, we need to go to the opposite direction, and this is not as simple. The difficulty (and also its resolution) is similar to the acoustic case, so we are rather brief in describing it. If one knows the far-fields of all waves scattered by plane waves, one also knows, using Rellich's argument, the scattered fields in the complement of the union of the supports of $\gamma - \varepsilon_0$ and $\mu - \mu_0$, and thus one also knows their boundary values on Γ, i.e., one knows the restriction of the impedance map to all total fields corresponding to incoming plane waves. The problem is to show that this data determines the impedance map. This was shown to be true for Maxwell's equations in [19] (Section 6.4.) and hence the scattering problem is reduced to IBP. The argument is a modification of the idea of Nachman [13] and Ramm [18]. In fact, the argument in [19] deals with the acoustic and electromagnetic cases simultaneously. Using this argument, however, one has to assume that the interface Γ is chosen so that ω is not a magnetic resonance frequency. Of course one is free to choose the interface so that this is avoided, but in practice it is not easy to determine when one is close to a resonant frequency.

One can also deal with the inverse scattering problem directly, without reducing it to the IBP: For Maxwell's equations this was done in [4] assuming that $\mu = \mu_0$, and stability results were obtained in [5]. The crucial part is again the construction of the exponentially increasing solutions, and in these proofs one does not need any assumptions on ω.

3. The Scaled System

In this section we follow the idea of [17] and rescale the electromagnetic forms in such a way that we only have to deal with one metric and complete this system to an elliptic system of Dirac type. In [17] this was only done for the case of Euclidean background metric, but the principle remains the same in a more general setting, see [9]. The basic idea is of course old and well-known: Even though the divergence conditions (1–6) and (1–8) are implied by the two other equations (1–5) and (1–7), they make it possible to reduce the system to an elliptic system that in a homogenous medium coincides with the Helmholtz equation for E and H, respectively. Also, the divergence conditions are crucial when analyzing the low-frequency limit, since they single out the right limit value (remember that Maxwell's equations have an infinite dimensional kernel when $\omega = 0$). The approach we follow is a modification of the argument originally due to R. Picard (see [15]). The idea is to get a first order elliptic and symmetric system (at least in the principal part) that under some conditions reduces to Maxwell's equations. The ellipticity is achieved by including the divergence conditions to the system, but to make it symmetric one needs to modify it further. We start with the system (1–10) to (1–13) and introduce 3-forms Φ and Ψ by

$$i\omega\Phi = d(\gamma * E), \qquad i\omega\Psi = d(\mu * H). \tag{3–1}$$

Of course, if E and H satisfy Maxwell's equations, these forms vanish. Now we modify (1–10) and (1–12) to

$$dH - \frac{1}{\mu} * d * \left(\frac{1}{\gamma}\Phi\right) + i\omega\gamma * E = 0 \tag{3–2}$$

$$dE + \frac{1}{\gamma} * d * \left(\frac{1}{\mu}\Psi\right) - i\omega\mu * H = 0. \tag{3–3}$$

The principal part of this system still depends on γ and μ, and in order to make it more simple we scale the unknown fields: Let

$$e = \gamma^{1/2}E, \quad h = \mu^{1/2}H, \quad \phi = \frac{1}{\gamma\mu^{1/2}}\Phi, \quad \psi = \frac{1}{\gamma^{1/2}\mu}\Psi. \tag{3–4}$$

This makes the physical dimensions of the unknown 1- and 3-forms equal, and it also makes the principal part of the system depend only on the background metric g as we shall see. To get a full graded algebra, let us further define the 0- and 2-forms φ and b as

$$\varphi = *\phi, \qquad b = *h.$$

We introduce the notation

$$\mathbf{\Omega}M = \Omega^0 M \times \Omega^1 M \times \Omega^2 M \times \Omega^3 M,$$

for the full Grassmannian bundle and endow $\mathbf{\Omega}M$ with the obvious inner product: If $u = (u^0, u^1, u^2, u^3)$, $v = (v^0, v^1, v^2, v^3) \in \mathbf{\Omega}M$, we set

$$(u, v) = \sum_{j=0}^{3} \int_M u^j \wedge *\overline{v^j}.$$

Define a graded form

$$X = (\varphi, e, b, \psi) \in \mathbf{\Omega}M.$$

A straightforward insertion into the augmented equations (3–1) to (3–3) along with the identities $\delta = (-1)^k * d *$ and $** = (-1)^{k(n-k)} = 1$ for forms of degree k in $\mathbb{R}^n$ with $n = 3$ give that X satisfies the system

$$(P - i\kappa)X + VX = 0, \tag{3–5}$$

where the principal part is

$$P = \begin{pmatrix} 0 & -\delta & 0 & 0 \\ d & 0 & -\delta & 0 \\ 0 & d & 0 & -\delta \\ 0 & 0 & d & 0 \end{pmatrix},$$

the scalar $\kappa = \kappa(x)$ is the nonconstant wave number,

$$\kappa = \omega(\gamma\mu)^{1/2},$$

and V is a local potential given by

$$V = \begin{pmatrix} 0 & *(d\alpha \wedge * \,\cdot\,) & 0 & 0 \\ d\beta \wedge \,\cdot & 0 & *(d\beta \wedge * \,\cdot\,) & 0 \\ 0 & -d\alpha \wedge \,\cdot & 0 & *(d\alpha \wedge * \,\cdot\,) \\ 0 & 0 & d\beta \wedge \,\cdot & 0 \end{pmatrix}.$$

Here

$$\alpha = \tfrac{1}{2} \ln \gamma, \qquad \beta = \tfrac{1}{2} \ln \mu.$$

The first order operator has several important properties. First, Stokes' formula (2–1) implies that

$$(Pu, v) + (u, Pv) = \langle \boldsymbol{t}u, \boldsymbol{n}v\rangle + \overline{\langle \boldsymbol{t}v, \boldsymbol{n}u\rangle}, \tag{3–6}$$

where we introduced the shorthand notation $\boldsymbol{t}v$, $\boldsymbol{n}v \in \boldsymbol{\Omega}\partial M = \Omega^0\partial M \times \Omega^1\partial M \times \Omega^2\partial M$,

$$\boldsymbol{t}v = (\boldsymbol{t}v^0, \boldsymbol{t}v^1, \boldsymbol{t}v^2), \qquad \boldsymbol{n}v = (\boldsymbol{n}v^3, \boldsymbol{n}v^2, \boldsymbol{n}v^1),$$

and

$$\langle \boldsymbol{t}u, \boldsymbol{n}v\rangle = \langle \boldsymbol{t}u^0, \boldsymbol{n}v^1\rangle + \langle \boldsymbol{t}u^1, \boldsymbol{n}v^2\rangle + \langle \boldsymbol{t}u^2, \boldsymbol{n}v^3\rangle.$$

Observe that this expression does not define an inner product on ∂M.

Second, we observe immediately that

$$P^2 = -\boldsymbol{\Delta} = -\mathrm{diag}(\Delta^0, \Delta^1, \Delta^2, \Delta^3),$$

where $\Delta^k = \delta d + d\delta$ is the Laplace–Beltrami operator for k-forms. But more is true. Namely, splitting the Grassmann algebra into its even and odd degree parts,

$$\boldsymbol{\Omega}M = \boldsymbol{\Omega}^+M \oplus \boldsymbol{\Omega}^-M,$$

where $\boldsymbol{\Omega}^+M = \Omega^0M \times \Omega^2M$ and $\boldsymbol{\Omega}^-M = \Omega^1M \times \Omega^3M$, we can write the potential as an off-diagonal block matrix

$$V = \begin{pmatrix} 0 & V_- \\ V_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & *(d\alpha \wedge * \,\cdot\,) & 0 \\ 0 & 0 & -d\alpha\wedge\,\cdot & *(d\alpha \wedge * \,\cdot\,) \\ d\beta\wedge\,\cdot & *(d\beta\wedge * \,\cdot\,) & 0 & 0 \\ 0 & d\beta\wedge\,\cdot & 0 & 0 \end{pmatrix},$$

with $V_\pm : \boldsymbol{\Omega}^\pm M \to \boldsymbol{\Omega}^\mp M$. In this representation, the operator P becomes just the Dirac type operator $\boldsymbol{D} = d - \delta : \boldsymbol{\Omega}^\pm M \to \boldsymbol{\Omega}^\mp M$. Hence we can write (3–5) equivalently as

$$(\boldsymbol{D} - i\kappa + V)X = 0.$$

Now let $\tilde{V}$ be the adjoint of V:

$$\tilde{V} = \begin{pmatrix} 0 & \tilde{V}_- \\ \tilde{V}_+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & *(d\beta \wedge * \,\cdot\,) & 0 \\ 0 & 0 & -d\beta\wedge\,\cdot & *(d\beta \wedge * \,\cdot\,) \\ d\alpha\wedge\,\cdot & *(d\alpha\wedge * \,\cdot\,) & 0 & 0 \\ 0 & d\alpha\wedge\,\cdot & 0 & 0 \end{pmatrix}.$$

LEMMA 3.1. *The first order terms of the product*

$$(\boldsymbol{D} - i\kappa(x) + V(x))(\boldsymbol{D} + i\kappa(x) - \tilde{V}(x)) \tag{3–7}$$

vanish, *i.e.*, *the product is of the form* $-\boldsymbol{\Delta} + k^2 + Q(x)$, *where* Q *is a zeroth order pointwise multiplier*.

Proof. The nontrivial part of the lemma is of course the vanishing of the first order derivatives of the commutator-like term $V\boldsymbol{D} - \boldsymbol{D}\tilde{V}$. Since both $\boldsymbol{D}$ and V change the parity of the degree, we may consider only the even degree part. The odd degree case is handled similarly. By a direct computation we get for $u^+ = (u^0, u^2) \in \boldsymbol{\Omega}^+ M$ that

$$V_-(d-\delta)u^+ = \begin{pmatrix} *(d\alpha \wedge *(du^0 - \delta u^2)) \\ -d\alpha \wedge (du^0 - \delta u^2) + *(d\alpha \wedge *d\omega^2) \end{pmatrix}, \tag{3–8}$$

and similarly,

$$(d-\delta)\tilde{V}_+ u^+ = \begin{pmatrix} -\delta(d\alpha \wedge u^0 + *(d\alpha \wedge *u^2)) \\ d(d\alpha \wedge u^0 + *(d\alpha \wedge *u^2)) - \delta(d\alpha \wedge u^2) \end{pmatrix}. \tag{3–9}$$

By using a local orthonormal coframe, a straightforward computation shows that the differences are of order zero in u^+. For later use, we demonstrate this explicitly for the 0-form component. The first term in the upper component of (3–9) is

$$-\delta(d\alpha \wedge u^0) = *d*(d\alpha \wedge u^0) = *(d\alpha \wedge *du^0) - u^0 \Delta^0 \alpha,$$

and the second term gives

$$-\delta * (d\alpha \wedge *u^2) = *d(d\alpha \wedge *u^2) = -*(d\alpha \wedge d * u^2) = -*(d\alpha \wedge *\delta u^2)$$

By comparing to the dirst component of (3–8), we observe that

$$(V_-(d-\delta)u^+ - (d-\delta)\tilde{V}_+ u^+)^0 = u^0 \Delta^0 \alpha. \tag{3–10}$$

The calculation of the second component is slightly more tedious but straightforward.

Observe also that the potential $V\tilde{V}$ is diagonal. Indeed, we have

$$V\tilde{V} = \operatorname{diag}\big(|d\alpha|^2, |d\alpha|^2, |d\beta|^2, |d\beta|^2\big),$$

where $|d\alpha|^2 = *(d\alpha \wedge *d\alpha)$. This result is used later. □

4. Green's Function

In this section we derive exponentially growing (or Faddeev's) Green's function for the complete Maxwell system (3–5) treated in the previous section. Although the discussion of the previous section can be carried out in more general metric, here we have to confine ourselves to the Euclidian case. Thus, we shall assume that $g = g_e$ is the Euclidian metric, and the Euclidian normal coordiantes are denoted by (x^1, x^2, x^3).

We start by recaling the definition of scalar Faddeev's Green's function: For any $\zeta \in \mathbb{C}^3$, set

$$G(x) = G_\zeta(x) = e^{i\langle x,\zeta\rangle} g_\zeta(x), \qquad g_\zeta(x) = \Big(\frac{1}{2\pi}\Big)^3 \int_{\mathbb{R}^3} \frac{e^{i\langle x,\xi\rangle}}{|\xi|^2 + 2\langle \xi,\zeta\rangle}\, d\xi,$$

where the inner products are the real inner products, i.e., no complex conjugation is included. When ζ is chosen in such a way that

$$\langle \zeta, \zeta \rangle = k^2, \tag{4–1}$$

the function G is indeed Green's function for the Helmholtz operator,

$$(\Delta - k^2)G(x) = \delta(x).$$

Note that our Δ is now the geometer's Laplacian $d\delta + \delta d$, which is a positive operator.

This scalar Green's function has the following important asymptotic property as $|\zeta| \to \infty$ along the variety $\{\zeta \in \mathbb{C}^3; \langle \zeta, \zeta \rangle = k^2\}$. Letting L^2_δ be the weighted L^2-space with norm

$$\|f\|^2_\delta = \int_{\mathbb{R}^3} |f|^2 (1 + |x|^2)^\delta \, dx \tag{4–2}$$

we have the following estimate due to Sylvester and Uhlmann ([23]):

Proposition 4.1. *For* $|\zeta|$ *large*, *we have*

$$\|g_\zeta * f\|_\delta \le \frac{C}{|\zeta|} \|f\|_{\delta+1},$$

where $-1 < \delta < 0$.

By using this scalar Green's function, we define an exponentially growing Green's tensor for $\boldsymbol{\Delta} - k^2$ by setting

$$\begin{aligned} \boldsymbol{G}(x-y) &= G(x{-}y)\left(1,\ \sum_{j=1}^{3} dx^j \otimes dy^j,\ \sum_{j=1}^{3} \theta_j \otimes \nu_j, dV_x \otimes dV_y\right) \\ &= G(x{-}y)\boldsymbol{I}, \end{aligned}$$

where $\theta_j = \frac{1}{2}\varepsilon_{jk\ell} dx^k \wedge dx^\ell$ and $\nu_j = \frac{1}{2}\varepsilon_{jk\ell} dy^k \wedge dy^\ell$ and $dV_x = dx^1 \wedge dx^2 \wedge dx^3$, $dV_y = dy^1 \wedge dy^2 \wedge dy^3$. Oberve that

$$\boldsymbol{I} \wedge *(\lambda_j dy^j) = \lambda_j dx^j.$$

For later reference, note that $\boldsymbol{I}$ can be written componentwise as

$$\boldsymbol{I} = \sum_{j=1}^{8} \omega_x^j \otimes \omega_y^j, \tag{4–3}$$

where $\omega_x^1 = \omega_y^1 = (1,0,0,0)$, $\omega_x^2 = (0, dx^1, 0, 0)$, $\omega_y^2 = (0, dy^1, 0, 0)$ and so on.

With the help of this Green's tensor, we define now a graded form that could be called a *generalized Sommerfeld potential*. Let $Y_0 \in \boldsymbol{\Omega} M$ be any graded form satisfying

$$(-\boldsymbol{\Delta} + k^2)Y_0 = 0 \tag{4–4}$$

in $\mathbb{R}^3$. We seek to solve for the potential $Y \in \boldsymbol{\Omega} M$ from the Lippmann–Schwinger type equation

$$Y(x) = Y_0(x) - \int_M \boldsymbol{G}(x-y) \wedge *(Q(y)Y(y)). \tag{4–5}$$

The existence of such a solution for large $|\zeta|$ is guaranteed by Proposition 4.1. Also, we observe that Y satisfies the Schrödinger equation

$$(-\boldsymbol{\Delta} + k^2 + Q(x))Y(x) = 0. \tag{4–6}$$

From Proposition 4.1, we obtain also the important information of the asymptotic behaviour of Y for large $|\zeta|$.

THEOREM 4.2. *For $|\zeta|$ large enough, $-1 < \delta < 0$, and for any constant coefficient form y_0 which is bounded in ζ, the equation* (4–5) *has a unique solution $Y_\zeta = e^{i\langle x,\zeta\rangle}(y_0 + w_\zeta)$, where $\|w_\zeta\|_\delta < C/|\zeta|$.*

For later use, we fix already here the form Y_0 and require that it is of the form

$$Y_0(x) = e^{i\langle x,\zeta\rangle} y_0,$$

where we assume that $\zeta \in \mathbb{C}^3$ satisfies the condition (4–1), guaranteeing equation (4–4) to be valid. Furthermore, the constant graded form $y_0 = (y^0, y^1, y^2, y^3)$ is required to satisfy

$$ky^0 = -*(\zeta \wedge *y^1), \qquad ky^3 = -\zeta \wedge y^2, \tag{4–7}$$

where we identified ζ with a complex 1-form by $\zeta(u) = \langle \zeta, u\rangle$. The conditions above imply that

$$((P+ik)Y_0)^0 = 0, \qquad ((P+ik)Y_0)^3 = 0, \tag{4–8}$$

i.e., the 0-form and 3-form components of $(P+ik)Y_0(x)$ vanish.

Now we use the decomposition property of Lemma 3.1. By setting

$$X_\zeta(x) = (P + i\kappa(x) - \tilde{V}(x))Y_\zeta(x),$$

we find that X_ζ satisfies the complete Maxwell's system

$$(P - i\kappa(x) + V(x))X_\zeta(x) = 0. \tag{4–9}$$

We call this solution the *exponentially growing solution* of the complete Maxwell's system.

From the foregoing definition, it is not obvious that $X_\zeta = (X_\zeta^0, X_\zeta^1, X_\zeta^2, X_\zeta^3)$ is indeed a solution to the original Maxwell's system, i.e., that $X_\zeta^0 = 0$ and $X_\zeta^3 = 0$ as they should in order that the pair (X_ζ^1, X_ζ^2) would represent scaled electric and magnetic fields. However, one can prove the following result.

LEMMA 4.3. *Assume that Y_0 is chosen so that the conditions* (4–8) *are satisfied. Then, for large $|\zeta|$, we have $X_\zeta^0 = 0$ and $X_\zeta^3 = 0$.*

PROOF. In view of what was said, it only remains to check that for $|\zeta|$ large enough the 0- and 3-form components of X_ζ vanish. We show this for the first component, the last component being handled similarly.

Since X_ζ satisfies the complete Maxwell's system (4–9), we have

$$(P + i\kappa - \tilde{V})(P - i\kappa + V)X_\zeta = 0.$$

A calculation similar to the one in the proof of Lemma 3.1 shows now that X_ζ^0 satisfies

$$(-\Delta + k^2)X_\zeta^0 + qX_\zeta^0 = 0 \tag{4–10}$$

where the potential $q(x)$ is given by

$$q = \Delta^0\beta - |d\beta|^2 + (\kappa^2 - k^2). \tag{4–11}$$

On the other hand, by using the particular form of the solution Y_ζ defined in Theorem 4.2, we can decompose X_ζ by a straightworward substitution of Y_ζ into the definition X_ζ as

$$X_\zeta = (P + ik)Y_0 + e^{i\langle x,\zeta\rangle}w_\zeta,$$

where $w_\zeta \in L^2_{-\delta}$. Furthermore, from the equation (4–10), it follows further that w_ζ^0 must satisfy the integral equation

$$w_\zeta^0 = w_0^0 - g_\zeta * (qw_\zeta^0),$$

with

$$w_0^0 = e^{-i\langle x,\zeta\rangle}((P + ik)Y_0)^0 = 0$$

by the assumption of Y_0. It follows from Proposition 4.1 that for large $|\zeta|$, $w_\zeta^0 = 0$.

For details, see [17]. □

5. From Inside to Boundary

In this section, we derive a formula that relates the material parameters inside M to the boundary values of the exponentially growing solution. Here, M is bounded and Euclidean with a smooth boundary, and $\gamma = \varepsilon_0$, $\mu = \mu_0$ near the boundary. The formula is related to the energy integral appearing in electrical impedance tomography, but due to the complexity of the complete Maxwell's system it is more involved.

To begin with, let $Y_0^* \in \mathbf{\Omega}M$ be any solution of the homogenous space problem,

$$(P - ik)Y_0^* = 0. \tag{5–1}$$

By using equation (4–6) and the decomposition of $\mathbf{\Delta}$, we have

$$\begin{aligned}(QY_\zeta, Y_0^*) &= -((-\mathbf{\Delta} + k^2)Y_\zeta, Y_0^*) = -((P - ik)(P + ik)Y_\zeta, Y_0^*)\\ &= -((P - ik)\tilde{X}_\zeta, Y_0^*),\end{aligned} \tag{5–2}$$

where we denoted $\tilde{X}_\zeta = (P + ik)Y_\zeta$. Observe that when $\gamma = \varepsilon_0$ and $\mu = \mu_0$, we have $\tilde{X}_\zeta = X_\zeta$ by the definition of X_ζ. By using Stokes' formula (3–6) for P and equation (5–1), we find that

$$(QY_\zeta, Y_0^*) = -\langle \boldsymbol{t}X_\zeta, \boldsymbol{n}Y_0^*\rangle - \overline{\langle \boldsymbol{t}Y_0^*, \boldsymbol{n}X_\zeta\rangle}.$$

Here, we used the fact that at ∂M, $X_\zeta = \tilde{X}_\zeta$. Hence, if we know the boundary data $\{\boldsymbol{t}X_\zeta, \boldsymbol{n}X_\zeta\}$, we obtain an integral involving the potential Q over M.

To understand the significance of this relation better, we look at the linearization of the left hand side of (5–2) with a particular choice of the form Y_0^*. The linearization means the approximation

$$(QY_\zeta, Y_0^*) \approx (QY_0, Y_0^*). \tag{5–3}$$

In view of Theorem 4.2, this approximation is asymptotically valid as $|\zeta| \to \infty$. Following the original ideas of Calderón, we choose $Y_0(x)$ as in the previous section. Similarly, we set

$$Y_0^*(x) = e^{i\langle x, \zeta^*\rangle} y_0^*.$$

where

$$\zeta - \overline{\zeta^*} = \xi,$$

$\xi \in \mathbb{R}^3$ being a fixed vector. We require further that

$$\langle \zeta, \zeta\rangle = \langle \zeta^*, \zeta^*\rangle = k^2.$$

As we shall see in Section 7, in $\mathbb{C}^3$ there is enough space to make such a choice. The constant graded form y_0^* must be chosen again in such a way that equation (5–1) holds. It is easy to see that such a choice is obtained if we set, e.g.,

$$y_0^* = \frac{1}{|\zeta|}(P(i\zeta^*) + ik)z,$$

where $z = (z^0, z^1, z^2, z^3)$ is an arbitrary constant coefficient graded form and $P(i\zeta^*)$ is the symbol of the operator P, i.e.,

$$P(i\zeta^*) = e^{-i\langle x, \zeta^*\rangle} P e^{i\langle x, \zeta^*\rangle}.$$

With these choices, we obtain

$$(QY_0, Y_0^*) = \int_M e^{i\langle x, \xi\rangle}(Q(x)y_0, y_0^*). \tag{5–4}$$

Hence, we see that within the linearization, the boundary values of X determine the Fourier transform of $(Q(x)y_0, y_0^*)$ and thus the function itself. In Section 7, we show how the material parameters $\mu(x)$ and $\gamma(x)$ can be recovered from this data.

6. From Λ to Boundary Values of X

In the previous section, we showed how the boundary values of X determine the integral (5–4). In this section, we show that the knowledge of the admittance map determines the boundary values of X.

The idea is to derive a version of the Stratton–Chu representation formula for the field X. To this end, we start with the Lippmann–Schwinger type equation for Y, and by writing

$$\tilde{X}(y) = (P + ik)Y(y),$$

we have

$$\begin{aligned} Y(x) &= Y_0(x) - \int_M \boldsymbol{G}(x-y) \wedge *Q(y)Y(y) \\ &= Y_0(x) + \int_M \boldsymbol{G}(x-y) \wedge *(-\boldsymbol{\Delta} + k^2)Y(y) \\ &= Y_0(x) + \int_M \boldsymbol{G}(x-y) \wedge *(P - ik)\tilde{X}(y). \end{aligned}$$

By writing $\boldsymbol{G}(x-y)$ in terms of the components of $\boldsymbol{I}$ as in (4–3), we obtain through integration by parts the equation

$$\begin{aligned} Y(x) &= Y_0(x) + \sum \omega_x^j \int_M G(x{-}y)\omega_y^j \wedge *(P - ik)\tilde{X}(y) \\ &= Y_0(x) + \sum \omega_x^j \int_M (-P_y - ik)G(x{-}y)\omega_y^j \wedge *\tilde{X}(y) \\ &\qquad + \sum \omega_x^j \biggl(\int_{\partial M} \boldsymbol{t}G(x{-}y)\omega_y^j \wedge \boldsymbol{n}X(y) + \int_{\partial M} \boldsymbol{t}X(y) \wedge \boldsymbol{n}G(x{-}y)\omega_y^j \biggr). \end{aligned}$$

Here we used the fact that at the boundary, $\tilde{X}(y) = X(y)$. We substitute the integral representation of $Y(x)$ in this formula and use the fact that for $x \neq y$, we have

$$(P_x + ik)(-P_y - ik)G(x{-}y)\boldsymbol{I} = 0,$$

and we arrive at the identity

$$\begin{aligned} X(x) = X_0(x) + (P + ik) \sum \omega_x^j \biggl(&\int_{\partial M} \boldsymbol{t}G(x{-}y)\omega_y^j \wedge \boldsymbol{n}X(y) \\ &+ \int_{\partial M} \boldsymbol{t}X(y) \wedge \boldsymbol{n}G(x{-}y)\omega_y^j \biggr), \end{aligned}$$

where

$$X_0(x) = (P + ik)Y_0(x).$$

Now assume that $|\zeta|$ is large. Then, by Lemma 4.3, we have $X = (0, e, h, 0)$, and the boundary integral above takes the form

$$X(x) = X_0(x)+(P+ik)\bigg(\int_{\partial M} G(x-y)\boldsymbol{n}e,\ \sum_{j=1}^{3} dx^j \int_{\partial M} G(x-y)\boldsymbol{t}dy^j\wedge\boldsymbol{n}b,$$
$$\sum_{j=1}^{3}\theta_j \int_{\partial M} G(x-y)\boldsymbol{t}e\wedge\boldsymbol{n}\nu_j,\ dV\int_{\partial M} G(x-y)\boldsymbol{t}h\bigg). \quad (6\text{–}1)$$

Letting the point x approach the boundary ∂M from the exterior domain $\mathbb{R}^3\setminus M$ we obtain an integral equation for the boundary values of X. However, assuming that the impedance map Λ is known, it suffices to solve the tangential component of the electric field. Indeed, we have

$$\boldsymbol{n}b = \Lambda\boldsymbol{t}e,$$

and assuming that $|\zeta|$ is large, from Maxwell's equations

$$-\delta b + ike = 0, \qquad de + ikb = 0,$$

we find that

$$\boldsymbol{n}e = \frac{1}{ik}\boldsymbol{n}\delta b = \frac{1}{ik}\boldsymbol{t}d * b.$$

Since the exterior derivative and the tangential trace commute, we have further

$$\boldsymbol{n}e = \frac{1}{ik}d_\partial\boldsymbol{n}b = \frac{1}{ik}d_\partial\Lambda\boldsymbol{t}e.$$

Here, d_∂ denotes the exterior derivative on ∂M. Similarly, we have

$$\boldsymbol{t}b = -\frac{1}{ik}\boldsymbol{t}de = -\frac{1}{ik}d_\partial\boldsymbol{t}e.$$

Summarizing,

$$\boldsymbol{t}X = (0,\ \boldsymbol{t}e,\ \boldsymbol{t}b) = \Big(0, \boldsymbol{t}e, -\frac{1}{ik}d_\partial\boldsymbol{t}e\Big),$$
$$\boldsymbol{n}X = (0, \boldsymbol{n}b, \boldsymbol{n}e) = \Big(0, \Lambda\boldsymbol{t}e, \frac{1}{ik}d_\partial\Lambda\boldsymbol{t}e\Big).$$

Thus, we shall consider only the 1-form component of the system (6–1) and solve it for $\boldsymbol{t}e$. Denoting by e_0 the 1-form component of X_0 we have, for $x\in\mathbb{R}^3\setminus M$,

$$e = e_0 + d\int_{\partial M} G\boldsymbol{n}e - \delta\sum_{j=1}^{3}\theta_j\int_{\partial M} G\boldsymbol{t}e\wedge\boldsymbol{n}\nu_j + ik\sum_{j=1}^{3}dx^j\int_{\partial M} G\boldsymbol{t}dy^j\wedge\boldsymbol{n}b$$
$$= e_0 + \frac{1}{ik}d\int_{\partial M} Gd\Lambda\boldsymbol{t}e - \delta\sum_{j=1}^{3}\theta_j\int_{\partial M} G\boldsymbol{t}e\wedge\boldsymbol{n}\nu_j + ik\sum_{j=1}^{3}dx^j\int_{\partial M} G\boldsymbol{t}dy^j\wedge\Lambda\boldsymbol{t}e.$$

Here the arguments of the functions are suppressed for brevity. Now we need to apply the tangential boundary trace from the exterior domain to both sides of this equation. To get a boundary integral equation, we need to take into account the jump relations of the layer potentials. Consider first the second integral on the right. By using the identites

$$\boldsymbol{n}\nu_j = \boldsymbol{t}dy^j, \qquad \delta\theta_j f(x) = *df(x)\wedge dx^j,$$

we obtain

$$\delta\sum_{j=1}^{3}\theta_j\int_{\partial M} G\boldsymbol{t}e\wedge\boldsymbol{n}\nu_j = \sum_{j=1}^{3}\left(\int_{\partial M}\frac{\partial G}{\partial x^k}\boldsymbol{t}e\wedge dy^j\right) * (dx^k\wedge dx^j).$$

For simplicity, assume for a while that we use tangent-normal coordinates such that $M=\{x^3\geq 0\}$. Then $\boldsymbol{t}e\wedge dy^3=0$, while $\boldsymbol{t}(*(dx^k\wedge dx^j))=0$ for $j=1,2$ and $k\neq 3$, so finally

$$\begin{aligned}\boldsymbol{t}\sum_{j=1}^{3}&\left(\int_{\partial M}\frac{\partial G}{\partial x^k}\boldsymbol{t}e\wedge dy^j\right)*(dx^k\wedge dx^j)\\ &=-\left(\int_{\partial M}\frac{\partial G}{\partial x^3}e_1dy^1\wedge dy^2\right)\Big|_{\partial M}^{+}dx^1-\left(\int_{\partial M}\frac{\partial G}{\partial x^3}e_2dy^1\wedge dy^2\right)\Big|_{\partial M}^{+}dx^2\\ &=\tfrac12\boldsymbol{t}e-\sum_{j=1}^{3}\boldsymbol{n}\left(\int_{\partial M}d_xG\boldsymbol{t}e\wedge dy^j\right)\wedge dx^j,\end{aligned}$$

the normal trace of the singular integral being understood in the sense of the principal value.

In a similar fashion we treat the first integral. Here we observe that since the tangential trace and the exterior derivative commute, the integral kernel has no derivatives of Green's function in the normal direction and hence the jump relations produce no extra terms besides the principal value integral.

By combining the terms, we reach the identity

$$\tfrac12\boldsymbol{t}e=\boldsymbol{t}e_0+D\Lambda\boldsymbol{t}e-K\boldsymbol{t}e, \tag{6–2}$$

where the operators D and K are given as

$$\begin{aligned}D\omega(x)&=\frac{1}{ik}\left(d_\partial\boldsymbol{t}\int_{\partial M}G(x-y)d\omega(y)+k^2\sum_{j=1}^{3}\int_{\partial M}G(x-y)\boldsymbol{t}dy_j\wedge\omega(y)\right),\\ K\omega(x)&=\sum_{j=1}^{3}\boldsymbol{n}\left(\int_{\partial M}d_xG(x-y)\omega(y)\wedge dy^j\right)\wedge dx^j,\end{aligned}$$

and where $x\in\partial M$, $\omega\in\Omega^1\partial M$ and the singular integrals are understood in the sense of principal values. Introduce the spaces

$$\begin{aligned}H(d,\Omega^kM)&=\left\{f\in L^2(\Omega^kM):df\in L^2(\Omega^{k+1}M)\right\},\\ H(\delta,\Omega^kM)&=\left\{f\in L^2(\Omega^kM):\delta f\in L^2(\Omega^{k-1}M)\right\}\end{aligned}$$

and on the boundary

$$H^{-1/2}(d,\Omega^k\partial M)=\big\{g\in H^{-1/2}(\Omega^k\partial M);d_\partial g\in H^{-1/2}(\Omega^{k+1}\partial M)\big\}.$$

Then we have the bounded trace maps

$$\begin{aligned}\boldsymbol{t}&:H(d,\Omega^kM)\to H^{-1/2}(d,\Omega^k\partial M),\\ \boldsymbol{n}&:H(\delta,\Omega^kM)\to H^{-1/2}(d,\Omega^{3-k}\partial M),\end{aligned}$$

and these maps are onto. Also, K maps $H^{-1/2}(d,\Omega^k\partial M)$ compactly to itself, and D just boundedly. For more details on this the reader is referred to [14]. Also there is large literature on layer potential techniques on (subdomains) of Riemannian manifolds, even with Lipschitz boundaries, see [12] and references therein. Of course, it is not known what is the analogue of the exponentially increasing Green's function in the general metric case.

It turns out that the equation (6–2) is of Fredholm type and has a unique solution exactly when ω is not an eigenfrequency for the interior Maxwell problem with vanishing tangential electric field. We shall not go into details here but refer to the (vector version) of this equation in the references [16] and [17].

7. From (Y_0^*,QY_0) to γ and μ

It turns out that for the reconstruction of γ and μ in the interior one does not need to recover the whole matrix Q. Indeed, since we do not know the relevant boundary data for the second order system, this cannot be done starting from the impedance map. However, as remarked in Section 5, one can still hope to extract information on Q. We start by making some explicit choices for the constant form Y_0. Fix $\xi\in\mathbb{R}^3$, and choose coordinates so that $\xi=(|\xi|,0,0)$. Then for $R>0$ let

$$\begin{aligned}\zeta&=\zeta(R)=(|\xi|/2,i(|\xi|^2/4+R^2)^{1/2},(R^2+k^2)^{1/2}),\\ \zeta^*&=\bar\zeta-\xi.\end{aligned}$$

Now $\langle\zeta,\zeta\rangle=\langle\zeta^*,\zeta^*\rangle=k^2$, and for some constant 1- and 2-forms y^1 and y^2 to be chosen later, let

$$y_0=\frac{1}{|\zeta|}\big(-{*}(\zeta\wedge{*}y^1),ky^1,ky^2,-\zeta\wedge y^2\big).$$

This choice guarantees the conditions (4–7). Further, let

$$y_0^*=\frac{1}{|\zeta|}(P(i\zeta^*)+ik)z.$$

Now, as $|\zeta|\to\infty$ we have the limits

$$\begin{aligned}\lim_{|\zeta|\to\infty}y_0&=-\big({*}(\hat\zeta\wedge{*}y^1),0,0,\hat\zeta\wedge y^2\big),\\ \lim_{|\zeta|\to\infty}y_0^*&=P(i\hat\zeta)z,\end{aligned}$$

where $\hat{\zeta} = \lim \zeta/|\zeta| = 1/\sqrt{2}(0, i, 1)$. By choosing y^1 so that $-{*}(\hat{\zeta} \wedge {*}y^1) = 1$ and y^2, we have

$$\lim_{|\zeta|\to\infty} y_0 = (1, 0, 0, 0).$$

On the other hand, by choosing $z = (0, z^1, 0, 0)$, we have

$$\lim_{|\zeta|\to\infty} y_0^* = (-i * (\hat{\zeta} \wedge *z^1), 0, i\hat{\zeta} \wedge z^1, 0).$$

If z^1 satisfies $-i\,{*}(\hat{\zeta} \wedge {*}z^1) = 1$, we use the equality $Q_{0,j} = 0$ for $j \neq 0$, where Q is the potential of Lemma 3.1, to obtain

$$\lim_{|\zeta|\to\infty} (QY_\zeta, Y_0^*) = \hat{Q}_{0,0}(\xi),$$

where $Q_{0,0}$ denotes the component of the potential that maps 0-forms to 0-forms. Similarly, we may choose the forms to yield

$$\lim_{|\zeta|\to\infty} (QY_\zeta, Y_0^*) = \hat{Q}_{3,3}(\xi).$$

From Lemma 3.1, we find that

$$Q_{0,0} = \Delta^0 \alpha - |d\alpha|^2 + (\kappa^2 - k^2),$$

and similarly

$$Q_{3,3} = \Delta^0 \beta - |d\beta|^2 + (\kappa^2 - k^2).$$

By denoting $u = \gamma^{1/2}$, $v = \mu^{1/2}$ so that $\alpha = \log u$, $\beta = \log v$ and $\kappa = kuv$, the equations above simplify further as

$$Q_{0,0} = \frac{1}{u}(\Delta^0 u - k^2 u(uv - 1)),$$
$$Q_{0,0} = \frac{1}{v}(\Delta^0 v - k^2 v(uv - 1)).$$

We assumed that in a neighbourhood of ∂M, u and v are known constants. An application of the unique continuation principle for elliptic equations then shows that u and v, i.e., μ and γ are uniquely determined by the admittance map.

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Petri Ola
University of Oulu
Department of Mathematical Sciences
University of Oulu
PO Box 3000
FIN–90014
Finland
Petri.Ola@oulu.fi

Lassi Päivärinta
University of Oulu
Department of Mathematical Sciences
University of Oulu
PO Box 3000
FIN–90014
Finland
lassi@tols16.oulu.fi

Erkki Somersalo
Helsinki University of Technology
Institute of Mathematics
PO Box 1100
FIN–02015
Finland
Erkki.Somersalo@hut.fi