New Directions in Hopf Algebras
MSRI Publications
Volume **43**, 2002

Hopf Algebra Extensions and Monoidal Categories

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Abstract. Tannaka reconstruction provides a close link between monoidal categories and (quasi-)Hopf algebras. We discuss some applications of the ideas of Tannaka reconstruction to the theory of Hopf algebra extensions, based on the following construction: For certain inclusions of a Hopf algebra into a coquasibialgebra one can consider a natural monoidal category consisting of Hopf modules, and one can reconstruct a new coquasibialgebra from that monoidal category.

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1. Introduction

In most applications of Hopf algebras, it is a key property that one can form tensor products of representations of Hopf algebras. A most fruitful — if incorrect — truism in quantum group and Hopf algebra theory says that the converse holds: For every nice category with abstract tensor product there has to be some kind of bialgebra or Hopf algebra whose representations (modules or comodules) are (or are related to) that category.

In reality, additional data and conditions are needed to transform this rather vague statement into precise mathematical facts — however, we would like to maintain the general idea that once we find a monoidal category in nature, there is, with some luck, a Hopf algebra behind it, and we only have to compute or "reconstruct" what it looks like. An early example for this "philosophy" is in [37].

The correspondence between Hopf algebras and their representation categories is often a good way to understand Hopf algebraic constructions. A good example is Drinfeld's construction of the quantum double of a Hopf algebra, which has a very conceptual explanation in the construction of the center of a monoidal category. It is the author's firm (though easily refutable) belief that for every reasonable construction in Hopf algebra theory there ought to be a natural construction on the level of categories with tensor product that explains the Hopf algebra construction.

This paper presents some applications of that general principle to the theory of Hopf algebra extensions. In the broadest sense of the term, an extension of Hopf algebras is perhaps just some injective Hopf algebra map $K \to H$ (or, dually, a Hopf algebra surjection, which is the viewpoint taken in [35]). We will be much stricter in our terminology by calling an extension only a short exact sequence $K \to H \to Q$ of Hopf algebra maps, which is in addition cleft. However, we will also be interested in more general inclusions $K \subset H$ of Hopf algebras, which we shall always require to be (co)cleft. The central result in our considerations, taken from [47], is a construction that assigns to each such inclusion another inclusion $K^* \subset \tilde{H}$, provided K is finite. Actually it will turn out that $\tilde{H}$ is often just a coquasibialgebra (the dual notion to Drinfeld's quasibialgebras), and it will be natural to also allow coquasibialgebras for H — in this setting, the construction is often an involution, and it can be made into a functor, which we denote by $H \mapsto \mathfrak{F}(H) := \tilde{H}$.

The description of short exact sequences of Hopf algebras is quite complicated: In any such sequence $K \to H \to Q$ the middle term is a bicrossproduct $H = K\,{}^{\tau}\#_{\sigma}\,Q$ described in terms of a weak action of Q on K, a weak coaction of K on Q, a two-cocycle $\sigma\colon Q \otimes Q \to K$, and a two-cycle $\tau\colon Q \to K \otimes K$. These data have to fulfill a rather large list of axioms to ensure that the multiplication and comultiplication on $K \otimes Q$ built from them will result in an honest bialgebra. Worse than the size of the list is its lack of a conceptual interpretation. If

K is commutative and Q is cocommutative the axioms have a cohomological description, although the responsible cohomology theory is in its turn hard to handle; if K and Q are arbitrary, even the cohomological interpretation fails, and the equations become merely a rather complicated combinatorical description of a bialgebra structure on $K \otimes Q$. Matters can only get worse if we allow more general inclusions $K \subset H$, or even such inclusions in which H is only a coquasibialgebra. But many prominent constructions in the theory of Hopf algebras and quantum groups do yield cocleft Hopf algebra inclusions: besides bicrossproducts, these are double crossproducts, including the Drinfeld double as an example, and Radford biproducts of a Hopf algebra with a Yetter–Drinfeld Hopf algebra. The combinatorics of these, and of more general cases, have been analyzed in the literature.

We would like to emphasize that our general construction does not require the knowledge of most of the combinatorial data involved in an extension or inclusion. Rather, it is quite intrinsic, and proceeds according to the general creed formulated at the beginning. To any inclusion $K \subset H$, there is naturally associated a monoidal category, namely ${}^H_K\mathcal{M}_K$, the category of K-K-bimodules within the monoidal category of H-comodules. It only remains to reconstruct the bialgebra, or, in this case, the coquasibialgebra, responsible for this monoidal category structure. This turns out to be possible if H is cocleft over K; some additional conditions have to be met if the coquasibialgebra H is not an ordinary bialgebra.

In several instances the construction has well-known special cases. For example, one can apply it to a bismash product Hopf algebra $K\#Q$, and obtains a double crossproduct Hopf algebra $Q \bowtie K^*$. Equally well, one can apply the construction to an inclusion $L \subset Q \bowtie L$ of a Hopf algebra L into a double crossproduct, and obtains a bismash product $L^*\#Q$. On one hand, this is nothing new: The combinatorial data (termed a Singer pair by Masuoka) needed for the construction of a bismash product have long been known to be in bijection with the combinatorial data (called a matched pair) needed for the construction of a double crossproduct. In fact, one of the primary sources for Singer pairs is matched pairs arising naturally from groups that are the product of two subgroups. On the other hand, some aspects are new after all: We have found an intrinsic connection between a bismash product Hopf algebra and the double crossproduct Hopf algebra built from the same data. This gives an explanation for the bijection between Singer pairs and matched pairs which refers to the actual Hopf algebras in consideration, rather than the combinatorial data used to construct them. It is clear that one may hope to also gain insights through categorical considerations into properties of the two Hopf algebras thus related, although we can at the moment only offer a rather strange application to their Drinfeld doubles.

For most Hopf algebra inclusions $K \subset H$, the resulting inclusion $K^* \subset \tilde{H} = \mathfrak{F}(H)$ has $\tilde{H}$ a coquasibialgebra rather than an ordinary bialgebra. However, this turns out to have as a special case a result on ordinary cocommutative Hopf algebras and their cohomology: Let F, G be finite groups, $L = k[G]$ and $Q = k[F]$ the group algebras, and $K = k^G$ the dual of L. We assume given a group $F \bowtie G$ having F and G as subgroups satisfying $FG = F \bowtie G$. This also gives rise to a double crossproduct $Q \bowtie L \cong k[F \bowtie G]$, hence to a Singer pair (K, Q). We denote by $\mathcal{H}^n(L)$ the Sweedler cohomology group with coefficients in the base field k, which is the same as group cohomology of G with coefficients in the multiplicative group of k. The long exact sequence

$$0 \to \mathcal{H}^1(Q \bowtie L) \stackrel{\text{res}}{\to} \mathcal{H}^1(Q) \oplus \mathcal{H}^1(L) \to \operatorname{Aut}(K\#Q) \to \mathcal{H}^2(Q \bowtie L) \stackrel{\text{res}}{\to}$$
$$\stackrel{\text{res}}{\to} \mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Opext}(Q, K) \to \mathcal{H}^3(Q \bowtie L) \stackrel{\text{res}}{\to} \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$$

connecting a certain group $\operatorname{Opext}(Q, K)$ of Hopf algebra extensions $K \to H \to Q$ and the automorphism group of the special extension $K\#Q$ with the group cohomologies of F, G and $F \bowtie G$ was discovered by Kac [22] (in a slightly different setting, cf. [32]). The Kac sequence is the long exact sequence arising from a short exact sequence of double complexes: One of these is the intricate cohomological description of bicrossproducts that we alluded to above, while the other two turn out to compute the three group cohomologies involved in the sequence. We note that the notations chosen above for the Kac sequence are deceivingly Hopf algebraic: the sequence does not at first make sense when we consider general cocommutative Hopf algebras L, Q in place of group algebras. In Kac' work the cohomology groups are group cohomology with coefficients in the multiplicative group of the field (which is naturally isomorphic to Sweedler cohomology), and the techniques leading to the sequence are specific to the case of group algebras. In [47], we have replaced these techniques by arguments using the functor $\mathfrak{F}$: One of the maps in Kac' sequence, the one from $\operatorname{Opext}(Q, K)$ to $\mathcal{H}^3(Q \bowtie L)$, assigns to a Hopf algebra (one of the extensions of Q by K) another Hopf algebra (the double crossproduct $Q \bowtie L$) and a three-cocycle. As it turns out, this map is a special case of the functor $\mathfrak{F}$, when we interpret $Q \bowtie L$ together with the cocycle as a coquasibialgebra. Exactness of Kac' sequence means in particular that the map from $\operatorname{Opext}(Q, K)$ to $\mathcal{H}^3(Q \bowtie L)$ has a partial inverse (back from a subgroup of its codomain to a quotient of its domain). As it turns out, this partial inverse is also a special case of the functor $\mathfrak{F}$. Besides a new proof, these facts give a new interpretation to Kac' result: The correspondence between certain Hopf algebras, and certain Hopf algebras with cocycles, which originally arose from cohomological data and their interpretation as a combinatorical description of Hopf algebras, can now be described more intrinsically. Moreover, the new explanation generalizes immediately to the case where Q and L are cocommutative Hopf algebras, L is finite, and $K = L^*$. It generalizes even further to the case of arbitrary Hopf algebras Q and L with L finite, if we replace the cohomology groups by suitable sets of classes of coquasibialgebras. Unfortunately, it does not generalize to cover the analog of the Kac sequence proved by Masuoka for Lie algebras and their enveloping algebras in place of groups and their group algebras, although some things can be said about this case as well.

The plan of the paper is as follows: In the first section we recall the basic machinery of reconstruction that allows us to find Hopf algebras associated to monoidal categories, and we review the relevant notions of Hopf modules that will furnish the necessary supply of examples underlying our application.

In the second section we recall various notions of extensions of Hopf algebras. Only very little emphasis is placed on the cohomological descriptions, since these are treated in more detail in Akira Masuoka's report [31] in this volume. We will pay equally little attention to the precise combinatorical description of the various constructions, in accordance with the fact that our constructions need hardly any of this information.

In the third section we first review the basic construction of the functor $\mathfrak{F}$ for the case of inclusions $K \subset H$ with finite K, as found in [47]. After discussing some examples and properties of $\mathfrak{F}$, we turn to two extensions of the theory: First, we discuss what remains of the construction if we drop finiteness of K (but work over a field). Second, we get into the question when $\mathfrak{F}(H)$ has an antipode, provided H is an ordinary Hopf algebra; this is closely related to the question when the category ${}^H_K\mathcal{M}_K$ has duals. Further, we discuss how the functor $\mathfrak{F}$ specializes to a construction treated earlier by Yongchang Zhu [61], when applied to an inclusion of groups. Zhu's results inspired the considerations on antipodes in the present paper, and a byproduct is Example 4.5.1, a counterexample to the following (very desirable) statement: If the category of finite dimensional comodules over a coquasibialgebra H is rigid (that is, all objects have dual objects), then H is a coquasi-Hopf algebra.

In the final section we treat Kac' sequence and its variations. We review the results in [47], adding two features: One is a short discussion of some of Masuoka's results (cf. [31]). These are not covered by the general constructions in subsection 4.1, but we shall investigate in some detail where this really fails. The other new feature concerns a result of Kreimer [25] on Galois objects over tensor products. We show how this derives easily from the generalized Kac sequence.

Preliminaries and notation. Throughout, we work over a commutative base ring k, all algebras, coalgebras, etc. are over k, and most maps are tacitly supposed to be k-linear maps. We write $\otimes$ and Hom for tensor product over k and the set of k-linear maps. We use Sweedler notation without summation symbol for comultiplication, $\Delta(c) = c_{(1)} \otimes c_{(2)}$, and the versions $\delta(v) = v_{(-1)} \otimes v_{(0)}$ and $\delta(v) = v_{(0)} \otimes v_{(1)}$ for left and right comodules, respectively. There will be much need for variations in the shape of the parentheses to distinguish various structures. We denote the opposite algebra of an algebra A by A^{op}, the opposite coalgebra of a coalgebra C by C^{cop}, and the opposite algebra and opposite coalgebra of a bialgebra B by B^{bop}. We write $\triangleright$ for the usual left action of an algebra A on its dual space A^*, and use $\triangleleft$ for the right action. For an algebra A over a field k, we denote its finite dual by A°. We denote the category of left A-modules by ${}_A\mathcal{M}$, with similar meanings of $\mathcal{M}_A$, ${}_A\mathcal{M}_B$. We denote the category of right C-comodules by $\mathcal{M}^C$, with the obvious variations ${}^C\mathcal{M}$, ${}^C\mathcal{M}^D$. The notations ${}_f\mathcal{M}_A$ and $\mathcal{M}^C_f$ (with the obvious variations) denote the full subcategories of those modules or comodules that are finitely generated projective k-modules. Our general references for Hopf algebra theory are [55; 1; 34].

2. Hopf Algebras and Monoidal Categories

The topic of this section is mainly the translation procedure between Hopf algebras and monoidal categories. Its basis is the simple observation that there is a tensor product of representations (modules or comodules) of a bialgebra B. In this way ${}_B\mathcal{M}$ and $\mathcal{M}^B$ become monoidal categories. In the reverse direction, one hopes to associate to any monoidal category a bialgebra having the objects of that category as its representations. Specifically, if the category of representations of an algebra or coalgebra has a monoidal category structure, one wants to reconstruct a bialgebra that is responsible for the tensor product in the category. While this last step seems almost trivial, one should not forget that rather special circumstances are necessary to make it work: The tensor product of two representations (be it modules or comodules) of a bialgebra is formed as the tensor product over k, carrying a representation defined using the bialgebra structure. Stated in a fancy manner, the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is a monoidal functor. But we would like to point out that even when this rather strict requirement is not fulfilled, the general scheme by which every monoidal category should come from a bialgebra is still valid to some extent: When the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is not quite as nice, it often turns out that A fulfills a weakened set of axioms compared to those defining a bialgebra. Drinfeld's notion of a (co)quasibialgebra [14] is central to this paper: it occurs when the underlying functor does preserve tensor products, but in an incoherent fashion; thus the representations of a (co)quasibialgebra form a monoidal category, with the ordinary tensor product over k, but the associativity of tensor products is modified. The representations of a Hopf face algebra in the sense of Hayashi [18] form a monoidal category with respect to a "truncated" tensor product. The representations of a weak Hopf algebra in the sense of Böhm, Nill and Szlachányi [6] form a monoidal category with tensor product a certain submodule of the tensor product over k. The representations of a $\times_R$-bialgebra in the sense of Takeuchi [56] form a monoidal category with the tensor product over the k-algebra R [43].

2.1. Monoidal categories. The relevant notion of a category with a nice tensor product is called a tensor category or monoidal category. A monoidal category consists of a category $\mathcal{C}$, a bifunctor $\otimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, a neutral object $I\in\mathcal{C}$ and isomorphisms $\alpha\colon (X\otimes Y)\otimes Z\to X\otimes(Y\otimes Z)$, $\lambda\colon I\otimes X\to X$ and $\rho\colon X\otimes I\to X$ which are natural in $X,Y,Z\in\mathcal{C}$ and required to be *coherent.* The latter means by definition that the pentagonal diagrams

$$\begin{array}{ccc} & (W\otimes X)\otimes(Y\otimes Z) & \\ {\scriptstyle\alpha}\nearrow & & \searrow{\scriptstyle\alpha} \\ ((W\otimes X)\otimes Y)\otimes Z & & W\otimes(X\otimes(Y\otimes Z)) \\ \downarrow{\scriptstyle\alpha\otimes Z} & & \nearrow{\scriptstyle W\otimes\alpha} \\ (W\otimes(X\otimes Y))\otimes Z & \xrightarrow{\ \alpha\ } & W\otimes((X\otimes Y)\otimes Z) \end{array}$$

and the diagrams

$$\begin{array}{ccc} (X\otimes I)\otimes Y & \xrightarrow{\ \alpha\ } & X\otimes(I\otimes Y) \\ & {\scriptstyle\rho}\searrow \qquad \swarrow{\scriptstyle\lambda} & \\ & X\otimes Y & \end{array}$$

commute for all $W,X,Y,Z\in\mathcal{C}$. We say that a monoidal category $\mathcal{C}$ is strict if α,λ, and ρ are identity morphisms.

The meaning of coherence is in Mac Lane's coherence theorem [26], which says that every diagram formally composed from instances of α,λ, and ρ (like the two in the definition) commutes. Informally, one can then 'identify' any multiple tensor products (like the source and the sink of the pentagon) that only differ in the way that parentheses are set — just like one usually writes multiple tensor products of vector spaces without ever using parentheses, or just like in a strict monoidal category.

With the notion of a monoidal category comes the notion of a monoidal functor between them. A monoidal functor $(\mathcal{F},\xi,\xi_0)\colon\mathcal{C}\to\mathcal{D}$ consists of a functor $\mathcal{F}\colon\mathcal{C}\to\mathcal{D}$, an isomorphism $\xi\colon\mathcal{F}(X)\otimes\mathcal{F}(Y)\to\mathcal{F}(X\otimes Y)$, natural in $X,Y\in\mathcal{C}$, and an isomorphism $\xi_0\colon I\to\mathcal{F}(I)$, required to fulfill the coherence conditions

$$\begin{array}{ccccc} (\mathcal{F}(X)\otimes\mathcal{F}(Y))\otimes\mathcal{F}(Z) & \xrightarrow{\xi\otimes 1} & \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) & \xrightarrow{\ \xi\ } & \mathcal{F}((X\otimes Y)\otimes Z) \\ \downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\mathcal{F}(\alpha)} \\ \mathcal{F}(X)\otimes(\mathcal{F}(Y)\otimes\mathcal{F}(Z)) & \xrightarrow[1\otimes\xi]{} & \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) & \xrightarrow[\xi]{} & \mathcal{F}(X\otimes(Y\otimes Z)) \end{array}$$

along with the two coherence conditions

$$\mathcal{F}(\lambda)\xi(\xi_0 \otimes \mathrm{id}) = \lambda\colon I \otimes \mathcal{F}(X) \to \mathcal{F}(X)$$
$$\mathcal{F}(\rho)\xi(\mathrm{id} \otimes \xi_0) = \rho\colon \mathcal{F}(X) \otimes I \to \mathcal{F}(X)$$

involving the neutral objects. Considering the case where $\mathcal{F}$ is a forgetful functor, this means that the coherence morphisms 'up' in the category $\mathcal{C}$ are the same as 'down' in the category $\mathcal{D}$.

We will have need for a more relaxed version of tensor product preserving functor, in which this need not be true. By a tensor functor (or incoherent tensor functor) we will mean a functor $\mathcal{F}$ equipped with functorial isomorphisms ξ and ξ_0 as above, which now do not need to satisfy the coherence conditions imposed for a monoidal functor. We call a tensor functor $(\mathcal{F}, \xi, \xi_0)$ strict if ξ and ξ_0 are identity morphisms.

Throughout the paper, we will avoid discussions involving the unit objects of monoidal categories, and treat them as if the coherence isomorphisms λ, ρ dealing with the unit objects were identities. In keeping with this, we will only be dealing with incoherent tensor functors that at least satisfy the coherence conditions involving unit objects, which we will call neutral tensor functors; we will never deal with unit objects explicitly, and thus suppress all considerations involving ξ_0. Consequently, we will refer to monoidal categories $(\mathcal{C}, \otimes, \alpha)$ and monoidal (or tensor) functors $(\mathcal{F}, \xi)$.

Tensor functors can be composed by the rule

$$(\mathcal{F}', \xi')(\mathcal{F}, \xi) = (\mathcal{F}'\mathcal{F}, \mathcal{F}'(\xi) \circ \xi'(\mathcal{F} \times \mathcal{F}))$$

i. e. by giving $\mathcal{F}'\mathcal{F}$ the tensor functor structure defined by

$$\mathcal{F}'\mathcal{F}(X) \otimes \mathcal{F}'\mathcal{F}(Y) \xrightarrow{\xi'} \mathcal{F}'(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \xrightarrow{\mathcal{F}'(\xi)} \mathcal{F}'\mathcal{F}(X \otimes Y).$$

By a morphism $\varphi\colon (\mathcal{F}, \xi) \to (\mathcal{F}', \xi')$ between tensor functors we mean a natural morphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$ compatible with the tensor functor structure in the sense that all diagrams

$$\begin{array}{ccc} \mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\xi} & \mathcal{F}(X \otimes Y) \\ {\scriptstyle \varphi_X \otimes \varphi_Y}\downarrow & & \downarrow{\scriptstyle \varphi_{X \otimes Y}} \\ \mathcal{F}'(X) \otimes \mathcal{F}'(Y) & \xrightarrow{\xi'} & \mathcal{F}'(X \otimes Y) \end{array}$$

commute. A monoidal equivalence between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor $(\mathcal{F}, \xi)\colon \mathcal{C} \to \mathcal{D}$ such that the functor $\mathcal{F}$ is a category equivalence. One can show that there is then a monoidal functor $(\mathcal{G}, \zeta)\colon \mathcal{D} \to \mathcal{C}$ such that $(\mathcal{G}, \zeta)(\mathcal{F}, \xi) \cong (\mathcal{I}d, \mathrm{id})$ and $(\mathcal{F}, \xi)(\mathcal{G}, \zeta) \cong (\mathcal{I}d, \mathrm{id})$ as monoidal functors. Thus a monoidal equivalence roughly speaking establishes a one-to-one correspondence between any statements or concepts expressed in terms of the monoidal category structure of $\mathcal{C}$, and the same statements or concepts in $\mathcal{D}$. Mac Lane's coherence theorem can be expressed nicely as saying that every monoidal category is monoidally equivalent to a strictly monoidal category [23, XI.5]. This gives support to the following informal statement used widely in the literature: To prove a general claim about monoidal categories, it will always suffice to treat the case of a strict monoidal category.

We will already follow this principle when we now define the notion of a dual object of an object X in a monoidal category $\mathcal{C}$. A left dual $X^\vee$ is by definition a triple $(X^\vee, \mathrm{db}, \mathrm{ev})$ in which $\mathrm{db}\colon I \to X \otimes X^\vee$ and $\mathrm{ev}\colon X^\vee \otimes X \to I$ are morphisms satisfying

$$\left(X \xrightarrow{\mathrm{db}\otimes X} X \otimes X^\vee \otimes X \xrightarrow{X\otimes \mathrm{ev}} X\right) = \mathrm{id}_X,$$
$$\left(X^\vee \xrightarrow{X^\vee\otimes \mathrm{db}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}\otimes X^\vee} X^\vee\right) = \mathrm{id}_{X^\vee}.$$

A right dual ${}^\vee X$ is defined similarly with morphisms $\mathrm{db}\colon k \to {}^\vee X \otimes X$ and $\mathrm{ev}\colon X \otimes {}^\vee X \to k$. For a k-module to have a dual in ${}_k\mathcal{M}$ means to be finitely generated projective. More generally $M \in {}_R\mathcal{M}_R$ has a left dual iff it is finitely generated projective as right R-module, and the same holds with left and right interchanged. We say that a category $\mathcal{C}$ is left (or right) rigid if every object in $\mathcal{C}$ has a left (or right) dual. By definition, a right inner hom-functor $\underline{\mathrm{hom}}(X, -)$ is a right adjoint to the functor $\mathcal{C} \ni Y \mapsto Y \otimes X \in \mathcal{C}$; that is, it is defined by an adjunction

$$\mathcal{C}(Y \otimes X, Z) \cong \mathcal{C}(Y, \underline{\mathrm{hom}}(X, Z))$$

natural in $Y, Z \in \mathcal{C}$. A left inner hom-functor is a right adjoint for $X \otimes -$. If $\mathcal{C}$ has inner hom-functors $\underline{\mathrm{hom}}(X, -)$ for all X, then we say that $\mathcal{C}$ is right closed. If X has a dual, then an inner hom-functor $\underline{\mathrm{hom}}(X, -)$ exists: one can take $\underline{\mathrm{hom}}(X, Z) = Z \otimes X^\vee$. In particular, left rigid categories are right closed. Conversely, assume that a right inner hom-functor $\underline{\mathrm{hom}}(X, -)$ exists. Then from the adjointness defining $\underline{\mathrm{hom}}(X, -)$ one can construct natural morphisms $Y \otimes \underline{\mathrm{hom}}(X, Z) \to \underline{\mathrm{hom}}(X, Y \otimes Z)$ for all $Y, Z \in \mathcal{C}$. If all these are isomorphisms, then X has a dual, namely $\underline{\mathrm{hom}}(X, I)$.

It is important to note that monoidal functors automatically preserve duals in the following sense: If $X \in \mathcal{C}$ has a left dual $X^\vee$, and $(\mathcal{F}, \xi)\colon \mathcal{C} \to \mathcal{D}$ is a monoidal functor, then $\mathcal{F}(X^\vee)$ is a dual to $\mathcal{F}(X)$ endowed with the maps

$$\mathcal{F}(X^\vee) \otimes \mathcal{F}(X) \xrightarrow{\xi} \mathcal{F}(X^\vee \otimes X) \xrightarrow{\mathcal{F}(\mathrm{ev})} \mathcal{F}(I) = I$$
$$I = \mathcal{F}(I) \xrightarrow{\mathcal{F}(\mathrm{db})} \mathcal{F}(X \otimes X^\vee) \xrightarrow{\xi^{-1}} \mathcal{F}(X) \otimes \mathcal{F}(X^\vee)$$

If both $\mathcal{C}$ and $\mathcal{D}$ are rigid, this results in isomorphisms $\mathcal{F}(X^\vee) \cong \mathcal{F}(X)^*$ (where $(-)^*$ denotes the dual in $\mathcal{D}$), which can be chosen to be functorial in X. In general, monoidal functors need not preserve inner hom-functors, and it will turn out that incoherent tensor functors need not preserve duals.

2.2. Bialgebras, quasibialgebras, and their representations. Key examples of monoidal categories arise from representations of bialgebras. For simplicity we shall pretend throughout that the category ${}_k\mathcal{M}$ of k-modules with the tensor product over k is strictly monoidal. When B is a bialgebra, then the category ${}_B\mathcal{M}$ of left B-modules is (strictly) monoidal: the tensor product of two B-modules V, W is their tensor product over k with the diagonal module structure $b(v \otimes w) = b_{(1)}v \otimes b_{(2)}w$. Dually, the category $\mathcal{M}^B$ of right B-comodules is monoidal: the tensor product of two B-comodules V, W is their tensor product over k, endowed with the codiagonal comodule structure $V \otimes W \ni v \otimes w \mapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)}w_{(1)}$.

In all of these examples the obvious underlying functors to the category of k-modules are strictly monoidal. If we are given an algebra A, and a structure of monoidal category on ${}_A\mathcal{M}$ such that the underlying functor to ${}_k\mathcal{M}$ is a strict monoidal functor, then we can actually reconstruct a unique bialgebra structure on A inducing the given monoidal category structure on ${}_A\mathcal{M}$. A similar statement holds for the category ${}^C\mathcal{M}$ of comodules over a coalgebra C: If ${}^C\mathcal{M}$ is monoidal, and the underlying functor to ${}_k\mathcal{M}$ is strictly monoidal, then there is a unique map $\nabla\colon C \otimes C \to C$ such that $(v \otimes w)_{(-1)} \otimes (v \otimes w)_{(0)} = v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}$ for all $V, W \in {}^C\mathcal{M}$, $v \in V$, and $w \in W$, and ∇ makes C into a bialgebra. Throughout this section we will state 'reconstruction' results like this without a hint of a proof, before sketching some of the techniques in the background in the next section.

The situation is *not* essentially worse if we assume that the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ (or ${}^C\mathcal{M} \to {}_k\mathcal{M}$) is a non-strict monoidal functor. In that case we can also find a monoidal category structure on ${}_A\mathcal{M}$ such that the identity is a (non-strict) monoidal equivalence between the two monoidal category structures on ${}_A\mathcal{M}$, and the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is strictly monoidal for the new structure.

An essential change of the situation occurs if we assume the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ to be a (neutral) tensor functor instead of a monoidal one.

Drinfeld [14] has defined a quasibialgebra A to be an algebra equipped with a not necessarily coassociative comultiplication $\Delta\colon A \to A \otimes A$ and a counit $\varepsilon\colon A \to k$ for Δ, both of which are algebra maps, and an invertible element $\phi \in A \otimes A \otimes A$, the associator, satisfying $(A \otimes \varepsilon \otimes A)(\phi) = 1_A \otimes 1_A \in A \otimes A$, $(A \otimes \varepsilon)\Delta = \mathrm{id}_A = (\varepsilon \otimes A)\Delta$,

$$(A \otimes \Delta)\Delta(a) \cdot \phi = \phi \cdot (\Delta \otimes A)\Delta(a) \in A \otimes A \otimes A$$

for all $a \in A$, and

$$(A \otimes A \otimes \Delta)(\phi) \cdot (\Delta \otimes A \otimes A)(\phi) = (1 \otimes \phi) \cdot (A \otimes \Delta \otimes A)(\phi) \cdot (\phi \otimes 1)$$

in $A \otimes A \otimes A \otimes A$.

The meaning of the definition is that the category ${}_A\mathcal{M}$ of A-modules over a quasibialgebra (A, ϕ) is a monoidal category. The tensor product of $V, W \in {}_A\mathcal{M}$

is formed just as in the case of ordinary bialgebras, by setting $a(v \otimes w) = a_{(1)}v \otimes a_{(2)}w$. The essential change compared to the bialgebra case is that the underlying functor ${}_A\mathcal{M} \to {}_k\mathcal{M}$ is no longer monoidal, but only a strict neutral tensor functor. This means that the associator morphism in ${}_A\mathcal{M}$ is not the same as the ordinary one for k-modules. Instead, one defines $\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for $U, V, W \in {}_A\mathcal{M}$ as left multiplication by $\phi \in A \otimes A \otimes A$, that is $\alpha(u \otimes v \otimes w) = \phi^{(1)}u \otimes \phi^{(2)}v \otimes \phi^{(3)}w$, if we write formally $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$. One can check that the converse holds: Every structure of monoidal category for ${}_A\mathcal{M}$ for which the underlying functor is a strict neutral tensor functor arises from a quasibialgebra structure on A.

The dual notion was used first by Majid [28]: A coquasibialgebra H is a coalgebra equipped with a not necessarily associative multiplication $\nabla\colon H \otimes H \to H$, which is a coalgebra map, a grouplike element $1_H \in H$ which is a unit for ∇, and a convolution invertible trilinear form $\phi\colon H \otimes H \otimes H \to k$, the coassociator, satisfying $\phi(g \otimes 1 \otimes h) = \varepsilon(g)\varepsilon(h)$,

$$(f_{(1)}g_{(1)})h_{(1)}\phi(f_{(2)} \otimes g_{(2)} \otimes h_{(2)}) = \phi(f_{(1)} \otimes g_{(1)} \otimes h_{(1)})f_{(2)}(g_{(2)}h_{(2)}),$$

and

$$\begin{aligned}\phi(d_{(1)}f_{(1)} \otimes g_{(1)} \otimes h_{(1)})\phi(d_{(2)} \otimes f_{(2)} \otimes g_{(2)}h_{(2)})\\ = \phi(d_{(1)} \otimes f_{(1)} \otimes g_{(1)})\phi(d_{(2)} \otimes f_{(2)}g_{(2)} \otimes h_{(1)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(2)})\end{aligned}$$

for $d, f, g, h \in H$. For any coquasibialgebra (H, ϕ) the equations $\phi(1 \otimes g \otimes h) = \phi(g \otimes h \otimes 1) = \varepsilon(gh)$ hold for all $g, h \in H$.

If (H, ϕ) is a coquasibialgebra, then the category ${}^H\mathcal{M}$ has the following structure of a monoidal category: The tensor product of $V, W \in {}^H\mathcal{M}$ is $V \otimes W$ with the codiagonal comodule structure $v \otimes w \mapsto v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}$ as in the case of an ordinary bialgebra, and the associator isomorphisms $\alpha\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for $U, V, W \in {}^H\mathcal{M}$ are given by $\alpha(u \otimes v \otimes w) = \phi(u_{(-1)} \otimes v_{(-1)} \otimes w_{(-1)})u_{(0)} \otimes v_{(0)} \otimes w_{(0)}$. Conversely, if ${}^H\mathcal{M}$ is a monoidal category, and the underlying functor to ${}_k\mathcal{M}$ is a strict neutral tensor functor, then there are unique maps $\nabla\colon H \otimes H \to H$ and $\phi \in (H \otimes H \otimes H)^*$ such that $(v \otimes w)_{(-1)} \otimes (v \otimes w)_{(0)} = v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}$ and $\alpha(u \otimes v \otimes w) = \phi(u_{(-1)} \otimes v_{(-1)} \otimes w_{(-1)})u_{(0)} \otimes v_{(0)} \otimes w_{(0)}$ hold for all $U, V, W \in {}^H\mathcal{M}$, $u \in U$, $v \in V$, and $w \in W$. With the structures ∇ and ϕ, H is a coquasibialgebra.

We note that for any coquasibialgebra $H = (H, \phi)$ there are opposite, coopposite, and biopposite coquasibialgebras $H^{\mathrm{op}}, H^{\mathrm{cop}}, H^{\mathrm{bop}}$ in which multiplication, comultiplication, or both are opposite, and the coassociators are given by $\phi^{\mathrm{op}}(f \otimes g \otimes h) = \phi^{-1}(h \otimes g \otimes f)$, $\phi^{\mathrm{cop}} = \phi^{-1}$, and $\phi^{\mathrm{bop}}(f \otimes g \otimes h) = \phi(h \otimes g \otimes f)$.

It goes without saying that coquasibialgebras form a category (which, incidentally, is even monoidal): There is an obvious notion of morphism between two coquasibialgebras (H, ϕ) and (H', ϕ'), namely, this should be a multiplicative, unit-preserving coalgebra map $f\colon H \to H'$ satisfying $\phi'(f \otimes f \otimes f) =$

$\phi\colon H^{\otimes 3} \to k$. If $f\colon H \to H'$ is a coquasibialgebra map, then the functor ${}^f\mathcal{M}\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ is a strictly monoidal functor. Conversely, any strict monoidal functor $\mathcal{F}\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ that commutes with the underlying functors to ${}_k\mathcal{M}$ is of this form. Thus maps of coquasibialgebras classify strict monoidal functors commuting with underlying functors, and this statement makes the need for a more relaxed notion obvious: There should be morphisms between coquasibialgebras that classify non-strict monoidal functors. The following definition of such morphisms is clearly folklore, although only the cases $F = \mathrm{id}$ or $\theta = \varepsilon$ appear to be in the literature:

Definition 2.2.1. Let (H,ϕ) and (H',ϕ') be two coquasibialgebras. A coquasimorphism $(F,\theta)\colon (H,\phi) \to (H',\phi')$ consists of a unital coalgebra map $F\colon H \to H'$ and a convolution invertible $\theta\colon H \otimes H \to k$ satisfying

$$\theta(g_{(1)} \otimes h_{(1)})F(g_{(2)}h_{(2)}) = F(g_{(1)})F(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})$$

and

$$\begin{aligned}\theta(f_{(1)} \otimes g_{(1)})&\theta(f_{(2)}g_{(2)} \otimes h_{(1)})\phi(f_{(3)} \otimes g_{(3)} \otimes h_{(2)})\\ &= \phi'(F(f_{(1)}) \otimes F(g_{(1)}) \otimes F(h_{(1)}))\theta(g_{(2)} \otimes h_{(2)})\theta(f_{(2)} \otimes g_{(3)}h_{(3)})\end{aligned}$$

for all $f,g,h \in H$.

When $F\colon H \to H'$ is a coalgebra map between two coquasibialgebras (H,ϕ) and (H',ϕ'), then a bijection between forms $\theta\colon H \otimes H \to k$ making (F,θ) a coquasimorphism, and monoidal functor structures on ${}^F\mathcal{M}\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ is given as follows: When (F,θ) is a coquasimorphism, then ${}^{(F,\theta)}\mathcal{M} := ({}^F\mathcal{M},\xi)\colon {}^H\mathcal{M} \to {}^{H'}\mathcal{M}$ is a monoidal functor with the isomorphism $\xi : {}^F\mathcal{M}(V) \otimes {}^F\mathcal{M}(W) \to {}^F\mathcal{M}(V \otimes W)$ given, for $V,W \in {}^H\mathcal{M}$, by

$$V \otimes W \ni v \otimes w \mapsto \theta(v_{(-1)} \otimes w_{(-1)})v_{(0)} \otimes w_{(0)} \in V \otimes W.$$

If we define the composition of two coquasimorphisms $(F,\theta)\colon H \to H'$ and $(F',\theta')\colon H' \to H''$ by $(F',\theta')(F,\theta) = (F'F, \theta'(F \otimes F) * \theta)$, then we have

$$\big({}^{(F',\theta')}\mathcal{M}\big)\big({}^{(F,\theta)}\mathcal{M}\big) = {}^{(F',\theta')(F,\theta)}\mathcal{M}$$

as monoidal functors.

A special case of coquasimorphisms are cotwists dual to the twists defined in [14]: These can be considered as coquasimorphisms in which the underlying coalgebra map is the identity. Let (H,∇,ϕ) be a coquasibialgebra, and $\theta\colon H \otimes H \to k$ a convolution invertible map satisfying $\theta(1 \otimes h) = \theta(h \otimes 1) = \varepsilon(h)$ for all $h \in H$. Then there is a unique coquasibialgebra structure $(H,\nabla,\phi)^\theta = (H^\theta,\nabla^\theta,\phi^\theta)$ on the coalgebra $H^\theta := H$ such that $(\mathrm{id}_H,\theta)\colon H \to H^\theta$ is a coquasimorphism: One can easily solve the equations defining a coquasimorphism for ∇^θ and ϕ^θ. By the above, a cotwist (id_H,θ) gives rise to a monoidal equivalence $\mathcal{I}d\colon {}^H\mathcal{M} \to {}^{H^\theta}\mathcal{M}$, whose monoidal functor structure is induced by θ.

Note that every coquasimorphism $(F,\theta)\colon H \to H'$ in which θ is invertible factors into a cotwist and a coquasibialgebra map:

$$(F,\theta) = \bigl(H \xrightarrow{(\mathrm{id}_H,\theta)} H^\theta \xrightarrow{(F,\varepsilon)} H'\bigr).$$

If H is a bialgebra and $\theta\colon H \otimes H \to k$ is a cotwist, then H^θ is a bialgebra (with trivial coassociator) if and only if $\theta(f_{(1)} \otimes g_{(1)})\theta(f_{(2)}g_{(2)} \otimes h) = \theta(g_{(1)} \otimes h_{(1)})\theta(f \otimes g_{(2)}h_{(2)})$ holds for all $f,g,h \in H$. We say that θ is a (normalized) two-cocycle on H. If H is a Hopf algebra, then any two-cocycle (co)twist H^θ is also a Hopf algebra. Cotwists by two-cocycles appear in [12].

The dual (and older, cf. [14]) notion to a cotwist is the twist of a quasibialgebra H by an invertible element $t \in H \otimes H$. We will only need the case where H is an ordinary bialgebra, and t is a two-cycle, that is, satisfies $t_{23}(\mathrm{id}\otimes\Delta)(t) = t_{12}(\Delta\otimes\mathrm{id})(t) \in H \otimes H \otimes H$, where $t_{23} = 1 \otimes t$ and $t_{12} = t \otimes 1$, and $(\varepsilon\otimes\mathrm{id})(t) = (\mathrm{id}\otimes\varepsilon)(t) = 1$. Then H_t is a bialgebra, with underlying algebra the same as H, and comultiplication $\Delta_t(h) = t\Delta(h)t^{-1}$. If H is a Hopf algebra, then so is H_t. Note that if H is finite, then t is a two-cycle if and only if t, considered as a map $H^* \otimes H^* \to k$, is a two-cocycle on H^*.

There is also a way of (co)twisting coquasimorphisms: If H and H' are coquasibialgebras, $(F,\theta)\colon H \to H'$ is a coquasimorphism, and $t\colon H \to k$ satisfies $t(1) = 1$, then $(F,\theta)^t := (F^t,\theta^t)\colon H \to H'$, defined by

$$\theta^t(g \otimes h) = t(g_{(1)})t(h_{(1)})\theta(g_{(2)} \otimes h_{(2)})t^{-1}(g_{(3)}h_{(3)}),$$
$$F^t(h) = t(h_{(1)})F(h_{(2)})t^{-1}(h_{(3)})$$

for $g,h \in H$, is also a coquasimorphism, called the cotwist of (F,θ) by t. This procedure has its interpretation in terms of the comodule category as well: Two coquasomorphisms are each other's cotwists if and only if the corresponding monoidal functors are isomorphic as monoidal functors.

A coquasibialgebra is called a coquasi-Hopf algebra if it has a coquasiantipode. By definition, this is a triple (S,β,γ) consisting of a coalgebra antiautomorphism $S\colon H \to H$ and linear forms $\beta,\gamma \in H^*$ such that

$$h_{(1)}\beta(h_{(2)})S(h_{(3)}) = \beta(h)1_H,$$
$$S(h_{(1)})\gamma(h_{(2)})h_{(3)} = \gamma(h)1_H,$$
$$\phi(S(h_{(1)}) \otimes \gamma(h_{(2)})h_{(3)}\beta(h_{(4)}) \otimes S(h_{(5)})) = \varepsilon(h),$$
$$\phi^{-1}(h_{(1)} \otimes \beta(h_{(2)})S(h_{(3)})\gamma(h_{(4)}) \otimes h_{(5)}) = \varepsilon(h)$$

hold for all $h \in H$. The meaning of the definition (whose dual, a quasi-Hopf algebra, appears in [14]) is that — just as for ordinary Hopf algebras — the category of k-finitely generated projective H-comodules is left and right rigid. The right dual of $V \in {}^H\mathcal{M}_f$ is V^* equipped with the comodule structure defined by

$\varphi_{(-1)}\varphi_{(0)}(v) = S(v_{(-1)})\varphi(v_{(0)})$ for all $v \in V$ and $\varphi \in V^*$, and the maps

$$\mathrm{ev}\colon V \otimes V^* \ni v \otimes \varphi \mapsto \beta(v_{(-1)})\varphi(v_{(0)}) \in k,$$
$$\mathrm{db}\colon k \ni 1 \mapsto v^i \otimes \gamma(v_{i(-1)})v_{i(0)} \in V^* \otimes V.$$

The first two equations in the definition of a quasi-antipode express colinearity of the maps ev and db, while the other two say that the maps fulfill the axioms for a dual object in the comodule category.

Conversely, if we are given a coquasibialgebra H such that the category of finite projective H-comodules is left and right rigid, then we would like to be able to reconstruct a coquasi-antipode for H; we can only hope for this to work if k is a field or H itself is finite; otherwise finite comodules give us too little information on H. This reconstruction result for coquasi-Hopf algebras is indeed claimed to hold, at least over a field, in [29, Sec. 9.4.1]. In fact the same belief has been held (independently, as it were) by the author for considerable time, but we shall present a counterexample in subsection 4.5. The analogous statement for ordinary bialgebras was proved by Ulbrich [59]. The key difference between the cases is that monoidal functors (such as the underlying functor from the comodule category over a bialgebra to the category of vector spaces) preserve duals, while incoherent tensor functors need not. Majid's proof relies on an (explicitly given, [29, (9.37)]) isomorphism between the dual object of an H-comodule V in the monoidal category of H-comodules (which is assumed to exist) and the dual vector space of V. However, we shall see that the dual object of V needs not even have the same dimension as V. Nevertheless, we shall without giving further details refer to [29] for the rest of the proof of the following fact: If H is a coquasibialgebra over a field k, if the category ${}^H\mathcal{M}_f$ is left and right rigid, and if the underlying functor ${}^H\mathcal{M}_f \to {}_k\mathcal{M}$ preserves duals, then H has a coquasiantipode.

2.3. Coendomorphism coalgebras. In the preceding section we have discussed how to get from a (coquasi)bialgebra to a monoidal category, and stated without proof that one can go the other way. In this section we shall discuss in a little more detail how the reconstruction process works. This is largely an elaboration of work of Ulbrich [59; 58]; see also [29; 21; 41; 40].

Assume first that we are given a coalgebra C, and consider the underlying functor $\mathcal{U}\colon \mathcal{M}^C \to {}_k\mathcal{M}$. When we want to reconstruct properties of C (like being a bialgebra), the key fact is that the coaction of C on each of its comodules defines a natural transformation $\delta\colon \mathcal{U} \to \mathcal{U} \otimes C$ which has a universal property: For every k-module T, every natural transformation $\phi\colon \mathcal{U} \to \mathcal{U} \otimes T$ factors as $\phi = (\mathcal{U} \otimes f)\delta$ for a unique k-linear map $f\colon C \to T$. This describes an isomorphism

$$\mathrm{Hom}(C,T) \ni f \mapsto (\mathcal{U} \otimes f)\delta \in \mathrm{Nat}(\mathcal{U}, \mathcal{U} \otimes T),$$

whose inverse maps a natural transformation ψ to $(\varepsilon \otimes T)\psi_C$.

When we are given an abstract monoidal category $\mathcal{C}$ and want to reconstruct a Hopf algebra H such that $\mathcal{C}$ is the category of modules (or comodules) over H, the least additional data we need is a specified functor $\omega\colon \mathcal{C} \to {}_k\mathcal{M}$ to the category of k-modules, which will play the role of the underlying functor above. Then a coendomorphism coalgebra of ω is by definition a k-module $\operatorname{coend}(\omega)$ together with a universal natural transformation $\delta\colon \omega \to \omega \otimes \operatorname{coend}(\omega)$ giving an isomorphism

$$\operatorname{Hom}(\operatorname{coend}(\omega), T) \ni f \mapsto (\omega \otimes f)\delta \in \operatorname{Nat}(\omega, \omega \otimes T)$$

for every k-module T. We have seen that any coalgebra C can be recovered as the coendomorphism coalgebra of the underlying functor $\mathcal{M}^C \to {}_k\mathcal{M}$. If k is a field, then C can equally well be recovered from the underlying functor $\mathcal{M}^C_f \to {}_k\mathcal{M}$, due to the finiteness theorem for comodules. This is one of the reasons why coendomorphism coalgebras are often to be preferred over just taking the endomorphism algebra of ω, which we can always do if we are not too worried about set theoretical complications, and relating the category $\mathcal{C}$ to the modules over $\operatorname{End}(\omega)$. Another reason is that if ω has a coendomorphism coalgebra, then the natural morphism

$$\begin{aligned} \operatorname{Hom}(\operatorname{coend}(\omega) \otimes M, T) &\cong \operatorname{Nat}(\omega \otimes M, \omega \otimes T) \\ f &\mapsto (\omega \otimes f)(\delta \otimes M) \end{aligned} \tag{2–1}$$

is an isomorphism for all $M \in {}_k\mathcal{M}$, by the slightly sketchy calculation

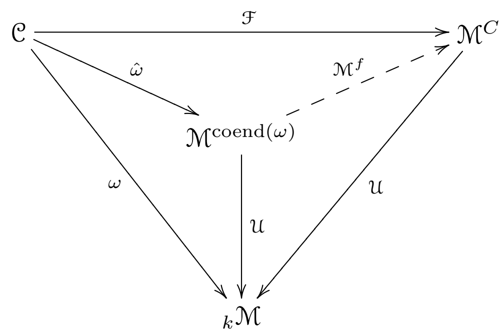
$$\begin{aligned} \operatorname{Hom}(\operatorname{coend}(\omega) \otimes M, T) &\cong \operatorname{Hom}(M, \operatorname{Hom}(\operatorname{coend}(\omega), T)) \\ &\cong \operatorname{Hom}(M, \operatorname{Nat}(\omega, \omega \otimes T)) \cong \operatorname{Nat}(\omega \otimes M, \omega \otimes T). \end{aligned}$$

This generalization of the defining property of coend is important when we try to reconstruct additional properties of $\operatorname{coend}(\omega)$ from properties of $\mathcal{C}$ and ω, such as being monoidal.

The basis for such reconstructions is the universal property of $\operatorname{coend}(\omega)$ and the natural transformation δ. The universal property first defines a unique coalgebra structure Δ for $\operatorname{coend}(\omega)$ such that the natural transformation δ endows every $\omega(X)$ with a $\operatorname{coend}(\omega)$-comodule structure. This defines a functor $\hat{\omega}\colon \mathcal{C} \to \mathcal{M}^{\operatorname{coend}(\omega)}$ that lifts ω as in the following diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\hat{\omega}} & \mathcal{M}^{\operatorname{coend}(\omega)} \\ & {\scriptstyle\omega}\searrow \quad \swarrow{\scriptstyle\mathcal{U}} & \\ & {}_k\mathcal{M} & \end{array} \tag{2–2}$$

in which $\mathcal{U}$ denotes the underlying functor. The functor $\hat{\omega}$ in its turn has a universal property: Every other functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{M}^C$ compatible with the underlying functor as in the outer triangle of the following diagram



(2–3)

factors through a dashed arrow induced by a coalgebra map $f\colon \operatorname{coend}(\omega) \to C$, which is the unique k-linear map making all the diagrams

$$\begin{array}{ccc} \omega(X) & \xrightarrow{\delta} & \omega(X) \otimes \operatorname{coend}(\omega) \\ \Vert & & \downarrow{\scriptstyle \mathrm{id}\otimes f} \\ \mathcal{U}\mathcal{F}(X) & \xrightarrow{\delta} & \mathcal{U}\mathcal{F}(X) \otimes C \end{array}$$

commute. In particular, if $\omega\colon \mathcal{C} \to {}_k\mathcal{M}$ as well as $\nu\colon \mathcal{D} \to {}_k\mathcal{M}$ have coendomorphism coalgebras, then every functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ satisfying $\nu\mathcal{F} = \omega$ induces a coalgebra map $\operatorname{coend}(\mathcal{F})\colon \operatorname{coend}(\omega) \to \operatorname{coend}(\nu)$ making the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\mathcal{F}} & \mathcal{D} \\ {\scriptstyle \hat{\omega}}\downarrow & & \downarrow{\scriptstyle \hat{\nu}} \\ \mathcal{M}^{\operatorname{coend}(\omega)} & \xrightarrow{\mathcal{M}^f} & \mathcal{M}^{\operatorname{coend}(\nu)} \end{array}$$

commute, and f is the unique k-linear map making all the diagrams

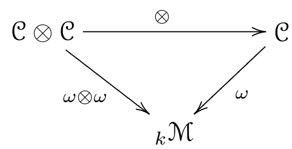
$$\begin{array}{ccc} \omega(X) & \xrightarrow{\delta} & \omega(X) \otimes \operatorname{coend}(\omega) \\ \Vert & & \downarrow{\scriptstyle \mathrm{id}\otimes f} \\ \nu\mathcal{F}(X) & \xrightarrow{\delta} & \nu\mathcal{F}(X) \otimes \operatorname{coend}(\nu) \end{array}$$

commute

The reconstruction process for morphisms needs a slight refinement, since one usually finds the outer triangle in (2–3) to commute only up to a specified isomorphism $\zeta\colon \omega \to \mathcal{U}$. However, it is sufficiently general as stated to distill a bialgebra structure on $\operatorname{coend}(\omega)$ out of a monoidal category $\mathcal{C}$ and *strict* monoidal underlying functor ω: We have to apply the procedure to the functor $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, with respect to the functors

$$\omega \otimes \omega\colon \mathcal{C} \times \mathcal{C} \ni (X, Y) \mapsto \omega(X) \otimes \omega(Y) \in {}_k\mathcal{M},$$

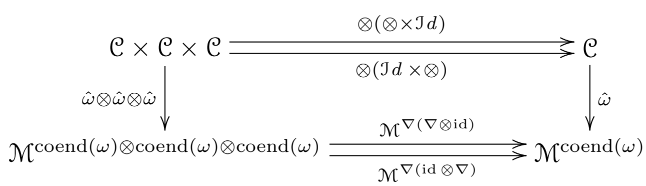
and $\omega\colon \mathcal{C} \to {}_k\mathcal{M}$, which should work since ω being strictly monoidal means a commutative triangle



We only have to know what coalgebra is attached to the functor $\omega\otimes\omega$. More generally, for $\omega\otimes\nu\colon \mathcal{C}\otimes\mathcal{D} \to {}_k\mathcal{M}$ the stronger property (2–1) of the coendomorphism coalgebra allows us to calculate

$$\begin{aligned}
\operatorname{Nat}(\omega\otimes\nu, \omega\otimes\nu\otimes T) &\cong \operatorname{Nat}(\omega, \operatorname{Nat}(\nu, \omega\otimes\nu\otimes T))\\
&\cong \operatorname{Nat}(\omega, \operatorname{Hom}(\operatorname{coend}(\nu), \omega\otimes T))\\
&\cong \operatorname{Nat}(\omega\otimes\operatorname{coend}(\nu), \omega\otimes T)\\
&\cong \operatorname{Hom}(\operatorname{coend}(\omega)\otimes\operatorname{coend}(\nu), T),
\end{aligned}$$

so that in particular $\operatorname{coend}(\omega\otimes\omega) = \operatorname{coend}(\omega)\otimes\operatorname{coend}(\omega)$. Thus a strict monoidal $\omega\colon \mathcal{C} \to {}_k\mathcal{M}$ gives rise to a coalgebra map $\nabla\colon \operatorname{coend}(\omega)\otimes\operatorname{coend}(\omega) \to \operatorname{coend}(\omega)$, and ∇ makes $\operatorname{coend}(\omega)$ a bialgebra such that (2–2) is a commutative diagram of monoidal functors. To see that ∇ is associative, we only need to look at the diagram



which commutes both with the upper and the lower horizontal arrows. But the two arrows at the top are identical, so ∇ is associative.

The reconstruction of a multiplication ∇ on $\operatorname{coend}(\omega)$ can be carried over directly to the case where ω is a strict incoherent tensor functor, but the last step fails in this situation. But it still follows, as first observed by Majid [28], that there is a unique coquasibialgebra structure on $\operatorname{coend}(\omega)$ such that ω lifts to a monoidal functor $\hat\omega$ making (2–2) a commutative diagram of incoherent tensor functors. To see this, one has to reconstruct a trilinear form $\phi\colon \operatorname{coend}(\omega)^{\otimes 3} \to k$ from the associativity isomorphism in $\mathcal{C}$. More generally, let $\mathcal{F}, \mathcal{F}'\colon \mathcal{C} \to \mathcal{M}^C$ be two functors with $\mathcal{U}\mathcal{F} = \omega = \mathcal{U}\mathcal{F}'$ and let $f, f'\colon \operatorname{coend}(\omega) \to C$ be the corresponding coalgebra maps. Assume further that there is some isomorphism $\alpha\colon \mathcal{F} \to \mathcal{F}'$. Then $\mathcal{U}(\alpha)$ is an endomorphism of ω, which corresponds to a linear form $\phi\colon \operatorname{coend}(\omega) \to k$ by the universal property of $\operatorname{coend}(\omega)$. Note that if $\mathcal{U}(\alpha) = \mathrm{id}$, then $\phi = \varepsilon$. One can show that $f * \phi = \phi * f'$. In particular, if $\mathcal{F}, \mathcal{F}'\colon \mathcal{C} \to \mathcal{D}$ are functors with $\nu\mathcal{F} = \omega = \nu\mathcal{F}'$, and $\alpha\colon \mathcal{F} \to \mathcal{F}'$ is a natural isomorphism, then there is a unique (invertible) linear form ϕ on $\operatorname{coend}(\omega)$ making the diagrams

$$\begin{array}{ccc} \omega(X) & \xrightarrow{\delta} & \omega(X)\otimes\operatorname{coend}(\omega) \\ \Big\| & & \Big\downarrow{\scriptstyle \operatorname{id}\otimes\phi} \\ \nu\mathcal{F}(X) & \xrightarrow{\nu(\alpha)} & \nu\mathcal{F}'(X) \end{array}$$

commute, and $f*\phi=\phi*f'$, if f and f' are obtained from $(\mathcal{F},\zeta)$ and $(\mathcal{F}',\zeta')$, respectively. A coquasibialgebra structure can be reconstructed by applying this to the two functors $\mathcal{F},\mathcal{F}'\colon\mathcal{C}^3\to\mathcal{C}$ composed from tensor product, and the associativity isomorphism α between them. The defining equations of a coquasibialgebra can be deduced from the coherence axioms for α. In fact there is a rather general principle why axioms for categories, functors, and natural transformations give rise to 'analogous' axioms for coalgebras, coalgebra maps, and linear forms. The reconstruction is a two-functor assigning, on the object level, coalgebras to categories with functors to ${}_k\mathcal{M}$; on the morphism level, coalgebra maps to functors commuting with underlying functors; on the two-morphism level, linear forms to natural transformations. The two-functor laws say that composition of functors corresponds to compositon of coalgebra maps, and that composition of natural transformations corresponds to convolution of linear forms. The reconstruction two-functor is even monoidal: we have seen that it assigns the tensor product of $\operatorname{coend}(\omega)$ and $\operatorname{coend}(\nu)$ to a 'tensor product' $(\mathcal{C},\omega)\otimes(\mathcal{D},\nu):=(\mathcal{C}\times\mathcal{D},\omega\otimes\nu)$. We refer the interested reader to [40, Sec. 2] for more details.

Similar reasoning shows that if $\mathcal{C},\mathcal{D}$ are monoidal categories with strict neutral tensor functors $\omega\colon\mathcal{C}\to{}_k\mathcal{M}$ and $\nu\colon\mathcal{D}\to{}_k\mathcal{M}$, then any monoidal functor $(\mathcal{F},\xi)\colon\mathcal{C}\to\mathcal{D}$ such that $\nu\mathcal{F}=\omega$ as functors, yields a coquasimorphism $(F,\theta)\colon\operatorname{coend}(\omega)\to\operatorname{coend}(\nu)$: The coalgebra map F is reconstructed from the functor $\mathcal{F}$, and the bilinear form θ is reconstructed from the isomorphism

$$\xi\colon\ \otimes\circ(\mathcal{F}\times\mathcal{F})\to\mathcal{F}\circ\otimes\colon\mathcal{C}\otimes\mathcal{C}\to\mathcal{D}.$$

As a particular case, we see how changing the tensor functor structure of a functor $\omega\colon\mathcal{C}\to{}_k\mathcal{M}$ affects the coquasibialgebra attached to it: If ξ,ξ' are two tensor functor structures for ω, and if H, H' are the coquasibialgebra structures on $\operatorname{coend}(\omega)$ obtained from them, then one can regard the identity functor with the trivial monoidal functor structure as the functor $(\mathcal{F},\beta)$ giving rise to a coquasimorphism $(F,\theta)\colon H\to H'$, in which F is the identity; in short, changing the tensor functor structure of ω amounts to changing $\operatorname{coend}(\omega)$ by a cotwist.

It remains to discuss some cases where coendomorphism coalgebras exist. First of all, any coalgebra C is the coendomorphism coalgebra of the underlying functor $\mathcal{M}^C\to{}_k\mathcal{M}$. If k is a field, then C is also the coendomorphism coalgebra of the underlying functor from the category of finite dimensional comodules over C. Generally, if $\omega\colon\mathcal{C}\to{}_k\mathcal{M}$ is a functor that takes values in finitely generated projective k-modules, then ω has a coendomorphism coalgebra. If k is a field, $\mathcal{C}$

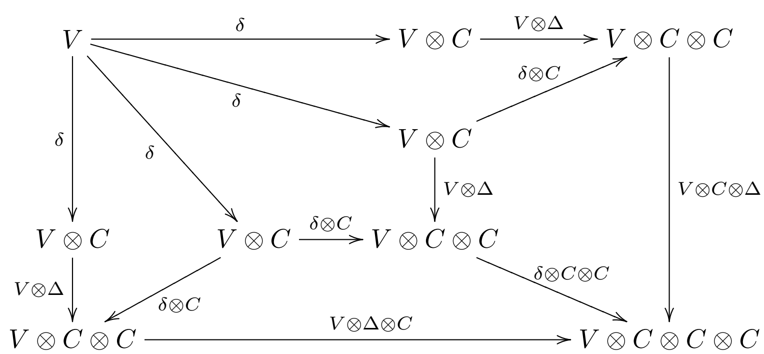
is a k-linear abelian category, and ω is an exact faithful k-linear functor taking values in finite dimensional vector spaces, then the functor $\hat{\omega}\colon \mathcal{C} \to \mathcal{M}^{\mathrm{coend}(\omega)}$ in (2–2) induces an equivalence with the category of finite dimensional coend(ω)-comodules.

Even when general reasons guarantee the existence of a coendomorphism coalgebra, it may be hard to figure out its structure. The following observation will prove useful to justify an educated guess. The situation we have in mind is that we can guess the coalgebra coend(ω) and even show that we have a category equivalence $\mathcal{C} \cong \mathcal{M}^{\mathrm{coend}(\omega)}$ — except we don't know that our candidate for coend(ω) is really a coassociative coalgebra. The rather simle idea is that, dually, if we have a nonassociative algebra that has a faithful associative representation, then the algebra is associative after all.

LEMMA 2.3.1. *Let C be a k-module endowed with maps $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$. We define the category $\mathcal{M}^C$ of C-comodules to have objects all pairs (V, ρ) with $V \in {}_k\mathcal{M}$ and $\rho\colon V \to V \otimes C$ satisfying $(V \otimes \Delta)\rho = (\rho \otimes C)\rho\colon V \to V \otimes C \otimes C$.*

If for every $c \in C$ there exist $V \in \mathcal{M}^C$, elements $v_i \in V$, and linear forms $\varphi_i \in V^*$ such that $c = \sum_i (\varphi_i \otimes C)\delta(v_i)$, then C is a coalgebra.

PROOF. For any $V \in \mathcal{M}^C$ with "comodule" structure map δ the diagram



shows that for $f = (\Delta \otimes C)\Delta - (C \otimes \Delta)\Delta\colon C \to C \otimes C \otimes C$ at least the equation $(V \otimes f)\delta = 0$ holds. But this entails, for $c \in C$ and V, v_i, φ_i as above, that $f(c) = f(\sum(\varphi_i \otimes \mathrm{id})\delta(v_i)) = \sum(\varphi_i \otimes \mathrm{id})(V \otimes f)\delta(v_i) = 0$. Hence C is a coassociative coalgebra, and ε is a counit by a similar reasoning. □

The following Lemma is useful when we try to reconstruct a coalgebra over a field k from its finite dimensional comodules. One should think of the case where the finite dual A° of an algebra A is found as a coalgebra which is a subset of A^* with comultiplication the restriction of the map $A^* \to (A \otimes A)^*$ dual to multiplication. The subset A° consists of all the entries of the finite matrix representations of A, or of all the elements in A^* that "occur" in the maps $V \to V \otimes A^*$ associated to finite dimensional A-modules V.

Lemma 2.3.2. *Let k be a field, C^1 and C^2 two vector spaces, $\iota\colon C^1 \otimes C^1 \to C^2$ an injection, $\Delta^1\colon C^1 \to C^2$ and $\varepsilon^1\colon C^1 \to k$ linear maps.*

Define the category $\mathcal{M}^{C^1}$ of right C^1-comodules to have as objects vector spaces V equipped with a map $\rho^1\colon V \to C^1$ such that the diagrams

$$\begin{array}{ccc}
V & \xrightarrow{\rho^1} & V \otimes C^1 \\
 & & \downarrow{\scriptstyle \rho^1 \otimes C^1} \\
\downarrow{\scriptstyle \rho^1} & & V \otimes C^1 \otimes C^1 \\
 & & \downarrow{\scriptstyle V \otimes \iota} \\
V \otimes C^1 & \xrightarrow{V \otimes \Delta^1} & V \otimes C^2
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
V & \xrightarrow{\rho^1} & V \otimes C^1 \\
 & \searrow{=} & \downarrow{\scriptstyle V \otimes \varepsilon^1} \\
 & & V
\end{array}$$

commute.

For a C^1-comodule V, element $v \in V$, and linear form $\varphi \in V^*$ put $[\varphi|v] := \langle \varphi, v_{(0)} \rangle v_{(1)} \in C^1$, where $v_{(0)} \otimes v_{(1)} = \rho^1(v)$.

Put $C := k\text{-}\operatorname{span}\{[\varphi|v] | V \in \mathcal{M}_f^{C^1}, v \in V, \varphi \in V^*\}$.

Then C is a coalgebra, and a category equivalence $\mathcal{M}_f^C \to \mathcal{M}_f^{C^1}$ is induced by the inclusion $C \subset C^1$.

Proof. We have $\Delta^1(C) \subset \iota(C \otimes C)$, since for every $V \in \mathcal{M}_f^{C^1}$, $v \in V$, and $\varphi \in V^*$ we have, denoting by v_i the elements of a basis of V, and v^i the elements of the dual basis of V^*:

$$\begin{aligned}
\Delta^1([\varphi|v]) = \langle \varphi, v_{(0)} \rangle \Delta^1(v_{(1)}) &= \langle \varphi, v_{(0)(0)} \rangle \iota(v_{(0)(1)} \otimes v_{(1)}) \\
&= \langle \varphi, v_{i(0)} \rangle \langle v^i, v_{(0)} \rangle \iota(v_{i(1)} \otimes v_{(1)}) = \iota([\phi|v_i] \otimes [v^i|v])
\end{aligned}$$

Thus, C is a not necessarily coassociative coalgebra with comultiplication Δ defined by $\iota\Delta = \Delta^1|_C$, and counit $\varepsilon|_C$. Moreover, by definition every C^1-comodule is actually a C-comodule in the sense used in the preceding Lemma. By definition of C, the preceding Lemma applies to show that C is coassociative. □

2.4. Coquasibialgebras and cohomology. So far we have dealt with the interpretation of coquasibialgebras in terms of monoidal categories of comodules. There is another, cohomological interpretation coming from the cocommutative case. A cocommutative coquasibialgebra is the same as an ordinary bialgebra, equipped in addition with a three-cocycle of Sweedler cohomology [54]: The modified associativity in a coquasibialgebra specializes to ordinary associativity in the cocommutative case, and the equation corresponding to Mac Lane's pentagonal axiom is precisely the condition for $\phi\colon H \otimes H \otimes H \to k$ to be a Sweedler three-cocycle with coefficients in the trivial H-module algebra k. A coquasimorphism $(F, \theta)\colon H \to H'$ between cocommutative coquasibialgebras (H, ϕ) and (H', ϕ') is just an ordinary bialgebra homomorphism F, with, in addition, a normalized invertible $\theta\colon H \otimes H \to k$ such that its Sweedler coboundary is the quotient $\phi'(F \otimes F \otimes F) * \phi^{-1}$. In particular, cotwisting a cocommutative coquasibialgebra does not affect its multiplication, but only its coassociator, and two coassociators for a cocommutative bialgebra H are cotwists of each other if and only if they are cohomologous Sweedler cocycles. Finally, if $(F,\theta)\colon H \to H'$ is a coquasimorphism between cocommutative coquasibialgebras, and $t \in H^*$ is normalized and convolution invertible, then $(F,\theta)^t = (F^t,\theta^t)$ has $F^t = F$, and θ^t differs from θ by the Sweedler coboundary of t. Motivated by this, we shall, for non-cocommutative coquasibialgebras as well, say that two coquasimorphisms (F,θ) and (F',θ') are cohomologous if there exists t with $(F',\theta') = (F,\theta)^t$. As a special case, we have a notion of when two cotwists on a coquasibialgebra H are cohomologous — but usually multiplication of cotwists, while trivially well-defined (it is just convolution), loses its meaning in the non-cocommutative case, since a cotwist will usually have a codomain different from its domain. However, we can consider the group of cohomology classes of "self-twists" $(\mathrm{id},\theta)\colon (H,\phi) \to (H,\phi)$, that is, of twists that do not affect the coquasibialgebra (H,ϕ) they act on. The conditions for this to happen are easily worked out: If the cocycle ϕ is trivial, they mean that θ is a two-cocycle (this is the condition for ϕ^θ to be trivial if ϕ is trivial), but also that

$$\theta(g_{(1)} \otimes h_{(1)})g_{(2)}h_{(2)} = g_{(1)}h_{(1)}\theta(g_{(2)} \otimes h_{(2)})$$

holds for all $g,h \in H$ (this is the condition that twisted multiplication is the same as the original one). We will call such two-cocycles *central*, and denote the group of cotwisting classes of central two-cocycles by $\mathcal{H}^2_c(H)$. We will use the same notation for the group of self-twists of a coquasibialgebra H with nontrivial coassociator.

2.5. Hopf modules. The most conceptual definition of a Hopf module is that it is a module (or comodule) over a certain algebra (or coalgebra) within a certain monoidal category. Although all Hopf modules in this paper will be of this type, it should be noted that there are meaningful Hopf modules that do not fit easily into this scheme, namely Doi–Koppinen Hopf modules [11; 24], and more generally entwined modules [7]. An entwining structure consists of an algebra A, a coalgebra C, and a map $\psi\colon C \otimes A \to A \otimes C$ satisfying the identities $\psi(C \otimes \nabla) = (\nabla \otimes C)(A \otimes \psi)(\psi \otimes A)$ and $(A \otimes \Delta)\psi = (\psi \otimes C)(C \otimes \psi)(\Delta \otimes A)$ along with some identities involving (co)units. An entwined module is an A-module and C-comodule, with the module structure μ and comodule structure ρ satisfying a compatibility condition governed by ψ, namely $\rho\mu = (\mu \otimes C)(M \otimes \psi)(\rho \otimes A)\colon M \otimes A \to M \otimes C$. Doi–Koppinen Hopf modules can be seen as special cases: When C is an H-module coalgebra and A is an H-comodule algebra, then an entwining structure can be defined by $\psi(c \otimes a) = a_{(0)} \otimes c \leftharpoonup a_{(1)}$; a Doi–Koppinen Hopf module is by definition a right A-module as well as right C-comodule satisfying $(ma)_{(0)} \otimes (ma)_{(1)} = m_{(0)}a_{(0)} \otimes m_{(1)} \leftharpoonup a_{(1)} \in M \otimes C$ for all $m \in M$ and $a \in A$.

We will now turn to the more conceptual versions of Hopf modules defined in terms of monoidal categories. This is based on the theory of algebras within monoidal categories, as developed in [36].

Suppose that $\mathcal{C}$ is a monoidal category. There is a natural notion of associative algebra in $\mathcal{C}$. This is most obvious if $\mathcal{C}$ is strictly monoidal. Then an algebra (A,∇) within $\mathcal{C}$ consists of an object A in $\mathcal{C}$, equipped with a multiplication $\nabla\colon A\otimes A\to A$ satisfying associativity $\nabla(\nabla\otimes A)=\nabla(A\otimes\nabla)\colon A\otimes A\otimes A\to A$, and admitting a unit $\eta\colon I\to A$ satisfying $\nabla(A\otimes\eta)=\mathrm{id}_A=\nabla(\eta\otimes A)\colon A\to A$.

When $(\mathcal{C},\otimes,\alpha)$ is a no longer strict monoidal category, an algebra (A,∇,η) in $\mathcal{C}$ is an object A together with a multiplication ∇ and unit η as above, now satisfying associativity in the form

$$\nabla(\nabla\otimes A)=\nabla(A\otimes\nabla)\alpha\colon (A\otimes A)\otimes A\to A$$

and the unit conditions as above. Obviously the instance of α was introduced into the associativity condition as the only conceivable way of making it make sense; one cannot compare $\nabla(A\otimes\nabla)$ and $\nabla(\nabla\otimes A)$ directly as they have distinct domains, but the domains are isomorphic via the morphism α making part of the definition of the monoidal category $\mathcal{C}$.

There are more instances where one has an obvious notion generalizing a well-known concept of ring and module theory to ring and module theory within a *strict* monoidal category, and a perhaps slightly less obvious version in a non-strict monoidal category, obtained by adorning the strict version with suitable copies of α. We will tacitly maintain each time that it is clear how this has to be done. Although this is quite informal, at least a hint may be given as to how one might try to formalize the idea: Each monoidal category $\mathcal{C}$ is equivalent as a monoidal category to a strict monoidal category $\hat{\mathcal{C}}$. Thus we may define an algebra in $\mathcal{C}$ to be a pair (A,∇) which is mapped by that monoidal equivalence to an algebra in $\hat{\mathcal{C}}$. However, we should stress once again that there appears to be no rigorous metatheorem in the literature that would allow us to work only in the strict case as long as a certain type of notions and statements is concerned.

Given an algebra A in a monoidal category $\mathcal{C}$, a right A-module is an object M of $\mathcal{C}$ with a morphism $\mu\colon M\otimes A\to M$ satisfying $\mu(M\otimes\nabla)=\mu(\mu\otimes A)\colon M\otimes A\otimes A\to M$. The notion of a left A-module is analogous. Given two algebras A,B, an A-B-bimodule is a triple (M,μ_ℓ,μ_r) such that (M,μ_ℓ) is a left A-module, (M,μ_r) is a right B-module, and $\mu_r(\mu_\ell\otimes B)=\mu_\ell(A\otimes\mu_r)\colon A\otimes M\otimes B\to M$. We denote the categories of left (right, bi-) modules by ${}_A\mathcal{C},\mathcal{C}_A,{}_A\mathcal{C}_B$.

For any algebra A in $\mathcal{C}$, the underlying functor $\mathcal{C}_A\to\mathcal{C}$ has a left adjoint; that is, there exists a free right A-module over any object V of $\mathcal{C}$. This can be constructed as $V\otimes A$, equipped with the right A-module structure $V\otimes\nabla\colon V\otimes A\otimes A\to V\otimes A$. More generally, if B is another algebra, then the underlying functor ${}_A\mathcal{C}_B\to{}_A\mathcal{C}$ has a left adjoint assigning to a left A-module N the free A-B-bimodule over N, which can be constructed as $N\otimes B$ with right B-module structure $N\otimes\nabla$ and left A-module structure $\mu\otimes B$, where μ is the A-module structure of N. If the category $\mathcal{C}$ has coequalizers, then one can define the tensor product of $M \in \mathcal{C}_A$ and $N \in {}_A\mathcal{C}$ by the coequalizer

$$M \otimes A \otimes N \underset{M\otimes\mu}{\overset{\mu\otimes N}{\rightrightarrows}} M \otimes N \longrightarrow M \underset{A}{\otimes} N$$

Provided that the tensor product in $\mathcal{C}$ preserves coequalizers, one can endow the tensor product $M \underset{A}{\otimes} N$ of $M \in {}_B\mathcal{C}_A$ and $N \in {}_A\mathcal{C}_R$ with an A-R-bimodule structure. In particular, the category ${}_A\mathcal{C}_A$ of A-A-bimodules in $\mathcal{C}$ is a monoidal category. We note [36, sec. 3] that if $\mathcal{C}$ is closed and has equalizers, then the category ${}_A\mathcal{C}_A$ is closed as well. If $M = V \otimes A$ is the free right A-module generated by $V \in \mathcal{C}$, then for any $N \in {}_A\mathcal{C}$ we have a canonical isomorphism $M \underset{A}{\otimes} N \cong V \otimes N$.

If the category $\mathcal{C}$ is the category of representations of a bialgebra H, there is standard terminology for the concepts above (see e. g. [34]): An algebra A in $\mathcal{M}^H$ is called an H-comodule algebra, A right A-module M in the category $\mathcal{M}^H$ is called a (relative) Hopf module; we'll denote the category of these by $\mathcal{M}^H_A$, with obvious variants ${}_A\mathcal{M}^H$, ${}_A\mathcal{M}^H_A$, etc. A coalgebra in the monoidal category of right H-modules, which is supposed to mean an algebra in the opposite category, is called an H-module coalgebra, and a comodule over C in $\mathcal{M}_H$ is also called a relative Hopf module, belonging to the category $\mathcal{M}^C_H$. The first Hopf modules in the literature are examples of both situations at the same time: If H is a bialgebra, then the regular right coaction of H on itself makes H an H-comodule algebra, and the regular right action of H on itself makes it a right H-module coalgebra. Both definitions of a Hopf module in $\mathcal{M}^H_H$ as given above agree in this situation. More generally, if $\iota\colon K \to H$ is a bialgebra map, which we consider an inclusion for simplicity, then H is a K-module coalgebra, and K an H-comodule algebra, via ι. Both notions of a Hopf module in $\mathcal{M}^H_K$ defined in these situations agree: Such an object is a right K-module as well as right H-comodule M satisfying $(mx)_{(0)} \otimes (mx)_{(1)} \in M \otimes H$ for all $m \in M$ and $x \in K$.

The conceptual definition of Hopf module allows the effortless generalization to cases involving (co)quasibialgebras in place of bialgebras. This means that we have to introduce instances of the associator maps in various places in the definitions and constructions we wrote down above for algebras and modules in strict categories; this being done, we can specialize to the case of comodules over coquasibialgebras. Thus if (H, ϕ) is a coquasibialgebra, then a left H-comodule algebra A is defined to be an algebra in the category of left H-comodules, which is by definition a left H-comodule A, equipped with a multiplication $A \otimes A \to A$ and unit $1 \in A$. The multiplication is required to be a map in ${}^H\mathcal{M}$, that is, to satisfy $(ab)_{(-1)} \otimes (ab)_{(0)} = a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)}$ for all $a, b \in A$, and to be associative in the sense that $(xy)z = \phi(x_{(-1)} \otimes y_{(-1)} \otimes z_{(-1)})x_{(0)}(y_{(0)}z_{(0)})$ holds. The unit has to satisfy $1_{(-1)} \otimes 1_{(0)} = 1 \otimes 1 \in H \otimes A$ and is an ordinary unit for multiplication. A left A-module in ${}^H\mathcal{M}$ is an H-comodule M with a map $A \otimes M \ni a \otimes m \mapsto am \in M$ of H-comodules satisfying $(xy)m = \phi(x_{(-1)} \otimes$

$y_{(-1)} \otimes m_{(-1)})x_{(0)}(y_{(0)}m_{(0)})$; similarly for right modules and bimodules. The free right A-module generated by $V \in {}^H\mathcal{M}$ is $V \otimes A$ with the right A-module structure defined by $(v \otimes x)y = \phi(v_{(-1)} \otimes x_{(-1)} \otimes y_{(-1)})v_{(0)} \otimes x_{(0)}y_{(0)}$. If V was a left B-module, then $V \otimes A$ is a B-A-bimodule with $b(v \otimes x) = \phi^{-1}(b_{(-1)} \otimes v_{(-1)} \otimes x_{(-1)})b_{(0)}v_{(0)} \otimes x_{(0)}$. The tensor product of a right A-module M and a left A-module N in ${}^H\mathcal{M}$ is by definition the quotient of $M \otimes N$ by the relations $mx \otimes n = \phi(m_{(-1)} \otimes x_{(-1)} \otimes n_{(-1)})m_{(0)} \otimes x_{(0)}n_{(0)}$ for all $m \in M, n \in N$, and $x \in A$.

3. Hopf Algebra Extensions

3.1. Cleft extensions and crossed products. Let H be a bialgebra. An H-comodule algebra A is said to be cleft if there exists a convolution invertible H-colinear map $j\colon H \to A$ (a cleaving map). One can show that such a j can always be chosen to satisfy $j(1) = 1$ (otherwise replace it with $\tilde{j}$ defined by $\tilde{j}(h) = j^{-1}(1)j(h)$). Assume given a cleaving map j for the H-comodule algebra A; put $R := A^{\operatorname{co} H} := \{x \in A | \rho(x) = x \otimes 1\}$. One can show that $\pi\colon A \ni x \mapsto x_{(0)}j^{-1}(x_{(1)}) \in B$ is well-defined (i. e. takes values in the coinvariants R), and one has an isomorphism $\psi\colon R \otimes H \to A$ of left R-modules and right H-comodules, given by $\psi(r \otimes h) = rj(h)$ and $\psi^{-1}(x) = \pi(x_{(0)}) \otimes x_{(1)}$. Hence classifying cleft H-comodule algebras with coinvariant subalgebra R amounts to classifying algebra structures on $R \otimes H$ such that $R \ni r \mapsto r \otimes 1 \in R \otimes H$ is an algebra map, and $R \otimes H$ is an H-comodule algebra with the comodule structure induced by that of H. One can show that any such multiplication is given by a formula

$$(r \otimes g)(s \otimes h) = r(g_{(1)} \rightharpoonup s)\sigma(g_{(2)}|h_{(1)}) \otimes g_{(3)}h_{(2)},$$

in which $\rightharpoonup\colon H \otimes R \to R$ and $\sigma\colon H \otimes H \to R$ are linear maps. Certain conditions have to be met by $\rightharpoonup$ and σ to ensure that multiplication is associative with unit $1 \otimes 1$. We make a point of not recalling these conditions, since we will never need them explicitly. We do mention, however, that in the case where H is cocommutative, and R is commutative, there is a reasonable cohomological interpretation of the axioms: They say that R is an H-module algebra, and σ is a two-cocycle in the Sweedler cohomology [54] of H with coefficients in R. Hence the term cocycle for σ, which is also used in the general situation where there is no cohomology theory behind it.

We will call an algebra $A := R\#_\sigma H := R \otimes H$ with the multiplication according to the above formula a crossed product of R and H, not quite in accordance with the literature, where crossed products are usually required to have σ invertible. If H is a Hopf algebra, this is equivalent to the H-comodule algebra $R\#_\sigma H$ being cleft; the map $j\colon H \ni h \mapsto 1 \otimes h \in R \otimes H$ can serve as a cleaving map.

If H is a Hopf algebra, then a crossed product $R\#_\sigma H$ can also be characterized as a Hopf–Galois extension with normal basis. Here a Hopf–Galois extension

A/R is an H-comodule algebra A with coinvariants $A^{\operatorname{co} H} = R$, such that the Galois map $\beta\colon A \underset{B}{\otimes} A \to A \otimes H$ given by $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is a bijection. A normal basis for an H-Galois extension A/B is an isomorphism $A \cong B \otimes H$ of left B-modules and right H-comodules. Any cleft extension with cleaving map j is a Galois extension (obviously with a normal basis) with $\beta^{-1}(x \otimes h) = xj^{-1}(h_{(1)}) \otimes j(h_{(2)})$. Conversely, Hopf–Galois extensions with normal basis are cleft.

The theory of cleft extensions, crossed products, and Hopf–Galois extensions with normal basis is developed in [13; 4; 5], see also [34]. There is a dual counterpart, which was worked out in detail in [9]. A crossed coproduct of a coalgebra C and a bialgebra H is a coalgebra structure on $H \otimes C$ making $H \otimes C$ an H-module coalgebra with the obvious left H-module structure, such that the map $\varepsilon \otimes C\colon H \otimes C \to C$ is a coalgebra map. One can show that any such comultiplication is necessarily of the form

$$\Delta(h \otimes c) = h_{(1)}\tau^{(1)}(c_{(1)}) \otimes c_{(2)[0]} \otimes h_{(2)}\tau^{(2)}(c_{(2)})c_{(2)[1]} \otimes c_{(3)},$$

involving maps $\rho\colon C \ni c \mapsto c_{[0]} \otimes c_{[1]} \in C \otimes H$ and $\tau\colon C \ni c \mapsto \tau^{(1)}(c) \otimes \tau^{(2)}(c) \in H \otimes H$, subject to a list of axioms that we'll skip once more. If the "two-cycle" τ is invertible (if C is cocommutative and H is commutative it is then an honest two-cycle in a suitable homology theory), then $C \otimes H$ is a cocleft H-module coalgebra with cocleaving map $\pi = (\varepsilon \otimes H)\colon C \otimes H \to H$. Here, a right H-module coalgebra D is called cocleft if there exists a convolution invertible right H-module map (a cocleaving) $\pi\colon D \to H$. Such a cocleaving may always be chosen to be counital, i. e. to satisfy $\varepsilon\pi = \varepsilon$. A cocleft H-module coalgebra is always an H-Galois coextension; this notion appears (without a name) in [48]. By definition, an H-Galois coextension is a surjection $\nu\colon D \to C$ of an H-module coalgebra D onto the factor coalgebra $C := D/DH^+$, such that the Galois map

$$\beta\colon D \otimes H \ni d \otimes h \mapsto d_{(1)} \otimes d_{(2)}h \in D \underset{C}{\square} D$$

is a bijection. In the case where D is cocleft, one may write $\beta^{-1}(\sum d_i \otimes d_i') = d_i\pi^{-1}(d_{i(1)}) \otimes \pi(d_{i(2)})$. Any H-Galois coextension D with a normal basis, that is, such that $D \cong C \otimes H$ as left C-comodules and right H-comodules, is cocleft.

A very important tool in our constructions will be Schneider's structure theorem for Hopf modules, in the version for Galois coextensions [48]. It states that when D is an H-Galois coextension of C such that H is flat and D is faithfully coflat as C-comodule (for example, this is fulfilled if D has a normal basis), then the functor

$$\mathcal{M}^C \ni V \mapsto V \underset{C}{\square} D \in \mathcal{M}^D_H$$

is a category equivalence with inverse mapping $M \in \mathcal{M}_H^D$ to M/MH^+. Here $V \underset{C}{\Box} D \subset V \otimes D$ is the cotensor product

$$V \underset{C}{\Box} D := \left\{ \sum v_i \otimes d_i | \sum v_{i(0)} \otimes v_{i(1)} \otimes d_i = \sum v_i \otimes \nu(d_{i(1)}) \otimes d_{i(2)} \right\}$$

(incidentally, a special case of the tensor product in monoidal categories, namely the tensor product over the algebra C in the opposite of the category of k-modules.) The Hopf module structures of $V \underset{C}{\Box} D$ are induced by the structures of the right tensor factor, and the comodule structure of M/MH^+ is that of a factor comodule of M. We are most interested in the case that D is cocleft. Then $V \underset{C}{\Box} D \cong V \otimes H$ as right H-modules; the isomorphism $M \cong M/MH^+ \otimes H$ for $M \in \mathcal{M}_H^D$ that makes part of Schneider's category equivalence is then given by $m \mapsto \overline{m_{(0)}} \otimes m_{(1)}$, and its inverse maps $\overline{m} \otimes h$ to $m_{(0)}\pi^{-1}(m_{(1)})h$.

3.2. Short exact sequences. One of the main objects of interest in this paper will be extensions of Hopf algebras, or short exact sequences. We refer to [50] for a detailed analysis of this notion.

For the purpose of this paper, a sequence of Hopf algebras and Hopf algebra maps $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ is exact if it fulfills the conditions stated to be equivalent in the following Proposition.

Proposition 3.2.1. *Let K, H, Q be Hopf algebras with bijective antipodes, $\iota\colon K \to H$ and $\nu\colon H \to Q$ Hopf algebra maps. The following are equivalent*:

(1) (i) *ν is surjective and conormal*,
 (ii) *ι is the kernel of ν, and*
 (iii) *H is cleft as a right Q-comodule algebra.*

(2) (i) *ι is injective and normal*,
 (ii) *ν is the cokernel of ι, and*
 (iii) *H is cocleft as a left K-module coalgebra.*

(3) (i) *ν is surjective, $\nu\iota = \eta\varepsilon$, and*
 (ii) *there is a unital and counital right Q-comodule map $j\colon Q \to H$ such that $\nabla(\iota \otimes j)\colon K \otimes Q \to H$ is an isomorphism.*

(4) (i) *ι is injective, $\nu\iota = \eta\varepsilon$, and*
 (ii) *there is a unital and counital left K-module map $\pi\colon H \to K$ such that $(\pi \otimes \nu)\Delta\colon H \to K \otimes Q$ is an isomorphism.*

Even if we don't know the precise definitions involved, it seems obvious that (1) and (2) as well as (3) and (4) are mutually dual. It is also obvious that (1) contains a naïve definition of short exact sequence, which would state that ν is surjective and ι the kernel of ν; the meaning of the remaining conditions in (1) is perhaps less obvious, and we shall also be very short in even explaining the ingredients.

From [50] we recall that the (Hopf) kernel of a Hopf algebra map ν (i. e. the equalizer of ν and the trivial morphism $\eta\varepsilon$ in the category of Hopf algebras) can be computed by $\mathrm{HKer}(\nu) = H^{\mathrm{co}\,Q} = {}^{\mathrm{co}\,Q}H$ if ν is conormal, and $\mathrm{HKer}(\nu)$ will then be a Hopf subalgebra. Dually, the Hopf cokernel of a normal morphism ι is $\mathrm{HCoker}(\iota) = H/K^+H = H/HK^+$.

The equivalence of (1) and (2) is essentially contained in [50]; however, only faithful (co)flatness conditions are required there in place of the (co)cleftness conditions, giving two weaker, mutually dual and equivalent sets of conditions (by the way, bijectivity of the antipodes is not needed for the equivalence of (1) and (2)). Conditions (c) should be seen as analogs of the fact that, in an extension $N \hookrightarrow G \to G/N$ of groups one can always choose a section for the surjection by just choosing coset representatives. For the equivalence of conditions (c) we just note that, given a cleaving j, a cocleaving π can be obtained just like the map π for general cleft extensions at the beginning of the previous section, and a cleaving can be obtained from a cocleaving in the dual fashion. We will not show the equivalence with (3) and (4), but point out that they state that H has normal basis as Q-comodule algebra and K-module coalgebra.

By a result of Schneider [49], the (co)cleftness conditions on a short exact sequence are automatic whenever the Hopf algebras involved are finite dimensional over a base field. More generally, whenever H is a finite dimensional Hopf algebra over the field k, and $K \subset H$ is a Hopf subalgebra H is cocleft as a left K-module coalgebra, and, dually, when $I \subset H$ is a Hopf ideal, then H is cleft as right comodule algebra over H/I. As an immediate consequence we have:

Corollary 3.2.2. *Let K, H, Q be finite dimensional Hopf algebras, $\iota\colon K \to H$ an injective, and $\nu\colon H \to Q$ a surjective Hopf algebra map with $\nu\iota = \eta\varepsilon$.*

The sequence $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$ is exact if and only if $\dim H = \dim K \cdot \dim Q$.

Although this result in finite dimensions makes the notion of short exact sequence look very natural, we should note that in the infinite dimensional case cleftness as we required it does rule out important examples of short exact sequences, for instance of affine algebraic groups.

Given two extensions $K \xrightarrow{\iota} H \xrightarrow{\nu} Q$, and $K \xrightarrow{\iota'} H' \xrightarrow{\nu'} Q$ an isomorphism of extensions is an isomorphism $f\colon H \to H'$ with $\nu' f = \nu$ and $f\iota = \iota'$.

We note that the following situation arises from the conditions in (3) and (4) in Proposition 3.2.1, which are equivalent even if we are dealing with bialgebras in place of Hopf algebras: We are given a mapping system

$$K \underset{\pi}{\overset{\iota}{\rightleftarrows}} H \underset{j}{\overset{\nu}{\rightleftarrows}} Q$$

in which K, H, Q are bialgebras, ι and ν are bialgebra maps, π is a unital counital left K-module map, j is a unital and counital Q-comodule map, and the conditions $\pi\iota = \mathrm{id}_K$, $\nu j = \mathrm{id}_Q$, $\pi j = \eta\varepsilon$, $\nu\iota = \eta\varepsilon$, and $\mathrm{id}_H = \iota\pi * j\nu$ are satisfied.

Assume given such a situation (where we can now drop the assumption that k is a field, but assume that all the bialgebras are k-flat). We can identify $H = K \otimes Q$ with $\pi = K \otimes \varepsilon$, $\nu = \varepsilon \otimes Q$, $j = \eta \otimes Q$ and $\iota = K \otimes \eta$.

Then $H = K \#_\sigma Q$ is a crossed product algebra with respect to $\rightharpoonup\colon Q \otimes K \to K$ and $\sigma\colon Q \otimes Q \to K$, and simultaneously a crossed coproduct coalgebra $H = K\,{}^{\tau}\!\# \, Q$ with $\rho\colon Q \ni q \mapsto q_{[0]} \otimes q_{[1]} \in Q \otimes K$ and $\tau\colon Q \to K \otimes K$. Such a bialgebra is called a bicrossproduct, and was introduced by Majid, see [29], generalizing the previously known cases where K is commutative and Q is cocommutative [53; 20]. An important special case are bismash products, in which σ and τ are trivial. This case can be characterized by saying that the map j is an algebra map, and π is a coalgebra map. Masuoka has called a collection $(K, Q) = (K, Q, \rightharpoonup, \rho)$ with maps $\rightharpoonup\colon Q \otimes K \to K$ and $\rho\colon Q \to Q \otimes K$ a Singer pair, if the data fulfill the necessary conditions to build such a bismash product. If K is commutative and Q cocommutative, we will speak of an abelian Singer pair. If $K\,{}^{\tau}\!\#_\sigma Q$ is a bicrossproduct with K commutative and Q cocommutative, then the maps $\rightharpoonup$ and ρ in the bicrossproduct form a Singer pair, and they only depend on the isomorphism class of the extension $K \to K\,{}^{\tau}\!\#_\sigma Q \to Q$. The set of extensions giving rise to a fixed abelian Singer pair is denoted $\operatorname{Opext}(Q, K)$. It has a group structure (given by a kind of Baer product), and it has a cohomological description [53; 20] through a double complex built from the Singer pair.

3.3. More general inclusions. Assume that $\iota\colon K \to H$ is a map of flat bialgebras such that H is a cocleft left K-module algebra (for example, that H and K are finite dimensional Hopf algebras over a field). If we put $Q := H/K^+H$, then Q is a quotient coalgebra and right H-module of H, and we have an isomorphism $(\pi \otimes \nu)\Delta\colon H \to K \otimes Q$, where ν denotes the canonical surjection and $\pi\colon H \to K$ is a cocleaving map, which we may assume unital and counital. If we define $j\colon Q \to H$ by $j(\nu(h)) = h_{(1)}\pi^{-1}(h_{(2)})$, then the inverse isomorphism is given by $\nabla(\iota \otimes j)\colon K \otimes Q \to H$. Hence we have a mapping system of the following type:

$$K \underset{\pi}{\overset{\iota}{\rightleftarrows}} H \underset{j}{\overset{\nu}{\rightleftarrows}} Q \ , \tag{3–1}$$

where K and H are bialgebras, ι is a bialgebra map, π is a unital counital left K-module map, Q is a right H-module coalgebra, ν is a right H-module coalgebra map, $j\colon Q \to H$ is a unital counital right Q-comodule map, and the equations $\pi\iota = \mathrm{id}_K$, $\nu j = \mathrm{id}_Q$, $\pi j = \eta\varepsilon$, $\nu\iota = \eta\varepsilon$, and $\mathrm{id}_H = \iota\pi * j\nu$ are satisfied. In any such system we can again identify $H = K \otimes Q$ with $\nu = \varepsilon \otimes Q$, $\iota = K \otimes \eta$, $\pi = K \otimes \varepsilon$, and $j = \eta \otimes Q$.

Several well-known constructions in quantum group theory lead to mapping systems of this type.

One of these is the bicrossproduct construction discussed above; here Q is a bialgebra and ν is a bialgebra map. A special case of this is the bismash product construction, in which both of the "cocycles" σ and τ are trivial; this is characterized by π being a coalgebra map, and j being an algebra map. The even more special case of a tensor product Hopf algebra is characterized by all four maps ι, π, ν, and j being bialgebra maps.

Another type of mapping system is furnished by the double crossproduct construction [27], where ι and j are bialgebra maps (and Q a bialgebra), whereas π and ν are coalgebra maps. The coalgebra structure of a double crossproduct $H = K \bowtie Q := K \otimes Q$ is the tensor product coalgebra, whereas multiplication is given by

$$(x \otimes p)(y \otimes q) = x(p_{(1)} \rightharpoonup y_{(1)}) \otimes (p_{(2)} \leftharpoonup y_{(2)})q,$$

in terms of certain maps $\rightharpoonup\colon Q \otimes K \to K$ and $\leftharpoonup\colon Q \otimes K \to Q$. A collection $(K, Q) = (K, Q, \rightharpoonup, \leftharpoonup)$ of data fulfilling the necessary conditions to build a double crossproduct is called a matched pair. If K and Q are cocommutative, we speak of an abelian matched pair; this case is treated in [57]. One can characterize double crossproducts as the general form of a bialgebra H equipped with bialgebra maps $\iota\colon K \to H$ and $j\colon Q \to H$ such that $\nabla(\iota \otimes j)\colon K \otimes Q \to H$ is a bijection. From this description it should be obvious that a matched pair of group algebras arises whenever two groups F, G are subgroups of a third group $F \bowtie G$ such that multiplication induces a bijection $F \times G \to F \bowtie G$.

Another class of mapping systems of type (3–1) arises from Radford's biproduct construction [38]. This is the case in which both ι and π are bialgebra maps. In a biproduct $K \star Q$, comultiplication and multiplication are given by

$$\Delta(x \otimes q) = x_{(1)} \otimes q_{(1)[0]} \otimes x_{(2)} q_{(1)[1]} \otimes q_{(2)}$$
$$(x \otimes p)(y \otimes q) = xy_{(1)} \otimes (p \leftharpoonup y_{(2)})q,$$

that is, $K \star Q$ is a cosmash product coalgebra, and a smash product algebra, but on the other side compared to a bismash product. The combinatorics of the action and coaction have, for once, a conceptual interpretation in this case: In more modern language they say that Q is a Hopf algebra within the braided monoidal category of Yetter–Drinfeld modules over K.

Biproducts, bismash products, and double crossproducts were unified in the construction of trivalent products in [2], and independently but later in [8] and [44]. In particular, the papers [2; 8] describe how one can construct bialgebra structures on a tensor product $K \otimes Q$ in which both K and Q are algebras and coalgebras (but not necessarily bialgebras). This is precisely the case of a mapping system (3–1) in which π is a coalgebra map and j an algebra map. The coalgebra structure in such a trivalent product is a cosmash product as in all the examples before, and the multiplication has the same form as for a double crossproduct.

The construction in [44] is less general in that the first factor is always a bialgebra, but more general in that it allows the second factor Q to be a nonassociative algebra. This will perhaps seem more natural after subsection 4.2 below. Nonassociative algebras are unavoidable in the construction, which completely classifies mapping systems of the type discussed above, with ι a bialgebra map and π a left K-module coalgebra map.

A still more (and perhaps most) general version can be found in [3]. Here tensor products $K\otimes Q$ are covered in which K is an algebra and non-coassociative coalgebra, and Q is a coalgebra and nonassociative algebra; moreover, everything is set within a braided monoidal category.

4. Coquasibialgebras Reconstructed from Hopf Modules

In this section we will use the concepts of section 2 to give a general construction of coquasibialgebras from cocleft inclusions of Hopf algebras (and in fact more general inclusions involving coquasibialgebras). The construction will have a conceptual description that, at the utmost level of compression, fits in a few lines; the outcome can be computed quite explicitly, and we will discuss several interesting special cases.

4.1. The general construction. Assume $K \subset H$ is an inclusion of Hopf algebras such that H is cocleft as a left K-module coalgebra with cocleaving π. We consider the monoidal category ${}^H_K\mathcal{M}_K = {}_K({}^H\mathcal{M})_K$ of K-bimodules in the category of H-comodules (which we can do since K is an H-comodule algebra via the inclusion). Since H is cocleft, we have $M \cong K\otimes M/K^+M$ for each $M \in {}^H_K\mathcal{M}$, hence the functor ${}^H_K\mathcal{M}_K \ni M \mapsto M/K^+M \in {}_k\mathcal{M}$ preserves tensor products (by the calculation $M \underset{K}{\otimes} N = M \underset{K}{\otimes} (K \otimes N/K^+N) \cong M \otimes N/K^+N$, hence $(M \underset{K}{\otimes} N)/K^+(M \underset{K}{\otimes} N) \cong M/K^+M \otimes N/K^+N)$. Thus if there is a coendomorphism coalgebra of the functor ω, then it is naturally a coquasibialgebra, which we denote $\tilde{H}$ (it depends, of course, on H, K, the inclusion, and the cleaving map).

Directly from the abstract description of $\tilde{H}$ we can see some things about its structure; we assume that K is finite. Every Hopf module in ${}^K_K\mathcal{M}_K$ is trivially a Hopf module in ${}^H_K\mathcal{M}_K$. Thus we get a functor ${}^K_K\mathcal{M}_K \to {}^H_K\mathcal{M}_K$. Now the former category is equivalent to $\mathcal{M}_K \cong {}^{K^*}\mathcal{M}$ by sending a right K-module V to $K \otimes V$ with the obvious structure of a Hopf module in ${}^K_K\mathcal{M}$, and the diagonal right K-module structure [42]. Observe that the composition $\mathcal{M}_K \to {}^K_K\mathcal{M}_K \to {}^H_K\mathcal{M}_K \xrightarrow{\omega} {}_k\mathcal{M}$ is just the underlying functor. Thus we have a natural map $K^* \to \tilde{H}$; upon closer inspection, this turns out to be a coquasibialgebra map, and to turn $\tilde{H}$ into a right K^*-module coalgebra. Next, we also have a natural map $\tilde{H} \to K^*$: By the universal property of $\tilde{H}$, this amounts to giving a natural transformation $\omega \to K^* \otimes \omega$, or $\omega \otimes K \to \omega$. Such a transformation can easily be assembled from the right K-module structures of the objects in ${}^H_K\mathcal{M}_K$, that is, we can define $\omega \otimes K \to \omega$ by $\overline{m} \otimes x \mapsto \overline{mx}$ for $m \in M$ and $x \in K$; upon closer inspection, the resulting map $\tilde{H} \to K^*$ will turn out to be a cocleaving map for the right K^*-module coalgebra $\tilde{H}$.

All in all, we have (up to existence of the necessary coendomorphism coalgebras) a natural construction that requires an inclusion of a Hopf algebra K

into a bialgebra H, such that H is left K-cocleft. The construction yields as its result an inclusion of K^* into a coquasibialgebra $\tilde{H}$, such that $\tilde{H}$ is a right K^*-module coalgebra, and cocleft as such. Stated rather sloppily, the output of the construction is similar to, but not quite as good as, its input.

We would like to have a generalization that can do with input of the same type as its output.

So assume that H is only a coquasibialgebra, and $K \subset H$ a sub-coquasibialgebra, and let us investigate what additional hypotheses we need to arrive at a similar construction $\tilde{H}$ as above. We will endeavor to make Definition 4.1.1 below look inevitable, though the discussion will not be sufficiently rigorous to yield necessary conditions; in fact, we will sketch in subsection 4.6 a situation where it appears one can get a result from weaker conditions.

To be able to consider ${}_K({}^H\mathcal{M})_K$, we needed to have K an algebra in the category ${}^H\mathcal{M}$, with respect to the comodule structure induced by the inclusion. This means that for x, y, z the associativity in the coquasibialgebra H has to coincide with the modified associativity

$$(xy)z = \phi(x_{(1)} \otimes y_{(1)} \otimes z_{(1)})x_{(2)}(y_{(2)}z_{(2)})$$

for an algebra in ${}^H\mathcal{M}$. Upon applying ε, this entails that $\phi|_{K\otimes K\otimes K} = \varepsilon$, and K is hence an ordinary bialgebra. To apply the structure theorem for Hopf modules, we used that K is a Hopf algebra, and H a K-module coalgebra with cocleaving map π. To have H a K-module means that

$$x(yh) = (xy)h = \phi(x_{(1)} \otimes y_{(1)} \otimes h_{(1)})x_{(2)}(y_{(2)}h_{(2)})$$

for all $x, y \in K$ and $h \in H$, which is true if and only if $\phi|_{K\otimes K\otimes H} = \varepsilon$. As above, we have a natural functor ${}^K_K\mathcal{M}_K \to {}^H_K\mathcal{M}_K$ that yields a natural map $K^* \to \tilde{H}$. Assume that we know it is a bialgebra map, and that it makes $\tilde{H}$ a right K^*-module coalgebra. Then a cocleaving map $\tilde{\pi}\colon \tilde{H} \to K^*$ was constructed above by using the natural transformation $\omega \otimes K \to \omega$ given by $\overline{m} \otimes y \mapsto \overline{my}$ for each $m \in M \in {}^H_K\mathcal{M}_K$ and $y \in K$. If H is a coquasibialgebra, this is no longer obviously well-defined, since objects in ${}^H_K\mathcal{M}_K$ are not ordinary bimodules; it is well-defined, however, if we assume that right multiplication by $y \in K$ on $M \in {}^H_K\mathcal{M}_K$ is a left K-module map; this means $x(my) = (xm)y = \phi(x_{(1)} \otimes m_{(-1)} \otimes y_{(1)})x_{(2)}(m_{(0)}y_{(2)})$ for all $x, y \in K$ and $m \in M$. Assuming there are enough objects in ${}^H_K\mathcal{M}_K$, this means that $\phi|_{K\otimes H\otimes K}$ should be trivial. Another vital part of the construction was using the structure theorem for Hopf modules to assemble the isomorphism

$$\theta\colon M \underset{K}{\otimes} N \cong M \underset{K}{\otimes} (K \otimes N/K^+N) \cong M \otimes N/K^+N$$

with $\theta(m \otimes n) = m\pi(n_{(-1)}) \otimes \overline{n_{(0)}}$. To maintain this in the case where H is a coquasibialgebra, we need to remember that the tensor product $M \underset{K}{\otimes} N$ is now defined in the category ${}^H\mathcal{M}$. We need to have, abusing notation

$$\theta(mx \otimes n) = \theta(\phi(m_{(-1)} \otimes x_{(1)} \otimes n_{(-1)})m_{(0)} \otimes x_{(2)}n_{(0)})$$

for $m \in M$, $x \in K$, and $n \in N$. Now

$$\begin{aligned}\theta(mx \otimes n) &= (mx)\pi(n_{(-1)}) \otimes \overline{n_{(0)}} \\ &= \phi(m_{(-1)} \otimes x_{(1)} \otimes \pi(n_{(-1)})_{(1)})m_{(0)}(x_{(2)}\pi(n_{(-1)})_{(2)}) \otimes \overline{n_{(0)}}\end{aligned}$$

and

$$\begin{aligned}&\theta(\phi(m_{(-1)} \otimes x_{(1)} \otimes n_{(-1)})m_{(0)} \otimes x_{(2)}n_{(0)}) \\ &\qquad = \phi(m_{(-1)} \otimes x_{(1)} \otimes n_{(-2)})m_{(0)}\pi(x_{(2)}n_{(-1)}) \otimes \overline{x_{(3)}n_{(0)}} \\ &\qquad = \phi(m_{(-1)} \otimes x_{(1)} \otimes n_{(-2)})m_{(0)}(x_{(2)}\pi(n_{(-1)})) \otimes \overline{n_{(0)}}.\end{aligned}$$

We can only expect these to be equal for all objects M, N, if

$$\phi(g \otimes x \otimes h_{(1)})\pi(h_{(2)}) = \phi(g \otimes x \otimes \pi(h)_{(1)})\pi(h)_{(2)}$$

for all $g, h \in H$ and $x \in K$. Next, θ was used above to endow the functor $\omega\colon {}^H_K\mathcal{M}_K \to {}_k\mathcal{M}$ with an incoherent tensor functor structure, which was easy since θ was a left K-module map. In our generalized situation, we have $\theta(x(m \otimes n)) = \theta(\phi^{-1}(x_{(1)} \otimes m_{(1)} \otimes n_{(1)})x_{(2)}m_{(0)} \otimes n_{(0)}) = \theta^{-1}(x_{(1)} \otimes m_{(-1)} \otimes n_{(-2)})x_{(2)}m_{(0)}\pi(n_{(-1)}) \otimes \overline{n_{(0)}}$ and $x\theta(m \otimes n) = xm\pi(n_{(1)}) \otimes \overline{n_{(0)}}$. To have these equal for all $x \in K$, $m \in M \in {}^H_K\mathcal{M}_K$ and $n \in N \in {}^H_K\mathcal{M}_K$ should imply $\phi|_{K \otimes H \otimes H}$ is trivial.

We have arrived at the set of conditions for objects in the following definition [47, Def. 3.3.3]:

Definition 4.1.1. Let K be a k-flat Hopf algebra. The category $\mathfrak{E}_\ell(K)$ is defined as follows:

An object of $\mathfrak{E}_\ell(K)$ is

(1) a k-flat coquasibialgebra H together with
(2) a coquasibialgebra map $\iota\colon K \to H$ such that $\phi(\iota \otimes H \otimes H) = \varepsilon$, and
(3) a convolution invertible, unit and counit preserving left K-module map $\pi\colon H \to K$ such that $\phi(g \otimes x \otimes h_{(1)})\pi(h_{(2)}) = \phi(g \otimes x \otimes \pi(h)_{(1)})\pi(h)_{(2)}$ holds for all $g, h \in H$ and $x \in K$.

A morphism in $\mathfrak{E}_\ell(K)$ is a coquasimorphism $(F, \theta)\colon H \to H'$ such that $F\iota = \iota'$ and $\theta(\iota \otimes H) = \varepsilon$

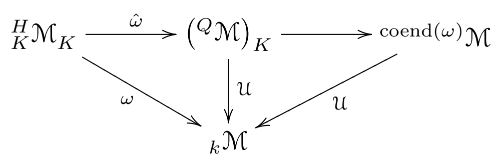
We note once more that condition (3) makes sense since condition (2) implies that H is a K-module coalgebra. We also note that the cocleftness assumption implies that ι is injective; we shall treat it as an inclusion.

There is an obvious left-right switched version of $\mathfrak{E}_\ell(K)$ which we denote by $\mathfrak{E}_r(K)$. It is defined so that $H \in \mathfrak{E}_\ell(K)$ means the same as $H^{\mathrm{bop}} \in \mathfrak{E}_r(K^{\mathrm{bop}})$.

An object $H \in \mathfrak{E}_\ell(K)$ is really a sextuple of data $(H, \Delta, \nabla, \phi, \iota, \pi)$; we shall always use these notations, and similar ones for other objects of $\mathfrak{E}_\ell(K)$, say $H' = (H', \Delta', \nabla', \phi', \iota', \pi')$. Also, we will always use the following conventions: We define the functor $\omega\colon {}^H_K\mathcal{M}_K \to {}_k\mathcal{M}$ by $\omega(M) := \overline{M} := M/K^+M$, writing

$\nu\colon M \to \overline{M}$ for the canonical surjection. For any $M \in {}^H_K\mathcal{M}$ we define $j\colon \overline{M} \to M$ by $j(\overline{m}) = \pi^{-1}(m_{(-1)})m_{(0)}$. We note that $M \ni m \mapsto \pi(m_{(-1)})\otimes\overline{m_{(0)}} \in K\otimes\overline{M}$ is an isomorphism of left K-modules for every $M \in {}^H_K\mathcal{M}$. We write $Q = H/K^+H$, a quotient coalgebra of H, endowed with a canonical right action of K defined by $\overline{h} \leftharpoonup x := \overline{h}x := \overline{hx}$. We endow the functor ω with the tensor functor structure $\xi\colon \overline{M}\otimes\overline{N} \to \overline{M \underset{K}{\otimes} N}$ given by $\xi(\overline{m}\otimes v) = \overline{m \otimes j(v)}$, with $\xi^{-1}(\overline{m \otimes n}) = \overline{m\pi(n_{(-1)})} \otimes \overline{n_{(0)}}$.

Our task in the rest of the section is to see that $\operatorname{coend}(\omega)$ exists, and to find its coquasibialgebra structure. To describe the coquasibialgebra structure, it is convenient to introduce an intermediate step in the general procedure described in Sections 2.2 and 2.3. We define [47, Def. 3.2.4] the category $\left({}^Q\mathcal{M}\right)_K$ to consist of left Q-comodules V equipped with a map $V \otimes K \ni v \otimes x \mapsto vx \in V$ satisfying $(vx)_{(-1)} \otimes (vx)_{(0)} = v_{(-1)}x_{(1)} \otimes v_{(0)}x_{(2)}$ and $(vx)y = \phi(v_{(-1)} \otimes x_{(1)} \otimes y_{(1)})v_{(0)}(x_{(2)}y_{(2)})$, along with $v1_K = v$, for all $v \in V$ and $x, y \in K$; the modified associativity condition makes sense since ϕ induces a well-defined map $\phi\colon Q \otimes H \otimes H \to k$ by [47, Lem.3.2.1]. Schneider's structure theorem for Hopf modules says that ω induces a category equivalence ${}^H_K\mathcal{M} \cong {}^Q\mathcal{M}$, and this extends [47, Prop. 3.2.5] to a category equivalence ${}^H_K\mathcal{M}_K \cong \left({}^Q\mathcal{M}\right)_K$. This splits the problem of lifting ω to an equivalence $\hat{\omega}\colon {}^H_K\mathcal{M}_K \cong {}^{\operatorname{coend}(\omega)}\mathcal{M}$ of monoidal categories into two parts:



where the two underlying functors, denoted $\mathcal{U}$, can be made into strict neutral tensor functors in such a way that the two triangles are commutative triangles of tensor functors: The monoidal category structure of $\left({}^Q\mathcal{M}\right)_K$ achieving this for the left hand triangle is obtained by transporting the structures of ${}^H_K\mathcal{M}_K$ through the equivalence $\hat{\omega}$. For example, we have to determine the action and coaction on the tensor product of $V, W \in \left({}^Q\mathcal{M}\right)_K$ in such a way that the isomorphism $\xi\colon \omega(M \underset{K}{\otimes} N) \to \omega(M) \otimes \omega(N)$ are compatible with these actions and coactions. This yields

$$(v \otimes w)x = \phi(j(v_{(-1)}) \otimes j(w_{(-1)})_{(1)} \otimes x_{(1)})v_{(0)}\pi(j(w_{(-1)})_{(2)}x_{(2)}) \otimes w_{(0)}x_{(3)}$$
$$(v \otimes w)_{(-1)} \otimes (v \otimes w)_{(0)} = \overline{j(v_{(-1)})j(w_{(-1)})_{(1)}} \otimes v_{(0)}\pi(j(w_{(-1)})_{(2)}) \otimes w_{(0)}$$

both of which can be determined from the equation

$$\begin{aligned}(v \otimes w)_{(-1)} \otimes (v \otimes w)_{(0)}x = \overline{j(v_{(-2)})j(w_{(-2)})_{(1)}}&\phi(v_{(-1)} \otimes j(w_{(-2)})_{(2)} \otimes x_{(1)}) \\ &\otimes v_{(0)}\pi(j(v_{(-2)})_{(3)})(v_{(-1)} \rightharpoonup x_{(2)}) \otimes w_{(0)}x_{(3)}\end{aligned}$$

in which we have written $q \rightharpoonup x = \pi(j(q)x)$ for $q \in Q$ and $x \in K$. The transport of structures also means that, since $\hat{\omega}$ has to be a monoidal functor, the diagrams

$$\begin{array}{ccccc} (\omega(L) \otimes \omega(M)) \otimes \omega(N) & \xrightarrow{\xi \otimes \mathrm{id}} & \omega(L \underset{K}{\otimes} M) \otimes \omega(N) & \xrightarrow{\xi} & \omega((L \underset{K}{\otimes} M) \underset{K}{\otimes} N) \\ \downarrow{\scriptstyle \tilde{\alpha}} & & & & \downarrow{\scriptstyle \omega(\alpha)} \\ \omega(L) \otimes (\omega(M) \otimes \omega(N)) & \xrightarrow{\mathrm{id} \otimes \xi} & \omega(L) \otimes \omega(M \underset{K}{\otimes} N) & \xrightarrow{\xi} & \omega(L \underset{K}{\otimes} (M \underset{K}{\otimes} N)) \end{array}$$

have to commute, which leads to the formula

$$\tilde{\alpha}(u \otimes v \otimes w) = \phi(j(u_{(-1)}) \otimes j(v_{(-1)})_{(1)} \otimes j(w_{(-1)})_{(1)}) \\ u_{(0)}\pi(j(v_{(-1)})_{(2)} j(w_{(-1)})_{(2)}) \otimes v_{(0)}\pi(j(w_{(-1)})_{(3)}) \otimes w_{(0)}$$

The category equivalence ${}^H_K\mathcal{M}_K \cong \left({}^Q\mathcal{M}\right)_K$ and the monoidal category structure of the latter category did not depend as yet on finiteness of K. The next step is to assume K finite, and to find a coalgebra $\tilde{H}$ with ${}^{\tilde{H}}\mathcal{M} \cong \left({}^Q\mathcal{M}\right)_K$. It is easy to understand how we may be led to the idea that $\tilde{H}$ can be modelled on $Q \otimes K^*$. After all, an object in $\left({}^Q\mathcal{M}\right)_K$ is determined by maps $V \to Q \otimes V$ and $V \to K^* \otimes V$, the latter corresponding to the action $V \otimes K \to V$ — but of course this is not a sufficient explanation; we shall sketch one now.

Define $\tilde{H} := Q \otimes K^*$, and define $\Delta\colon \tilde{H} \to \tilde{H} \otimes \tilde{H}$ by

$$\Delta(q \rtimes \varphi) = q_{(1)} \rtimes q_{(2)[-\tilde{1}]}\tilde{\tau}^{(1)}(q_{(3)})\varphi_{(1)} \otimes q_{(2)[\tilde{0}]} \rtimes \tilde{\tau}^{(2)}(q_{(3)})\varphi_{(2)},$$

where the map $\tilde{\rho}\colon Q \ni q \mapsto q_{[-\tilde{1}]} \otimes q_{[\tilde{0}]} \in K^* \otimes Q$ is defined to correspond to the right action $Q \otimes K \to Q$, and

$$\tilde{\tau}\colon Q \ni q \mapsto \tilde{\tau}^{(1)}(q) \otimes \tilde{\tau}^{(2)}(q) \in K^* \otimes K^*$$

is defined by $\tilde{\tau}(q)(x \otimes y) = \phi(j(q) \otimes x \otimes y)$. Define $\varepsilon\colon \tilde{H} \to k$ by $\varepsilon(q \rtimes \varphi) = \varepsilon(q)\varphi(1)$, $\tilde{\nu}\colon \tilde{H} \to Q$ by $\tilde{\nu}(q \rtimes \varphi) = q\varphi(1)$, and $\tilde{\pi} := \varepsilon \otimes K^*\colon \tilde{H} \to Q$.

We claim that $\tilde{H}$ is the object we are looking for, and a category equivalence between ${}^{\tilde{H}}\mathcal{M}$ and $\left({}^Q\mathcal{M}\right)_K$ is given in the 'obvious' way, that is, by the fact that a map $V \to \tilde{H} \otimes V$ induces maps $V \to Q \otimes V$ and $V \to K^* \otimes V$, the latter corresponding to a map $V \otimes K \to V$.

For the time being, we do not know whether $\tilde{H}$ is a coalgebra, but it is obvious that $\tilde{\nu}$ is a morphism of coalgebras, one of which is not necessarily coassociative. In [47] we have shown explicitly that $\tilde{H}$ is coassociative, and we will repeat essentially the same calculations in subsection 4.3 for a more general case. However, it seems worthwhile to indicate a trick that lets us get away without any explicit calculations with the cocycle ϕ. After all, the construction of $\tilde{H}$ has been intrinsic so far in only referring to the representation categories coming from H; we already used the defining axioms for a coquasibialgebra H by saying that ${}^H\mathcal{M}$ is a monoidal category, and since this characterizes coquasibialgebras, nothing else 'should' be needed. We will apply Lemma 2.3.1: $\tilde{\nu}$ being comultiplicative and counital, any $\tilde{H}$-comodule V (in the sense of Lemma 2.3.1) with comodule structure Λ is a Q-comodule via $\lambda = (\tilde{\nu} \otimes V)\Lambda$. In addition, the obvious left Q-comodule structure on $\tilde{H}$ coincides with $(\tilde{\nu} \otimes \tilde{H})\tilde{\Delta}$. Thus, applying $\tilde{\nu} \otimes \tilde{H} \otimes V$ to the coassociativity condition of a $\tilde{H}$-comodule V yields that the comodule structure map $\Lambda\colon V \to \tilde{H} \otimes V$ is left Q-colinear, hence can be written in the form $\Lambda(v) = v_{(-1)} \otimes \mu'(v_{(-0)})$ for some map $\mu'\colon V \to K^* \otimes V$, which is the same as a map $\mu\colon V \otimes K \to V$. Conversely, given a left Q-comodule V and a map $\mu\colon V \otimes K \to V$, let μ' denote the corresponding map $V \to K^* \otimes V$. Then we can define

$$\Lambda\colon V \ni v \mapsto v_{(-1)} \otimes \mu'(v_{(-0)}) \in \tilde{H} \otimes V.$$

To have a category equivalence $({}^Q\mathcal{M})_K \cong {}^{\tilde{H}}\mathcal{M}$ of the claimed form, we need to show that Λ is a comodule structure if and only if μ makes V an object of $({}^Q\mathcal{M})_K$. The condition $(\tilde{H} \otimes \Lambda)\Lambda(v) = (\tilde{\Delta} \otimes V)\Lambda(v) \in \tilde{H} \otimes \tilde{H} \otimes V$ needs only be verified after applying the maps $Q \otimes x^* \otimes Q \otimes y^* \otimes V\colon Q \otimes K^* \otimes Q \otimes K^* \otimes V \to Q \otimes Q \otimes V$ for all $x, y \in K$ (where $x^*\colon K^* \ni \varphi \mapsto \varphi(x) \in k$.) Now $(Q \otimes x^* \otimes Q \otimes y^* \otimes V)(\tilde{H} \otimes \Lambda)\Lambda(v) = v_{(-1)} \otimes (v_{(0)}x)_{(-1)} \otimes (v_{(0)}x)_{(0)}y$ and $(Q \otimes x^* \otimes Q \otimes y^*)\Delta(q \otimes \varphi) = q_{(1)} \otimes q_{(2)}x_{(1)}\phi(q_{(3)} \otimes x_{(2)} \otimes y_{(1)})\varphi(x_{(3)}y_{(2)})$ and hence $(Q \otimes x^* \otimes Q \otimes y^* \otimes V)(\tilde{\Delta} \otimes V)\Lambda(v) = v_{(-3)} \otimes v_{(-2)}x_{(1)} \otimes \phi(v_{(-1)} \otimes x_{(2)} \otimes y_{(1)})v_{(0)}(x_{(3)}y_{(2)})$ and thus V is a Λ-comodule if and only if $V \in ({}^Q\mathcal{M})_K$ (for the "only if" part apply once $\varepsilon \otimes \varepsilon \otimes V$, and once apply $\varepsilon \otimes Q \otimes V$ after specializing $y = 1$). This establishes the category equivalence $\left({}^Q\mathcal{M}\right)_K \cong {}^{\tilde{H}}\mathcal{M}$, where the latter is still in the sense of Lemma 2.3.1; we will use the notation $v \mapsto v_{(-\tilde{1})} \otimes v_{(\tilde{0})}$ for $\tilde{H}$-comodule structures below. To show that $\tilde{H}$ is, after all, coassociative, it remains to apply Lemma 2.3.1. We consider $H \in {}^H_K\mathcal{M}$, and the free K-K-bimodule $H \otimes K$ generated by the left K-module H in ${}^H\mathcal{M}$. The corresponding object in $({}^Q\mathcal{M})_K$ is $\overline{H \otimes K} \cong Q \otimes K$ with left comodule structure $(q \otimes x)_{(-1)} \otimes (q \otimes x)_{(0)} = q_{(1)}x_{(1)} \otimes q_{(2)} \otimes x_{(2)}$ and K-action $(q \otimes x)y = \phi(q_{(1)} \otimes x_{(1)} \otimes y_{(1)})q_{(2)} \otimes x_{(2)}y_{(2)}$. We claim that simply $q \otimes \varphi = (q \otimes 1)_{(-\tilde{1})}(\varepsilon \otimes \varphi)((q \otimes 1)_{(\tilde{0})})$, for which it suffices to check that

$$\begin{aligned}(Q \otimes x^*)(\tilde{H} \otimes \varepsilon \otimes \varphi)\Lambda(q \otimes 1) &= (Q \otimes \varepsilon \otimes \varphi)((q \otimes 1)_{(-1)} \otimes (q \otimes 1)_{(0)}x) \\ &= (Q \otimes \varepsilon \otimes \varphi)(q_{(1)} \otimes q_{(2)} \otimes x) = q\varphi(x) = (Q \otimes x^*)(q \otimes \varphi)\end{aligned}$$

for all $x \in K$.

This finishes the proof that $\tilde{H}$ is a coalgebra and we have a category equivalence ${}^{\tilde{H}}\mathcal{M} \cong \left({}^Q\mathcal{M}\right)_K$. Since we know the monoidal category structure on $\left({}^Q\mathcal{M}\right)_K$, we know it on ${}^{\tilde{H}}\mathcal{M}$, and can assemble the corresponding coquasibialgebra structure: Multiplication is given by

$$\begin{aligned}(p \otimes \varphi)(q \otimes \psi) &= \tilde{\nabla}(p \otimes \varphi \otimes q \otimes \psi) \\ &:= \overline{j(p_{(1)})j(q_{(1)})_{(1)}} \otimes \kappa(j(p_{(2)}) \otimes j(q_{(1)})_{(2)})((\varphi \triangleleft \pi(j(q_{(1)})_{(3)}))\leftharpoondown q_{(2)})\psi\end{aligned}$$

where $\kappa\colon H \otimes H \to K^*$ is given by $\kappa(g \otimes h)(x) = \phi(g \otimes h \otimes x)$, and $\leftharpoondown\colon K^* \otimes Q \to K^*$ is defined by $\langle \varphi \leftharpoondown q, x\rangle = \langle \varphi, q \rightharpoonup x\rangle$; for if we define $\tilde{\nabla}\colon \tilde{H} \otimes \tilde{H} \to \tilde{H}$ by this formula, then

$$(Q\otimes x^*)\tilde\nabla(p\otimes\varphi\otimes q\otimes\psi)=\overline{j(p_{(1)})j(q_{(1)})_{(1)}}\phi(j(p_{(2)})\otimes j(q_{(1)})_{(2)}\otimes x_{(1)})\\ \langle\varphi,\pi(j(q_{(1)})_{(3)})(q_{(2)}\rightharpoonup x_{(2)})\rangle\langle\psi,x_{(3)}\rangle$$

for $x\in K$, and we see immediately that

$$(Q\otimes x^*)\tilde\nabla(v_{(-\tilde 1)}\otimes w_{(-\tilde 1)})\otimes v_{(\tilde 0)}\otimes w_{(\tilde 0)}=(v\otimes w)_{(-1)}\otimes(v\otimes w)_{(0)}x\\ =(Q\otimes x^*)((v\otimes w)_{(-\tilde 1)})\otimes(v\otimes w)_{(\tilde 0)}$$

holds for all $v\in V\in{}^{\tilde H}\mathcal{M}$ and $w\in W\in{}^{\tilde H}\mathcal{M}$.

The coassociator $\tilde\phi$ is read off from the associator in $\left({}^Q\mathcal{M}\right)_K$ to be

$$\tilde\phi(p\otimes\varphi\otimes q\otimes\psi\otimes r\otimes\vartheta)=\\ \phi(j(p)\otimes j(q)_{(1)}\otimes j(r)_{(1)})\langle\varphi|\pi(j(q)_{(2)}j(r)_{(2)})\rangle\langle\psi|\pi(j(r)_{(3)})\rangle\varepsilon(\vartheta)$$

for all $p,q,r\in Q$, $\varphi,\psi,\vartheta\in K^*$.

It turns out that $\tilde H:=\mathfrak{F}(H)$ is naturally an object in $\mathfrak{E}_r(K^*)$. We will skip the details showing that $\mathfrak{F}$ can be made into a functor from $\mathfrak{E}_\ell(K)\to\mathfrak{E}_r(K^*)$. The idea is that a morphism $H\to H'$ in $\mathfrak{E}_\ell(K)$ leads to a functor ${}^H_K\mathcal{M}_K\to{}^{H'}_K\mathcal{M}_K$, which induces a coquasimorphism $\tilde H\to\tilde H'$ by reconstruction. We will denote the version of the functor $\mathfrak{F}\colon\mathfrak{E}_\ell(K)\to\mathfrak{E}_r(K^*)$ obtained by switching sides by $\mathfrak{F}^{\mathrm{bop}}\colon\mathfrak{E}_r(K)\to\mathfrak{E}_\ell(K^*)$.

4.2. First examples. The least complicated examples of the functor $\mathfrak{F}$ occur when we require both $H\in\mathfrak{E}_\ell(K)$ and $\tilde H\in\mathfrak{E}_r(K^*)$ to be ordinary bialgebras with trivial coassociators ϕ and $\tilde\phi$. When we inspect the formula for the coassociator of $\tilde H$ in subsection 4.1, we find (by specializing $q=1$) that we have to have $\pi(j(r)_{(1)})\otimes\pi(j(r)_{(2)})=\varepsilon(r)1\otimes 1$ for all $r\in Q$, from which it is easy to see that π is a coalgebra map. By specializing $\psi=\varepsilon$, we find $\pi(j(q)j(r))=\varepsilon(q)\varepsilon(r)$, from which one can deduce that j is multiplicative. This means that H is a trivalent product of K and Q as recalled in subsection 3.3, that is, comultiplication and multiplication in $H\cong K\otimes Q$ have to be given by

$$(x\otimes p)(y\otimes q)=x(p_{(1)}\rightharpoonup y_{(1)})\otimes(p_{(2)}\leftharpoonup y_{(2)})q,\\ \Delta(x\otimes q)=x_{(1)}\otimes q_{(1)[0]}\otimes x_{(2)}q_{(1)[1]}\otimes q_{(3)}$$

for some maps $\rightharpoonup\colon Q\otimes K\to K$, $\leftharpoonup\colon Q\otimes K\to Q$, and $\rho\colon Q\ni q\mapsto q_{[0]}\otimes q_{[1]}\in Q\otimes K$, which are necessarily given by $q\rightharpoonup x=\pi(j(q)x)$, $q\leftharpoonup x=\nu(j(q)x)$, and $\rho(q)=\nu(j(q)_{(1)})\otimes\pi(j(q)_{(2)})$. When we now inspect multiplication and comultiplication in $\tilde H$, we find that they are given by

$$(p\otimes\varphi)(q\otimes\psi)=p(\varphi\tilde\rightharpoonup q_{(1)})\otimes(\varphi_{(2)}\tilde\leftharpoonup q_{(2)})\psi,\\ \Delta(q\otimes\varphi)=q_{(1)}\otimes q_{(1)[-\tilde 1]}\varphi_{(1)}\otimes q_{(2)[\tilde 0]}\otimes\varphi_{(2)},$$

where $\tilde{\rightharpoonup}\colon K^*\otimes Q\to Q$, $\tilde{\leftharpoonup}\colon K^*\otimes Q\to Q$ and $\tilde{\rho}\colon Q\ni q\mapsto q_{[-\tilde{1}]}\otimes q_{[\tilde{0}]}\in K^*\otimes Q$ are induced in the obvious way by ρ, $\rightharpoonup$, and $\leftharpoonup$, respectively. In particular, $\tilde{H}$ is a trivalent product as well (on the other side), with the necessary actions and coactions obtained in essence by permuting the data that led to H. Note that we never had to check explicitly that the new data fulfill the axioms that they have to in order that a trivalent product can be constructed. This is not particularly hard to do, but we are still happy to have a new insight into why it works. Obviously applying $\mathfrak{F}^{\mathrm{bop}}$ to $\tilde{H}$ gives us H back, so we have established a bijection between trivalent products $K\otimes Q$ and trivalent products $Q\otimes K^*$. As a particular case we have recovered the well-known bijection between Singer pairs $(K,Q,\rho,\rightharpoonup)$ and matched pairs $(Q,K^*,\tilde{\rightharpoonup},\tilde{\leftharpoonup})$. This bijection is a major source for Singer pairs, since — as we already recalled briefly in subsection 3.3 — matched pairs arise naturally from groups composed from two subgroups; see [31] for examples. With a view towards our next application, we note a rare example where the bijection can be used in the other direction: If K is a finite Hopf algebra and $Q=K^{\mathrm{op}}$, then the tensor product Hopf algebra $K\otimes Q$ is obviously a bismash product extension of Q by K. We can endow it with the not so obvious cocleaving π defined by $\pi(x\otimes q)=xq$. This leads to $j(q)=S(q_{(1)})\otimes q_{(2)}$, which is multiplicative, while π is a coalgebra map. Hence we have written $K\otimes K^{\mathrm{op}}$ as a bismash product in a nontrivial way. As it turns out, the associated double crossproduct $K^{\mathrm{op}}\bowtie K^*$ is (a version of) the Drinfeld double of K.

Free with our approach to the bijection betwen Singer and matched pairs, we get an intrinsic connection between the bismash product $K\#Q$ constructed from a Singer pair (K,Q), and the double crossproduct $Q\bowtie K^*$ constructed from the associated matched pair (Q,K^*) in the shape of category equivalences

$$ {}^{K\#Q}_{K}\mathcal{M}_K\cong {}^{Q\bowtie K^*}\mathcal{M}\text{ and }{}_{K^*}\mathcal{M}^{Q\bowtie K^*}_{K^*}\cong\mathcal{M}^{K\#Q}. $$

Admittedly, it is not obvious what kind of information on the Hopf algebras might be drawn from this connection, but at least we can give a rather strange application to Drinfeld doubles via the center construction. Recall that the category of modules over the Drinfeld double $D(H)$ of a finite dimensional Hopf algebra H is equivalent to the category ${}^H_H\mathcal{YD}$ of (left-left) Yetter–Drinfeld modules, which is in turn equivalent to the center $\mathcal{Z}({}^H\mathcal{M})$ of the monoidal category ${}^H\mathcal{M}$, as well as to the center of the monoidal category ${}_H\mathcal{M}$. Since our application will not appear again elsewhere in this paper, we just refer to [23] for more details on these facts (with slightly different conventions), and proceed to do the following calculation, with the assumption that we are given a Singer pair (K,Q) with K finite:

$$ {}^{Q\bowtie K^*}_{Q\bowtie K^*}\mathcal{YD}\cong\mathcal{Z}\left({}^{Q\bowtie K^*}\mathcal{M}\right)\cong\mathcal{Z}\left({}^{K\#Q}_{K}\mathcal{M}_K\right)\cong\mathcal{Z}\left({}^{K\#Q}\mathcal{M}\right)\cong{}^{K\#Q}_{K\#Q}\mathcal{YD}. $$

The only nontrivial part is the third category equivalence in the chain. This is the fact that (under additional hypotheses of exactness of certain tensor products) for any algebra K in any monoidal category $\mathcal{C}$, the center of ${}_K\mathcal{C}_K$ is equivalent to the center of $\mathcal{C}$; the proof is quite complicated [46], but purely categorical and not at all specific to the Hopf algebra situation. As a consequence we get, if Q is also finite, an equivalence ${}_{D(Q\bowtie K^*)}\mathcal{M} \cong {}_{D(K\#Q)}\mathcal{M}$ of braided monoidal categories, and we conclude that the Drinfeld doubles $D(K\#Q)$ and $D(Q \bowtie K^*)$ are isomorphic up to a Drinfeld twist. Note that they should not be expected to be isomorphic, since Q is not (at least not in the obvious way) a subcoalgebra of $D(K\#Q)$, but Q is a Hopf subalgebra of $D(Q \bowtie K^*)$. A special case of the above occurs if the double crossproduct $A = Q \bowtie K^*$ is itself a Drinfeld double, which can be arranged with the associated bismash product $K \otimes K^{\mathrm{op}}$. Thus $D(A) \cong D(K \otimes K^{\mathrm{op}}) \cong D(K) \otimes D(K^{\mathrm{op}}) \cong A \otimes A$ up to a twist. This was stated more generally for factorizable quasitriangular A by Reshetikhin and Semenov–Tian–Shansky [39], see [51].

Trivalent products exhaust the examples where both H and $\mathfrak{F}(H)$ are ordinary bialgebras *with trivial coassociators* (while it may occur that $\mathfrak{F}(H)$ is a nontrivial coquasibialgebra, but also happens to be a bialgebra with the same multiplication, typically when $\mathfrak{F}(H)$ is cocommutative; we shall look at this case in more detail in section 5 below).

But of course one can apply $\mathfrak{F}$ to any inclusion $K \subset H$ of a Hopf algebra into a bialgebra with cocleaving $\pi\colon H \to K$. This situation yields a mapping system (3–1) in which Q is a coalgebra and nonassociative algebra, satisfying axioms that are very unpleasant even in the case [44] where π is a coalgebra map. One can characterize precisely [47, Sec.5.1] which mapping systems

$$K^* \underset{\tilde{\pi}}{\overset{\tilde{\iota}}{\rightleftarrows}} \tilde{H} \underset{j}{\overset{\tilde{\nu}}{\rightleftarrows}} Q$$

arise under $\mathfrak{F}$ from bialgebras H. What's more, one can show that the functor $\mathfrak{F}^{\mathrm{bop}}$ is an inverse mapping for $\mathfrak{F}$ as a map between the class of bialgebras in $\mathfrak{E}_\ell(K)$, and the corresponding subclass of $\mathfrak{E}_r(K^*)$.

In general the functor $\mathfrak{F}$ has a vague tendency to be an involution in this sense: When one lists all the combinatorial data involving only K and Q rather than the whole object $H \in \mathfrak{E}_\ell(K)$ (such data are the restrictions of ϕ to combinations of copies of K and Q in each argument, the mutual actions and coactions between K and Q, or the cocycles in a bicrossproduct), then it turns out that $\mathfrak{F}$ will look like an involution on all these data, acting on them essentially by reinterpreting maps involving K as maps involving K^* through duality. However, there is no reason to expect that the combinatorial data determines H completely (as opposed to the ordinary bialgebra case), and we know of no intrinsic arguments leading to a general isomorphism $\mathfrak{F}^{\mathrm{bop}}(\mathfrak{F}(H)) \cong H$. See [47, Sec. 4].

4.3. Remedies for infinite dimensional cases. The general construction reviewed in subsection 4.1 applies only when the subobject $K \subset H$ is finitely generated projective. In this section we shall investigate how much can be done in the case where K is infinite. We assume throughout this section that the base ring k is a field.

THEOREM 4.3.1. *Let K be a Hopf algebra, and $H \in \mathfrak{E}_\ell(K)$. Let ${}^H_K\underline{\mathcal{M}}_K$ be the full subcategory of ${}^H_K\mathcal{M}_K$ whose objects are finitely generated left K-modules. ${}^H_K\underline{\mathcal{M}}_K$ is a monoidal subcategory of ${}^H_K\mathcal{M}_K$. The functor $\omega\colon {}^H_K\underline{\mathcal{M}}_K \to {}_k\mathcal{M}$ with $\omega(M) = \overline{M} = M/K^+M$ factors over an equivalence $\hat{\omega}\colon {}^H_K\underline{\mathcal{M}}_K \to {}^{\tilde{H}}\mathcal{M}_f$, where $\tilde{H}$ is a coquasibialgebra that can be realized as a subspace of $Q\otimes K^*$ with coquasibialgebra structure determined by*

$$(Q\otimes x^*\otimes Q\otimes y^*)\Delta(q\otimes\varphi) = q_{(1)}\otimes q_{(2)} \leftharpoonup x_{(1)}\phi(q_{(3)}\otimes x_{(2)}\otimes y_{(1)})\langle\varphi, x_{(3)}y_{(2)}\rangle,$$

$$(p\otimes\varphi)(q\otimes\psi) = \overline{j(p_{(1)})j(q_{(1)})_{(1)}}\otimes\kappa(j(p_{(2)})\otimes j(q_{(1)})_{(2)})((\varphi\triangleleft\pi(j(q_{(1)})_{(3)}))\leftharpoondown q_{(2)})\psi,$$

and

$$\tilde{\phi}(p\otimes\varphi\otimes q\otimes\psi\otimes r\otimes\vartheta) = \phi(j(p)\otimes j(q)_{(1)}\otimes j(r)_{(1)})\langle\varphi|\pi(j(q)_{(2)}j(r)_{(2)})\rangle\langle\psi|\pi(j(r)_{(3)})\rangle\varepsilon(\vartheta)$$

for all $p,q,r\in Q$, $\varphi,\psi,\vartheta\in K^*$, and $x,y\in K$, where $x^*,y^*\colon K^*\to k$ correspond to $x,y\in K$.

PROOF. Although the tensor product in ${}^H_K\mathcal{M}_K$ is not an ordinary tensor product, it is still true that if $M,N\in{}^H_K\mathcal{M}_K$ are finitely generated as left K-modules, then so is $M\underset{K}{\otimes}N$: Clearly the image of $M\otimes V$ in $M\underset{K}{\otimes}N$ is finitely generated whenever V is a finite dimensional subspace of N. Now since N is finitely generated, we can choose a finite dimensional H-subcomodule of N generating N as left K-module. Then $M\underset{K}{\otimes}N$ is the image of $M\otimes V$, since for $m\in M$, $x\in K$ and $v\in V$ we have

$$m\otimes xv = \phi(m_{(-1)}\otimes x_{(1)}\otimes v_{(-1)})m_{(0)}x_{(2)}\otimes v_{(0)}$$

and V is a subcomodule.

With the aim of applying Lemma 2.3.2, we define $\tilde{H}^1 := Q\otimes K^*$, $\tilde{H}^2 := \operatorname{Hom}(K\otimes K, Q\otimes Q)$, and $\iota\colon Q\otimes K^*\otimes Q\otimes K^*\to\operatorname{Hom}(K\otimes K,Q\otimes Q)$ by

$$\iota(p\otimes\varphi\otimes q\otimes\psi)(x\otimes y) = p\otimes q\langle\varphi,x\rangle\langle\psi,y\rangle,$$

that is $\iota(F)(x\otimes y) = (Q\otimes x^*\otimes Q\otimes y^*)(F)$. Further, we define $\Delta^1\colon\tilde{H}^1\to\tilde{H}^2$ by

$$\Delta^1(q\otimes\varphi)(x\otimes y) = q_{(1)}\otimes q_{(2)}\leftharpoonup x_{(1)}\phi(q_{(3)}\otimes x_{(2)}\otimes y_{(1)})\langle\varphi,x_{(3)}y_{(2)}\rangle,$$

and $\varepsilon^1(q\otimes\varphi) = \varepsilon(q)\varphi(1)$.

Now ${}^H_K\underline{\mathcal{M}}_K$ is equivalent to the full subcategory $\left({}^Q\mathcal{M}_f\right)_K$ of $\left({}^Q\mathcal{M}\right)_K$ consisting of finite-dimensional objects. We have an equivalence $\left({}^Q\mathcal{M}_f\right)_K\cong{}^{\tilde{H}^1}\mathcal{M}_f$, where the latter category is to be understood in the sense of Lemma 2.3.2: This can be proved in essentially the same manner as in subsection 4.1. In fact the form of comultiplication we have given here was already used in the proof there. The only difference is that this time the action $V\otimes K\to V$ on $V\in\left({}^Q\mathcal{M}\right)_K$ gives rise to $V\to K^*\otimes V$ since V is finite-dimensional. Now we can apply Lemma 2.3.2. The formulas for multiplication and coassociator are proved as in subsection 4.1. □

The inclusion functor ${}^K_K\mathcal{M}_K\to{}^H_K\mathcal{M}_K$ restricts to those Hopf modules that are finitely generated left K-modules to give an inclusion ${}_f\mathcal{M}_K\to{}^{\tilde{H}}\mathcal{M}_f$ from finite dimensional K-modules to finite dimensional $\tilde{H}$-comodules. Hence we have a natural map $K^\circ\to\tilde{H}$, which is easily seen to have the obvious form when we consider $\tilde{H}\subset Q\otimes K^*$. We also have an obvious coalgebra map $\tilde{\nu}\colon\tilde{H}\to Q$, and a natural map $\tilde{\pi}\colon H\to K^*$ coming from the natural transformations $\omega\to\omega$ that arise from right multiplication with an element of K.

However, we do not know if $\tilde{H}$ contains all of $Q\otimes K^\circ$, nor if it is contained in $Q\otimes K^\circ$, nor if $\tilde{\pi}$ takes values in K°. When H is an ordinary bialgebra with trivial coquasibialgebra structure ϕ, then all of the $\omega(M)$ for $M\in{}^H_K\underline{\mathcal{M}}_K$ are ordinary right K-modules, of finite dimension, so all elements of $\tilde{H}$ are contained in $Q\otimes K^\circ$, and $\tilde{\pi}$ takes values in K°. The same conclusion can be obtained under the weaker assumption that the map $\tilde{\tau}\colon H\to(K\otimes K)^*$ induced by ϕ takes values in $K^\circ\otimes K^\circ$. We then write

$$\tilde{\tau}(q):=\tilde{\tau}^{(1)}(q)\otimes\tilde{\tau}^{(2)}(q)\in K^\circ\otimes K^\circ$$

and denote the convolution inverse of $\tilde{\tau}$ by

$$\tilde{\tau}^{-1}(q):=\tilde{\tau}^{-(1)}(q)\otimes\tilde{\tau}^{-(2)}(q).$$

To show that $v_{[-\tilde{1}]}\otimes v_{[\tilde{0}]}\in K^\circ\otimes V$ for all $v\in V\in\left({}^Q\mathcal{M}_f\right)_K$ we calculate

$$\begin{aligned}\langle v_{[-\tilde{1}]},xy\rangle v_{[\tilde{0}]}&=v(xy)=\phi^{-1}(v_{(-1)}\otimes x_{(1)}\otimes y_{(1)})(v_{(0)}x_{(2)})y_{(2)}\\&=\phi^{-1}(v_{(-1)}\otimes x_{(1)}\otimes y_{(1)})\langle v_{(0)[-\tilde{1}]},x_{(2)}\rangle\langle v_{(0)[\tilde{0}][-\tilde{1}]},y_{(2)}\rangle v_{(0)[\tilde{0}][\tilde{0}]}\\&=\langle\tilde{\tau}^{-(1)}(v_{(-1)})v_{(0)[-\tilde{1}]},x\rangle\langle\tilde{\tau}^{-(2)}(v_{(-1)})v_{(0)[\tilde{0}][-\tilde{1}]},y\rangle v_{(0)[\tilde{0}][\tilde{0}]}\end{aligned}$$

for $x,y\in K$, showing $\Delta(v_{[-\tilde{1}]})\otimes v_{[\tilde{0}]}\in K^*\otimes K^*\otimes V$.

To show that, under additional hypotheses, $\tilde{H}$ contains all of $Q\otimes K^\circ$, we need a sufficient supply of objects in $\left({}^Q\mathcal{M}_f\right)_K$. This time, unfortunately, we cannot get away without a calculation involving cocycle identities. It is unfortunate that there appears to be no intrinsic reason to expect the object $H\otimes K^*$ to be defined below to be in ${}^H_K\mathcal{M}_K$ (as opposed to the object $H\otimes K$ used in subsection 4.1.) If K is finite, the object $\overline{H\otimes K^*}$ is just the regular left $\tilde{H}$-comodule $\tilde{H}$.

Define $\beta\colon H\otimes K\to K^\circ$ by $\beta(h\otimes x)(y)=\phi(h\otimes x\otimes y)$ for $h\in H$ and $x,y\in K$. We note that $\beta(xh\otimes y)=\varepsilon(x)\beta(h\otimes y)$, a reformulation of [47, Lem. 3.2.1]. Moreover

$$\beta(h_{(1)}x_{(1)}\otimes y_{(1)})(\beta(h_{(2)}\otimes x_{(2)})\triangleleft y_{(2)})=\phi(h_{(1)}\otimes x_{(1)}\otimes y_{(1)})\beta(h_{(2)}\otimes x_{(2)}y_{(2)})$$

as a special case of the cocycle identity of ϕ, taking $\phi|_{K\otimes K\otimes K} = \varepsilon$ into account. It follows that $H\otimes K^* \in {}^H_K\mathcal{M}_K$ with the left H-comodule and K-module structures induced by those of the left tensor factor, and the right K-action defined by $(h\otimes\varphi)x = h_{(1)}x_{(1)} \otimes \beta(h_{(2)}\otimes x_{(2)})(\varphi \triangleleft x_{(3)})$. In fact

$$\begin{aligned}\phi(h_{(1)}\otimes x_{(1)}\otimes y_{(1)})&(h_{(2)}\otimes\varphi)(x_{(2)}y_{(2)})\\ &= \phi(h_{(1)}\otimes x_{(1)}\otimes y_{(1)})h_{(2)}(x_{(2)}y_{(2)})\otimes\beta(h_{(3)}\otimes x_{(3)}y_{(3)})(\varphi\triangleleft x_{(4)}y_{(4)})\\ &= (h_{(1)}x_{(1)})y_{(1)}\otimes\phi(h_{(2)}\otimes x_{(2)}\otimes y_{(2)})\beta(h_{(3)}\otimes x_{(3)}y_{(3)})(\varphi\triangleleft x_{(4)}y_{(4)})\\ &= (h_{(1)}x_{(1)})y_{(1)}\otimes\beta(h_{(2)}x_{(2)}\otimes y_{(2)})(\beta(h_{(3)}\otimes x_{(3)})\triangleleft y_{(3)})(\varphi\triangleleft x_{(4)}y_{(4)})\\ &= (h_{(1)}x_{(1)})y_{(1)}\otimes\beta(h_{(2)}x_{(2)}\otimes y_{(2)})((\beta(h_{(3)}\otimes x_{(3)})(\varphi\triangleleft x_{(4)}))\triangleleft y_{(3)})\\ &= (h_{(1)}x_{(1)}\otimes\beta(h_{(2)}\otimes x_{(2)})(\varphi\triangleleft x_{(3)}))y = ((h\otimes\varphi)x)y\end{aligned}$$

and

$$\begin{aligned}(xh\otimes\varphi)y &= x_{(1)}h_{(1)}y_{(1)}\otimes\beta(x_{(2)}h_{(2)}\otimes y_{(2)})(\varphi\triangleleft y_{(3)})\\ &= xh_{(1)}y_{(1)}\otimes\beta(h_{(2)}\otimes y_{(2)})(\varphi\triangleleft y_{(3)}) = x((h\otimes\varphi)y)\end{aligned}$$

for all $x, y \in K$, $h \in H$, and $\varphi \in K^*$, while $H\otimes K^* \in {}^H_K\mathcal{M}$ holds trivially.

THEOREM 4.3.2. *Let k be a field, K a Hopf algebra, and $H \in \mathfrak{E}_\ell(K)$. Assume that H is an ordinary right K-module (for example, ϕ is trivial or H is cocommutative), and that the action of K on Q is locally finite. Assume that the map $\tilde{\tau}\colon Q \to (K\otimes K)^*$ defined by ϕ takes values in $K^\circ\otimes K^\circ$.*

Then $\tilde{H} = Q\otimes K^\circ \in \mathfrak{E}_r(K^\circ)$.

PROOF. We have already shown $\tilde{H} \subset Q\otimes K^\circ$. Since $\beta\colon Q\otimes K \to K^*$ takes values in K°, and $K^\circ \subset K^*$ is stable under the right action of K, we see that $H\otimes K^\circ \subset H\otimes K^*$ is a subobject in ${}^H_K\mathcal{M}_K$. We consider the object $\omega(H\otimes K^*) \cong Q\otimes K^* \in \left({}^Q\mathcal{M}\right)_K$. We claim that $Q\otimes K^\circ$ is the union of its finite dimensional subobjects. In fact by local finiteness of the K-module Q, any $q \in Q$ is contained in a finite dimensional subcoalgebra $C \subset Q$ that is in addition stable under the action of K. The image of $\beta|_{C\otimes K}$ is contained in some finite dimensional subspace of K°, and for any $\varphi \in K^\circ$ we know that $\varphi\triangleleft K$ is finite dimensional. We conclude that $(q\otimes\varphi)K$ is finite dimensional, and of course a subobject of $Q\otimes K^\circ$ in $\left({}^Q\mathcal{M}\right)_K$. We conclude the proof by observing that $q\otimes\varphi = (q\otimes\varphi)_{(-\tilde{1})}(\varepsilon\otimes\eta^*)((q\otimes\varphi)_{(\tilde{0})})$. □

4.4. Hopf algebra inclusions and antipodes. So far, we have never treated the question when the coquasibialgebra associated to an inclusion $K \subset H$ is a coquasi-Hopf algebra. We shall turn to this question now, but only derive some criteria for the case of an inclusion of Hopf algebras.

If $K \subset H$ is an inclusion of Hopf algebras with bijective antipodes, and $\pi\colon H \to K$ is a cocleaving map for the left K-module coalgebra H, then it is straightforward to check that $\pi' = S_K\pi S_H^{-1}\colon H \to K$ is a cocleaving for the right

K-module coalgebra H, so we can simply say below that H is K-cocleft in this case.

Obviously a cocleaving map for the left K-module coalgebra H is also a cocleaving map for the right K^{bop}-module coalgebra H^{bop}, so that H is K-cocleft if and only if H^{bop} is K^{bop}-cocleft, and H^{op} is K^{op}-cocleft if and only if H^{cop} is K^{cop}-cocleft.

For the rest of the section we assume that $K \subset H$ is an inclusion of Hopf algebras with bijective antipodes and H is K-cocleft, that is, $H \in \mathfrak{C}_\ell(K)$.

To see whether $\tilde{H}$ has a coquasi-antipode, we shall first investigate dual objects in the Hopf module category ${}_K^H\mathcal{M}_K$.

If $M \in {}_K^H\mathcal{M}_K$ has a right (resp. left) dual, then, since monoidal functors preserve duals, M is necessary finitely generated projective as left (resp. right) K-module, and the underlying K-K-bimodule of the right (resp. left) dual of M is ${}^\vee M = \mathrm{Hom}_{K-}(M, K)$ (resp. $M^\vee = \mathrm{Hom}_{-K}(M, K)$.) On the other hand, when M is finitely generated projective as left (resp. right) K-module, Ulbrich [60] has given a comodule structure on $\mathrm{Hom}_{K-}(M, N)$ (resp. $\mathrm{Hom}_{-K}(M, N)$) for every Hopf module $N \in {}_K^H\mathcal{M}_K$ that defines an inner hom-functor in the category ${}_K^H\mathcal{M}_K$. The comodule structure is defined by

$$f_{(-1)} \otimes f_{(0)}(m) = S(m_{(-1)})f(m_{(0)})_{(-1)} \otimes f(m_{(0)})_{(0)}$$

for $f \in \mathrm{Hom}_{K-}(M, N)$, and

$$f_{(-1)} \otimes f_{(0)}(m) = f(m_{(0)})_{(-1)}S^{-1}(m_{(-1)}) \otimes f(m_{(0)})_{(0)}$$

for $f \in \mathrm{Hom}_{-K}(M, N)$. (More generally, if k is a field, then a certain submodule of $\mathrm{Hom}_{K-}(M, N)$ is is an inner hom-functor if M is not finitely generated. In fact, from the results in [36] cited in subsection 2.5, we can expect ${}_K^H\mathcal{M}_K$ to be closed even if H is only a coquasi-Hopf algebra, since then ${}^H\mathcal{M}$ is closed; however, the inner hom-functors have a complicated description which we have not been able to use.) Since the canonical maps ${}^\vee M \otimes N \to \mathrm{Hom}_{K-}(M, N)$ are bijective if ${}_K M$ is finitely generated projective, it follows that ${}^\vee M$ is a right dual of M in the category ${}_K^H\mathcal{M}_K$. Similarly $M^\vee$ is a left dual of M if M is finitely generated projective as right K-module. All these considerations do not give us any left or right rigid subcategory of ${}_K^H\mathcal{M}_K$, since, for example, when M is left finitely generated projective, the same needs not hold for ${}^\vee M$. We do get two subcategories of ${}_K^H\mathcal{M}_K$ that are in full duality with each other by taking left, resp. right duals.

Thus, we are interested in cases where the two finiteness conditions on a Hopf module in ${}_K^H\mathcal{M}_K$ coincide. In fact only finite generation will be an issue: If k is a field, then by our general cocleftness assumption every Hopf module in ${}_K^H\mathcal{M}$ is a free K-module. In addition every Hopf module in ${}^H\mathcal{M}_K$ is a projective K-module by [48, Rem. 4.3]. In Theorem 4.4.3 we shall assume moreover that H^{cop} is K^{cop}-cocleft, so that every Hopf module in ${}^H\mathcal{M}_K \cong {}_{K^{\mathrm{op}}}^{H^{\mathrm{op}}}\mathcal{M}$ will even be a free K-module by more elementary reasons. The key to dealing with finite generation is the following simple observation, which follows immediately from the fact that Hopf modules in ${}^H_K\mathcal{M}$ are free K-modules:

Remark 4.4.1. Let k be a field and K a Hopf algebra, and H a cocleft left K-module coalgebra. Then $M \in {}^H_K\mathcal{M}$ is a finitely generated K-module if and only if M/K^+M is finite dimensional. If this is the case, then every Hopf submodule of M is also a finitely generated K-module.

For the proof of Theorem 4.4.3 it is convenient to provide

Lemma 4.4.2. *Let k be a field and $H \in \mathfrak{C}_\ell(K)$, where H is a bialgebra, and K is a Hopf algebra. If $Q := H/K^+H$ is a locally finite right K-module, then H is the union of those of its K-K-bimodule subcoalgebras that are finitely generated as left K-modules.*

Proof. Let $h \in H$, and choose a finite dimensional subcoalgebra $E \subset H$ with $h \in E$. Put $D := KEK \ni h$, which is by construction a K-K-bimodule subcoalgebra of H, and in particular a subobject of H in the Hopf module category ${}^H_K\mathcal{M}_K$. By local finiteness $\nu(D) = \nu(E)K \subset Q$ is finite dimensional, hence D is a finitely generated left K-module by the preceding remark. □

Theorem 4.4.3. *Let k be a field, and $H \in \mathfrak{C}_\ell(K)$, where H and K are Hopf algebras with bijective antipodes.*

Assume that H^{cop} is cocleft as a K^{cop}-module coalgebra, and $Q := H/K^+H$ is a locally finite right K-module.

Then the category ${}^H_K\underline{\mathcal{M}}_K$ is left and right rigid.

Proof. The cocleftness assumption on H^{cop} is equivalent to $H^{\mathrm{op}} \in \mathfrak{C}_\ell(K^{\mathrm{op}})$ (we have used $H^{\mathrm{cop}} \in \mathfrak{C}_\ell(K^{\mathrm{cop}})$ for the statement of the theorem since H will be cocommutative in our application in subsection 4.5). Using the bijective antipodes one sees that H/HK^+ is a locally finite left K-module, or stated differently $H^{\mathrm{op}}/(K^{\mathrm{op}})^+H^{\mathrm{op}}$ is a locally finite right K^{op}-module. In other words, the inclusion $K^{\mathrm{op}} \subset H^{\mathrm{op}}$ satisfies the same hypotheses as $K \subset H$.

We will show that these hypotheses imply that $M \in {}^H_K\mathcal{M}_K$ is a finitely generated left K-module if and only if it is a finitely generated right K-module, which is all we need by the discussion preceding Remark 4.4.1.

Due to the identification ${}^H_K\mathcal{M}_K \cong {}^{H^{\mathrm{op}}}_{K^{\mathrm{op}}}\mathcal{M}_{K^{\mathrm{op}}}$ and the remarks at the beginning of the proof we need only show one of the implications. So let $M \in {}^H_K\mathcal{M}_K$ be finitely generated as left K-module. We can apply Remark 4.4.1 to the left K^{op}-module structure of M and need only show that M can be embedded in a larger Hopf module $N \in {}^H\mathcal{M}_K$ such that N is a finitely generated right K-module. To construct N, let C be a finite dimensional subcoalgebra of Q such that the Q-comodule $\overline{M}$ is a C-comodule. Then $m_{(-1)} \otimes \overline{m_{(0)}} = \pi(m_{(-2)})j(\overline{m_{(-1)}}) \otimes \overline{m_{(0)}} \in Kj(C) \otimes \overline{M}$ for all $m \in M$. We can apply Lemma 4.4.2 to the left K^{op}-module coalgebra H^{op} to conclude that $j(C)$ and hence $Kj(C)$ is contained in a K-K-bimodule subcoalgebra $D \subset H$ that is finitely generated as right K-module, say by a finite-dimensional subspace $D' \subset D$. We have $M \cong H\underset{Q}{\square}\overline{M} \cong D\underset{Q}{\square}\overline{M} \subset {}^{\cdot}D_{\cdot} \otimes \overline{M}_{\cdot}$ as Hopf modules in ${}^H\mathcal{M}_K$, and $N := D \otimes \overline{M}$ is generated as right K-module by $D' \otimes \overline{M}$, since for all $d \in D'$, $x \in K$, and $v \in \overline{M}$ we have $dx \otimes v = dx_{(1)} \otimes vS^{-1}(x_{(3)})x_{(2)} = (d \otimes vS^{-1}(x_{(2)}))x_{(1)} \in (D' \otimes \overline{M})K$. □

Remarks 4.4.4. (1) If H is finite, and a Hopf algebra, then the conditions in Theorem 4.4.3 are always satisfied.

(2) Under the assumptions of Theorem 4.4.3, $\tilde{H} \cong Q \otimes K^\circ$ is a cosmash product as a coalgebra.

(3) If k is a commutative base ring, but we assume that K is finitely generated projective, and H^{cop} is cocleft as K^{cop}-module coalgebra, then we can also conclude that ${}^H_K\underline{\mathcal{M}}_K$ is left and right rigid, because now every Hopf module $M \in {}^H_K\mathcal{M}_K$ can be written as $K \otimes (M/K^+M)$ as well as $(M/MK^+) \otimes K$; thus if M is finite projective as left K-module, then M/K^+M is finitely generated projective over k, hence so is M, hence M/MK^+ is finitely generated projective over k, and M finitely generated projective as right K-module.

Now for $\tilde{H}$ to have a coquasi-antipode, we not only need ${}^H_K\underline{\mathcal{M}}_K$ to have duals, but we also need the underlying functor ${}^{\tilde{H}}\mathcal{M} \to {}_k\mathcal{M}$, hence the functor $\omega\colon {}^H_K\mathcal{M}_K \to {}_k\mathcal{M}$, to preserve these duals.

Theorem 4.4.5. *Let $H \in \mathfrak{E}_\ell(K)$, where H and K are Hopf algebras with bijective antipodes. Assume that K and H are finitely generated projective, or that k is a field.*

If there is an isomorphism $H \cong K_{\cdot} \otimes Q^{\cdot}_{\cdot}$ of right Q-comodules and K-modules, then $\tilde{H}$ is a quasi-Hopf algebra.

Proof. The isomorphism $H \cong K_{\cdot} \otimes Q^{\cdot}_{\cdot}$ yields an isomorphism of right K-modules, natural in $M \in {}^H_K\mathcal{M}_K$,

$$M \cong H\underset{Q}{\square}\overline{M} \cong (K_{\cdot} \otimes Q^{\cdot}_{\cdot})\underset{Q}{\square}\overline{M}_{\cdot} \cong K_{\cdot} \otimes \overline{M}_{\cdot} \cong \overline{M} \otimes K.$$

(where the last isomorphism maps $x \otimes v \mapsto vS^{-1}(x_{(2)}) \otimes x_{(1)}$). In particular, $M \in {}^H_K\underline{\mathcal{M}}_K$ is a finitely generated left K-module, iff it is a finitely generated projective right K-module, iff $\overline{M}$ is a finitely generated projective k-module. In particular, ${}^H_K\underline{\mathcal{M}}_K \cong {}^{\tilde{H}}\mathcal{M}_f$ is left and right rigid. Moreover, one has a natural isomorphism $\operatorname{Hom}_{K-}(M,K) \cong \operatorname{Hom}_{K-}(K \otimes \overline{M}, K) \cong \overline{M}^* \otimes K$ of right K-modules, and thus a natural isomorphism $\omega(\operatorname{Hom}_{K-}(M,K)) \cong \operatorname{Hom}_{K-}(M,K)/\operatorname{Hom}_{K-}(M,K)K^+ \cong \overline{M}^*$ of k-modules. Thus, the underlying functor ${}^{\tilde{H}}\mathcal{M}_f \to {}_k\mathcal{M}$ preserves duals, and $\tilde{H}$ is a quasi-Hopf algebra. □

The criterion given in Theorem 4.4.5 is sharp whenever K is finite:

Theorem 4.4.6. *Let k be a field, K a finite dimensional Hopf algebra, and $H \in \mathfrak{E}_\ell(K)$ a Hopf algebra with bijective antipode. If $\tilde{H}$ is a quasi-Hopf algebra, then there is an isomorphism $H \cong K_{\cdot} \otimes Q^{\cdot}_{\cdot}$ of right Q-comodules and K-modules.*

Proof. By assumption the functor $\omega\colon {}^H_K\underline{\mathcal{M}}_K \to {}_k\mathcal{M}$ preserves duals. Now for any finite dimensional $M \in {}^H_K\mathcal{M}_K$ the right dual in ${}^H_K\mathcal{M}_K$ is

$$^\vee M = \operatorname{Hom}_{K-}(M, K) \cong \overline{M}^* \otimes K$$

as right K-modules, and hence we have a natural isomorphism $\omega(^\vee M) \cong \overline{M}^* \cong {}^\vee M/(^\vee M)K^+$ for $M \in {}^H_K\underline{\mathcal{M}}_K$. Since $^\vee(-)$ is a full duality, we have a natural isomorphism $M/K^+M \cong M/MK^+$ for $M \in {}^H_K\underline{\mathcal{M}}_K$, which specializes for $M = H \otimes K$ to an isomorphism $f\colon Q \otimes K \to H$. For $x \in K$ an automorphism of $H \otimes K$ in ${}^H_K\mathcal{M}_K$ is given by $h \otimes y \mapsto hx_{(2)} \otimes S^{-1}(x_{(1)})y$. Applying naturality of f to this automorphism yields $f(qx_{(2)} \otimes S^{-1}(x_{(1)})y) = f(q \otimes y)x$. For any $\gamma \in Q^*$, an automorphism of $H \otimes K$ in ${}^H_K\mathcal{M}_K$ is given by $h \otimes x \mapsto h_{(1)}\gamma(\overline{h_{(2)}}) \otimes x$. Applying naturality of f yields $f(q_{(1)}\gamma(q_{(2)}) \otimes y) = f(q \otimes y)_{(1)} \otimes \overline{f(q \otimes y)_{(2)}}$. From this we conclude that $K \otimes Q \ni y \otimes q \mapsto f(q \otimes S(y)) \in H$ is a Q-comodule and K-module map as required. □

The following Lemma gives a sufficient condition to have an isomorphism as needed in Theorem 4.4.5. It is designed to apply to the examples in subsection 4.5 below, and asks essentially for H to be right K-cocleft in a very particular way.

Lemma 4.4.7. *Let $K \subset H$ be an inclusion of Hopf algebras with bijective antipodes, such that H is cleft as a left K-module coalgebra.*

Denote by $\underline{h}$ the class in H/HK^+ of $h \in H$. Assume that there exists a coalgebra section $\gamma\colon H/HK^+ \to H$ for the canonical surjection, such that

$$\vartheta\colon H/HK^+ \otimes K \ni \underline{h} \otimes x \mapsto \gamma(\underline{h})x \in H$$

and $\nu\gamma\colon H/HK^+ \to Q$ are bijections.

Then there is an isomorphism $H \cong K_{.} \otimes Q^{\cdot}_{.}$ of right K-modules and right Q-comodules.

Proof. Being a coalgebra map, γ is convolution invertible with inverse $\gamma^{-1} = S\gamma$.

Since γ is a coalgebra map, ϑ and hence its inverse are left H/HK^+-comodule maps, and thus ϑ^{-1} can be written in the form $\vartheta^{-1}(h) = \underline{h_{(1)}} \otimes \kappa(h_{(2)})$ for some right K-module map $\kappa\colon H \to K$. By definition $\vartheta(\underline{h_{(1)}})\kappa(h_{(2)}) = h$ for all $h \in H$, hence $\kappa(h) = \vartheta^{-1}(\underline{h_{(1)}})h_{(2)}$. Since $\Delta\gamma^{-1}(h) = \gamma^{-1}(\underline{h_{(2)}}) \otimes \gamma^{-1}(\underline{h_{(1)}})$, we have $\Delta(\kappa(h)) = \vartheta^{-1}(\underline{h_{(2)}})h_{(3)} \otimes \vartheta^{-1}(\underline{h_{(1)}})h_{(4)} = \kappa(h_{(2)}) \otimes \vartheta^{-1}(\underline{h_{(1)}})h_{(3)}$. Now we claim that the (bijective) map

$$\ell\colon H \ni h \mapsto \nu(\gamma(\underline{h_{(1)}})) \otimes \kappa(h_{(2)}) \in Q^{\cdot} \otimes K^{\cdot}_{.}$$

is a map of right Q-comodules and K-modules as indicated. From this, the claim follows since

$$Q^{\cdot} \otimes K^{\cdot}_{.} \ni q \otimes x \mapsto x_{(1)} \otimes qx_{(2)} \in K_{.} \otimes Q^{\cdot}_{.}$$

is an isomorphism of right Q-comodules and K-modules. Now ℓ is obviously bijective (since $\nu\gamma$ is), and right K-linear. Moreover, the right comodule structure on $Q \otimes K$ gives

$$\begin{aligned}\ell(h)_{(0)} \otimes \ell(h)_{(1)} &= (\nu\gamma(\underline{h_{(1)}}) \otimes \kappa(h_{(2)}))_{(0)} \otimes (\nu\gamma(\underline{h_{(1)}}) \otimes \kappa(h_{(2)}))_{(1)} \\ &= \nu(\gamma(\underline{h_{(1)}}))_{(1)} \otimes \kappa(h_{(2)})_{(1)} \otimes \nu(\gamma(\underline{h_{(1)}}))_{(2)}\kappa(h_{(2)})_{(2)} \\ &= \nu(\gamma(\underline{h_{(1)}})) \otimes \kappa(h_{(4)}) \otimes \nu(\gamma(\underline{h_{(2)}})\gamma(\underline{h_{(3)}})h_{(5)}) \\ &= \nu(\gamma(\underline{h_{(1)}})) \otimes \kappa(h_{(2)}) \otimes \nu(h_{(3)}) = \ell(h_{(1)}) \otimes \overline{h_{(2)}}\end{aligned}$$

so that ℓ is Q-colinear. □

4.5. Yongchang Zhu's example. A special case of our construction assigning a coquasibialgebra to any inclusion of Hopf algebras was previously obtained by Yongchang Zhu [61] who constructed a quasi-Hopf algebra $A(G, B)$ associated to any inclusion $B \subset G$ of finite groups. Some indications are also given for the case of an inclusion of infinite groups. The motivation in [61] is a close relation between the category of representations of $A(G, B)$ and the Hecke algebra $H(G//B)$ associated to the inclusion. In this section we shall give a brief overview how [61] relates to the construction in [47] cited above.

Plainly, if $B \subset G$ is an inclusion of finite groups, we can consider $k[G]$ as an object in $\mathfrak{E}_\ell(k[B])$ once we choose a cocleaving map $\pi\colon k[G] \to k[B]$. To have a result that compares easily to [61], we do this in the following rather roundabout manner: Choose a set R of representatives for the set G/B of right cosets. Then every $g \in G$ can be written uniquely in the form $g = [g]\{g\}^{-1}$ for $[g] \in R$ and $\{g\} \in B$. We choose the cocleaving $\pi\colon k[G] \to k[B]$ with $\pi(g) = \{g^{-1}\}$ and find $g = \{g^{-1}\}[g^{-1}]^{-1}$, hence $j(\overline{g}) = [g^{-1}]^{-1}$ for the associated map $j\colon Q = k[G]/k[B]^+k[G] \to k[G]$. We identify Q with $k[R]$ by $\overline{g} = [g^{-1}]$, so that we have $j(r) = r^{-1}$ for $r \in R$. We denote by $\hat{b}$ for $b \in B$ the elements of the basis of k^B dual to the basis B of $k[B]$. The coquasibialgebra $\tilde{H}$ from subsection 4.1 identifies with $k[R] \otimes k^B$, with canonical basis consisting of the $r \otimes \hat{b}$ with $r \in R$ and $b \in B$, and it remains to specialize the formulas for the coquasibialgebra structure. The right action of $k[B]$ on Q translates as $r \leftharpoondown b = \overline{r^{-1}b} = [b^{-1}r]$. The action $Q \otimes k[B] \to k[B]$ translates as $r \rightharpoonup b = \pi(j(r)b) = \{b^{-1}r\}$. Then the coaction of k^B on $k[R]$ is given by $r_{[-\tilde{1}]} \otimes r_{[\tilde{0}]} = \sum_{b \in B} \hat{b} \otimes [b^{-1}r]$, and the dualized action of $k[R]$ on k^B satisfies $(\hat{b} \mathbin{\tilde{\leftharpoondown}} r)\hat{c} = (\hat{b} \mathbin{\tilde{\leftharpoondown}} r)(c)\hat{c} = \hat{b}(\{c^{-1}r\})\hat{c} = \delta_{b,\{c^{-1}r\}}\hat{c}$. Finally we note that j is a coalgebra map, $j(r)j(s) = r^{-1}s^{-1}$ and hence $\overline{j(r)j(s)} = [sr]$ and $\pi(j(r)j(s)) = \{sr\}$. Now we can calculate

$$\Delta(r \otimes \hat{b}) = \sum_{c,d \in B} r \otimes \hat{c}\hat{d} \otimes [c^{-1}r] \otimes \widehat{d^{-1}b} = \sum_{c \in B} r \otimes \hat{c} \otimes [c^{-1}r] \otimes \widehat{c^{-1}b},$$

$$(r \otimes \hat{b})(s \otimes \hat{c}) = \overline{j(r)j(s)} \otimes (\hat{b} \mathbin{\tilde{\leftharpoondown}} s)\hat{c} = \delta_{b,\{c^{-1}s\}}[sr] \otimes \hat{c},$$

and

$$\phi(r \otimes \hat{a} \otimes s \otimes \hat{b} \otimes t \otimes \hat{c}) = \langle \hat{a}, \pi(j(s)j(t))\rangle \delta_{b,e}\delta_{c,e} = \langle \hat{a}, \{ts\}\rangle \delta_{b,e}\delta_{c,e}.$$

We can dualize these formulas to get a quasibialgebra structure on $\tilde{H}^* \cong k^R \otimes k[B]$ with its canonical basis of elements $\hat{r} \otimes b$ for $r \in R$ and $b \in B$, which is also the dual basis to the basis $r \otimes \hat{b}$ of $\tilde{H}$. We compute

$$\begin{aligned}(\hat{r} \otimes b)(\hat{s} \otimes c) &= \sum_{\substack{t \in R\\ a,d \in B}} \delta_{rt}\delta_{bd}\delta_{s,[d^{-1}t]}\delta_{c,d^{-1}a}\hat{t} \otimes a\\ &= \delta_{s,[b^{-1}r]}\hat{r} \otimes bc = \delta_{r,[bs]}\hat{r} \otimes bc,\end{aligned}$$

$$\begin{aligned}\Delta(\hat{r} \otimes b) &= \sum_{\substack{s,t \in R\\ c,d \in B}} \delta_{c,\{d^{-1}t\}}\delta_{r,[ts]}\delta_{bd}\hat{s} \otimes c \otimes \hat{t} \otimes d\\ &= \sum_{s,t \in R} \delta_{[t^{-1}r],s}\hat{s} \otimes \{b^{-1}t\} \otimes \hat{t} \otimes b = \sum_{t \in R} \widehat{[t^{-1}r]} \otimes \{b^{-1}t\} \otimes \hat{t} \otimes b,\end{aligned}$$

$$\phi = \sum_{r,s,t \in R} \hat{r} \otimes \{ts\} \otimes \hat{e} \otimes s \otimes \hat{e} \otimes t.$$

Comparing with [61] we see that $\tilde{H}^* \cong A(G,B)^{\mathrm{cop}}$ as quasibialgebras (note that between the conventions in [61] and the present paper ϕ is replaced by its inverse).

It goes without saying that the category $\mathcal{C}(G,B)$ constructed by Zhu is equivalent to the category ${}^{k[G]}_{k[B]}\underline{\mathcal{M}}_{k[B]}$ of finite-dimensional of G-graded vector spaces with a compatible two-sided B-action; after all, the former is shown to be equivalent to the category of finite dimensional $A(G,B)$-modules in [61], while the latter is, by definition of $\tilde{H}$, the category of finite-dimensional $\tilde{H}$-comodules. We will sketch a direct proof: Let $D \in B\backslash G/B$ be a double coset in G. Then for every $M \in {}^{k[G]}_{k[B]}\mathcal{M}_{k[B]} = {}_{k[B]}\mathcal{M}^{k[G]}_{k[B]}$ the subspace $M_D := \bigoplus_{g \in D} M_g = \{m \in M | m_{(0)} \otimes m_{(-1)} \in M \otimes k[D]\}$ is a $k[B]$-subbimodule, hence a subobject in ${}^{k[G]}_{k[B]}\mathcal{M}_{k[B]}$, and $M = \bigoplus_{D \in B\backslash G/B} M_D$. It remains to describe the objects in ${}_{k[B]}\mathcal{M}^{k[D]}_{k[B]}$ when D is a double coset. We know that this subcategory of ${}_{k[B]}\mathcal{M}^{k[G]}_{k[B]}$ is equivalent to the category $\mathcal{M}^{k[B\backslash D]}_{k[B]}$. If $S \subset B$ is the stabilizer of an element of $B\backslash D$, then $B\backslash D \cong S\backslash B$ as right B-sets. But a special case of Schneider's structure theorem [48, Thm. 3.7] for Hopf modules gives a category equivalence $\mathcal{M}_{k[S]} \cong \mathcal{M}^{k[S\backslash B]}_{k[B]}$ mapping V to $V \underset{k[S]}{\otimes} k[B]$. To wrap up, objects of ${}_{k[B]}\mathcal{M}^{k[G]}_{k[B]}$ decompose as direct sums of objects of ${}_{k[B]}\mathcal{M}^{k[D]}_{k[B]}$ for each double coset D, and the latter category is equivalent to the category of representations of the stabilizer S of some element in $B\backslash D$, matching the description of the category $\mathcal{C}(G,B)$ by Zhu [61].

The motivation for $\mathcal{C}(G,B)$ in [61] is that its Grothendieck ring $\mathcal{G}(\mathcal{C}(G,B))$ maps surjectively onto the Hecke ring $H(G//B)$ associated to the inclusion $B \subset G$. We shall reproduce this now for the category ${}^{k[G]}_{k[B]}\underline{\mathcal{M}}_{k[B]}$. Let $H(G//B)$ be the free $\mathbb{Z}$-module over the set $G//B = B\backslash G/B$ of double cosets. We choose a set W of representatives for the double cosets and write D_w for the double coset containing $w \in W$. The universal property of the Grothendieck ring implies immediately that we have a well-defined group homomorphism

$$T\colon \mathcal{G}\left({}^{k[G]}_{k[B]}\underline{\mathcal{M}}_{k[B]}\right) \ni M \mapsto \sum_{w\in W} \dim(M_w) D_w \in H(G//B),$$

which is onto since $T(k[D]) = D$. We wish to show that T is a homomorphism to the Hecke ring; as a byproduct we will obtain a proof that the usual multiplication on the Hecke ring really is a ring structure.

For $M, N \in {}^{k[G]}_{k[B]}\underline{\mathcal{M}}_{k[B]}$, double cosets D, D', and $w'' \in W$ we have

$$(M_D \otimes N_{D'})_{w''} = \bigoplus_{\substack{d\in D, d'\in D'\\ dd'=w''}} M_d \otimes N_{d'}.$$

We can write $M \underset{k[B]}{\otimes} N$ as the quotient of $M \otimes N$ by the right action of B through the automorphisms t_b defined by $t_b(m \otimes n) = mb \otimes b^{-1}n$ for $b \in B$. Since t_b preserves the grading and maps $M_x \otimes N_y$ bijectively onto $M_{xb} \otimes N_{b^{-1}y}$, we have

$$(M_D \underset{k[B]}{\otimes} N_{D'})_{w''} \cong \bigoplus_{\substack{r\in D\cap R, d'\in D'\\ rd'=w''}} M_r \otimes N_{d'} \cong \bigoplus_{\substack{r\in D\cap R, r'\in D'\cap R\\ rr'\in w''B}} M_r \otimes N_{r'}$$

and thus, for $D = D_w$ and $D' = D_{w'}$:

$$\dim((M_D \underset{k[B]}{\otimes} N_{D'})_{w''}) \\ = \left|\{(r,r') \in D\cap R \times D'\cap R | rr' \in w''B\}\right| \cdot \dim(M_w)\dim(N_{w'})$$

and finally $T(M \underset{k[B]}{\otimes} N) = T(M)T(N)$, if we define multiplication in $H(G//B)$ by

$$DD' = \sum_{w''\in W} \left|\{(r,r') \in D\cap R \times D'\cap R | rr' \in w''B\}\right| D_{w''}$$

In particular, it follows that this formula defines the structure of an associative unital ring on $H(G//B)$; it is the usual multiplication in the Hecke ring as given in [52]. It also follows that T is a ring homomorphism.

Thus far we have seen that Zhu's construction $A(G,B)$ for a finite group G and subgroup B is, as a quasibialgebra, contained as a special case in [47]. We postpone discussing the quasi-Hopf structure, and turn first to a generalization of $A(G,B)$ to the case of infinite groups G and B. This is sketched slightly informally in [61] by saying that all formulas in the construction of $A(G,B)$ make sense for infinite groups, provided one restricts the attention to finite dimensional representations of $A(G,B)$, while otherwise the formulas involve infinite sums. We believe that the generalized coquasibialgebra construction $\tilde{H}$ from subsection 4.3 for $H = k[G] \in \mathfrak{C}_\ell(k[B])$ is a more formal version of this. Recall that finite comodules of $\tilde{H}$ correspond to Hopf modules in ${}^{k[G]}_{k[B]}\mathcal{M}_{k[B]}$ that are finitely generated left B-modules. In view of the description of ${}^{k[G]}_{k[B]}\mathcal{M}_{k[B]}$ obtained above (which holds just as well for infinite groups), we see that no objects can occur that involve degrees in G whose double cosets have infinitely many orbits under the left (or right) action of B. Thus we assume for the rest of this section that we are given a group G and a subgroup $B \subset G$ such that every double coset $D \in B\backslash G/B$ contains only finitely many left (or, equivalently, finitely many right) cosets. In this case, the right $k[B]$-module $Q = k[G]/k[B]^+k[G]$ is locally finite as right $k[B]$-module, so that Theorem 4.3.2 applies, and we know that $\tilde{H}$ can be modelled on the k-space $k[R] \otimes k[B]^\circ$. We also note that the above construction of a surjective homomorphism T from the Grothendieck ring of ${}^{k[G]}_{k[B]}\underline{\mathcal{M}}_{k[B]}$ to the Hecke ring $H(G//B)$ works just as well in this situation.

The question remains how the antipode given by Zhu for $A(G,B)$ fits in our picture. (Co)quasiantipodes are not considered at all in [47]. In fact the author learned about [61] when visiting MSRI, and the construction of a quasiantipode in [61] motivated the constructions in subsection 4.4.

Zhu gives an antipode for $A(G,B)$ under the following conditions: There should exist a choice R of right coset representatives that is, at the same time, a set of left coset representatives. This is clearly true if G is finite. The criterion in Lemma 4.4.7 is designed to be applicable to this case, so that we can use Theorem 4.4.5 to arrive at the desired conclusion: Whenever there is a common set of representatives for the left and right cosets of B in G, then $\widetilde{H} = k[R]\otimes k[B]^\circ$ is a coquasi-Hopf algebra.

If $B \subset G$ is an inclusion of infinite groups, then one can show that there is a common set of representatives for the left and right cosets of B in G if and only if every double coset contains as many left as right cosets. It is certainly known that this may fail to be true; we shall nevertheless give an example: We consider the subgroup $2^{\mathbb{Q}}$ of the multiplicative group $\mathbb{R}^+$. We let an infinite cyclic group $\langle\sigma\rangle$ act on $2^{\mathbb{Q}}$ by $\sigma \rightharpoonup 2^x = 2^{2x}$, and let $G = 2^{\mathbb{Q}} \rtimes \langle\sigma\rangle$ be the semidirect product. We let B be the subgroup $2^{\mathbb{Z}} \subset 2^{\mathbb{Q}} \subset G$. The left coset of $2^x\sigma^k$ consists of all elements $2^{x+m}\sigma^k$ for $m \in \mathbb{Z}$, while the right coset consists of all $2^{x+2^k m}\sigma^k$ for $m \in \mathbb{Z}$. It follows that the double coset D of $2^x\sigma^k$ is a left coset containing 2^k right cosets when $k \geq 0$, and D is a right coset containing 2^{-k} left cosets when $k < 0$.

At any rate, if every double coset in G is the union of finitely many left cosets, then the quotient $k[G]/k[B]^+k[G]$ is a locally finite right $k[B]$-module, and since $k[G]$ is cocommutative, the inclusion $k[B] \subset k[G]$ satisfies the conditions of Theorem 4.4.3. It follows that ${}^{\tilde{H}}\mathcal{M}$ is left and right rigid. If D is a double coset, and $V = k[D]/k[B]^+k[D]$ the $\tilde{H}$-comodule corresponding to $k[D] \in {}^{k[G]}_{k[B]}\mathcal{M}_{k[B]}$, then the discussion preceding Remark 4.4.1 shows that $\dim V = |B\backslash D|$ is the number of left cosets in D, while $\dim({}^\vee V) = |D/B|$ is the number of right cosets in D, where ${}^\vee V$ is the right dual of V in ${}^{\tilde{H}}\mathcal{M}$. As we have seen, the two numbers, hence the dimensions of V and its dual can differ. Such a phenomenon cannot occur if $\tilde{H}$ is a coquasi-Hopf algebra, so we have seen that the sufficient condition considered by Zhu to construct a quasiantipode is in fact necessary to construct a coquasiantipode for $\tilde{H}$, and we have found the counterexample announced in subsection 2.2:

Example 4.5.1. There is a coquasibialgebra H such that the category ${}^H\mathcal{M}_f$ of finite dimensional left H-comodules is left and right rigid, but the dimension of the dual ${}^\vee V$ of an object $V \in {}^H\mathcal{M}_f$ is in general different from $\dim V$. In particular, H is not a coquasi-Hopf algebra

We will show in a separate paper (now available as a preprint [45]) that when a coquasibialgebra H is finite dimensional, and $V \in {}^H\mathcal{M}$ has a dual object ${}^\vee V$, then $\dim({}^\vee V) = \dim V$. One can deduce from this that if H is finite dimensional and cosemisimple, and ${}^H\mathcal{M}$ is left and right rigid, then H is a coquasi-Hopf algebra.

4.6. Not an example: Quantum doubles. Contrary to this section's title, we already explained in subsection 4.2 how the quantum double of a finite Hopf algebra K is contained as a special case in the constructions of subsection 4.1, namely, as the double crossproduct associated to $K \otimes K^{\mathrm{op}}$ and the cocleaving $\pi = \nabla\colon K \otimes K^{\mathrm{op}} \to K$. This means that we reconstruct $D(K)$ from its comodule category, which turns out to be ${}^{K\otimes K^{\mathrm{op}}}_{K}\mathcal{M}_K$. But the real meaning of $D(K)$ lies in its module category, which is braided due to the quasitriangular structure of $D(K)$.

If K is cocommutative, there is a 'better' way: We let $H = K \otimes K$, and consider $H \in \mathfrak{C}_\ell(K)$ with respect to the inclusion $\iota := \Delta\colon K \to H$, and the cocleaving $\pi := K \otimes \varepsilon\colon H \to K$. It is straightforward to check that ${}^H_K\mathcal{M}_K$ identifies canonically with ${}^K_K\mathcal{M}^K_K$, which is equivalent the category of Yetter–Drinfeld modules over K by [42], and hence equivalent to the category of modules over the Drinfeld double. Thus we expect that $\tilde{H}$ will be dual to the Drinfeld double of K. In detail we can identify Q with K according to the surjection $\nu : H \ni x \otimes y \mapsto S(x)y \in K$. We then find $j = \eta \otimes K\colon K \to H$. In particular, j is a bialgebra map, and the action of Q on K defined by $q \rightharpoonup x = \pi(j(q)\iota(x))$ turns out to be trivial. Hence the algebra $\tilde{H}$ is just $K \otimes K^*$. The coalgebra structure of $\tilde{H}$ is that of a cosmash product with respect to the coaction of K on K^* dual to the coadjoint coaction of K on itself.

As it turns out, the second way admits a generalization beyond the case of arbitrary Hopf algebras to the construction of the Drinfeld double of a quasi-Hopf algebra. The construction does not fit into the framework of subsection 4.1, but it is rather close in spirit, so I would like to sketch the approach, although it is unfinished work at present. (A variant of the constructions sketched below is now contained in [45].)

The origin of Drinfeld doubles of quasi-Hopf algebras is the construction by Dijkgraaf, Pasquier and Roche [10] of a variation $D^\omega(G)$ of the Drinfeld double $D(G) := D(k[G])$ of a finite group G, depending on a three-cocycle of the group G with values in the multiplicative group of the base field k. The result is a quasi-Hopf algebra, which was interpreted by Majid [30] as the Drinfeld double quasi-Hopf algebra of the quasi-Hopf algebra (k^G, ω), with the coassociator $\omega \in k^G \otimes k^G \otimes k^G$. Majid also announces the construction of a Drinfeld double for general quasi-Hopf algebras. It is surprising at first sight that a double quasi-Hopf algebra of a quasi-Hopf algebra H should exist: After all, the double of a Hopf algebra H is modelled on $H \otimes H^*$, with H and H^* as subalgebras. But if H is just a quasi-Hopf algebra, then H^* is not an associative algebra, so one is at a loss looking for an associative algebra structure for $H \otimes H^*$. However, as explained in [30], there is a good reason to expect that the construction works anyway: The module category over the Drinfeld double $D(H)$ of an ordinary Hopf algebra H is equivalent to the center of the category of H-modules. The center construction is a purely categorical procedure assigning a braided monoidal category to any monoidal category; of course it can be applied to the category of H-modules also when H is just a quasi-Hopf algebra. The result should be the module category for another quasi-Hopf algebra $D(H)$, the Drinfeld double of H — by reconstruction principles. While he gives some indications, Majid falls (in the author's opinion) short of showing that $D(H)$ can be realized on the vector space $H \otimes H^*$. This was achieved by Hausser and Nill in [15; 16] through explicit computations.

We shall sketch very briefly how one can derive the construction from ideas similar to those in subsection 4.1. To stay close to the formalism used so far, we dualize the problem and look for a coquasi-Hopf algebra analog $D^*(K)$ of the dual Drinfeld double of a coquasi-Hopf algebra (K, ϕ). First, we note that K, though not an associative algebra, is an algebra in the monoidal category ${}^K\mathcal{M}^K$ of K-bicomodules, due to the modified associativity of K as a coquasibialgebra. Thus, we can consider the "Hopf module" categories ${}^K_K\mathcal{M}^K = {}_K\left({}^K\mathcal{M}^K\right)$ and ${}^K_K\mathcal{M}^K_K = {}_K\left({}^K\mathcal{M}^K\right)_K$. The category ${}^K_K\mathcal{M}^K_K$ has a natural structure of monoidal category with a suitably modified tensor product over K. By comparing with [42] we expect ${}^K_K\mathcal{M}^K_K$ to be equivalent to the category of comodules over $D^*(K)$. To make the setup look more like that in subsection 4.1, we put $H = K \otimes K^{\mathrm{cop}}$, so that ${}^K_K\mathcal{M}^K_K \cong {}^H_K\mathcal{M}_K$, and we want to find $D^*(K) = \tilde{H}$ with ${}^H_K\mathcal{M}_K \cong {}^{\tilde{H}}\mathcal{M}$. The solution in subsection 4.1 was based on Schneider's category equivalence ${}^H_K\mathcal{M} \cong {}^Q\mathcal{M}$. For the present situation, another paper of Hausser and Nill [17] provides a category equivalence ${}^H_K\mathcal{M} \cong {}^K_K\mathcal{M}^K \cong \mathcal{M}^K \cong {}^{K^{\mathrm{cop}}}\mathcal{M}$ mapping $V \in \mathcal{M}^K$ to $K \otimes V \in {}^K_K\mathcal{M}^K$, the free left K-module in ${}^K\mathcal{M}^K$ generated by V. One can infer that objects of ${}^K_K\mathcal{M}^K_K$, which are objects of ${}^K_K\mathcal{M}^K$ with an additional right action of K, can be classified by objects of ${}^{K^{\mathrm{cop}}}\mathcal{M}$ with an additional right action of K (classically, the latter objects would just be Yetter–Drinfeld modules by [42]). From the necessary compatibility condition between the comodule structure and the action, one derives a (not a priori coassociative) comultiplication on $\tilde{H} = K \otimes K^*$ such that ${}^{\tilde{H}}\mathcal{M} \cong {}^{H}_{K}\mathcal{M}_{K}$. To show that it is coassociative after all, one applies Lemma 2.3.1 after furnishing 'enough' objects in ${}^{K}_{K}\mathcal{M}^{K}_{K}$, and for this last step one may use the construction of free modules within the monoidal category ${}^{K}\mathcal{M}^{K}$, which provides the free right K-module over the free left K-module generated by the object $K \in {}^{K}\mathcal{M} \subset {}^{K}\mathcal{M}^{K}$.

While this approach to constructing the (dual) Drinfeld double is a spitting image of the procedure in subsection 4.1, we should stress again that it is not at all a special case. Rather, it poses the question for a common generalization.

5. Kac' Exact Sequence

In [22], Kac describes the following exact sequence (we have adopted notations from [32])

$$\begin{aligned} 0 \to&\mathcal{H}^1(F \bowtie G, k^\times) \overset{\mathrm{res}}{\to} \mathcal{H}^1(F, k^\times) \oplus \mathcal{H}^1(G, k^\times) \to \mathrm{Aut}(k^G \# kF) \to \\ \to&\mathcal{H}^2(F \bowtie G, k^\times) \overset{\mathrm{res}}{\to} \mathcal{H}^2(F, k^\times) \oplus \mathcal{H}^2(G, k^\times) \to \mathrm{Opext}(kF, k^G) \to \\ \to&\mathcal{H}^3(F \bowtie G, k^\times) \overset{\mathrm{res}}{\to} \mathcal{H}^3(F, k^\times) \oplus \mathcal{H}^3(G, k^\times) \to \cdots \end{aligned}$$

Here $\mathcal{H}^\bullet(G, A)$ stands for the cohomology of a group G with coefficients in a G-module A, and $k^\times$ is the multiplicative group of a fixed base field (in Kac' work $k = \mathbb{C}$, but see [32]), with the trivial group action. The group $F \bowtie G$ is assumed to be a group containing F and G as subgroups such that the map $F \times G \to F \bowtie G$ given by multiplication is a bijection. By the remarks in subsection 4.2 this leads to a Singer pair between the Hopf algebras k^G and kF, hence to a bismash product Hopf algebra $k^G \# kF$. By $\mathrm{Aut}(k^G \# kF)$ we mean the group of automorphisms of the extension $k^G \# kF$ of kF by k^G. By $\mathrm{Opext}(kF, k^G)$ we denote the group of extensions of kF by k^G giving rise to the same Singer pair.

We refer to Akira Masuoka's paper [31] in this volume for more information on Kac' sequence beyond the following remarks: There are in essence three steps leading to the sequence. First, one has to translate the Opext group (and the automorphism group) in the sequence into cohomological data. This part of Kac' work considerably predates the more general cohomological description of extensions of cocommutative by commutative Hopf algebras as carried out by Singer [53] and Hofstetter [19; 20], who do not seem to have been aware of [22]. The description involves a certain double complex. Second, one exhibits this double complex as the middle term of a short exact sequence of double complexes, and obtains a long exact cohomology sequence. Third, one interprets the cohomology groups of the end terms of the short exact sequence of double complexes. One of them is trivially related to the bar resolutions computing the group cohomology of F and G, while it is more intricate to show that the third term is a non-standard resolution of $\mathbb{Z}$ that can be used to compute the cohomology of $F \bowtie G$. It is particularly this last step which is specific to the group case.

5.1. A Kac sequence for general Hopf algebras. Of course we can pass from the groups F, G, and $F \bowtie G$ in the Kac sequence to their group algebras $Q := kF, L := kG$, and $k[F \bowtie G] \cong kF \bowtie kG$. From [54] we know that group cohomology with coefficients in the multiplicative group of the field k is the same as Sweedler cohomology of the group algebra with coefficients in the trivial module algebra k. Abbreviating the latter by $\mathcal{H}^\bullet(H) = \mathcal{H}^\bullet(H, k)$, and writing $K := L^*$, we obtain the form

$$0 \to \mathcal{H}^1(Q \bowtie L) \stackrel{\text{res}}{\to} \mathcal{H}^1(Q) \oplus \mathcal{H}^1(L) \to \operatorname{Aut}(K\#Q) \to \mathcal{H}^2(Q \bowtie L) \stackrel{\text{res}}{\to}$$
$$\stackrel{\text{res}}{\to} \mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Opext}(Q, K) \to \mathcal{H}^3(Q \bowtie L) \stackrel{\text{res}}{\to} \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$$

of Kac' sequence. It shows no explicit reference any more to the groups involved, so one may write this down equally well for cocommutative Hopf algebras Q and L, with L finite, and $K = L^*$. However, from [22] or [32] we cannot draw a definition of all the maps in the sequence, much less infer its exactness; specifically the maps from the automorphism group to the second cohomology, and from the Opext group to the third cohomology, will need a new explanation. We will not transfer Kac' techniques to the situation with general cocommutative Hopf algebras, but rather give an explanation through the functor $\mathfrak{F}$; this generalizes beyond the cohomological picture to an analog of the Kac sequence for non-cocommutative Hopf algebras. However, to formulate this 'quantum' analog, we have to alter the appearance of the sequence quite completely. If Q and K are noncommutative and non-cocommutative, then the congruence class of an extension $K \to H \to Q$ no longer determines the action of Q on K and the coaction of K on Q uniquely, and in the description of H as a bicrossproduct $K\,{}^{\tau}\#_{\sigma}\, Q$ with cocycle σ and cycle τ it is no longer possible to obtain a bismash product from the same action and coaction with trivial (co)cycles. The only solution appears to be to consider, instead of the Opext groups for each Singer pair separately, the set $\operatorname{Ext}(Q, K)$ of congruence classes of all extensions of Q by K; in the abelian case, this is the disjoint union of all the Opext groups for various Singer pairs. Next, we have to consider what happens to the map from the Opext group to degree three cohomology in the general case. We have seen that a good non-cocommutative replacement for a three-cocycle is a coquasibialgebra structure. If we are given a double crossproduct of cocommutative Q and L, then for a three-cocycle ϕ on it to be in the kernel of the homomorphism $\mathcal{H}^3(Q \bowtie L) \to \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$ induced by the restrictions means that the restrictions of ϕ to $Q \otimes Q \otimes Q$ as well as to $L \otimes L \otimes L$ are trivial, and this in turn means that Q and L, ordinary bialgebras with trivial coassociators, are subcoquasbibialgebras of $(Q \bowtie L, \phi)$. In the non-cocommutative case, we cannot fix our attention on a Singer pair at a time, and thus it also makes no sense to consider only one double crossproduct at a time. Thus, we define the set $\mathfrak{P}(Q, L)$ to consist of the cohomology classes of generalized product coquasibialgebras of Q and L. Here a generalized product coquasibialgebra is by definition a coquasibialgebra (P, ϕ) with injective coquasibialgebra maps $Q \to P$ and $L \to P$, such that multiplication in P induces an isomorphism $Q \otimes L \to P$. We call two generalized product coquasibialgebras P, P' cohomologous if there exists a coquasiisomorphism $(F, \theta)\colon P \to P'$ whose underlying map F commutes with the respective inclusion maps from L and Q. One can show that in the case where both Q and L are cocommutative, $\mathfrak{P}(Q, L)$ is in natural bijection with the sum of the kernels of all the homomorphism $\mathcal{H}^3(Q \bowtie L) \to \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$ induced by the restrictions, for all possible matched pairs between Q and L: the bijection is obtained by interpreting a three-cocycle ϕ on a double crossed product $Q \bowtie L$ as a coquasibialgebra $(Q \bowtie L, \phi)$. Thus, a good replacement for the exact sequences

$$\mathrm{Opext}(Q, K) \to \mathcal{H}^3(Q \bowtie L) \to \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$$

is a surjection

$$\mathrm{Ext}(Q, K) \to \mathfrak{P}(Q, L).$$

We can describe such a surjection in terms of the functor $\mathfrak{F}$: By choosing a cocleaving, an extension $K \to H \to Q$ can be regarded as an element of $\mathfrak{E}_\ell(K)$, and gives rise to $\tilde{H} := \mathfrak{F}(H) \in \mathfrak{E}_r(L)$. One can specialize the formulas in subsection 4.1 to find that Q is a subcoquasibialgebra of $\tilde{H}$, with $\tilde{\phi}|_{Q \otimes \tilde{H} \otimes \tilde{H}} = \varepsilon$. The obvious isomorphism $Q \otimes L \to \tilde{H}$ is given by multiplication. Finally, modifying the cleaving in H leads to a different result $\tilde{H}$, but one that only differs by a twist. Thus, the functor $\mathfrak{F}$ induces a well-defined map $[\mathfrak{F}]\colon \mathrm{Ext}(Q, K) \to \mathfrak{P}(Q, L)$. It appears to be far from surjective at first sight, however, since the coassociator $\tilde{\phi}$ on $\tilde{H}$ satisfies stronger conditions than just $\tilde{\phi}|_{Q \otimes Q \otimes Q} = \varepsilon$ and $\tilde{\phi}|_{L \otimes L \otimes L} = \varepsilon$. But one can show [47, Prop. 6.2.3] that any element of $\mathfrak{P}(Q, L)$ does contain a representative (P, ϕ) that fulfills the stronger conditions $\phi|_{Q \otimes P \otimes P} = \varepsilon$ and $\phi|_{P \otimes P \otimes L}$. This not only removes the obvious obstacle to surjectivity of $[\mathfrak{F}]$, but also allows to prove it: The special representative (P, ϕ) is, by the second condition, an element of $\mathfrak{E}_r(L)$, and it turns out that the functor $\mathfrak{F}^{\mathrm{bop}}\mathfrak{E}_r(L) \to \mathfrak{E}_\ell(K)$, applied to such representatives, defines a section for $[\mathfrak{F}]$. This is a special case of the fact, mentioned in subsection 4.2, that $\mathfrak{F}$ is an involution on certain classes of objects of $\mathfrak{E}_\ell(K)$.

We turn now to the part

$$\mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \mathrm{Opext}(Q, K) \to \mathcal{H}^3(Q \bowtie L)$$

of the cocommutative Kac sequence. Taking the union over all possible Singer pairs, we obtain an exact sequence

$$\mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Ext}(Q, K) \to \mathfrak{P}(Q, L),$$

with a certain sloppiness: since neither $\operatorname{Ext}(Q, K)$ nor $\mathfrak{P}(Q, L)$ are groups, we have to say what exact means. Now via the group homomorphisms $\mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Opext}(Q, K)$, the group $\mathcal{H}^2(Q) \oplus \mathcal{H}^2(L)$ acts on each Opext group, hence on the set $\operatorname{Ext}(Q, K)$, and exactness means that the fibers of the map $\operatorname{Ext}(Q, K) \to \mathfrak{P}(Q, L)$ are the orbits of this action. For the non-cocommutative case, we now have to find a replacement for this action on $\operatorname{Ext}(Q, K)$. In fact this is quite easy: Recall that any two-cocycle $\theta \colon Q \otimes Q \to k$ can be used to cotwist Q to give a different Hopf algebra Q^θ. Of course we can pull θ back to give a two-cocycle on any extension H in $\operatorname{Ext}(Q, K)$, and thus twist H to give an extension H^θ in $\operatorname{Ext}(Q^\theta, K)$. Similarly, if $t \colon L \otimes L \to k$ is a two-cocycle, we can consider it as a cycle $t \in K \otimes K$, and twist K to obtain a new Hopf algebra K_t. Surely t is also a two-cycle in any extension H in $\operatorname{Ext}(Q, K)$, and we can twist H by it to get a new extension $H_t \in \operatorname{Ext}(Q, K_t)$. Now for our purposes we are not interested in modifying the end terms of an extension, but only the middle term. Thus, we need to restrict our attention to those cocycles θ and t that do not affect Q and K, or, in the terminology introduced in subsection 2.4, we have to consider central cocycles. It turns out [47, Lem. 6.3.1] that one has indeed a well-defined action of $\mathcal{H}^2_c(Q) \oplus \mathcal{H}^2_c(L)$ on $\operatorname{Ext}(Q, K)$ as just described, and the orbits of this action are [47, Thm. 6.3.6] precisely the fibers of $[\mathfrak{F}] \colon \operatorname{Ext}(Q, K) \to \mathfrak{P}(Q, L)$.

Next, we consider the portion

$$\mathcal{H}^2(Q \bowtie L) \to \mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Opext}(Q, K)$$

of the cocommutative Kac sequence. For the non-cocommutative version, we have already replaced the right hand map by an action of $\mathcal{H}^2_c(Q) \times \mathcal{H}^2_c(L)$ on $\operatorname{Ext}(Q, K)$, so now exactness amounts to determining the stabilizers of the action. In the cocommutative case, the stabilizer of an extension H in $\operatorname{Ext}(Q, K)$ is the image of the second cohomology of the associated double crossproduct $Q \bowtie L$. In our situation it turns out that the stabilizer is the image of the self-twist group $\mathcal{H}^2_c(\mathfrak{F}(H))$ of the associated coquasibialgebra [47, Thm. 6.3.5].

We have thus discussed the non-cocommutative replacements for exactness of the portion

$$\begin{aligned}\mathcal{H}^2(Q \bowtie L) \to \mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to \operatorname{Opext}(Q, K) \to& \\ \to \mathcal{H}^3(Q \bowtie L) \to \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)&\end{aligned}$$

of Kac' exact sequence. We will skip discussing the lower order terms [47, Sec. 6.5], although we will present an application in the next section. It is worthwhile noting that the cocommutative Kac sequence that we obtain as a special case is in fact essentially the same as the original Kac sequence for groups [47, Sec. 6.4]

5.2. Masuoka's sequence for Lie algebras. In [33], Masuoka gives variants of Kac' exact sequence that apply to Lie algebras and their enveloping algebras instead of groups and their group algebras.

The purpose of this section is not to show how these derive from the author's sequence for cocommutative Hopf algebras. The results from subsection 5.1 simply do not apply because the enveloping algebras are always infinite-dimensional. Even though we showed in subsection 4.3 how to modify the construction underlying the generalized Kac sequence to the case of infinite Hopf subalgebras, we will find that this does not explain Masuoka's version of Kac' sequence. We shall indicate here which parts appear to generalize smoothly, and where the real difficulties particular to the Lie case come in to obstruct the way. Throughout the section we assume the base ring k is a field.

We refer the reader to [31], in this volume, for details on Masuoka's sequences. We will discuss here only the variant in [33, Cor. 4.12.], which is an exact sequence

$$\begin{aligned} 0 \to \mathcal{H}^1(\mathfrak{f} \bowtie \mathfrak{g}) \to \mathcal{H}^1(\mathfrak{f}) \oplus \mathcal{H}^1(\mathfrak{g}) \to \operatorname{Aut}((U\mathfrak{g})^\circ \# U(\mathfrak{f})) \to \mathcal{H}^2(\mathfrak{f} \bowtie \mathfrak{g}) \\ \to \mathcal{H}^2(\mathfrak{f}) \oplus \mathcal{H}^2(\mathfrak{g}) \to \operatorname{Opext}(U(\mathfrak{f}), (U\mathfrak{g})^\circ) \to \mathcal{H}^3(\mathfrak{f} \bowtie \mathfrak{g}) \to \mathcal{H}^3(\mathfrak{f}) \oplus \mathcal{H}^3(\mathfrak{g}) \end{aligned}$$

in which $\mathfrak{f}$ and $\mathfrak{g}$ are Lie algebras, with $\mathfrak{g}$ finite dimensional, $\mathfrak{f} \bowtie \mathfrak{g}$ is a double crossproduct Lie algebra, and the cohomology groups are Lie algebra cohomology with coefficients in the trivial module k. The universal enveloping algebra of $\mathfrak{f} \bowtie \mathfrak{g}$ is a double crossproduct Hopf algebra $U(\mathfrak{f} \bowtie \mathfrak{g}) \cong U(\mathfrak{f}) \bowtie U(\mathfrak{g})$ with respect to a matched pair of Hopf algebras in which the action of $U(\mathfrak{g})$ on $U(\mathfrak{f})$ is locally finite. This matched pair gives rise to a Singer pair $(U(\mathfrak{g})^\circ, U(\mathfrak{f}))$ which features in the smash product and the Opext group that appear in the sequence.

By [54] we can replace these cohomology groups by Sweedler cohomology groups of the associated universal enveloping algebras. If we then put $Q = U\mathfrak{f}$, $L = U\mathfrak{g}$, and $K = L^\circ$, then Masuoka's sequence looks just the same as the generalized cocommutative Kac sequence in subsection 5.1. However, since L has infinite dimension, the sequence in subsection 5.1 does not contain Masuoka's sequence as a special case. We will now discuss how far the results from the previous sections apply to Masuoka's situation.

Assume first that we are given a matched pair $(Q, L, \rightharpoonup, \leftharpoonup)$ of Hopf algebras. Assume that the action $\rightharpoonup$ of L on Q is locally finite. Then by the left-right switched version of Theorem 4.3.2 we obtain $\mathfrak{F}^{\mathrm{bop}}(Q \bowtie L) \cong L^\circ \otimes Q$. The formulas obtained for the multiplication and comultiplication on $\mathfrak{F}^{\mathrm{bop}}(Q \bowtie L)$ specialize to say that $L^\circ \otimes Q = L^\circ \# Q$ is a bismash product with respect to the coaction of L° on Q dual to $\rightharpoonup$, and the action of Q on L° dual to $\leftharpoonup$. In particular, it follows that the action of Q on L^* stabilizes L°, and that we have a Singer pair (Q, K) with $K := L^\circ$ (cf. [33, Lem. 4.1]). On the other hand, let K be any Hopf algebra and assume given a bicrossproduct extension $K\,{}^\tau\#_\sigma Q$.

Since the action of K on Q is trivial, Theorem 4.3.2 applies again to show that $\mathfrak{F}(K\,{}^{\tau}\#_{\sigma}\,Q) = Q \otimes K^\circ \in \mathfrak{P}(Q, K^\circ)$ is a generalized product coquasibialgebra.

Assume further that L is cocommutative, $K = L^\circ$, and the abelian Singer pair underlying the bicrossproduct $K\,{}^{\tau}\#_{\sigma}\,Q$ is the same as that arising from the matched pair in the double crossproduct $Q \bowtie L$. Then the natural map $L \to K^\circ$ induces a bialgebra map $Q \bowtie L \to Q \bowtie K^\circ$, and we can restrict the coquasibialgebra structure on $Q \bowtie K^\circ$ arising from $\mathfrak{F}(K\,{}^{\tau}\#_{\sigma}\,Q)$ to give a three-cocycle on $Q \bowtie L$. All in all, we have defined a map $[\mathfrak{F}]\colon \operatorname{Opext}(Q,K) \to \mathcal{H}^3(Q \bowtie L)$ which takes values in the kernel of the map $\mathcal{H}^3(Q \bowtie L) \to \mathcal{H}^3(Q) \oplus \mathcal{H}^3(L)$ induced by the restrictions. Precisely as in [47], one can also show that any element of the latter kernel contains a representative ϕ that fulfills $\phi|_{(Q\bowtie L)\otimes(Q\bowtie L)\otimes L} = \varepsilon$ and $\phi|_{Q\otimes(Q\bowtie L)\otimes(Q\bowtie L)} = \varepsilon$, so that $(Q \bowtie L, \phi) \in \mathfrak{E}_r(L)$. We can apply the functor $\mathfrak{F}^{\mathrm{bop}}$ to obtain some bialgebra $H = \widetilde{Q \bowtie L}$ equipped with maps $K \to H \to Q$ of bialgebras. However, we do not know that $H = K \otimes Q$ is a bicrossproduct in $\operatorname{Opext}(Q,K)$ unless the map $Q \to (L \otimes L)^*$ induced by ϕ takes values in $K \otimes K$; this is what we would need to apply Theorem 4.3.2 once more.

Of course we can specialize the results above to the case where Q and L are enveloping algebras as in Masuoka's sequence. We have not checked that the map $\operatorname{Opext}(U(\mathfrak{f}), (U\mathfrak{g})^\circ) \to \mathcal{H}^3(\mathfrak{f} \bowtie \mathfrak{g})$ we obtain is the same as that obtained by Masuoka. If this is the case, then Masuoka's sequence seems to indicate that every cohomology class in the kernel of the map

$$\mathcal{H}^3(U\mathfrak{f} \bowtie U\mathfrak{g}) \to \mathcal{H}^3(U\mathfrak{f}) \oplus \mathcal{H}^3(U\mathfrak{g})$$

has a representative ϕ which is trivial on $U(\mathfrak{f} \bowtie \mathfrak{g}) \otimes U(\mathfrak{f} \bowtie \mathfrak{g}) \otimes U(\mathfrak{g})$ as well as $U(\mathfrak{f}) \otimes U(\mathfrak{f} \bowtie \mathfrak{g}) \otimes U(\mathfrak{f} \bowtie \mathfrak{g})$, and in addition induces a map $U(\mathfrak{f} \bowtie \mathfrak{g}) \to U(\mathfrak{g})^\circ \otimes U(\mathfrak{g})^\circ$.

5.3. Galois extensions over tensor products. In this final section we sketch how the generalized Kac sequence contains a result of Kreimer [25, Thm. 3.7] on Galois objects over tensor products of Hopf algebras. It states that when Q and L are finitely generated projective cocommutative Hopf algebras, then the group of Galois objects $\operatorname{Gal}(Q \otimes L)$ can be computed as

$$\operatorname{Gal}(Q \otimes L) \cong \operatorname{Gal}(Q) \oplus \operatorname{Gal}(L) \oplus \operatorname{Hopf}(Q, L^*),$$

the last term being the group of Hopf algebra homomorphisms under convolution.

We will prove this result from the low order terms of the generalized Kac sequence; one should note that this cannot adequately be called a short proof of Kreimer's result, since we just shifted the complications to the proof of the Kac sequence. However, we believe that the connection between the two results is of some interest.

We first recall some background: If H is a cocommutative finitely generated projective Hopf algebra, then the H-Galois extensions A/k that are faithfully flat k-modules form an abelian group $\operatorname{Gal}(H)$ under cotensor product over H. The abelian group $\operatorname{Gal}(H)$ is a contravariant functor of H, again by cotensor product: For a k-split Hopf algebra map $f\colon H \to F$, the associated group homomorphism maps $A \in \operatorname{Gal}(F)$ to $A \underset{F}{\Box} H$; in case f is an injection, we can identify this with $A(H) := \{a \in A | a_{(0)} \otimes a_{(1)} \in A \otimes H\}$.

If $H = Q \otimes L$ is a tensor product of cocommutative Hopf algebras, it follows that the map $\operatorname{Gal}(H) \to \operatorname{Gal}(Q) \oplus \operatorname{Gal}(L)$ induced by the projections of H to Q and L is a split surjection with splitting induced by the injections of Q and L into H. By the same reason the homomorphisms $\mathcal{H}^n(H) \to \mathcal{H}^n(Q) \oplus \mathcal{H}^n(L)$ are split epimorphisms. Thus the Kac sequence for the trivial double crossproduct $Q \otimes L$ reduces in low dimensions to a split short exact sequence

$$0 \to \operatorname{Aut}(L^* \otimes Q) \to \mathcal{H}^2(Q \bowtie L) \to \mathcal{H}^2(Q) \oplus \mathcal{H}^2(L) \to 0.$$

The first term is the group of such automorphisms of α of $L^* \otimes Q$ that fix L^* and induce the identity on the quotient Q. These are easily seen to be in bijection with Hopf algebra maps $f\colon Q \to L^*$ by the formula $\alpha(x \otimes q) = xf(q_{(1)}) \otimes q_{(2)}$.

To obtain Kreimer's result, we would like to replace the right half of the sequence by the split epimorphism $\operatorname{Gal}(Q \otimes L) \to \operatorname{Gal}(Q) \oplus \operatorname{Gal}(L)$, that is, we have to show that the latter has the same kernel. If we recall that the second Sweedler cohomology groups describe precisely Galois objects which are cleft, then we just have to see that if $A \in \operatorname{Gal}(Q \otimes L)$ becomes trivial under both maps to $\operatorname{Gal}(Q)$ and $\operatorname{Gal}(L)$, then A is cleft. But even when we only assume that $A(Q)$ is Q-cleft with cleaving map $j\colon Q \to A(Q)$, and that $A(L)$ is L-cleft with cleaving map $\gamma\colon L \to A(L)$, we arrive immediately at the desired conclusion that A is cleft with cleaving map $Q \otimes L \ni q \otimes \ell \mapsto j(q)\gamma(\ell) \in A$.

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