New Directions in Hopf Algebras
MSRI Publications
Volume **43**, 2002

Hopf Algebra Extensions and Cohomology

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Abstract. This is an expository paper on 'abelian' extensions of (quasi-) Hopf algebras, which can be managed by the abelian cohomology, with emphasis on the author's recent results which are motivated by an exact sequence due to George Kac. The cohomology plays here an important role in constructing and classifying those extensions, and even their cocycle deformations. We see also a strong connection of Hopf algebra extensions arising from a (matched) pair of Lie algebras with Lie bialgebra extensions.

Introduction

Let us first recall the theory of group extensions with abelian kernel [Mac, Chap. IV, Sections 3,4]. Each extension $M \to \Sigma \to \Pi$ of a group Π by an abelian group M gives rise to a Π-module structure on M. Those extensions which give rise to a fixed Π-module structure form an abelian group, $\mathrm{Opext}(\Pi, M)$, which is isomorphic to the cohomology $H^2(\Pi, M)$. The results were generalized by Singer [S] (1972) and Hofstetter [H] (1994) for those Hopf algebra extensions

$$K \to A \to H$$

which are abelian in the sense that H is cocommutative, K is commutative and A is cleft as an H-comodule algebra: each such extension gives rise to some structure, called a Singer pair structure (see Definition 2.2), on (H, K), and those extensions which give rise to a fixed Singer pair structure form an abelian group, $\mathrm{Opext}(H, K)$, which is isomorphic to some cohomology group. But, Kac [K] (1968) had already obtained these results in the case when $H = kF$ (group algebra), $K = k^G$ $(= (kG)^*)$ with F, G finite groups, and further proved an interesting, exact cohomology sequence involving $\mathrm{Opext}(kF, k^G)$, which we

The author gratefully acknowledges that many of his own results presented here were obtained while he was staying at the University of Munich as a Humboldt Research Fellow (November 1996 – September 1997 and June 1998) and visiting the National University of Córdoba, Argentina (October 1997). He thanks the referee for kind suggestions that improved the exposition.

call the Kac exact sequence. Unfortunately, Kac' work on extensions had long been overlooked by most of Hopf algebraists (perhaps, including Singer and Hofstetter), and especially his exact sequence had been peculiar to the restricted case as above. But, in these several years his work has been applied especially for the classification problem of semisimple Hopf algebras; see [M1], [N], [Ka], and also [IK]. In addition, the author [M] recently established the formulation of the Kac exact sequence in the Lie algebra case (see below), and then Schauenburg [Sb2] proved the sequence for general H and finite-dimensional K (see Remark 1.11 (3)).

This is an expository paper on abelian extensions of (quasi-)Hopf algebras with emphasis on the author's recent results which are motivated by the Kac exact sequence. We will see the cohomology plays an important role in constructing and classifying those extensions, and even their cocycle deformations.

The paper consists of two parts. Part I (Sections 1–4) begins with an elementary exposition of the Kac theory, which is followed by the generalized results from [S], [H]. Then Section 3 is devoted to the study of cocycle deformations of the middle term A by 2-cocycles for H (or by their liftings to A), with sample computations. In Section 4, we study quasi-Hopf algebra extensions of a similar form as above, but A is replaced by a quasi-Hopf algebra with Drinfeld associator in $K^{\otimes 3}$; parallel results to the Hopf algebra case are proved, including the modified Kac exact sequence. In Part II (Sections 5–8), we let $\mathfrak{f}$, $\mathfrak{g}$ denote finite-dimensional Lie algebras in characteristic zero, and see a strong connection between Hopf algebra extensions of the form $(U\mathfrak{g})^\circ \to \mathrm{A} \to U\mathfrak{f}$ and Lie bialgebra extensions of the form $\mathfrak{g}^* \to \mathfrak{l} \to \mathfrak{f}$; the result proves a Lie algebra version of the Kac exact sequence. Similar results are proved when $\mathfrak{g}$ is nilpotent and the Hopf dual $(U\mathfrak{g})^\circ$ of the universal envelope $U\mathfrak{g}$ is replaced by its irreducible component $(U\mathfrak{g})'$ containing 1. In the final section these parallel results are unified and generalized by introducing some topology onto $U\mathfrak{g}$.

This paper may be regarded as an enlarged version of the Córdoba lecture notes [M4], though some computational results are omitted here. Instead, proofs of the results in Sections 7 and 8 are included. Sections 3 and 4 are also added; they include unpublished results which are first presented with proofs. Following M. Takeuchi's suggestion, we emphasize categorical treatment of extensions, which seems new.

We work over a fixed ground field k. Tensor products $\otimes$ and exterior products $\wedge$ are taken over k, unless otherwise stated. For vector spaces V, W, we let $\mathrm{Hom}(V, W)$ denote the vector space of all linear maps $V \to W$, and write $V^* = \mathrm{Hom}(V, k)$, the dual vector space. By a module (resp., comodule), we mean a left module (resp., right comodule) unless otherwise stated. The coproduct, the counit and the antipode of a Hopf algebra are denoted, as usual, by Δ, ε and S, respectively. We use the Sweedler notation such as $\Delta(a) = \sum a_1 \otimes a_2$.

PART I: KAC THEORY, ITS GENERALIZATION AND VARIATION

In this part, F and G denote groups, which are supposed to be finite unless otherwise stated.

1. The Kac Theory

The theory will be reviewed from modern Hopf-algebraic view-point, but as elementarily as possible.

Definition 1.1 [T1, Def. 2.1]. A *matched pair* (*of groups*) is a pair (F, G) together with group actions $G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F$ on the sets such that

$$x \triangleright ab = (x \triangleright a)((x \triangleleft a) \triangleright b),$$
$$xy \triangleleft a = (x \triangleleft (y \triangleright a))(y \triangleleft a)$$

for $a, b \in F$, $x, y \in G$, or equivalently such that the cartesian product $F \times G$ forms a group under the product

$$(a, x)(b, y) = (a(x \triangleright b), (x \triangleleft b)y).$$

We denote this group by $F \bowtie G$, following [Mj]. (It was originally denoted by $F \between G$; see [T1, p. 842].)

The group $F \bowtie G$ includes subgroups $F = F \times 1$ and $G = 1 \times G$ so that the product map $F \times G \to F \bowtie G$ is a bijection. Conversely, if a group Σ includes F, G as subgroups so that the product $F \times G \to \Sigma$ is a bijection, then the actions $G \xleftarrow{\triangleleft} G \times F \xrightarrow{\triangleright} F$ determined by

$$ax = (a \triangleright x)(a \triangleleft x) \ (a \in F,\ x \in G)$$

make (F, G) matched so that $F \bowtie G \cong \Sigma$.

Let kG denote the group Hopf algebra in which each element in G is grouplike. Let k^G denote the dual Hopf algebra $(kG)^*$ of kG; it is spanned by the orthogonal idempotents e_x defined by $\langle e_x, y \rangle = \delta_{x,y}$, where $x, y \in G$. We may suppose that k^G is the algebra consisting of all maps $G \to k$ which has the pointwise product, and so that the abelian group $(k^G)^\times$ of units in k^G consists of all maps $G \to k^\times = k \setminus 0$.

By a G-module, we mean a module over the integral group ring $\mathbb{Z}G$, as usual.

Let $\triangleleft$: $G \times F \to G$ be an action on the set G, which corresponds to an action $\rightharpoonup$: $F \times k^G \to k^G$ of algebra automorphisms so that

$$a \rightharpoonup e_x = e_{x \triangleleft a^{-1}} \ (a \in F,\ x \in G).$$

Let $\sigma : F \times F \to (k^G)^\times$ be a 'normalized' 2-cocycle of the group F with coefficients in $(k^G)^\times$, which is an F-module under the action induced by $\rightharpoonup$. We identify σ naturally with the map $(x, a, b) \mapsto \sigma(a, b)(x)$, $G \times F \times F \to k^\times$, and denote the last value by $\sigma(x; a, b)$. Then the 2-cocycle condition for σ is given by

$$\sigma(x \lhd a; b, c)\sigma(x; a, bc) = \sigma(x; a, b)\sigma(x; ab, c),$$

while the normalization condition means here

$$\sigma(1; a, b) = \sigma(x; 1, b) = \sigma(x; a, 1) = 1,$$

where $a, b, c \in F$ and $x \in G$. The familiar construction of crossed product makes the tensor product $k^G \otimes kF = \bigoplus_{a\in F} k^G a$ into an algebra with unit $1 \otimes 1$, whose product is given by

$$(e_x a)(e_y b) = e_x(a \rightharpoonup e_y)\sigma(a, b)ab = \delta_{x\lhd a,y}\sigma(x; a, b)e_x ab,$$

where $a, b \in F$ and $x, y \in G$.

Suppose also that we are given an action $\rhd$: $G \times F \to F$ and a normalized 2-cocycle $\tau : G \times G \times F \to k^\times$ of G with coefficients in the right G-module $(k^F)^\times$. They make $kG \otimes k^F$ into an algebra of right crossed product, and so by duality make $k^G \otimes kF$ into a coalgebra. One sees that the coalgebra structure is given by

$$\begin{aligned}\Delta(e_x a) &= \sum_{y\in G} \tau(xy^{-1}, y; a)e_{xy^{-1}}(y \rhd a) \otimes e_y a,\\ \varepsilon(e_x a) &= \delta_{1,x}.\end{aligned}$$

Let $k^G \#_{\sigma,\tau} kF$ denote the tensor product $k^G \otimes kF$ with the described algebra and coalgebra structures.

Lemma 1.2. *$k^G \#_{\sigma,\tau} kF$ is a bialgebra if and only if $(F, G, \lhd, \rhd)$ is a matched pair and*

$$\begin{aligned}&\sigma(xy; a, b)\tau(x, y; ab)\\ &\qquad = \sigma(x; y \rhd a, (y \lhd a) \rhd b)\sigma(y; a, b)\tau(x, y; a)\tau(x \lhd (y \rhd a), y \lhd a; b)\end{aligned}$$

for all $a, b \in F$, $x, y \in G$. In this case, $k^G \#_{\sigma,\tau} kF$ is necessarily a Hopf algebra.

The proof is straightforward; see the proof of [M4, Prop. 4.7]. If the conditions given above are satisfied, the maps $\iota : k^G \to k^G \#_{\sigma,\tau} kF$, $\iota(e_x) = e_x 1$ and $\pi : k^G \#_{\sigma,\tau} kF \to kF$, $\pi(e_x a) = \delta_{1,x} a$ are obviously Hopf algebra maps. Further, we will see

$$(k^G \#_{\sigma,\tau} kF) = k^G \xrightarrow{\iota} k^G \#_{\sigma,\tau} kF \xrightarrow{\pi} kF \tag{1.3}$$

is a Hopf algebra extension.

Let $(A) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ be a sequence of finite-dimensional Hopf algebras. Let K^+ denote the kernel $\mathrm{Ker}(\varepsilon : K \to k)$ of the counit. Regarding A as a right (or left) H-comodule along π, let A^{coH} (or ${}^{coH}A$) denote the subalgebra of H-coinvariants. Thus, A^{coH} consists of $a \in A$ such that $\sum a_1 \otimes \pi(a_2) = a \otimes \pi(1)$.

Definition 1.4. Suppose ι is an injection and π is a surjection, so that we may regard ι as an inclusion and π as a quotient. The sequence (A) is called an *extension of H by K* if it satisfies the following equivalent conditions: (a) $A/K^+A = H$; (b) $A/AK^+ = H$; (c) $K = A^{coH}$; (d) $K = {}^{coH}A$. For two extensions (A), (A') of H by K, an *equivalence* $(A) \to (A')$ is a Hopf algebra map $f : A \to A'$ which induces the identity maps on H and K. If such exists, we say that (A) and (A') are *equivalent*.

One sees easily from [Sw2, Lemmas 16.0.2–3] that Conditions (a)–(d) are equivalent. We see easily that the sequence (1.3) is an extension of kF by k^G.

If (A) is an extension of H by K, then $A \supset K$ is a right H-Galois extension in the sense of [Mo, Def. 8.1.1]. Since A is right K-free by the Nichols-Zöller theorem, it follows that the map f giving an equivalence is necessarily an isomorphism, which justifies the term.

By [Sd, Thm. 2.4] or [MD, Thm. 3.5], a finite-dimensional extension (A) is necessarily cleft (see Definition 2.5 below) in the sense there is a K-linear and H-colinear isomorphism $A \cong K \otimes H$ which preserves unit and counit. This is easily proved in the special case when $H = kF$, $K = k^G$, since then A is a strictly graded F-algebra with neutral component k^G so that it is necessarily a crossed product. This proves also the following.

Proposition 1.5. *Any extension (A) of kF by k^G is equivalent to an extension of the form* (1.3).

For another choice $(A) \sim (k^G \#_{\sigma',\tau'} kF)$ of equivalence, the same matched pair $(F, G, \triangleleft, \triangleright)$ forms the Hopf algebras $k^G \#_{\sigma,\tau} kF$ and $k^G \#_{\sigma',\tau'} kF$, since the neutral components in A and in A^* are commutative.

Definition 1.6. We say that (A) *is associated with* the matched pair $(F, G, \triangleleft, \triangleright)$ thus uniquely determined by (A).

Two equivalent extensions of kF by k^G are associated with the same matched pair. In what follows, we fix a matched pair $(F, G, \triangleleft, \triangleright)$. We denote by

$$\operatorname{Opext}(kF, k^G)$$

the set of all equivalence classes of extensions associated with it. (The notation stems from $\operatorname{Opext}(\Pi, A, \varphi)$ [Mac, Chap. IV, Sect. 3] for the group extensions of a group Π by an abelian kernel A with fixed operators $\varphi : \Pi \to \operatorname{Aut} A$.)

We will give a cohomological description of $\operatorname{Opext}(kF, k^G)$. Let $\Sigma = F \bowtie G$ denote the group constructed by the fixed matched pair.

Let $0 \leftarrow \mathbb{Z} \leftarrow B.$ be the normalized bar resolution of the trivial F-module $\mathbb{Z}$. Thus,

$$B. = 0 \longleftarrow B_0 \overset{d_1}{\longleftarrow} B_1 \overset{d_2}{\longleftarrow} B_2 \overset{d_3}{\longleftarrow} \cdots$$

consists of the free F-modules B_p with basis $[a_1|\cdots|a_p]$, where $1 \neq a_i \in F$, and the differentials d_p are defined by

$$d_p[a_1|\cdots|a_p] = a_1[a_2|\cdots|a_p] + \sum_{i=1}^{p-1}(-1)^i[a_1|\cdots|a_ia_{i+1}|\cdots|a_p] + (-1)^p[a_1|\cdots|a_{p-1}].$$

The augmentation $\varepsilon : B_0 = \mathbb{Z}F \to \mathbb{Z}$ is given by $\varepsilon(a) = 1$ for $a \in F$. Define an action of G on the canonical $\mathbb{Z}$-free basis of B_p by

$$x(a[a_1|\cdots|a_p]) = x \triangleright a[(x \triangleleft a) \triangleright a_1|(x \triangleleft aa_1) \triangleright a_2|\cdots|(x \triangleleft aa_1\cdots a_{p-1}) \triangleright a_p],$$

where $x \in G$ and $1 \neq a_i$, $a \in F$. Then one sees that this together with the original F-action makes B_p into a Σ-module, and that d_p, ε are Σ-linear, where $\mathbb{Z}$ is a trivial Σ-module. So, $0 \leftarrow \mathbb{Z} \leftarrow B.$ turns to be a complex of Σ-modules.

The symmetric argument using a mirror makes the normalized bar resolution $0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon'} B'_.$ of the trivial right G-module $\mathbb{Z}$ into a complex of right Σ-modules. Regard it as a complex of left Σ-modules by twisting the action through the inverse of Σ, and tensor it with $B.$ over $\mathbb{Z}$. Then we obtain the double complex

$$B'_. \otimes_{\mathbb{Z}} B. = \quad \begin{array}{ccccc} \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \\ B'_1 \otimes_{\mathbb{Z}} B_0 & \xleftarrow{1\otimes d_1} & B'_1 \otimes_{\mathbb{Z}} B_1 & \longleftarrow & \cdots \\ \downarrow{\scriptstyle d'_1\otimes 1} & & \downarrow{\scriptstyle -d'_1\otimes 1} & & \\ B'_0 \otimes_{\mathbb{Z}} B_0 & \xleftarrow{1\otimes d_1} & B'_0 \otimes_{\mathbb{Z}} B_1 & \longleftarrow & \cdots \end{array}$$

of Σ-modules, where Σ acts diagonally on each term. Here and in what follows, when we construct a double complex, we resort such a trick (sign trick) that changes the sign of the differentials in odd columns (see above) unless otherwise stated. One sees that each Σ-module $B'_q \otimes_{\mathbb{Z}} B_p$ is free with basis $[x_q|\cdots|x_1] \otimes [a_1|\cdots|a_p]$, where $1 \neq a_i \in F$, $1 \neq x_i \in G$. This implies the following.

Lemma 1.7. *The total complex of $B'_. \otimes_{\mathbb{Z}} B.$ gives a Σ-free resolution of $\mathbb{Z}$ via the augmentation $\varepsilon' \otimes \varepsilon : B'_0 \otimes_{\mathbb{Z}} B_0 \to \mathbb{Z}$.*

Regard $k^\times$ as a trivial Σ-module, and form the double complex

$$D^{\cdot\cdot} = \mathrm{Hom}_\Sigma(B'_. \otimes_{\mathbb{Z}} B., k^\times).$$

Since $B'_q \otimes_{\mathbb{Z}} B_p$ has the Σ-free basis noted above, $\mathrm{Hom}_\Sigma(B'_q \otimes_{\mathbb{Z}} B_p, k^\times)$ is identified with the abelian group $\mathrm{Map}_+(G^q \times F^p, k^\times)$ of all maps $G^q \times F^p \to k^\times$ satisfying the normalization condition, where X^r denotes the cartesian product of r copies of $X = F, G$. Thus, $D^{\cdot\cdot}$ looks as follows.

$$
D^{\cdot\cdot} = \quad \begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
\mathrm{Map}_+(G, k^\times) & \longrightarrow & \mathrm{Map}_+(G \times F, k^\times) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \\
k^\times & \longrightarrow & \mathrm{Map}_+(F, k^\times) & \longrightarrow & \cdots
\end{array}
$$

Note that the edges in $D^{\cdot\cdot}$ consist of the standard complexes for computing the group cohomologies $H^{\cdot}(F, k^\times)$, $H^{\cdot}(G, k^\times)$. Remove these edges from $D^{\cdot\cdot}$ to obtain the following double complex.

$$
A^{\cdot\cdot} = \quad \begin{array}{ccccc}
\vdots & & \vdots & & \\
\uparrow & & \uparrow & & \\
\mathrm{Map}_+(G^2 \times F, k^\times) & \longrightarrow & \mathrm{Map}_+(G^2 \times F^2, k^\times) & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \\
\mathrm{Map}_+(G \times F, k^\times) & \longrightarrow & \mathrm{Map}_+(G \times F^2, k^\times) & \longrightarrow & \cdots
\end{array}
$$

For example, the horizontal and vertical differentials ∂, ∂' going into $\mathrm{Map}_+(G^2 \times F^2, k^\times)$ are given by

$$
\begin{aligned}
\partial\tau(x, y; a, b) &= \tau(x \triangleleft (y \triangleright a), y \triangleleft a; b)\tau(x, y; ab)^{-1}\tau(x, y; a) \\
\partial'\sigma(x, y; a, b) &= \sigma(y; a, b)\sigma(xy; a, b)^{-1}\sigma(x; y \triangleright a, (y \triangleleft a) \triangleright b),
\end{aligned}
$$

where $a, b \in F$, $x, y \in G$, $\sigma \in \mathrm{Map}_+(G \times F^2, k^\times)$ and $\tau \in \mathrm{Map}_+(G^2 \times F, k^\times)$. Let $\mathrm{Tot}\, A^{\cdot\cdot}$ denote the total complex of $A^{\cdot\cdot}$.

Proposition 1.8 (cf. [K, Thm. 5]). *For a total* 1*-cocycle* (σ, τ) *in* $A^{\cdot\cdot}$*, we have an extension* $(k^G \#_{\sigma,\tau} kF)$ *associated with the fixed matched pair* $(F, G, \triangleleft, \triangleright)$*. The assignment* $(\sigma, \tau) \mapsto (k^G \#_{\sigma,\tau} kF)$ *induces a bijection*

$$
H^1(\mathrm{Tot}\, A^{\cdot\cdot}) \cong \mathrm{Opext}(kF, k^G).
$$

Proof. By Lemma 1.2 and Proposition 1.5, the assignment gives a surjection $\mathrm{Z}^1(\mathrm{Tot}\, A^{\cdot\cdot}) \to \mathrm{Opext}(kF, k^G)$ from the group of total 1-cocycles.

Let (σ, τ), (σ', τ') be total 1-cocycles. If $\nu : G \times F \to k^\times$ is a 0-cochain such that $\sigma' = \sigma\partial\nu$, $\tau' = \tau\partial'\nu$, then $e_x a \mapsto \nu(x; a)e_x a$ gives an equivalence $(k^G \#_{\sigma',\tau'} kF) \xrightarrow{\cong} (k^G \#_{\sigma,\tau} kF)$. Conversely, one sees that any equivalence is given in this way by some ν, in which case we have $\sigma' = \sigma\partial\nu$, $\tau' = \tau\partial'\nu$ by simple computation. This proves the injectivity of the induced map. $\square$

The Baer product of group crossed products gives rise to a product on the set $\mathrm{Opext}(kF, k^G)$, which forms thereby an abelian semigroup; see Section 2 for more general treatment. One sees that the bijection in the last proposition preserves product, whence $\mathrm{Opext}(kF, k^G)$ is a group. The unit is represented by the extension $(k^G \#_{1,1} kF)$ which is given by the constant cocycles $(\sigma, \tau) = (1, 1)$ with value 1. We write simply $(k^G \# kF)$ for this extension, and let $\mathrm{Aut}(k^G \# kF)$ denote the group of its auto-equivalences.

Proposition 1.9. *For a total* 0*-cocycle* ν *in* $A^{\cdot\cdot}$, $e_x a \mapsto \nu(x; a)e_x a$ *gives an auto-equivalence of* $(k^G \# kF)$. *This gives an isomorphism*

$$H^0(\mathrm{Tot}\, A^{\cdot\cdot}) \cong \mathrm{Aut}(k^G \# kF).$$

Proof. This follows by the argument given in the second paragraph of the last proof if we suppose $\sigma = \sigma' = 1$, $\tau = \tau' = 1$. □

Theorem 1.10 (cf. [K, (3.14)]). *We have an exact sequence*

$$\begin{aligned} 0 &\to H^1(F \bowtie G, k^\times) \to H^1(F, k^\times) \oplus H^1(G, k^\times) \to \mathrm{Aut}(k^G \# kF) \\ &\to H^2(F \bowtie G, k^\times) \to H^2(F, k^\times) \oplus H^2(G, k^\times) \to \mathrm{Opext}(kF, k^G) \\ &\to H^3(F \bowtie G, k^\times) \to H^3(F, k^\times) \oplus H^3(G, k^\times), \end{aligned}$$

where $H^{\cdot}$ *denotes the group cohomology with coefficients in the trivial module* $k^\times$.

Proof. Since $A^{\cdot\cdot}$ is regarded (by dimension shift) as a double subcomplex of $D^{\cdot\cdot}$ such that the cokernel is the edges $E^{\cdot\cdot}$ in $D^{\cdot\cdot}$, we have a short exact sequence $0 \to A^{\cdot\cdot} \to D^{\cdot\cdot} \to E^{\cdot\cdot} \to 0$ of double complexes, which induces a long exact sequence of total cohomologies. It gives the desired sequence by Propositions 1.8 and 1.9, since we have also

$$\begin{aligned} H^n(\mathrm{Tot}\, D^{\cdot\cdot}) &= H^n(F \bowtie G, k^\times), && (n > 0), \\ H^n(\mathrm{Tot}\, E^{\cdot\cdot}) &= H^n(F, k^\times) \oplus H^n(G, k^\times) && (n > 0). \end{aligned}$$

□

Remark 1.11. (1) Using the Kac exact sequence just given, it can be proved that the abelian group $\mathrm{Opext}(kF, k^G)$ is torsion, but not necessarily finite; it is finite if k is algebraically closed. On the other hand, $\mathrm{Aut}(k^G \# kF)$ is always finite. See [M4, Props. 7.7–8].

(2) Suppose $k = \mathbb{C}$. Kac [K] actually worked on Hopf $*$-algebra (today called Kac algebra) extensions of $\mathbb{C}F$ of $\mathbb{C}^G$, where $*$-structures are given to $\mathbb{C}F$ by $a^* = a^{-1}$ $(a \in F)$, and to $\mathbb{C}^G$ by $(e_x)^* = e_x$ $(x \in G)$. See [IK] for new achievement. Let $\mathrm{Opext}^*(\mathbb{C}F, \mathbb{C}^G)$, $\mathrm{Aut}^*(\mathbb{C}^G \# \mathbb{C}F)$ denote the groups of all $*$-equivalence classes of Hopf $*$-algebra extensions, and of all $*$-auto-equivalences, respectively. These are described cohomologically by the double complex which modifies $A^{\cdot\cdot}$ with $\mathbb{C}^\times$ replaced by $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. In the same way as above, we have an exact sequence consisting of these groups and group cohomologies with coefficients in

$\mathbb{T}$. By comparing it with the original sequence, it follows that the natural group maps

$$\mathrm{Opext}^*(\mathbb{C}F, \mathbb{C}^G) \to \mathrm{Opext}(\mathbb{C}F, \mathbb{C}^G),$$
$$\mathrm{Aut}^*(\mathbb{C}^G \# \mathbb{C}F) \hookrightarrow \mathrm{Aut}(\mathbb{C}^G \# \mathbb{C}F)$$

are both isomorphisms, since the universal coefficient theorem implies that, for Γ a finite group, $H^n(\Gamma, \mathbb{T}) \cong H^n(\Gamma, \mathbb{C}^\times)$ for $n > 0$. See [M3, Remark 2.4].

(3) Recently, Schauenburg [Sb2] proved the Kac exact sequence for cleft Hopf algebra extensions $K \to A \to H$ (see Definition 2.5 below) at least when H is cocommutative and K is commutative and finite-dimensional; the group cohomologies were then replaced by the Sweedler cohomologies [Sw1] of the cocommutative Hopf algebras H, K^* and $H \bowtie K^*$ with coefficients in k. He actually deduced the sequence from nice, general results on monoidal equivalences of (generalized) Hopf bimodule categories; see also [Sb4], a very readable article. But, we do not discuss these interesting results any more in this paper.

2. Generalities on Hopf Algebra Extensions

Let H, J be cocommutative Hopf algebras.

Definition 2.1 (cf. [Kas, Def. IX 2.2]). A *matched pair of* (*cocommutative*) *Hopf algebras* is a pair (H, J) together with actions $J \xleftarrow{\triangleleft} J \otimes H \xrightarrow{\triangleright} H$ such that $(H, \triangleright)$ is a left J-module coalgebra, $(J, \triangleleft)$ is a right H-module coalgebra, and

$$x \triangleright ab = \sum (x_1 \triangleright a_1)((x_2 \triangleleft a_2) \triangleright b),$$
$$xy \triangleleft a = \sum (x \triangleleft (y_1 \triangleright a_1))(y_2 \triangleleft a_2)$$

for $a, b \in H$, $x, y \in J$. The conditions are equivalent to that the tensor product coalgebra $H \otimes J$ is a bialgebra, which is necessarily a Hopf algebra, with unit $1 \otimes 1$ under the product

$$(a \otimes x)(b \otimes y) = \sum a(x_1 \triangleright b_1) \otimes (x_2 \triangleleft b_2)y.$$

The Hopf algebra is denoted by $H \bowtie J$, whose antipode is given by

$$S(a \otimes x) = \sum (S(x_2) \triangleright S(a_2)) \otimes (S(x_1) \triangleleft S(a_1)).$$

If $H = kF$ and $J = kG$ are group Hopf algebras, then the structures of matched pair of groups on (F, G) are obviously in 1-1 correspondence with the structures of matched pair of Hopf algebras on (kF, kG).

In what follows, we suppose H is a cocommutative Hopf algebra and K is a commutative Hopf algebra.

Definition 2.2. A *Singer pair of Hopf algebras* is a pair (H, K) together with an action and a coaction,

$$\rightharpoonup : H \otimes K \to K \text{ and } \rho : H \to H \otimes K, \ \rho(a) = \sum a_H \otimes a_K,$$

such that $(K, \rightharpoonup)$ is an H-module algebra, (H, ρ) is a K-comodule coalgebra, and

$$\rho(ab) = \sum \rho(a_1)(b_H \otimes (a_2 \rightharpoonup b_K)),$$
$$\Delta(a \rightharpoonup t) = \sum((a_1)_H \rightharpoonup t_1) \otimes (a_1)_K(a_2 \rightharpoonup t_2)$$

for $a, b \in H$, $t \in K$.

The notion was introduced by Singer [S] under the name 'abelian matched pair'. We propose the term given above to avoid confusion with the notion defined by Definition 2.2.

Definition 2.3 [S, Def. 3.3]. Given a Singer pair $(H, K, \rightharpoonup, \rho)$, we define a category $\mathcal{C} = \mathcal{C}(H, K, \rightharpoonup, \rho)$ as follows. An object in $\mathcal{C}$ is an H-module M equipped with a K-comodule structure $\lambda : M \to M \otimes K$, $\lambda(m) = \sum m_0 \otimes m_1$ such that

$$\lambda(am) = \sum (a_1)_H m_0 \otimes (a_1)_K(a_2 \rightharpoonup m_1)$$

for $a \in H$, $m \in M$. A morphism in $\mathcal{C}$ is an H-linear and K-colinear map. In fact, $\mathcal{C}$ forms a k-abelian category.

Remark 2.4. Suppose J is a finite-dimensional cocommutative Hopf algebra. There is a 1-1 correspondence between the matched pair structures $(\triangleleft, \triangleright)$ on (H, J) and the Singer pair structures $(\rightharpoonup, \rho)$ on (H, J^*); it is given by the two familiar correspondences between module actions $\triangleleft$: $J \otimes H \to J$ and $\rightharpoonup$: $H \otimes J^* \to J^*$, and between module actions $\triangleright$: $J \otimes H \to H$ and comodule coactions $\rho : H \to H \otimes J^*$. Similarly we see that the module category $H \bowtie J$-Mod arising from a matched pair $(H, J, \triangleleft, \triangleright)$ is isomorphic to the category $\mathcal{C}$ arising from the corresponding Singer pair $(H, J^*, \rightharpoonup, \rho)$.

Definition 2.5. A sequence $(A) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ of Hopf algebras is called a *cleft extension of H by K* if the following equivalent conditions (see [MD, Prop. 3.2]) are satisfied, where A is regarded as a K-module along ι, and as an H-comodule along π.

(a) There is a left K-linear and right H-colinear isomorphism $\xi : A \xrightarrow{\cong} K \otimes H$;

(b) There is such an isomorphism ξ as in (a) which also preserves unit and counit;

(c) A is right H-cleft [Mo, Def. 7.2.1] in the sense that there is a (convolution-) invertible right H-colinear map $H \to A$ which preserves unit and counit, and ι induces an isomorphism $K \xrightarrow{\cong} A^{coH}$;

(d) A is left K-cocleft in the sense that there is an invertible left K-linear map $A \to K$ which preserves unit and counit, and π induces an isomorphism $A/K^+A \xrightarrow{\cong} H$.

An *equivalence* between cleft extensions of H by K is defined in the same way as in Definition 1.4.

Remark 2.6. (1) It follows by [Sb3] that, if (A) is a cleft extension of H by K, the antipode of A is necessarily bijective since those of H and K are. Hence, Conditions (a)–(d) are equivalent to those conditions obtained by exchanging 'left' and 'right', since the antipode gives a right K-linear and left H-colinear isomorphism $A \xrightarrow{\cong} A^{op,cop}$.

(2) Suppose a sequence $(A) = K \to A \to H$ is given. If (A) is a cleft extension, then A is injective as a right H-comodule and $K \cong A^{coH}$. One sees that the converse holds true if H is irreducible, or in particular if H is the universal envelope $U\mathfrak{f}$ of a Lie algebra $\mathfrak{f}$, as will be the case in Part II.

Let $\rightharpoonup: H \otimes K \to K$ be an action which makes K into an H-module algebra. Let $\sigma : H \otimes H \to K$ be an invertible linear map which satisfies the 2-cocycle condition that

$$\sum [a_1 \rightharpoonup \sigma(b_1, c_1)]\sigma(a_2, b_2c_2) = \sum \sigma(a_1, b_1)\sigma(a_2b_2, c) \tag{2.7}$$

and the normalization condition that

$$\sigma(1, a) = \varepsilon(a)1 = \sigma(a, 1),\ \ \varepsilon \circ \sigma(a, b) = \varepsilon(a)\varepsilon(b), \tag{2.8}$$

where $a, b, c \in H$. They make the tensor product $K \otimes H$ into an algebra of crossed product [Mo, Def. 7.1.1]; it has unit $1 \otimes 1$ and its product is given by

$$(s \otimes a)(t \otimes b) = \sum s(a_1 \rightharpoonup t)\sigma(a_2, b_1) \otimes a_3b_2, \tag{2.9}$$

where $a, b \in H$ and $s, t \in K$.

Dually, let $\rho : H \to H \otimes K$ be a coaction which makes H into a K-comodule coalgebra, and let $\tau : H \to K \otimes K$ be an invertible linear map satisfying the dual 2-cocycle condition and the normalization condition. They make $K \otimes H$ into a coalgebra of crossed coproduct. We denote by $K \#_{\sigma,\tau} H$ the tensor product $K \otimes H$ with the described algebra and coalgebra structures.

Lemma 2.10. *$K \#_{\sigma,\tau} H$ is a bialgebra if and only if $(H, K, \rightharpoonup, \rho)$ is a Singer pair and (σ, τ) is a total 1-cocycle in the double complex $A_0^{\cdot\cdot}$ defined below. In this case, $K \#_{\sigma,\tau} H$ is necessarily a Hopf algebra with the antipode S given by*

$$\begin{aligned} &S(t \# a) \\ &\quad = \sum(\sigma^{-1}(S(a_{1H1}), a_{1H2}) \# S(a_{1H3}))(S(ta_{1K})(1, S) \circ \tau^{-1}(a_2) \# 1), \end{aligned} \tag{2.11}$$

where $(1, S)(s \otimes t) = sS(t)$.

This follows by [H, Props. 3.3, 3.8 and 3.13]. If the conditions given above are satisfied, we obviously have a cleft extension

$$(K \#_{\sigma,\tau} H) = K \xrightarrow{\iota} K \#_{\sigma,\tau} H \xrightarrow{\pi} H \tag{2.12}$$

of H by K, where $\iota(t) = t \# 1$, $\pi(t \# a) = \varepsilon(t)a$. Conversely, it follows from [Mo, Prop. 7.2.3] and the dual result (see also [H, Prop. 3.6]) that any cleft extension (A) of H by K is equivalent to some extension of the form (2.12), since we have such an isomorphism $\xi : A \xrightarrow{\cong} K \otimes H$ as in Condition (b) in Definition 2.5.

Here the Singer pair $(H, K, \rightharpoonup, \rho)$ which together with σ, τ forms $K \#_{\sigma,\tau} H$ is uniquely determined by (A), being independent of choice of ξ.

Definition 2.13. In this case, we say that (A) *is associated with* the Singer pair $(H, K, \rightharpoonup, \rho)$. We denote by

$$\mathcal{O}pext(H, K) = \mathcal{O}pext(H, K, \rightharpoonup, \rho)$$

the category of cleft extensions associated with a fixed Singer pair $(H, K, \rightharpoonup, \rho)$, whose morphisms are equivalences between extensions so that this is a groupoid.

Remark 2.14. If $H = kF$, $K = k^G$, then the Singer pair structures on (kF, k^G) are in 1-1 correspondence with the matched pair structures on (F, G). A cleft extension associated with a Singer pair $(kF, k^G, \rightharpoonup, \rho)$ is precisely an extension associated with the corresponding matched pair $(F, G, \triangleleft, \triangleright)$ as defined by Definition 1.6. For the 1-1 correspondence above, F may be infinite. We see also that all results, except Remark 1.11, in the preceding section hold true even if F is infinite. In particular, Conditions (a)–(d) in Definition 1.4 are still equivalent for a sequence $(A) = k^G \to A \to kF$ with F infinite and, if they are satisfied, (A) is necessarily a cleft extension.

In what follows we fix a Singer pair $(H, K, \rightharpoonup, \rho)$. Recall H is cocommutative and K is commutative by assumption.

Let (A_1), (A_2) be in $\mathcal{O}pext(H, K)$. Form the tensor product $A_1 \otimes_K A_2$ of the left K-modules, on which two right H-comodule structures arise from the factors A_1 and A_2. Take the cotensor product of these comodule structures. Then we obtain the bi-tensor product $A_1 \otimes_K^H A_2$ as defined in [H, Sect. 4], which forms a Hopf algebra with the structure induced from the Hopf algebra $A_1 \otimes A_2$ of tensor product. Further, it forms naturally an extension $(A_1 \otimes_K^H A_2)$ in $\mathcal{O}pext(H, K)$. We write

$$(A_1) * (A_2) = (A_1 \otimes_K^H A_2).$$

One sees that $(K \#_{\sigma_1,\tau_1} H) * (K \#_{\sigma_2,\tau_2} H) = (K \#_{\sigma,\tau} H)$, where $\sigma = \sigma_1\sigma_2$, $\tau = \tau_1\tau_2$, convolution products.

If $\sigma : H \otimes H \to K$ and $\tau : H \to K \otimes K$ are trivial so that $\sigma(a, b) = \varepsilon(a)\varepsilon(b)1$, $\tau(a) = \varepsilon(a)1 \otimes 1$, then $K \#_{\sigma,\tau} H$ is the Hopf algebra of bi-smash product [T1, p.849], for which we write simply $K \# H$. This forms an extension $(K \# H)$ in $\mathcal{O}pext(H, K)$.

Proposition 2.15. *$\mathcal{O}pext(H, K)$ forms a symmetric monoidal groupoid with tensor product $*$ and unit object $(K \# H)$.*

This is essentially proved in [H, Sect. 5]. The associativity constraint and the symmetry are induced from the obvious isomorphisms $(A_1 \otimes A_2) \otimes A_3 \xrightarrow{\cong} A_1 \otimes (A_2 \otimes A_3)$ and $A_1 \otimes A_2 \xrightarrow{\cong} A_2 \otimes A_1$, respectively.

We denote by

$$\mathrm{Opext}(H, K) = \mathrm{Opext}(H, K, \rightharpoonup, \rho)$$

the set of all isomorphism (or equivalence) classes in $\mathscr{Opext}(H,K)$, which is a monoid under the product arising from $*$. Each object (A) in $\mathscr{Opext}(H,K)$ has inverse (A^{-1}) in the sense $(A)*(A^{-1})$ is isomorphic to the unit object $(K\,\#\,H)$, since we have $(A^{-1}) = (K\,\#_{\sigma^{-1},\tau^{-1}}\,H)$ if $(A) \sim (K\,\#_{\sigma,\tau}\,H)$. Hence, $\mathrm{Opext}(H,K)$ is in fact an abelian group.

We denote by $\mathrm{Aut}(K\,\#\,H)$ the group of auto-equivalences of $(K\,\#\,H)$. The group of auto-equivalences of any (A) in $\mathscr{Opext}(H,K)$ is canonically isomorphic to $\mathrm{Aut}(K\,\#\,H)$, since $(A)*$ gives a category equivalence.

In general, if $\mathcal{M}$ is a symmetric monoidal category with a small skeleton, the groups $K_0(\mathcal{M})$ and $K_1(\mathcal{M})$ of $\mathcal{M}$ are defined; see [B, Chap. VII, Sect. 1]. Suppose each object in $\mathcal{M}$ has inverse in the sense as above. This is equivalent to saying that all isomorphism classes of the objects in $\mathcal{M}$ form an abelian group under the product arising from the tensor product. Then, $K_0(\mathcal{M})$ is canonically isomorphic to this abelian group, while $K_1(\mathcal{M})$ is isomorphic to the automorphism group of the unit object (or any object). Therefore those groups of $\mathscr{Opext}(H,K)$ are given by

$$K_0 = \mathrm{Opext}(H,K),\ \ K_1 = \mathrm{Aut}(K\,\#\,H).$$

We follow Singer [S] to give cohomological description of these groups by technique of simplicial homology. Recall first the category $\mathcal{C} = \mathcal{C}(H,K,\rightharpoonup,\rho)$ is defined by Definition 2.3. We will denote by $V^{\otimes n} = V\otimes\cdots\otimes V$ the n-fold tensor product of a vector space V.

Let Comod-K denote the category of K-comodules. We define a functor

$$\mathbb{F} : \text{Comod-}K \to \mathcal{C},\ \ \mathbb{F}(P) = H\otimes P \tag{2.16}$$

by endowing the H-module $H\otimes P$ with the K-comodule structure $a\otimes p \mapsto \sum (a_1)_H \otimes p_0 \otimes (a_1)_K(a_2 \rightharpoonup p_1)$, $H\otimes P \to H\otimes P\otimes K$. Since one sees that this is left adjoint to the forgetful functor $\mathbb{U} : \mathcal{C} \to \text{Comod-}K$, it follows by [W, 8.6.2, p.280] that the functor $\mathbb{F}\circ\mathbb{U} : \mathcal{C}\to\mathcal{C}$, which we denote simply by $\mathbb{F}$, forms a cotriple $(\mathbb{F},\varepsilon,\delta)$ on $\mathcal{C}$, where $\varepsilon : \mathbb{F}\to \mathrm{id}$, $\delta : \mathbb{F}\to\mathbb{F}^2$ are the natural transformations defined by

$$\begin{aligned} &\varepsilon_M : H\otimes M \to M, && \varepsilon_M(a\otimes m) = am,\\ &\delta_M : H\otimes M \to H\otimes H\otimes M, && \delta_M(a\otimes m) = a\otimes 1\otimes m \end{aligned}$$

for $M\in\mathcal{C}$. Regard k as an object in $\mathcal{C}$ with the trivial structure. Then we have a simplicial object $\Phi.(k) = \{\mathbb{F}^{p+1}(k)\}_{p\geq 0}$ in $\mathbb{C}$; accompanied with the face and degeneracy operators determined by ε and δ, it looks like

$$\Phi.(k) = H \;\substack{\longleftarrow\\ \longleftarrow\\ \longrightarrow}\; H^{\otimes 2} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow\\ \longrightarrow}\; H^{\otimes 3} \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots \tag{2.17}$$

Dually we define a functor

$$\mathbb{G} : H\text{-Mod} \to \mathcal{C},\ \ \mathbb{G}(Q) = Q \otimes K \tag{2.18}$$

by endowing the K-comodule $Q \otimes K$ with an H-module structure via $a(q \otimes t) = \sum (a_1)_H q \otimes (a_1)_K (a_2 \rightharpoonup t)$, where $a \in H$, $q \otimes t \in Q \otimes K$. Since this is right adjoint to the forgetful functor $\mathbb{U} : \mathcal{C} \to H$-Mod, we have a triple $(\mathbb{G} = \mathbb{G} \circ \mathbb{U}, \eta, \mu)$ on $\mathcal{C}$, where $\eta : \mathrm{id} \to \mathbb{G}$, $\mu : \mathbb{G}^2 \to \mathbb{G}$ are the natural transformations defined by

$$\begin{aligned} &\eta_M : M \to M \otimes K, && \eta_M(m) = \textstyle\sum m_0 \otimes m_1, \\ &\mu_M : M \otimes K \otimes K \to M \otimes K, && \mu_M(m \otimes s \otimes t) = m \otimes \varepsilon(s) t. \end{aligned}$$

We have also a cosimplicial object $\Psi^{\cdot}(k) = \{\mathbb{G}^{q+1}(k)\}_{q \geq 0}$ in $\mathcal{C}$, which looks like

$$\Psi^{\cdot}(k) = K \;\substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow}\; K^{\otimes 2} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow}\; K^{\otimes 3} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow}\; \cdots \tag{2.19}$$

If $M, N \in \mathcal{C}$, then $M \otimes N$ is an object in $\mathcal{C}$ with the diagonal H-action and K-coaction. Thus, $\mathcal{C} = (\mathcal{C}, \otimes, k)$ forms a symmetric monoidal category with the obvious symmetry. Let $\mathcal{C}_c$ denote the category of cocommutative coalgebras in $\mathcal{C}$. Since H is in $\mathcal{C}_c$, it follows that for $C \in \mathcal{C}_c$, the coalgebra $\mathbb{F}(C) = H \otimes C$ of tensor product is in $\mathcal{C}_c$. Therefore, $(\mathbb{F}, \varepsilon, \delta)$ is regarded as a cotriple on $\mathcal{C}_c$ so that $\Phi.(k)$ is a simplicial object in $\mathcal{C}_c$. Similarly, $(\mathbb{G}, \eta, \mu)$ is regarded as a triple on the category $\mathcal{C}_a$ of commutative algebras in $\mathcal{C}$ so that $\Psi^{\cdot}(k)$ is a cosimplicial object in $\mathcal{C}_a$.

Let Reg_H^K (resp., Reg) denote the abelian group of (convolution-)invertible, H-linear and K-colinear (resp., k-linear) maps. If $C \in \mathcal{C}_c$, $A \in \mathcal{C}_a$, then an isomorphism

$$\mathrm{Reg}_H^K(\mathbb{F}(C), \mathbb{G}(A)) \cong \mathrm{Reg}(C, A) \tag{2.20}$$

is given by $f \mapsto (c \mapsto (1 \otimes \varepsilon) \circ f(1 \otimes c))$.

Form the double cosimplicial object $\mathrm{Reg}_H^K(\Phi.(k), \Psi^{\cdot}(k))$ in the category of abelian groups; by (2.20), it looks like

$$\begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow\uparrow\uparrow\downarrow\downarrow & & \uparrow\uparrow\uparrow\downarrow\downarrow & & \\ \mathrm{Reg}(k, K) & \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} & \mathrm{Reg}(H, K) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} & \cdots \\ \uparrow\uparrow\downarrow & & \uparrow\uparrow\downarrow & & \\ \mathrm{Reg}(k, k) & \substack{\longrightarrow \\ \longrightarrow \\ \longleftarrow} & \mathrm{Reg}(H, k) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longleftarrow \\ \longleftarrow} & \cdots \end{array}$$

Further, form the associated normalized double complex, whose term

$$\mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q})$$

in the (p,q)-th position consists of the invertible linear maps $H^{\otimes p} \to K^{\otimes q}$ satisfying the normalization condition such as given in (2.8). Remove the edges from the double complex just formed to obtain

$$A_0^{\cdot\cdot} = \begin{array}{ccccc} \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \\ \mathrm{Reg}_+(H, K^{\otimes 2}) & \longrightarrow & \mathrm{Reg}_+(H^{\otimes 2}, K^{\otimes 2}) & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \\ \mathrm{Reg}_+(H, K) & \longrightarrow & \mathrm{Reg}_+(H^{\otimes 2}, K) & \longrightarrow & \cdots \end{array}$$

One sees that, if $H = kF$, $K = k^G$ in particular, $A_0^{\cdot\cdot}$ is identified with $A^{\cdot\cdot}$ which was defined in the preceding section.

Proposition 2.20 [H, Props. 3.15, 6.5].
(1) *The assignment* $(\sigma, \tau) \mapsto (K \#_{\sigma,\tau} H)$, *where* (σ, τ) *is a total* 1*-cocycle in* $A_0^{\cdot\cdot}$, *induces an isomorphism*

$$H^1(\mathrm{Tot}\, A_0^{\cdot\cdot}) \cong \mathrm{Opext}(H, K).$$

(2) *For a total* 0*-cocycle* $\nu : H \to K$ *in* $A_0^{\cdot\cdot}$, $t \# a \mapsto \sum t\nu(a_1) \# a_2$ *gives an auto-equivalence of* $(K \# H)$. *This gives an isomorphism*

$$H^0(\mathrm{Tot}\, A_0^{\cdot\cdot}) \cong \mathrm{Aut}(K \# H).$$

3. Cocycle Deformations Arising in Extensions

Let A be a Hopf algebra. A (*normalized*) 2*-cocycle for* A is an invertible linear map $\sigma : A \otimes A \to k$ which satisfies the conditions given by (2.7) and (2.8) if we suppose therein $H = A$ and $K = k$, the trivial A-module algebra. The *cocycle deformation* A^σ by such σ is the coalgebra A endowed with the twisted product $\cdot$ defined by

$$a \cdot b = \sum \sigma(a_1, b_1) a_2 b_2 \sigma^{-1}(a_3, b_3),$$

where $a, b \in A$; this is in fact a Hopf algebra with the same unit 1 and the twisted antipode S^σ given by

$$S^\sigma(a) = \sum \sigma(a_1, S(a_2)) S(a_3) \sigma^{-1}(S(a_4), a_5),$$

where $a \in A$; see [D2, Thm. 1.6]. If $B = A^\sigma$, then σ^{-1} is regarded as a 2-cocycle for B, and we have $B^{\sigma^{-1}} = A$. This allows us to say that A and B are cocycle deformations of each other. If this is the case, the right (or equivalently left) comodule categories Comod-A and Comod-B are k-linearly monoidally equivalent. The converse holds true, if A or B, then necessarily both are finite-dimensional or pointed. See [Sb1, Sect. 5]. Here recall that the comodules over a bialgebra form a monoidal category in the obvious manner.

Let $(H, K, \rightharpoonup, \rho)$ be a Singer pair of Hopf algebras. Let $(A) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ be a cleft extension associated with the pair. A 2-cocycle $\theta : H \otimes H \to k$ for H is regarded as a 2-cocycle for A, composed with $\pi \otimes \pi$. We see that the cocycle deformation A^θ by such θ forms a cleft extension $(A^\theta) = K \xrightarrow{\iota} A^\theta \xrightarrow{\pi} H^\theta = H$ associated with the same Singer pair. Here note $H^\theta = H$, since H is cocommutative. (The deformation, as above, by lifted 2-cocycles dualizes the construction in [L], [EV].)

The 2nd Sweedler cohomology $H^2(H, k)$ [Sw1, Sect. 2] with coefficients in the trivial H-module algebra k is the 2nd cohomology of the bottom complex which was removed when we constructed $A_0^{\cdot\cdot}$; see Section 2. The removed vertical differential $\partial' : \mathrm{Reg}_+(H^{\otimes 2}, k) \to \mathrm{Reg}_+(H^{\otimes 2}, K)$ is given by

$$\partial'\theta(a, b) = \sum \theta(a_{1H}, b_{1H}) a_{1K} (a_2 \rightharpoonup b_{1K}) \theta^{-1}(a_3, b_2),$$

where $\theta \in \mathrm{Reg}_+(H^{\otimes 2}, k)$. We see that, if θ is a 2-cocycle for H, then $(\partial'\theta, \varepsilon)$, where ε is the identity in $\mathrm{Reg}_+(H, K^{\otimes 2})$, is a total 1-cocycle in $A_0^{\cdot\cdot}$, and that $\theta \mapsto (\partial'\theta, \varepsilon)$ induces a group map $H^2(H, k) \to H^1(\mathrm{Tot}\, A_0^{\cdot\cdot}) \cong \mathrm{Opext}(H, K)$, which we denote by

$$\delta : H^2(H, k) \to \mathrm{Opext}(H, K).$$

Proposition 3.1. *Let (A), (A') be in $\mathcal{O}\mathit{pext}(H, K)$. There is a 2-cocycle θ for H such that (A^θ) is equivalent to (A') if and only if (A) and (A') are equal in the cokernel* $\mathrm{Opext}(H, K)/\operatorname{Im}\delta$ *of* δ.

Proof. This follows since we see that, if (A) is given by a total 1-cocycle (σ, τ) in $A_0^{\cdot\cdot}$, then (A^θ) is given by $(\sigma\partial'\theta, \tau)$. □

We will give some results of sample computations in the special case when $H = kF$, $K = k^G$. The map δ is then identified with

$$\delta : H^2(F, k^\times) \to \mathrm{Opext}(kF, k^G)$$

which arises from $\partial' : \mathrm{Map}_+(F^2, k^\times) \to \mathrm{Map}_+(G \times F^2, k^\times)$ given by

$$\partial'\theta(x; a, b) = \theta(x \triangleright a, (x \triangleleft a) \triangleright b)\theta(a, b)^{-1},$$

where $a, b \in F$, $x \in G$ and $\theta \in \mathrm{Map}_+(F^2, k^\times)$. Note that this δ is involved in the Kac exact sequence.

If $\mathrm{triv} : G \times F \to G$ denotes the trivial action, any action $\triangleright : G \times F \to F$ by group automorphisms forms a matched pair $(F, G, \mathrm{triv}, \triangleright)$ of groups, so that $F \bowtie G = F \rtimes G$, the semi-direct product given by $\triangleright$.

Fix an integer $n > 1$. Suppose $F = \mathbb{Z}_n \oplus \mathbb{Z}_n$, where $\mathbb{Z}_n$ denotes the additive group of integers modulo n. Fix a matched pair $(F, \mathbb{Z}_2, \mathrm{triv}, \triangleright)$ of groups, where

$\triangleright\colon \mathbb{Z}_2 \times F \to F$ is defined by

$$0 \triangleright (i,j) = (i,j),\ 1 \triangleright (i,j) = (j,i).$$

Let $\mu_n(k)$ denote the group of all n-th roots of 1 in k. Let $\zeta \in \mu_n(k)$. Then,

$$\theta_\zeta : F \times F \to k^\times,\ \theta_\zeta((i,j),(k,l)) = \zeta^{il}$$

is a group 2-cocycle. We define a Hopf algebra A_ζ including $k^{\mathbb{Z}_2}$ as a central Hopf subalgebra as follows; A_ζ is generated by elements a_+, a_- over $k^{\mathbb{Z}_2}$, and is defined by the relations

$$a_\pm^n = 1,\ a_-a_+ = (e_0 + \zeta e_1)a_+a_-$$

together with the structures

$$\Delta(a_\pm) = a_\pm \otimes e_0 a_\pm + a_\mp \otimes e_1 a_\pm,\ \varepsilon(a_\pm) = 1,\ S(a_\pm) = e_0 a_\pm^{-1} + e_1 a_\mp^{-1}.$$

Thus, A_ζ is not cocommutative, and is commutative only if $\zeta = 1$. If $\pi : A_\zeta \to kF$ denotes the Hopf algebra map determined by $\pi(e_0) = 1$, $\pi(e_1) = 0$, $\pi(a_+) = (1,0)$ and $\pi(a_-) = (0,1)$, then we see that $(A_\zeta) = k^{\mathbb{Z}_2} \hookrightarrow A_\zeta \xrightarrow{\pi} kF$ is an extension associated with the fixed matched pair.

Proposition 3.2. *Suppose* $(k^\times)^n = k^\times$. *Then,* $\zeta \mapsto \theta_\zeta$ *and* $\zeta \mapsto (A_\zeta)$ *induce isomorphisms*

$$\mu_n(k) \cong H^2(F, k^\times),\ \mu_n(k) \cong \mathrm{Opext}(kF, k^{\mathbb{Z}_2}),$$

respectively. The map $\delta : H^2(F,k^\times) \to \mathrm{Opext}(kF, k^{\mathbb{Z}_2})$ *is induced by* $\theta_\zeta \mapsto (A_{\zeta^2})$. *Therefore, if* n *is odd, then* δ *is an isomorphism so that every* A_ζ *is a cocycle deformation of the commutative Hopf algebra* A_1.

Proof. It is easy to see the first isomorphism. The group $\mathrm{Opext}(k\mathbb{Z}_2, k^F)$ associated with the matched pair $(\mathbb{Z}_2, F, \triangleright, \mathrm{triv})$ is computed by [M2, Thm. 2.1 and corrigendum], whose proof gives the second isomorphism since $(A) \mapsto (A^*)$ induces an isomorphism $\mathrm{Opext}(kF, k^{\mathbb{Z}_2}) \cong \mathrm{Opext}(k\mathbb{Z}_2, k^F)$; see [M4, Exercise 5.5]. Since we compute

$$\partial'\theta_\zeta(1;(0,1),(1,0))/\partial'\theta_\zeta(1;(1,0),(0,1)) = \zeta^2,$$

it follows that $\delta\theta_\zeta$ is equivalent to (A_{ζ^2}). □

Next, we suppose $F = \mathbb{Z} \oplus \mathbb{Z}$. By a slight modification we define the matched pair $(F, \mathbb{Z}_2, \mathrm{triv}, \triangleright)$, the group 2-cocycle θ_ζ for F, and the extension (A_ζ) associated with the matched pair, where $\zeta \in k^\times$. The modification will be obvious except that we replace the relation $a_\pm^n = 1$ for A_ζ by the condition that $a_\pm$ are invertible. We see that $\zeta \mapsto \theta_\zeta$ induces an isomorphism $k^\times \cong H^2(F, k^\times)$. A slight modification of the last proof proves the following.

Proposition 3.3. *Suppose* $(k^\times)^2 = k^\times$. *Then*, $\zeta \mapsto (A_\zeta)$ *induces an isomorphism* $k^\times \cong \operatorname{Opext}(kF, k^{\mathbb{Z}_2})$. *The map* $\delta : H^2(F, k^\times) \to \operatorname{Opext}(kF, k^{\mathbb{Z}_2})$ *is a surjection, induced by* $\theta_\zeta \mapsto (A_{\zeta^2})$. *Therefore every* A_ζ *is a cocycle deformation of the commutative Hopf algebra* A_1.

Remark 3.4. Suppose $k = \mathbb{C}$, and $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. Then, A_ζ is a Hopf $*$-algebra with the $*$-structure $e_i^* = e_i$ $(i = 0, 1)$, $a_\pm^* = a_\pm^{-1}$. By (the proof of) the last proposition, A_ζ is isomorphic to the cocycle deformation $(A_1)^\sigma$ by $\sigma = \theta_{\sqrt{\zeta}}$ via $e_i \mapsto e_i$, $a_\pm \mapsto a_\pm$, which preserves $*$-structure, too. Suppose $\zeta = q^2$ with $q \in \mathbb{C}$. Then, A_{q^2} is isomorphic to the coordinate Hopf $*$-algebra $A(DT_q^2)$ of the quantum double torus DT_q^2 due to Hajac and Masuda [HM]: in fact, $e_0 \mapsto D^{-1}ad$, $a_+ \mapsto a + c$, $a_- \mapsto b + d$ give an isomorphism $A_{q^2} \cong A(DT_q^2)$. This concludes that their results [HM, Sect. 3] on unitary representations of DT_q^2 coincide completely with the results for the 'classical' double torus DT^2 with coordinate Hopf $*$-algebra A_1.

4. Quasi-Hopf Algebras Obtained by Extension

A *quasi-bialgebra* is a quadruple $(A, \Delta, \varepsilon, \Phi)$ which consists of an algebra A, algebra maps $\Delta : A \to A \otimes A$, $\varepsilon : A \to k$, and an invertible element $\Phi \in A \otimes A \otimes A$, called the *Drinfeld associator*, such that Conditions (2.1)–(2.4) in [BP, Def. 2.1] are fulfilled, or equivalently such that the module category A-Mod forms a monoidal category, where the tensor product is the usual one $V \otimes W$ of vector spaces on which A acts through Δ, the unit object is k on which A acts through ε, the left and right unit constraints are the obvious isomorphisms $k \otimes V = V = V \otimes k$, and the associativity constraint is given by $u \otimes v \otimes w \mapsto \Phi(u \otimes v \otimes w)$, $(U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$. Among the conditions just refered to, (2.2) is the usual counit property, and (2.1) is

$$(1 \otimes \Delta) \circ \Delta(a) = \Phi(\Delta \otimes 1) \circ \Delta(a)\Phi^{-1} \ (a \in A).$$

Hence a commutative quasi-bialgebra is necessarily an ordinary bialgebra.

The notion was first introduced by Drinfeld, which we define here in a stricter sense than original, assuming that the unit constraints are given by the obvious isomorphisms; cf [Kas, Prop. XV 1.2].

Let $A = (A, \Delta, \varepsilon, \Phi)$ be a quasi-bialgebra. A *gauge transformation* on A is an invertible element $\varphi \in A \otimes A$ satisfying the normalization condition

$$(1 \otimes \varepsilon)(\varphi) = 1 \otimes 1 = (\varepsilon \otimes 1)(\varphi).$$

By such φ, one constructs a new quasi-bialgebra $A_\varphi = (A, \Delta_\varphi, \varepsilon, \Phi_\varphi)$, where Δ_φ and Φ_φ are defined by

$$\Delta_\varphi(a) = \varphi\Delta(a)\varphi^{-1} \ (a \in A),$$
$$\Phi_\varphi = (1 \otimes \varphi)(1 \otimes \Delta)(\varphi)\Phi(\Delta \otimes 1)(\varphi^{-1})(\varphi^{-1} \otimes 1),$$

respectively, so that A-Mod and A_φ-Mod are monoidally equivalent in a natural manner; see [Kas, Prop. XV 3.2, Thm. XV 3.5]. This generalizes the dual notion of cocycle deformation (see Section 3) of an ordinary bialgebra.

A *map* $f : A \to A'$ *of quasi-bialgebras* is a linear map which preserves the structures, and so in particular $(f \otimes f \otimes f)(\Phi) = \Phi'$.

Let $(H, K, \rightharpoonup, \rho)$ be a Singer pair of Hopf algebras. Let

$$\sigma : H \otimes H \to K,\ \tau : H \to K \otimes K,\ \Phi \in K \otimes K \otimes K$$

be two invertible linear maps and an invertible element which all satisfy the normalization condition. On the tensor product $K \otimes H$, $\rightharpoonup$ and σ define such a product with unit $1 \otimes 1$ as defined by the formula (2.9); ρ and τ define a coproduct with counit $\varepsilon \otimes \varepsilon$ dually by the crossed coproduct construction. Let $K \#_{\sigma,\tau} H$ denote $K \otimes H$ with these structures. Identify Φ with its natural image $(\iota \otimes \iota \otimes \iota)(\Phi)$ in $(K \#_{\sigma,\tau} H)^{\otimes 3}$, where $\iota(t) = t \# 1$ for $t \in K$, and denote the image by Φ, too. We see directly the following.

Lemma 4.1 (cf. [BP, Remark 3.2]). *$K \#_{\sigma,\tau} H$ is a quasi-bialgebra with Drinfeld associator Φ, if and only if (σ, τ, Φ) is a total 2-cocycle in the double complex $A_1^{\cdot\cdot}$ given below. In this case, $K \#_{\sigma,\tau} H$ is necessarily a quasi-Hopf algebra in the sense defined by* [Kas, Def. XV 5.1], *whose requirement is fulfilled by the map S defined by the same formula as* (2.11), *and by $\alpha = \sum S(\Phi_1)\Phi_2 S(\Phi_3) \# 1$, $\beta = 1 \# 1$, if we write $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$.*

The double complex

$$
A_1^{\cdot\cdot} = \begin{array}{ccccccc}
\vdots & & & & & & \\
\uparrow & & & & & & \\
\mathrm{Reg}_+(k, K^{\otimes 3}) & \longrightarrow & \vdots\cdots & & & & \\
\uparrow & & \uparrow & & & & \\
\mathrm{Reg}_+(k, K^{\otimes 2}) & \longrightarrow & \mathrm{Reg}_+(H, K^{\otimes 2}) & \longrightarrow & \vdots\cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Reg}_+(k, K) & \longrightarrow & \mathrm{Reg}_+(H, K) & \longrightarrow & \mathrm{Reg}_+(H^{\otimes 2}, K) & \longrightarrow & \cdots
\end{array}
$$

enlarges $A_0^{\cdot\cdot}$ by joining the most left vertical complex which was removed when we constructed $A_0^{\cdot\cdot}$. Note that Φ is regarded as an element in $\mathrm{Reg}_+(k, K^{\otimes 3})$. The joined complex is the standard complex for computing the Doi cohomology $H^\cdot(k, K)$ [D1, Sect. 2.6], where k is the trivial K-comodule coalgebra. (Note $H^\cdot(k, K)$ is denoted by Coalg-$H^\cdot(k, K)$ in [D1].)

Since K is commutative, a Drinfeld associator of K is none other than an element in $\mathrm{Reg}_+(k, K^{\otimes 3})$ which vanishes through the vertical differential. Therefore, if (σ, τ, Φ) is a total 2-cocycle in $A_1^{\cdot\cdot}$, then $\iota : (K, \Phi) \to (K \#_{\sigma,\tau} H, \Phi)$, $\iota(t) = t \# 1$ is a quasi-bialgebra map. We regard H as a quasi-bialgebra with trivial Drinfeld associator, so that $\pi : K \#_{\sigma,\tau} H \to H$, $\pi(t \# a) = \varepsilon(t)a$ is a quasi-bialgebra map. We let

$$(K \#_{\sigma,\tau} H, \Phi) = K \xrightarrow{\iota} K \#_{\sigma,\tau} H \xrightarrow{\pi} H$$

denote the sequence of quasi-bialgebras thus obtained.

DEFINITION 4.2. A sequence $(A, \Phi) = K \to A \to H$ of quasi-bialgebras together with a Drinfeld associator Φ of K is called a *cleft extension of H by K*, if it is equivalent to some $(K \#_{\sigma,\tau} H, \Phi)$ in the sense that there is a quasi-bialgebra isomorphism $A \xrightarrow{\cong} K \#_{\sigma,\tau} H$ which induces the identity maps on H and K. We say that (A, Φ) *is associated with* the Singer pair $(H, K, \rightharpoonup, \rho)$ which forms $K \#_{\sigma,\tau} H$. (This is well-defined; see Proposition 4.5 below.)

If (A, Φ) is a cleft extension of H by K, then A is an (ordinary) H-comodule algebra as well as a K-module, and satisfies Conditions (a), (b) in Definition 2.5. Suppose $\Phi = 1$, the unit of $K^{\otimes 3}$. Then, $(A, 1)$ is a cleft quasi-bialgebra extension in the sense above if and only if it is a cleft Hopf algebra extension in the sense of Definition 2.5.

Let $(A, \Phi) = K \xrightarrow{\iota} A \xrightarrow{\pi} H$ be a cleft extension, and let φ be a gauge transformation on K. Then, $(\iota \otimes \iota)(\varphi)$ is a gauge transformation on A, which we denote by φ, too. We see easily the following.

LEMMA 4.3. *$(A_\varphi, \Phi_\varphi) = K \xrightarrow{\iota} A_\varphi \xrightarrow{\pi} H$ is a cleft extension associated with the same Singer pair as (A, Φ).* (Note $\Phi_\varphi = \Phi \partial' \varphi$.)

DEFINITION 4.4. A *quasi-equivalence* between cleft extensions is a pair $(f, \varphi) : (A, \Phi) \to (A', \Phi')$, where $f : A \to A'$ is a linear map and φ is a gauge transformation on K, such that f gives an equivalence $(A_\varphi, \Phi_\varphi) \xrightarrow{\cong} (A', \Phi')$. Its *composite* with another quasi-equivalence $(f', \varphi') : (A', \Phi') \to (A'', \Phi'')$ is the quasi-equivalence $(A, \Phi) \to (A'', \Phi'')$ defined by

$$(f', \varphi') \circ (f, \varphi) = (f' \circ f, \varphi' \varphi).$$

The quasi-equivalence defines a equivalence relation among all cleft extensions of H by K. It is easy to see the following.

PROPOSITION 4.5. *Two cleft extensions of H by K are associated with the same Singer pair if they are quasi-equivalent to each other.*

Let γ be an invertible element in K such that $\varepsilon(\gamma) = 1$. Its image under the vertical differential $\partial' : \mathrm{Reg}_+(k, K) \to \mathrm{Reg}_+(k, K^{\otimes 2})$ in $A_1^{\cdot\cdot}$ is given by

$$\partial' \gamma = (\gamma \otimes \gamma) \Delta(\gamma^{-1}),$$

which is hence a gauge transformation on K.

Lemma 4.6. *For any cleft extension* (A,Φ) *of* H *by* K, $(\operatorname{inn}\gamma, \partial'\gamma) : (A,\Phi) \xrightarrow{\cong} (A,\Phi)$ *is a quasi-auto-equivalence, where* $\operatorname{inn}\gamma(a) = \gamma^{-1}a\gamma$ *for* $a \in A$.

Proof. This follows easily if one notices that $\Phi_{\partial'\gamma} = \Phi\,\partial'^2\gamma = \Phi$. □

Definition 4.7 (cf. [Sb2, Def. 6.2.5]). Two quasi-equivalences $(f,\varphi), (f',\varphi') : (A,\Phi) \to (A',\Phi')$ are said to be *cohomologous* if there is $\gamma \in \mathrm{Reg}_+(k,K)$ such that $(f',\varphi') = (\operatorname{inn}\gamma, \partial'\gamma) \circ (f,\varphi)$.

This defines an equivalence relation among quasi-equivalences, which is compatible with the composition since we have $(f,\varphi)\circ(\operatorname{inn}\gamma,\partial'\gamma) = (\operatorname{inn}\gamma,\partial'\gamma)\circ(f,\varphi)$.

Definition 4.8. We denote by

$$\mathscr{O}\!pext'(H,K) = \mathscr{O}\!pext'(H,K,\rightharpoonup,\rho)$$

the groupoid of cleft extensions of quasi-bialgebras which are associated with a fixed Singer pair $(H,K,\rightharpoonup,\rho)$; the morphisms are cohomology classes of quasi-equivalences.

For (A_i,Φ_i) in $\mathscr{O}\!pext'(H,K)$, where $i = 1, 2$, we define by using the bi-tensor product $\otimes_K^H$

$$(A_1,\Phi_1) * (A_2,\Phi_2) = (A_1 \otimes_K^H A_2, \Phi_1\Phi_2),$$

which is naturally an object in $\mathscr{O}\!pext'(H,K)$. As a variation of Proposition 2.15 we have the following.

Proposition 4.9. $\mathscr{O}\!pext'(H,K)$ *forms a symmetric monoidal groupoid with tensor product* $*$ *and unit object* $(K\,\#\,H,1)$, *in which each object has inverse.*

Fix a Singer pair $(H,K,\rightharpoonup,\rho)$. We denote by

$$\mathrm{Opext}'(H,K),\ \mathrm{Aut}'(K\,\#\,H,1)$$

the group of all quasi-equivalence classes in $\mathscr{O}\!pext'(H,K)$ and the cohomology group of all quasi-auto-equivalences of $(K\,\#\,H,1)$, respectively. They give respectively the K_0 and K_1 groups of $\mathscr{O}\!pext'(H,K)$.

Proposition 4.10.

(1) *The assignment*

$$(\sigma,\tau,\Phi) \mapsto (K \underset{\sigma,\tau}{\#} H, \Phi),$$

where (σ,τ,Φ) *is a total* 2*-cocycle in* $A_1^{\cdot\cdot}$, *induces an isomorphism*

$$H^2(\operatorname{Tot} A_1^{\cdot\cdot}) \cong \mathrm{Opext}'(H,K).$$

(2) *To a total* 1*-cocycle* (ν,φ) *in* $A_1^{\cdot\cdot}$, *there is assigned a quasi-auto-equivalence* (f_ν,φ) *of* $(K\,\#\,H,1)$, *where* $f_\nu(t\,\#\,a) = \sum t\nu(a_1)\,\#\,a_2$. *This assignment induces an isomorphism*

$$H^1(\operatorname{Tot} A_1^{\cdot\cdot}) \cong \mathrm{Aut}'(K\,\#\,H,1).$$

The proof is straightforward.

Note that $A_0^{\cdot\cdot}$ is regarded as a double subcomplex of $A_1^{\cdot\cdot}$ such that the cokernel is the standard complex for computing the Doi cohomology $H^{\cdot}(k, K)$. The short exact sequence of complexes thus obtained gives rise to the following.

THEOREM 4.11. *We have an exact sequence*

$$\begin{aligned} 1 \to G(K)^H &\to H^1(k, K) \to \mathrm{Aut}(K \# H) \to \mathrm{Aut}'(K \# H, 1) \\ &\to H^2(k, K) \to \mathrm{Opext}(H, K) \to \mathrm{Opext}'(H, K) \to H^3(k, K), \end{aligned}$$

where $G(K)^H$ denotes the group of H-invariant grouplikes in K.

Let us call the last group map

$$\beta : \mathrm{Opext}'(H, K) \to H^3(k, K),$$

which is induced by $(K \#_{\sigma,\tau} H, \Phi) \mapsto \Phi$. We will see that β is a split surjection in some special case, though it is not even a surjection in general. Suppose H is finite-dimensional, and $K = H^*$. Let $(H, H, \triangleleft, \mathrm{triv})$ be the matched pair of Hopf algebras defined by the adjoint action $\triangleleft$: $H \otimes H \to H$, $x \triangleleft a = \sum S(a_1)xa_2$ and the trivial action $\mathrm{triv} : H \otimes H \to H$, $x \triangleright a = \varepsilon(x)a$. By Remark 2.4, it gives rise to a Singer pair $(H, H^*, \rightharpoonup, \mathrm{triv})$. We have an identification $\mathrm{Reg}_+(k, (H^*)^{\otimes 3}) = \mathrm{Reg}_+(H^{\otimes 3}, k)$. For $\Phi \in \mathrm{Reg}_+(H^{\otimes 3}, k)$, define $\sigma_\Phi, \tau_\Phi \in \mathrm{Reg}_+(H^{\otimes 3}, k)$ by

$$\begin{aligned} \sigma_\Phi(x; a, b) &= \sum \Phi^{-1}(x, a, b)\Phi(a, x \triangleleft a, b)\Phi^{-1}(a, b, x \triangleleft ab) \\ \tau_\Phi(x, y; a) &= \sum \Phi^{-1}(x, y, a)\Phi(x, a, y \triangleleft a)\Phi^{-1}(a, x \triangleleft a, y \triangleleft a), \end{aligned}$$

where $a, b, x, y \in H$. Here we wrote as $\Delta(a) = \sum a \otimes a$, omitting the subscripts of numbers; it would be allowed since H is cocommutative. Further, identify σ_Φ, τ_Φ with

$$\begin{aligned} &H \otimes H \to H^*, && a \otimes b \mapsto (x \mapsto \sigma_\Phi(x; a, b)), \\ &H \to H^* \otimes H^* = (H \otimes H)^*, && a \mapsto (x \otimes y \mapsto \tau_\Phi(x, y; a)), \end{aligned}$$

respectively. Then we see $\sigma_\Phi \in \mathrm{Reg}_+(H^{\otimes 2}, H^*)$, $\tau_\Phi \in \mathrm{Reg}_+(H, (H^*)^{\otimes 2})$.

PROPOSITION 4.12. *If Φ is a Drinfeld associator on H^*, then $(\sigma_\Phi, \tau_\Phi, \Phi)$ is a total 2-cocycle in the double complex $A_1^{\cdot\cdot}$ defined by the Singer pair $(H, H^*, \rightharpoonup, \mathrm{triv})$. The assignment $\Phi \mapsto (H^* \#_{\sigma_\Phi,\tau_\Phi} H, \Phi)$ induces a group map $\bar{\beta} : H^3(k, H^*) \to \mathrm{Opext}'(H, H^*)$ such that $\beta \circ \bar{\beta} = 1$.*

This is a reformulation of part of [BP, Theorem 3.1]; it proves further that the quasi-Hopf algebra $(H^* \#_{\sigma_\Phi,\tau_\Phi} H, \Phi)$ is quasi-triangular, generalizing [DPR] in which $H = kG$, a finite group Hopf algebra.

Suppose a matched pair $(F, G, \triangleleft, \triangleright)$ of groups, where G is finite, is given. By the same way of proving Theorem 1.10, we have the following variation of the Kac exact sequence.

THEOREM 4.13. *We have an exact sequence*

$$\begin{aligned} 1 \to \mathrm{X}(G)^F \quad &\to H^1(F \bowtie G, k^\times) \to H^1(F, k^\times) \\ \to \mathrm{Aut}'(k^G \# kF, 1) &\to H^2(F \bowtie G, k^\times) \to H^2(F, k^\times) \\ \to \mathrm{Opext}'(kF, k^G) \quad &\to H^3(F \bowtie G, k^\times) \to H^3(F, k^\times), \end{aligned}$$

where $\mathrm{X}(G)^F$ *denotes the group of the group maps* $f : G \to k^\times$ *such that* $f(x \triangleleft a) = f(x)$ *for all* $x \in G$, $a \in F$.

PART II: HOPF ALGEBRA EXTENSIONS ARISING FROM LIE ALGEBRAS

In this part, $\mathfrak{f}$ and $\mathfrak{g}$ denote finite-dimensional Lie algebras. The characteristic $\mathrm{ch}\, k$ of k will be supposed to be zero in Sections 6–8.

5. Lie Bialgebra Extensions

We show some results for Lie (bi)algebras that are parallel to those for groups given in Section 1.

DEFINITION 5.1 [Mj, Def. 8.3.1]. A *matched pair of Lie algebras* is a pair $(\mathfrak{f}, \mathfrak{g})$ together with Lie module actions $\mathfrak{g} \xleftarrow{\triangleleft} \mathfrak{g} \otimes \mathfrak{f} \xrightarrow{\triangleright} \mathfrak{f}$ such that

$$\begin{aligned} x \triangleright [a, b] &= [x \triangleright a, b] + [a, x \triangleright b] + (x \triangleleft a) \triangleright b - (x \triangleleft b) \triangleright a, \\ [x, y] \triangleleft a &= [x, y \triangleleft a] + [x \triangleleft a, y] + x \triangleleft (y \triangleright a) - y \triangleleft (x \triangleright a) \end{aligned}$$

for $a, b \in \mathfrak{f}$, $x, y \in \mathfrak{g}$, or equivalently such that the direct sum $\mathfrak{f} \oplus \mathfrak{g}$ of vector spaces forms a Lie algebra under the bracket

$$[a \oplus x, b \oplus y] = ([a, b] + x \triangleright b - y \triangleright a) \oplus ([x, y] + x \triangleleft b - y \triangleleft a).$$

This Lie bialgebra is denoted by $\mathfrak{f} \bowtie \mathfrak{g}$.

The universal envelope $U\mathfrak{f}$ of $\mathfrak{f}$ forms a cocommutative Hopf algebra in which each element in $\mathfrak{f}$ is primitive.

PROPOSITION 5.2 [M, Prop. 2.4]. *Actions* $\mathfrak{g} \xleftarrow{\triangleleft} \mathfrak{g} \otimes \mathfrak{f} \xrightarrow{\triangleright} \mathfrak{f}$ *which make* $(\mathfrak{f}, \mathfrak{g})$ *into a matched pair of Lie algebras are extended uniquely to actions* $U\mathfrak{g} \xleftarrow{\triangleleft} U\mathfrak{g} \otimes U\mathfrak{f} \xrightarrow{\triangleright} U\mathfrak{f}$ *which make* $(U\mathfrak{f}, U\mathfrak{g})$ *into a matched pair of Hopf algebras. The resulting Hopf algebra* $U\mathfrak{g} \bowtie U\mathfrak{f}$ *is naturally isomorphic to* $U(\mathfrak{g} \bowtie \mathfrak{f})$. *If* $\mathrm{ch}\, k = 0$, *any matched pair structure on* $(U\mathfrak{f}, U\mathfrak{g})$ *is obtained in this way.*

The last assertion follows, since in characteristic zero, the primitives in $U\mathfrak{g} \bowtie U\mathfrak{f}$ are exactly $(\mathfrak{g} \otimes k) \oplus (k \otimes \mathfrak{f}) = \mathfrak{g} \oplus \mathfrak{f}$, which forms hence a Lie algebra, and so $(\mathfrak{f}, \mathfrak{g})$ is matched in such a way that the Lie algebra equals $\mathfrak{g} \bowtie \mathfrak{f}$.

Let $\rightharpoonup : \mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ be an action by which $\mathfrak{g}^*$ is an $\mathfrak{f}$-Lie module. Then its transpose $\triangleleft : \mathfrak{g} \otimes \mathfrak{f} \to \mathfrak{g}$ makes $\mathfrak{g}$ into a (right) $\mathfrak{f}$-Lie module. Let $\rho : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{g}^*$, $\rho(a) = \sum a_{[0]} \otimes a_{[1]}$ be a coaction by which $\mathfrak{f}$ is a $\mathfrak{g}^*$-Lie comodule. This is equivalent to that the action $\triangleright : \mathfrak{g} \otimes \mathfrak{f} \to \mathfrak{f}$ defined by $x \triangleright a = \sum a_{[0]} \langle x, a_{[1]} \rangle$ ($x \in \mathfrak{g}$, $a \in \mathfrak{f}$) makes $\mathfrak{f}$ into a $\mathfrak{g}$-Lie module.

Definition 5.3. $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ is called a *Singer pair of Lie bialgebras* (though called a matched pair in [Mj, p.383]), if $(\mathfrak{f}, \mathfrak{g}, \triangleleft, \triangleright)$ is a matched pair of Lie algebras.

A finite-dimensional vector space $\mathfrak{l}$ is called a *Lie coalgebra* with co-bracket $\delta : \mathfrak{l} \to \mathfrak{l} \otimes \mathfrak{l}$, if the dual space $\mathfrak{l}^*$ is a Lie algebra with bracket $\delta^* : \mathfrak{l}^* \otimes \mathfrak{l}^* = (\mathfrak{l} \otimes \mathfrak{l})^* \to \mathfrak{l}^*$. $\mathfrak{l}$ is called a *Lie bialgebra* [Dr], if it is a Lie algebra and Lie coalgebra such that

$$\delta[a, b] = a\delta(b) + \delta(a)b \ (a, b \in \mathfrak{l}),$$

where $a(x \otimes y) = [a, x] \otimes y + x \otimes [a, y]$, $(x \otimes y)b = [x, b] \otimes y + x \otimes [y, b]$.

We regard $\mathfrak{f}$ as a Lie bialgebra with zero co-bracket. Naturally, $\mathfrak{g}^*$ is a Lie coalgebra, which we regard as a Lie bialgebra with zero bracket. By a *(Lie bialgebra) extension of* $\mathfrak{f}$ *by* $\mathfrak{g}^*$, we mean a sequence $(\mathfrak{l}) = \mathfrak{g}^* \to \mathfrak{l} \to \mathfrak{f}$ of Lie bialgebras and Lie bialgebra maps which is a short exact sequence of vector spaces. An *equivalence* between two such extensions is defined in the obvious way.

Given an $\mathfrak{f}$-Lie module action $\rightharpoonup : \mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ together with a (Lie) 2-cocycle $\sigma : \mathfrak{f} \wedge \mathfrak{f} \to \mathfrak{g}^*$ with coefficients in the $\mathfrak{f}$-Lie module $(\mathfrak{g}^*, \rightharpoonup)$, a Lie algebra $\mathfrak{g}^* >\!\!\triangleleft_\sigma \mathfrak{f}$ of crossed sum is constructed on the vector space $\mathfrak{g}^* \oplus \mathfrak{f}$ by the bracket

$$[s \oplus a, t \oplus b] = (a \rightharpoonup t - b \rightharpoonup s + \sigma(a, b)) \oplus [a, b].$$

Given also a right $\mathfrak{g}$-Lie module action $\leftharpoonup : \mathfrak{f}^* \otimes \mathfrak{g} \to \mathfrak{f}^*$ together with a 2-cocycle $\tau : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{f}^*$ with coefficients in $(\mathfrak{f}^*, \leftharpoonup)$, a Lie algebra $\mathfrak{g} \triangleright\!\!<_\tau \mathfrak{f}^*$ is constructed similarly, whose dual Lie coalgebra is denoted by $\mathfrak{g}^* \blacktriangleright\!\!<_\tau \mathfrak{f}$. Denote by $\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f}$ the Lie algebra and Lie coalgebra thus obtained.

Lemma 5.4 [M, Prop. 1.8]. *$\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f}$ is a Lie bialgebra if and only if $\rightharpoonup$ and the dual coaction $\rho = (\leftharpoonup)^* : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{g}^*$ of $\leftharpoonup$ make $(\mathfrak{f}, \mathfrak{g}^*)$ into a Singer pair and (σ, τ) is a total 1-cocycle in the double complex $C_0^{\cdot\cdot}$ defined below.*

If these conditions are satisfied, the Lie bialgebra forms a Lie bialgebra extension

$$(\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f}) = \mathfrak{g}^* \to \mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f} \to \mathfrak{f},$$

in which the maps are the natural inclusion and the projection.

Any extension $(\mathfrak{l})$ of $\mathfrak{f}$ by $\mathfrak{g}^*$ is equivalent to some $(\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f})$, since an identification $\mathfrak{l} = \mathfrak{g}^* \oplus \mathfrak{f}$ of vector spaces gives rise to a 'bicrossed sum' structure. Here the Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ which forms $\mathfrak{g}^* \blacktriangleright\!\!\triangleleft_{\sigma,\tau} \mathfrak{f}$ is uniquely determined by $(\mathfrak{l})$, being independent of the way of identification $\mathfrak{l} = \mathfrak{g}^* \oplus \mathfrak{f}$.

DEFINITION 5.5. In this case, we say that $(\mathfrak{l})$ *is associated with* the Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$. We denote by

$$\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*) = \mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$$

the groupoid of Lie bialgebra extensions associated with a fixed Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$, whose morphisms are equivalences.

In what follows we fix a Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ of Lie bialgebras.

Let $(\mathfrak{l}_1)$, $(\mathfrak{l}_2)$ be in $\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*)$. From the direct sum $(\mathfrak{l}_1 \oplus \mathfrak{l}_2)$, form first the pullback $(\mathfrak{l}')$ along the diagonal map $a \mapsto a \oplus a$, $\mathfrak{f} \to \mathfrak{f} \oplus \mathfrak{f}$, and then the pushout $(\mathfrak{l})$ along the addition $s \oplus t \mapsto s + t$, $\mathfrak{g}^* \oplus \mathfrak{g}^* \to \mathfrak{g}^*$, as follows.

$$\begin{array}{rcccc}
(\mathfrak{l}_1 \oplus \mathfrak{l}_2) = \mathfrak{g}^* \oplus \mathfrak{g}^* & \longrightarrow & \mathfrak{l}_1 \oplus \mathfrak{l}_2 & \longrightarrow & \mathfrak{f} \oplus \mathfrak{f} \\
\| & & \uparrow & \text{p.b.} & \uparrow \\
(\mathfrak{l}') = \mathfrak{g}^* \oplus \mathfrak{g}^* & \longrightarrow & \mathfrak{l}' & \longrightarrow & \mathfrak{f} \\
\downarrow & \text{p.o.} & \downarrow & & \| \\
(\mathfrak{l}) = \quad \mathfrak{g}^* & \longrightarrow & \mathfrak{l} & \longrightarrow & \mathfrak{f}
\end{array}$$

One sees that $(\mathfrak{l})$ is in $\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*)$, which we denote by $(\mathfrak{l}_1) * (\mathfrak{l}_2)$. If we form first pushout and then pullback, obtained is an extension of the same kind, which is equivalent to $(\mathfrak{l})$ through the isomorphism induced from the identity map on $\mathfrak{l}_1 \oplus \mathfrak{l}_2$. We see that $(\mathfrak{g}^* \blacktriangleright\!\!\lhd_{\sigma_1,\tau_1} \mathfrak{f}) * (\mathfrak{g}^* \blacktriangleright\!\!\lhd_{\sigma_2,\tau_2} \mathfrak{f}) = (\mathfrak{g}^* \blacktriangleright\!\!\lhd_{\sigma,\tau} \mathfrak{f})$, where $\sigma = \sigma_1 + \sigma_2$, $\tau = \tau_1 + \tau_2$.

If σ and τ are both zero maps, we write simply $\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f}$ for $\mathfrak{g}^* \blacktriangleright\!\!\lhd_{0,0} \mathfrak{f}$.

PROPOSITION 5.6. *$\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*)$ forms a symmetric monoidal groupoid with tensor product $*$ and unit object $(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f})$, in which each object has inverse.*

The associativity constraint and the symmetry are induced from the obvious isomorphisms $(\mathfrak{l}_1 \oplus \mathfrak{l}_2) \oplus \mathfrak{l}_3 \xrightarrow{\cong} \mathfrak{l}_1 \oplus (\mathfrak{l}_2 \oplus \mathfrak{l}_3)$, $\mathfrak{l}_1 \oplus \mathfrak{l}_2 \xrightarrow{\cong} \mathfrak{l}_2 \oplus \mathfrak{l}_1$, respectively. If we write $\mathfrak{l}_0 = \mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f}$, the projection $\mathfrak{l}_0 \oplus \mathfrak{l}_1 \to \mathfrak{l}_1$ induces the (left) unit constraint. If $(\mathfrak{l})$ is equivalent to $(\mathfrak{g}^* \blacktriangleright\!\!\lhd_{\sigma,\tau} \mathfrak{f})$, it has inverse $(\mathfrak{g}^* \blacktriangleright\!\!\lhd_{-\sigma,-\tau} \mathfrak{f})$.

We denote by

$$\mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*) = \mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$$

all isomorphism (or equivalence) classes in $\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*)$, which form naturally a group. We denote by $\mathrm{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f})$ the group of auto-equivalences of $(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f})$. The K_0 and K_1 groups of $\mathscr{O}\mathit{pext}(\mathfrak{f}, \mathfrak{g}^*)$ are given by

$$K_0 = \mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*), \ K_1 = \mathrm{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f}).$$

For cohomological description of the groups, note first that by Definition 5.3, a matched pair $(\mathfrak{f}, \mathfrak{g}, \lhd, \rhd)$ is obtained from the fixed Singer pair. Write $H = U\mathfrak{f}$,

$J = U\mathfrak{g}$. By Proposition 5.2, we have a matched pair $(H, J, \triangleleft, \triangleright)$ of Hopf algebras so that $H \bowtie J = U(\mathfrak{f} \bowtie \mathfrak{g})$.

Let

$$V.(\mathfrak{f}) = 0 \leftarrow H \leftarrow H \otimes \mathfrak{f} \leftarrow H \otimes \wedge^2 \mathfrak{f} \leftarrow \cdots$$

be the Chevalley-Eilenberg complex; the differentials are given by

$$\partial(u\langle a_1, \ldots, a_p\rangle) = \sum_{i=1}^{p} (-1)^{i+1} u a_i \langle a_1, \ldots, \hat{a_i}, \cdots, a_p\rangle \\ + \sum_{i<j} (-1)^{i+j} u\langle [a_i, a_j], a_1, \ldots, \hat{a_i}, \ldots, \hat{a_j}, \ldots, a_p\rangle$$

for $u \in H$, $\langle a_1, \ldots, a_p\rangle := a_1 \wedge \cdots \wedge a_p \in \wedge^p \mathfrak{f}$, where $\hat{a_i}$ denotes the omitted term. This gives an H-free resolution $0 \leftarrow k \leftarrow V.(\mathfrak{f})$ of the trivial H-module k, whose augmentation $H \to k$ is the counit ε of H. Regard each $P = \wedge^p \mathfrak{f}$ as a J-module with the diagonal action. As a general fact, $H \otimes P$ is an $H \bowtie J$-module, where H acts on the factor H and J acts by

$$x(a \otimes p) = \sum (x_1 \triangleright a_1) \otimes (x_2 \triangleleft a_2)p \quad (x \in J,\ a \otimes p \in H \otimes P).$$

It follows by [M, Lemma 2.6] that ∂ and ε are $H \bowtie J$-linear, where k is the trivial $H \bowtie J$-module.

Similarly the right version

$$V.'(\mathfrak{g}) = 0 \leftarrow J \leftarrow \mathfrak{g} \otimes J \leftarrow \wedge^2 \mathfrak{g} \otimes J \leftarrow \cdots$$

of the Chevalley-Eilenberg complex gives a right $H \bowtie J$-resolution of k. Regard $0 \leftarrow k \leftarrow V.'(\mathfrak{g})$ as a left $H \bowtie J$-resolution by twisting the action through the antipode, and form the double complex $V.'(\mathfrak{g}) \otimes V.(\mathfrak{f})$ with a sign trick applied as before. Each term in the double complex is of the form $(Q \otimes J) \otimes (H \otimes P)$ with $P = \wedge^p \mathfrak{f}$, $Q = \wedge^q \mathfrak{g}$; this is an $H \bowtie J$-free module whose free basis is given by any basis of the vector space $(Q \otimes k) \otimes (k \otimes P)$, so that we have

$$\mathrm{Hom}_{H \bowtie J}((Q \otimes J) \otimes (H \otimes P), M) = \mathrm{Hom}(Q \otimes P, M)$$

for an $H \bowtie J$-module M. Thus the total complex of $V.'(\mathfrak{g}) \otimes V.(\mathfrak{f})$ gives a non-standard $H \bowtie J$-free resolution of k.

Form the double complex $\mathrm{Hom}_{H \bowtie J}(V.'(\mathfrak{g}) \otimes V.(\mathfrak{f}), k)$, and then remove from it the edges, which consist of the standard complexes for computing the Lie algebra cohomologies $H^{\cdot}(\mathfrak{f},k)$, $H^{\cdot}(\mathfrak{g},k)$. We obtain the desired complex:

$$
\begin{array}{cccccc}
 & \vdots & & \vdots & & \\
 & \uparrow & & \uparrow & & \\
C_0^{\cdot\cdot} = & \mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\mathfrak{f},k) & \longrightarrow & \mathrm{Hom}(\wedge^2\mathfrak{g}\otimes\wedge^2\mathfrak{f},k) & \longrightarrow & \cdots \\
 & \uparrow & & \uparrow & & \\
 & \mathrm{Hom}(\mathfrak{g}\otimes\mathfrak{f},k) & \longrightarrow & \mathrm{Hom}(\mathfrak{g}\otimes\wedge^2\mathfrak{f},k) & \longrightarrow & \cdots
\end{array}
$$

Proposition 5.7 [M, Prop. 2.9].

(1) *Let* (σ,τ) *be a total* 1*-cocycle in* $C_0^{\cdot\cdot}$, *and identify* $\sigma:\mathfrak{g}\otimes\wedge^2\mathfrak{f}\to k$, $\tau:\wedge^2\mathfrak{g}\otimes\mathfrak{f}\to k$ *naturally with linear maps* $\wedge^2\mathfrak{f}\to\mathfrak{g}^*$, $\wedge^2\mathfrak{g}\to\mathfrak{f}^*$, *respectively, which are indeed Lie* 2*-cocycles. Then,* $(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft_{\sigma,\tau}\mathfrak{f})$ *is in* $\mathcal{O}pext(\mathfrak{f},\mathfrak{g}^*)$. *The assignment* $(\sigma,\tau)\mapsto(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft_{\sigma,\tau}\mathfrak{f})$ *induces an isomorphism*

$$H^1(\mathrm{Tot}\,C_0^{\cdot\cdot})\cong\mathrm{Opext}(\mathfrak{f},\mathfrak{g}^*).$$

(2) *Let* $\nu:\mathfrak{g}\otimes\mathfrak{f}\to k$ *be a total* 0*-cocycle in* $C_0^{\cdot\cdot}$, *and identify it naturally with a linear map* $\mathfrak{f}\to\mathfrak{g}^*$. *Then an auto-equivalence of* $(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft\,\mathfrak{f})$ *is given by* $s\oplus a\mapsto(s+\nu(a))\oplus a$. *The assignment gives an isomorphism*

$$H^0(\mathrm{Tot}\,C_0^{\cdot\cdot})\cong\mathrm{Aut}(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft\,\mathfrak{f}).$$

From the construction of $C_0^{\cdot\cdot}$, we see that the next result follows in the same way as in the group case.

Theorem 5.8 [M, Thm. 2.10]. *We have an exact sequence*

$$
\begin{aligned}
0 &\to H^1(\mathfrak{f}\bowtie\mathfrak{g},k)\to H^1(\mathfrak{f},k)\oplus H^1(\mathfrak{g},k)\ \to\mathrm{Aut}(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft\,\mathfrak{f})\\
&\to H^2(\mathfrak{f}\bowtie\mathfrak{g},k)\to H^2(\mathfrak{f},k)\oplus H^2(\mathfrak{g},k)\ \to\mathrm{Opext}(\mathfrak{f},\mathfrak{g}^*)\\
&\to H^3(\mathfrak{f}\bowtie\mathfrak{g},k)\to H^3(\mathfrak{f},k)\oplus H^3(\mathfrak{g},k),
\end{aligned}
$$

where $H^{\cdot}$ *denotes the Lie algebra cohomology with coefficients in the trivial Lie module* k.

This exact sequence and the Hochschild-Serre spectral sequence imply the following.

Corollary 5.9 [M, Cor. 2.11]. *Suppose* $\mathrm{ch}\,k=0$. *If either* (a) $\mathfrak{f}$ *is semisimple and* $\rightharpoonup$ *is zero or* (b) $\mathfrak{g}$ *is semisimple and* ρ *is zero, then the groups* $\mathrm{Opext}(\mathfrak{f},\mathfrak{g}^*)$ *and* $\mathrm{Aut}(\mathfrak{g}^*\blacktriangleright\!\!\triangleleft\,\mathfrak{f})$ *are both trivial.*

Throughout in the following sections we suppose $\mathrm{ch}\,k=0$.

6. Hopf Algebra Extensions of $U\mathfrak{f}$ by the Full Dual $(U\mathfrak{g})^\circ$

Let $(U\mathfrak{g})^\circ$ denote the Hopf dual of $U\mathfrak{g}$, which consists of the elements in $(U\mathfrak{g})^*$ annihilating some two-sided (or equivalently one-sided) ideal in $U\mathfrak{g}$ of cofinite dimension; see [Mo, Sect. 9.1]. Thus, $(U\mathfrak{g})^\circ$ is a commutative Hopf algebra; it is finitely generated if and only if $\mathfrak{g}$ is *perfect* in the sense $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$; see [Ho2, p.261]. If k is algebraically closed, it follows by [Ho1, Sect. 3] that a commutative Hopf algebra is of the form $(U\mathfrak{g})^\circ$ with $\mathfrak{g}$ perfect if and only if it is isomorphic to the coordinate Hopf algebra $O(G)$ of a simply connected affine algebraic group G with $G = [G, G]$.

Given a Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ of Lie bialgebras, we have by Definition 5.3 a matched pair $(\mathfrak{f}, \mathfrak{g}, \triangleleft, \triangleright)$ of Lie algebras and so by Proposition 5.2 such a pair $(U\mathfrak{f}, U\mathfrak{g}, \triangleleft, \triangleright)$ of Hopf algebras. Since the action $\triangleright\colon U\mathfrak{g} \otimes U\mathfrak{f} \to U\mathfrak{f}$ makes $U\mathfrak{f}$ into a locally finite $U\mathfrak{g}$-module, it gives rise to a comodule coaction $\rho' : U\mathfrak{f} \to U\mathfrak{f} \otimes (U\mathfrak{g})^\circ$. By [M, Lemma 4.1], the transpose $\rightharpoonup'\colon U\mathfrak{f} \otimes (U\mathfrak{g})^* \to (U\mathfrak{g})^*$ of the other action $\triangleleft\colon U\mathfrak{g} \otimes U\mathfrak{f} \to U\mathfrak{g}$ stabilizes $(U\mathfrak{g})^\circ$, and the induced action $\rightharpoonup'\colon U\mathfrak{f} \otimes (U\mathfrak{g})^\circ \to (U\mathfrak{g})^\circ$ together with ρ' makes $(U\mathfrak{f}, (U\mathfrak{g})^\circ)$ into a Singer pair of Hopf algebras.

Proposition 6.1 [M, Prop. 4.3]. *The assignment $(\rightharpoonup, \rho) \mapsto (\rightharpoonup', \rho')$ thus obtained gives an injection from the set of structures of Singer pair of Lie bialgebras on $(\mathfrak{f}, \mathfrak{g}^*)$ into the set of structures of such pair of Hopf algebras on $(U\mathfrak{f}, (U\mathfrak{g})^\circ)$. This is a bijection if $\mathfrak{g}$ is perfect.*

Write $K = (U\mathfrak{g})^\circ$, and let $\varpi : K^+ \to \mathfrak{g}^*$ denote the 'restriction' map induced from the inclusion $\mathfrak{g} \hookrightarrow U\mathfrak{g}$, which is a surjection by the Ado theorem. It follows from the proof of [M, Prop. 4.3] that a structure $(\rightharpoonup', \rho')$ arises from some $(\rightharpoonup, \rho)$ if and only if $\rightharpoonup'$ stabilizes the kernel $\operatorname{Ker}\varpi$ of ϖ, in which case the induced action $\mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ is the desired $\rightharpoonup$. (Note that $\rightharpoonup'$ necessarily stabilizes K^+, since $\varepsilon(a \rightharpoonup' t) = \varepsilon(a)\varepsilon(t)$ by [T1, Lemma 1.2].) This is always the case if $\mathfrak{g}$ is perfect, since then (and only then) $\operatorname{Ker}\varpi = (K^+)^2$.

Let $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ be a Singer pair of Lie bialgebras. It gives rise to a Singer pair $(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup', \rho')$ of Hopf algebras, as was seen above. Suppose that we are given a cleft extension (see Remark 2.6 (2))

$$(A) = (U\mathfrak{g})^\circ \xrightarrow{\iota} A \xrightarrow{\pi} U\mathfrak{f}$$

associated with the last pair. Regard A as a Lie algebra with the bracket $[a, b] = ab - ba$, as usual, and as a Lie coalgebra with the co-bracket $\delta(a) = \sum a_1 \otimes a_2 - \sum a_1 \otimes a_1$ (note that A is not necessarily a Lie bialgebra). Let L_A^+ denote the subspace of A consisting of the elements a such that $\varepsilon(a) = 0$ and $(1 \otimes \pi) \circ \Delta(a) = a \otimes 1 + 1 \otimes c$ for some $c \in \mathfrak{f}$. Then, L_A^+ is seen to be a Lie subalgebra and Lie subcoalgebra of A, and we have an extension $(L_A^+) = K^+ \xrightarrow{\iota} L_A^+ \xrightarrow{\pi} \mathfrak{f}$ of Lie algebras and at the same time of Lie coalgebras, where $K = (U\mathfrak{g})^\circ$. Since the restriction $\varpi : K^+ \to \mathfrak{g}^*$ is a surjection of Lie algebras and Lie coalgebras, we have an extension

$$(\mathfrak{l}_A) = \mathfrak{g}^* \to \mathfrak{l}_A \to \mathfrak{f},$$

where $\mathfrak{l}_A = L_A^+/\iota(\operatorname{Ker}\varpi)$. It is proved that $(\mathfrak{l}_A)$ is in $\mathscr{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$; see the paragraph preceding [M, Example 4.19].

Theorem 6.2. *The assignment* $(A) \mapsto (\mathfrak{l}_A)$ *gives a symmetric monoidal equivalence*

$$\mathscr{O}pext(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup', \rho') \xrightarrow{\approx} \mathscr{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho).$$

Let (A_1), (A_2) be in $\mathscr{O}pext(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup', \rho')$, and form $A = A_1 \otimes_H^K A_2$. From the extension

$$K^+ \oplus K^+ \to L_{A_1}^+ \oplus L_{A_2}^+ \to \mathfrak{f} \oplus \mathfrak{f},$$

construct an extension $(L_{A_1}^+) * (L_{A_2}^+)$ of $\mathfrak{f}$ by K^+, as before, by forming first a pullback along the diagonal map $\mathfrak{f} \to \mathfrak{f} \oplus \mathfrak{f}$ and then a pushout along the addition $K^+ \oplus K^+ \to K^+$. One sees that $a \oplus b \mapsto a \otimes 1 + 1 \otimes b$, $L_{A_1}^+ \oplus L_{A_2}^+ \to A_1 \otimes A_2$ induces an equivalence $(L_{A_1}^+) * (L_{A_2}^+) \xrightarrow{\cong} (L_A)$, and it in turn induces $(\mathfrak{l}_{A_1}) * (\mathfrak{l}_{A_2}) \xrightarrow{\cong} (\mathfrak{l}_A)$; this gives the monoidal structure of the functor $(A) \mapsto (\mathfrak{l}_A)$.

We see easily that $(A) \mapsto (\mathfrak{l}_A)$ gives a symmetric monoidal functor, and so that it induces group maps

$$\kappa_0 : \operatorname{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ) \to \operatorname{Opext}(\mathfrak{f}, \mathfrak{g}^*),$$
$$\kappa_1 : \operatorname{Aut}((U\mathfrak{g})^\circ \# U\mathfrak{f}) \to \operatorname{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f})$$

between the K_0, K_1 groups, which will be explicitly described by Proposition 6.5. We remark that κ_0, κ_1 just given coincide respectively with κ_1, κ_0 in [M, Thm. 4.11]; thus the notations are reverse. Since κ_0 and κ_1 are isomorphisms by [M, Thm. 4.11], we see from the following standard fact that the functor is equivalence, which proves Theorem 6.2: a symmetric monoidal functor between symmetric monoidal groupoids in which each object has inverse is an equivalence, if and only if the induced maps between the mutual K_0, K_1 groups are both isomorphisms. □

Since κ_0 and κ_1 are isomorphisms, the sequence given in Theorem 5.8 remains exact if the groups $\operatorname{Opext}(\mathfrak{f}, \mathfrak{g}^*)$, $\operatorname{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\lhd \mathfrak{f})$ are replaced by $\operatorname{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ)$, $\operatorname{Aut}((U\mathfrak{g})^\circ \# U\mathfrak{f})$, respectively. The exact sequence thus obtained cannot be covered by the generalized Kac exact sequence [Sb2] (see Remark 1.11 (3)), for which the kernel of extensions is supposed to be finite-dimensional, while we have the infinite-dimensional kernel $(U\mathfrak{g})^\circ$ unless $\mathfrak{g} = 0$.

Combined with Corollary 5.9, the isomorphisms prove also the following.

Corollary 6.3 [M, Cor. 4.13]. *The groups* $\operatorname{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ)$, $\operatorname{Aut}(U\mathfrak{f}, (U\mathfrak{g})^\circ)$ *associated with a Singer pair* $(U\mathfrak{f}, (U\mathfrak{g})^\circ, \rightharpoonup, \rho)$ *are both trivial, if either* (a) $\mathfrak{f}$ *is semisimple and* $\rightharpoonup$ *is trivial or* (b) $\mathfrak{g}$ *is semisimple and* ρ *is trivial.*

Combined with the last statement of Proposition 6.1, the theorem gives also the following corollary.

Corollary 6.4 (cf. [M, Thm. 4.14]). *If* $\mathfrak{g}$ *is perfect, we have a natural equivalence from the groupoid* $\mathscr{E}xt(U\mathfrak{f},(U\mathfrak{g})^\circ)$ *of all cleft extensions of* $U\mathfrak{f}$ *by* $(U\mathfrak{g})^\circ$ *to the groupoid* $\mathscr{E}xt(\mathfrak{f},\mathfrak{g}^*)$ *of all Lie bialgebra extensions of* $\mathfrak{f}$ *by* $\mathfrak{g}^*$.

As in Theorem 6.2, let $(U\mathfrak{f},(U\mathfrak{g})^\circ,\rightharpoonup',\rho')$ be a Singer pair which arises from such a pair $(\mathfrak{f},\mathfrak{g}^*,\rightharpoonup,\rho)$, and consider the double complexes $A_0^{\cdot\cdot}$, $C_0^{\cdot\cdot}$ defined by these pairs. Through the isomorphisms given by Propositions 2.21 and 5.7, let us regard κ_0, κ_1 as isomorphisms

$$\kappa_{1-n}: H^n(\operatorname{Tot} A_0^{\cdot\cdot}) \xrightarrow{\cong} H^n(\operatorname{Tot} C_0^{\cdot\cdot})\ (n=0,1).$$

Proposition 6.5.

(1) *Let* (σ,τ) *be a total* 1*-cocycle in* $A_0^{\cdot\cdot}$ *consisting of* $\sigma: U\mathfrak{f}\otimes U\mathfrak{f}\to(U\mathfrak{g})^\circ$ *and* $\tau: U\mathfrak{f}\to(U\mathfrak{g})^\circ\otimes(U\mathfrak{g})^\circ$. *Then the linear maps* $\bar\sigma:\mathfrak{f}\wedge\mathfrak{f}\to\mathfrak{g}^*$, $\bar\tau:\mathfrak{f}\to(\mathfrak{g}\wedge\mathfrak{g})^*$ *determined by*

$$\langle\bar\sigma(a\wedge b),x\rangle=\langle\sigma(a,b)-\sigma(b,a),x\rangle,$$
$$\langle\bar\tau(a),x\wedge y\rangle=\langle\tau(a),x\otimes y-y\otimes x\rangle,$$

where $a,b\in\mathfrak{f}$, $x,y\in\mathfrak{g}$, *form a total* 1*-cocycle* $(\bar\sigma,\bar\tau)$ *in* $C_0^{\cdot\cdot}$ (*under the natural identification as in Proposition* 5.7). *The isomorphism* κ_0 *is induced by* $(\sigma,\tau)\mapsto(\bar\sigma,\bar\tau)$.

(2) *Let* $\nu: U\mathfrak{f}\to(U\mathfrak{g})^\circ$ *be a total* 0*-cocycle in* $A_0^{\cdot\cdot}$. *Then the linear map* $\bar\nu:\mathfrak{f}\to\mathfrak{g}^*$ *determined by*

$$\langle\bar\nu(a),x\rangle=\langle\nu(a),x\rangle,$$

where $a\in\mathfrak{f}$, $x\in\mathfrak{g}$, *is a total* 0*-cocycle in* $C_0^{\cdot\cdot}$, *and* κ_1 *is given by* $\nu\mapsto\bar\nu$.

The result of Part 1 is given by the second paragraph following [M, Remark 4.15]. Part 2 is seen more easily.

7. Hopf Algebra Extensions of $U\mathfrak{f}$ by the Irreducible $(U\mathfrak{g})'$ with $\mathfrak{g}$ Nilpotent

Let $(U\mathfrak{g})'$ denote the largest irreducible subcoalgebra, necessarily a Hopf subalgebra, of $(U\mathfrak{g})^\circ$ containing 1, which consists of the elements in $(U\mathfrak{g})^*$ annihilating some power of $(U\mathfrak{g})^+$; see [Mo, Def. 9.2.2]. The next proposition follows from [Ho2, Thm. XVI 4.2].

Proposition 7.1. *For a commutative Hopf algebra* K, *the following conditions are equivalent*:

(a) K *is of the form* $(U\mathfrak{g})'$, *where* $\mathfrak{g}$ *is nilpotent*;
(b) K *is the coordinate Hopf algebra* $O(G)$ *of a unipotent affine algebraic group* G;
(c) K *is finitely generated as an algebra and irreducible as a coalgebra, and the intersection of all ideals in* K *of codimension* 1 *is zero* (*the last condition can be removed if* k *is algebraically closed*).

In what follows in this section, we suppose that $\mathfrak{g}$ is nilpotent. We will show parallel results of those in the preceding section, replacing $(U\mathfrak{g})^\circ$ by $(U\mathfrak{g})'$.

Proposition 7.2. *There is a natural* 1-1 *correspondence between* (a) *the set of structures* $(\rightharpoonup, \rho)$ *of Singer pair of Lie bialgebras on* $(\mathfrak{f}, \mathfrak{g}^*)$ *such that the action* $\triangleright\colon \mathfrak{g} \otimes \mathfrak{f} \to \mathfrak{f}$ *corresponding to* ρ *is nilpotent* (*in the sense* $\mathfrak{g}^l \triangleright \mathfrak{f} = 0$ *for some integer* $l > 0$) *and* (b) *the set of structures* $(\rightharpoonup', \rho')$ *of Singer pair of Hopf algebras on* $(U\mathfrak{f}, (U\mathfrak{g})')$.

Proof. Let $(\rightharpoonup, \rho)$ be in (a), and let $(\triangleleft, \triangleright)$ be the corresponding matched pair structure on $(\mathfrak{f}, \mathfrak{g})$, which gives rise to a matched pair structure on $(U\mathfrak{f}, U\mathfrak{g})$, by Proposition 5.2. Since $\triangleright\colon \mathfrak{g} \otimes \mathfrak{f} \to \mathfrak{f}$ is nilpotent, $\triangleright\colon U\mathfrak{g} \otimes U\mathfrak{f} \to U\mathfrak{f}$ is locally nilpotent. Since locally nilpotent module-actions $U\mathfrak{g} \otimes U\mathfrak{f} \to U\mathfrak{f}$ are in natural 1-1 correspondence with comodule coactions $U\mathfrak{f} \to U\mathfrak{f} \otimes (U\mathfrak{g})'$, a comodule coaction ρ' arises from the last $\triangleright$. Suppose $\mathfrak{g}^l \triangleright \mathfrak{f} = 0$, and write $I = (U\mathfrak{g})^+$. It follows by induction on n $(\geq l)$ that $I^{n-1} \triangleleft \mathfrak{f} \subset I^{n-l}$, and so that the transpose $\rightharpoonup'\colon U\mathfrak{f} \otimes (U\mathfrak{g})^* \to (U\mathfrak{g})^*$ of $\triangleleft$ satisfies $\mathfrak{f} \rightharpoonup' (U\mathfrak{g}/I^{n-l})^* \subset (U\mathfrak{g}/I^{n-1})^*$; this implies that $\rightharpoonup'$ stabilizes $(U\mathfrak{g})'$. One sees as in the proof of [M, Lemma 4.1] that the pair $(\rightharpoonup', \rho')$, where $\rightharpoonup'$ is the restricted action on $(U\mathfrak{g})'$, is in (b).

Let $(\rightharpoonup', \rho')$ be in (b). Reversing the procedure, we obtain a coaction $\rho : \mathfrak{f} \to \mathfrak{f} \otimes \mathfrak{g}^*$ from ρ'. Write $K = (U\mathfrak{g})'$. Since it follows from [Ho2, Thm. XVI 4.2] that the restriction map $K^+ \to \mathfrak{g}^*$ induces an isomorphism $K^+/(K^+)^2 \cong \mathfrak{g}^*$, one sees as in the proof of [M, Prop. 4.3] that $\rightharpoonup'$ induces an action $\rightharpoonup\colon \mathfrak{f} \otimes \mathfrak{g}^* \to \mathfrak{g}^*$, and that $(\rightharpoonup, \rho)$ is in (a).

We see easily that the correspondences thus defined are inverses of each other. □

Choose such $(\rightharpoonup, \rho)$, $(\rightharpoonup', \rho')$ respectively from (a), (b) that correspond to each other. Let (A) be in $\mathcal{O}pext(U\mathfrak{f}, (U\mathfrak{g})', \rightharpoonup', \rho')$. Since the restriction $K^+ \to \mathfrak{g}^*$ is a surjection as was seen above, we can construct $(\mathfrak{l}_A)$ in $\mathcal{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ in the same way as in the preceding section.

Theorem 7.3. *Suppose* $\mathfrak{g}$ *is nilpotent. The assignment* $(A) \mapsto (\mathfrak{l}_A)$ *gives a symmetric monoidal equivalence*

$$\mathcal{O}pext(U\mathfrak{f}, (U\mathfrak{g})', \rightharpoonup', \rho') \xrightarrow{\approx} \mathcal{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho).$$

A proof will be given in the next section in a more general context.

The induced isomorphisms between the K_0, K_1 groups together with Corollary 5.9 prove the following.

Corollary 7.4. *The groups* $\operatorname{Opext}(U\mathfrak{f}, (U\mathfrak{g})')$, $\operatorname{Aut}((U\mathfrak{g})' \# U\mathfrak{f})$ *associated with a Singer pair* $(U\mathfrak{f}, (U\mathfrak{g})', \rightharpoonup, \rho)$ *are both trivial if* $\mathfrak{f}$ *is semisimple*, $\mathfrak{g}$ *is nilpotent and* $\rightharpoonup$ *is trivial*.

Example 7.5. Let $\mathfrak{f} = ka$, $\mathfrak{g} = kx$ be 1-dimensional (abelian) Lie algebras. Then

$$U\mathfrak{f} = k[a],\ U\mathfrak{g} = k[x],\ (U\mathfrak{g})' = k[t],$$

polynomial Hopf algebras with a, x, t primitive. Here t is the element in $(U\mathfrak{g})^*$ determined by $\langle t, x^n\rangle = \delta_{1,n}$ $(n = 0, 1, \ldots)$; see [Mo, Example 9.1.7]. For arbitrary ξ, η in k, the actions $\triangleleft$, $\triangleright$ determined by

$$x \triangleleft a = \xi x, \;\; x \triangleright a = \eta a$$

make $(\mathfrak{f}, \mathfrak{g})$ matched. The action $\triangleright$ is nilpotent if and only if $\eta = 0$. The action $\rightharpoonup$: $k[a] \otimes k[t] \to k[t]$ arising from $\triangleleft$ is determined by

$$a \rightharpoonup t^n = n\xi t^n \;\; (n = 0, 1, \ldots).$$

Hence this $\rightharpoonup$ together with the trivial coaction $k[a] \to k[a]\otimes k[t]$ gives all possible Singer pair structures on $(k[a], k[t])$. Moreover, for each such pair the group $\mathrm{Opext}(k[a], k[t])$ is trivial since obviously $H^1(\mathrm{Tot}\, C_0^{\cdot\cdot}) = 0$. Thus the equivalence classes of all cleft extensions of $k[a]$ by $k[t]$ are in 1-1 correspondence with the elements ξ in k.

Example 7.6. Let $\mathfrak{sl}_2$ denote the special linear algebra of 2×2 matrices, with standard (Chevalley) basis x, y, h. The decomposition

$$\mathfrak{sl}_2 = \mathfrak{f} \oplus \mathfrak{g}$$

into the Lie subalgebras $\mathfrak{f} := kh + ky$ and $\mathfrak{g} := kx$ gives rise to a matched pair structure on $(\mathfrak{f}, \mathfrak{g})$, in which the action $\triangleright$ is nilpotent. We have $(U\mathfrak{g})' = k[t]$ as above, so $\langle t^m, x^n/n!\rangle = \delta_{m,n}$. The matched pair structure corresponds to the Singer pair structure on $(U\mathfrak{f}, (U\mathfrak{g})')$ given by

$$\frac{y^l}{l!}h^m \rightharpoonup t^n = \begin{cases} \binom{n-1}{l}(-1)^{l+m}2^m(n-l)^m t^{n-l} & (l < n) \\ 0 & (l \geq n), \end{cases}$$

$$\rho\left(\frac{y^l}{l!}h^m\right) = \sum_{i=0}^{l} \frac{y^{l-i}}{(l-i)!}\binom{h-l+i}{i} h^m \otimes t^i,$$

where l, m and n are non-negative integers, since we see from [Hum, Lemma 26.2] that in $U(\mathfrak{sl}_2)$,

$$\frac{x^n}{n!}\frac{y^l}{l!}h^m = \sum_{i=0}^{\min(l,n)} \frac{y^{l-i}}{(l-i)!}\binom{h-l-n+2i}{i}(h-2(n-i))^m \frac{x^{n-i}}{(n-i)!}.$$

By using the Kac exact sequence we compute

$$\mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})') \cong H^3(\mathfrak{sl}_2, k) = k,$$

since it is easy to see $H^n(\mathfrak{f}, k) = H^n(\mathfrak{g}, k) = 0$ for $n = 2, 3$.

REMARK 7.7. In their new paper [VV], Vaes and Vainerman study the same subject as this Part II in the framework of operator algebras, and especially give many examples. The computation of an Opext group given in [VV, Remark 5.7] (without detailed proof) seems essentially the same as our result in the preceding example.

8. Generalization by Introducing Topology

To unify and generalize the results in the preceding two sections, we let $\mathcal{I}$ be a set of (two-sided) ideals in $U\mathfrak{g}$ of cofinite dimension which satisfies the following conditions.

(i) For any $I_1, I_2 \in \mathcal{I}$, there exists $I \in \mathcal{I}$ such that $I \subset I_1 \cap I_2$;

(ii) For any $I \in \mathcal{I}$, there exists $I' \in \mathcal{I}$ such that $\Delta(I') \subset U\mathfrak{g} \otimes I + I \otimes U\mathfrak{g}$, $S(I') \subset I$;

(iii) There exists $I \in \mathcal{I}$ such that $\varepsilon(I) = 0$.

DEFINITION 8.1. Let $(U\mathfrak{g})^\circ_{\mathcal{I}}$ denote the subset of $(U\mathfrak{g})^*$ consisting of the elements f such that $f(I) = 0$ for some $I \in \mathcal{I}$. We see easily that this is a Hopf subalgebra of $(U\mathfrak{g})^\circ$.

EXAMPLE 8.2. (1) Suppose that $\mathcal{I}$ consists of only one ideal $(U\mathfrak{g})^+$. Then, $(U\mathfrak{g})^\circ_{\mathcal{I}} = k$.

(2) Suppose that $\mathcal{I}$ consists of all ideals of cofinite dimension. Then, $(U\mathfrak{g})^\circ_{\mathcal{I}} = (U\mathfrak{g})^\circ$.

(3) Suppose that $\mathcal{I}$ consists of the powers $I,\ I^2,\ \cdots$ of $I = (U\mathfrak{g})^+$. Then, $(U\mathfrak{g})^\circ_{\mathcal{I}} = (U\mathfrak{g})'$.

By a *topological vector space* [T2, p.507], we mean a vector space V with a topology such that for each $w \in V$, the translation $v \mapsto v + w$, $V \to V$ is continuous, and V has a basis of neighborhoods of 0 consisting of vector subspaces, which we call a *topological basis*.

Every vector space is a topological vector space with the discrete topology. We regard k as a discrete topological vector space.

The direct sum $\bigoplus_\lambda V_\lambda$ of topological vector spaces V_λ is a topological vector space with the direct sum topology [T2, 1.2]; it has a topological basis consisting of all $\bigoplus_\lambda W_\lambda$, where W_λ is in a fixed topological basis of V_λ. For a vector space X, we denote by $V \otimes (X)$ or $(X) \otimes V$ the topological vector space $V \otimes X$ or $X \otimes V$ which is given the direct sum topology, identified with the direct sum of $\dim X$ copies of V.

By Condition (i), $U\mathfrak{g}$ is a topological vector space with topological basis $\mathcal{I}$. Conditions (ii) and (iii) are equivalent to that the structure maps Δ, S and ε are continuous, where $U\mathfrak{g} \otimes U\mathfrak{g}$ is given the (tensor product) topology with topological basis consisting of all $U\mathfrak{g} \otimes I + I \otimes U\mathfrak{g}$ with $I \in \mathcal{I}$. We see that $(U\mathfrak{g})^\circ_{\mathcal{I}}$ equals the vector space $\mathrm{Hom}_c(U\mathfrak{g}, k)$ of continuous linear maps $U\mathfrak{g} \to k$.

We regard $U\mathfrak{f}$ as a discrete vector space, and $U\mathfrak{g}\otimes(U\mathfrak{f})$ as a topological vector space so as defined above.

Theorem 8.3. *Fix a Singer pair* $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ *of Lie bialgebras, which by definition corresponds to a matched pair* $(\mathfrak{f}, \mathfrak{g}, \triangleleft, \triangleright)$ *of Lie algebras. Suppose that the unique extensions*

$$U\mathfrak{g} \xleftarrow{\triangleleft} U\mathfrak{g}\otimes(U\mathfrak{f}) \xrightarrow{\triangleright} U\mathfrak{f}$$

of the actions $\mathfrak{g} \xleftarrow{\triangleleft} \mathfrak{g}\otimes\mathfrak{f} \xrightarrow{\triangleright} \mathfrak{f}$ *which make* $(U\mathfrak{f}, U\mathfrak{g})$ *matched (see Proposition 5.2) are both continuous.*

(1) *The extended actions give rise to an action* $\rightharpoonup' : U\mathfrak{f}\otimes(U\mathfrak{g})^\circ_{\mathcal{I}} \to (U\mathfrak{g})^\circ_{\mathcal{I}}$ *and a coaction* $\rho' : U\mathfrak{f} \to U\mathfrak{f}\otimes(U\mathfrak{g})^\circ_{\mathcal{I}}$ *such that* $(U\mathfrak{f}, (U\mathfrak{g})_{\mathcal{I}})^\circ, \rightharpoonup', \rho')$ *is a Singer pair.*
(2) *There exist natural group maps*

$$\kappa_0 : \mathrm{Opext}(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{I}}) \to \mathrm{Opext}(\mathfrak{f}, \mathfrak{g}^*),$$
$$\kappa_1 : \mathrm{Aut}((U\mathfrak{g})^\circ_{\mathcal{I}} \# U\mathfrak{f}) \to \mathrm{Aut}(\mathfrak{g}^* \blacktriangleright\!\!\triangleleft\, \mathfrak{f})$$

between the K_0, K_1 *groups of* $\mathcal{O}pext(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{I}}, \rightharpoonup', \rho')$ *and of* $\mathcal{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$.
(3) *Suppose* $H^1(\mathfrak{g}, (U\mathfrak{g})^\circ_{\mathcal{I}}) = 0$, *where* $(U\mathfrak{g})^\circ_{\mathcal{I}}$ *is a left (or equivalently right)* $\mathfrak{g}$*-Lie module with the transposed action of the right (or left) multiplication on* $U\mathfrak{g}$. *Then,* κ_1 *is an isomorphism.*
(4) *If in addition* $H^2(\mathfrak{g}, (U\mathfrak{g})^\circ_{\mathcal{I}}) = 0$, *then* κ_0 *is also an isomorphism.*
(5) *If in addition* $I \cap \mathfrak{g} = 0$ *for some* $I \in \mathcal{I}$, *there exists a symmetric monoidal equivalence*

$$\mathcal{O}pext(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{I}}, \rightharpoonup', \rho') \xrightarrow{\approx} \mathcal{O}pext(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$$

which induces the isomorphisms κ_0, κ_1.

Remark 8.4. We see that the following conditions including the assumption in Part 5 above are equivalent to each other.

(a) $I \cap \mathfrak{g} = 0$ for some $I \in \mathcal{I}$;
(b) $\bigcap_{I\in\mathcal{I}} I = 0$, or the topological space $U\mathfrak{g}$ is Hausdorff;
(c) The restriction map $(U\mathfrak{g})^\circ_{\mathcal{I}} \to \mathfrak{g}^*$ is a surjection;
(d) The canonical algebra map $U\mathfrak{g} \to [(U\mathfrak{g})^\circ_{\mathcal{I}}]^*$ is an injection.

To prove Part 1 of the theorem, we generalize the situation as follows. Let H, J be cocommutative Hopf algebras. Suppose we are given a set $\mathcal{I}$ of ideals in J of cofinite dimension which satisfies the same conditions as (i)–(iii) given above for $U\mathfrak{g}$; J is thus a topological vector space with topological basis $\mathcal{I}$. Let $K = J^\circ_{\mathcal{I}}$ denote the commutative Hopf algebra consisting of all continuous linear maps $J \to k$. We regard H as a discrete vector space. Suppose we are given also continuous actions

$$J \xleftarrow{\triangleleft} J\otimes(H) \xrightarrow{\triangleright} H$$

which make (H, J) matched.

Note that $H \otimes K = \mathrm{Hom}_c(J, H)$, the subspace of $\mathrm{Hom}(J, H)$ consisting of all continuous linear maps $J \to H$. Since $\triangleright$ is continuous, the image of the linear map $\rho : H \to \mathrm{Hom}(J, H)$ defined by

$$\rho(a)(x) = x \triangleright a \ (a \in H,\ x \in J)$$

is included in $H \otimes K$, so that we have a coaction $\rho : H \to H \otimes K$. Since $\triangleleft$ is continuous, $\triangleleft\, a : J \to J$ is continuous for each $a \in H$. By applying $\mathrm{Hom}_c(\ , k)$, we obtain a map $a \rightharpoonup : K \to K$, and hence also an action $\rightharpoonup : H \otimes K \to K$.

Lemma 8.5. $(H, K, \rightharpoonup, \rho)$ *forms a Singer pair.*

This follows easily as in the proof of [M, Lemma 4.1]. If J is Hausdorff, the correspondence $(\triangleleft, \triangleright) \mapsto (\rightharpoonup, \rho)$ between the sets of structures, given as above, is injective since then $\triangleleft$ is recovered from the transpose of $\rightharpoonup$ through the canonical injection $J \to K^*$; see the proof of [M, Cor. 4.2].

Part 1 of Theorem 8.3 follows from the last lemma. □

Let $(H, J, \triangleleft, \triangleright)$, $(H, K, \rightharpoonup, \rho)$ be as above. From the matched pair, the Hopf algebra $H \bowtie J$ and its module category $H \bowtie J$-Mod are constructed. From the Singer pair, the category $\mathcal{C} = \mathcal{C}(H, K, \rightharpoonup, \rho)$ is defined by Definition 2.3.

Generalizing the observation given above that the continuous action $\triangleright$ gives rise to ρ, we see that for a discrete vector space M, there is a natural 1-1 correspondence between the continuous module actions $J \otimes (M) \to M$ and the comodule coactions $M \to M \otimes K$. This proves the following.

Lemma 8.6. *$\mathcal{C}$ is regarded as a full subcategory of $H \bowtie J$-*Mod *which consists of the $H \bowtie J$-modules M such that the restricted action $J \otimes (M) \to M$ by J is continuous.*

Recall from (2.17), (2.19) the (co)simplicial objects $\Phi.(k)$, $\Psi^{\cdot}(k)$ in $\mathcal{C}$. The normalized (co)chain complexes associated with these objects coincide with the standard (co)free resolutions of k, if we forget K-coactions or H-actions. By removing the 0th terms H, K from them, we obtain (co)chain complexes

$$X.(H) = 0 \leftarrow H \otimes H_+ \leftarrow H \otimes H_+^{\otimes 2} \leftarrow \cdots,$$
$$Y^{\cdot}(K) = 0 \to K^+ \otimes K \to K^{+\otimes 2} \otimes K \to \cdots$$

in $\mathcal{C}$, where $H_+ = H/k1$.

Let Hom_H^K denote the vector space of H-linear and K-colinear maps. Form the double complex $B_0^{\cdot\cdot} = \mathrm{Hom}_H^K(X.(H), Y^{\cdot}(K))$. Here and in what follows when we form a double complex, we resort such a sign trick that changes the sign of differentials in even columns beginning with the 0th column. Note that each term in $X.(H)$ (resp., in $Y^{\cdot}(K)$) is of the form $\mathbb{F}(P)$ as given in (2.16) (resp., $\mathbb{G}(Q)$ as given in (2.18)). Since we have a natural isomorphism

$$\mathrm{Hom}_H^K(\mathbb{F}(P), \mathbb{G}(Q)) \cong \mathrm{Hom}(P, Q) \tag{8.7}$$

given in the same way of (2.20), $B_0^{\cdot\cdot}$ turns to be as follows.

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ & \uparrow & & \uparrow & \\ B_0^{\cdot\cdot} = & \mathrm{Hom}(H_+, K^{+\otimes 2}) & \longrightarrow & \mathrm{Hom}(H_+^{\otimes 2}, K^{+\otimes 2}) & \longrightarrow \cdots \\ & \uparrow & & \uparrow & \\ & \mathrm{Hom}(H_+, K^{+}) & \longrightarrow & \mathrm{Hom}(H_+^{\otimes 2}, K^{+}) & \longrightarrow \cdots \end{array}$$

PROPOSITION 8.8 [M, Prop. 3.14]. *Suppose $H = U\mathfrak{f}$ ($\mathfrak{f}$ can be of infinite dimension). An isomorphism $A_0^{\cdot\cdot} \cong B_0^{\cdot\cdot}$ between the double complexes of abelian groups is given by*

$$\log : \mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q}) \to \mathrm{Hom}(H_+^{\otimes p}, K^{+\otimes q}),\ \log f = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(f - \varepsilon)^n,$$

where ε is the identity in $\mathrm{Reg}_+(H^{\otimes p}, K^{\otimes q})$ and $(f-\varepsilon)^n = (f-\varepsilon)\cdots(f-\varepsilon)$ denotes the convolution product.

To prove Part 2 of Theorem 8.3, suppose in particular $H = U\mathfrak{f}$, $J = U\mathfrak{g}$, and so $K = (U\mathfrak{g})^{\circ}_{\mathcal{I}}$. We may suppose that the Singer pair $(H, K, \rightharpoonup, \rho)$ given by Lemma 8.5 arises from the Singer pair $(\mathfrak{f}, \mathfrak{g}^*, \rightharpoonup, \rho)$ of Lie bialgebras fixed in the theorem. Here we continue to denote the structure of the pair (H, K) by $\rightharpoonup$, ρ, instead of $\rightharpoonup'$, ρ'. The Singer pairs define the double complexes $A_0^{\cdot\cdot}$, $C_0^{\cdot\cdot}$. We will define natural group maps

$$\kappa_{1-n} : H^n(\mathrm{Tot}\, A_0^{\cdot\cdot}) \to H^n(\mathrm{Tot}\, C_0^{\cdot\cdot})\ (n = 0, 1),$$

which will prove Part 2 by Propositions 2.21 and 5.7.

We obtain from the Chevalley-Eilenberg complex $V_.(\mathfrak{f})$, by removing its term H, a chain complex

$$X_.(\mathfrak{f}) = 0 \leftarrow H \otimes \mathfrak{f} \leftarrow H \otimes \wedge^2\mathfrak{f} \leftarrow \cdots$$

in $H \bowtie J$-Mod, and so in $\mathcal{C}$ by Lemma 8.6. The well-known embedding $\varphi_. : X_.(\mathfrak{f}) \to X_.(H)$ given by

$$\varphi_{p-1} : X_{p-1}(\mathfrak{f}) = H \otimes \wedge^p\mathfrak{f} \to H \otimes H_+^{\otimes p} = X_{p-1}(H),$$
$$\varphi_{p-1}(u\langle a_1, \ldots, a_p\rangle) = \sum_{\sigma \in \mathfrak{S}_p} (\mathrm{sgn}\,\sigma) u \otimes \overline{a_{\sigma(1)}} \otimes \cdots \otimes \overline{a_{\sigma(p)}}$$

is a map of complexes in $\mathcal{C}$; see [M, Lemma 4.8]. The symmetric argument using a mirror gives chain complexes

$$X'_.(J) = 0 \leftarrow J_+ \otimes J \leftarrow J_+^{\otimes 2} \otimes J \leftarrow \cdots,$$
$$X'_.(\mathfrak{g}) = 0 \leftarrow \mathfrak{g} \otimes J \leftarrow \wedge^2\mathfrak{g} \otimes J \leftarrow \cdots$$

of right $H \bowtie J$-modules and an embedding $\varphi'_\cdot : X'_\cdot(\mathfrak{g}) \to X'_\cdot(J)$. Through the injection $(K^+)^{\otimes q} \otimes K \hookrightarrow (J_+)^{*\otimes q} \otimes J^* \subset (J_+^{\otimes q} \otimes J)^*$ induced from the inclusion $K \hookrightarrow J^*$, we can regard $Y^\cdot(K)$ as a subcomplex of the dual complex $X'_\cdot(J)^*$ in $H \bowtie J$-Mod.

In general, if Q is a right H-module of finite dimension, the right J-module $Q \otimes J$ is a right $H \bowtie J$-module, where H acts by

$$(q \otimes x)a = \sum q(x_1 \triangleright a_1) \otimes (x_2 \triangleleft a_2) \ (a \in H,\ q \otimes x \in Q \otimes J),$$

so that $(Q\otimes J)^* = Q^* \otimes J^*$ is a left $H \bowtie J$-module. The object $\mathbb{G}(Q^*) = Q^* \otimes K$ in $\mathcal{C}$ given by (2.18) is the largest $H \bowtie J$-submodule of $Q^* \otimes J^*$ which is an object in $\mathcal{C}$. Therefore we have a cochain complex in $\mathcal{C}$

$$Y^\cdot(\mathfrak{g}^*) = 0 \to \mathfrak{g}^* \otimes K \to (\wedge^2 \mathfrak{g})^* \otimes K \to \cdots$$

which is a subcomplex of the dual complex $X'_\cdot(\mathfrak{g})^*$ in $H \bowtie J$-Mod.

Lemma 8.9. $Y^\cdot(\mathfrak{g}^*)$ *is naturally isomorphic to the complex obtained by removing the term K from the standard complex*

$$0 \to K \to \mathrm{Hom}(\mathfrak{g}, K) \to \mathrm{Hom}(\wedge^2 \mathfrak{g}, K) \to \cdots$$

for computing the cohomology $H^\cdot(\mathfrak{g}, K)$ with coefficients in the right $\mathfrak{g}$-Lie module K.

Proof. The canonical isomorphism $\mathrm{Hom}_J(Q \otimes J, J^*) \cong Q^* \otimes J^*$ induces $\mathrm{Hom}_J(Q \otimes J, K) \cong Q^* \otimes K$, which gives rise to an isomorphism between the complexes. □

Form the double complex $\mathrm{Hom}_H^K(X_\cdot(\mathfrak{f}), Y^\cdot(\mathfrak{g}^*))$, in which each term is of the form $\mathrm{Hom}(\wedge^p \mathfrak{f}, (\wedge^q \mathfrak{g})^*)$ by (8.7).

Lemma 8.10. *The double complex just formed is naturally identified with $C_0^{\cdot\cdot}$.*

Proof. Write $P = \wedge^p \mathfrak{f}$, $Q = \wedge^q \mathfrak{g}$. The natural maps

$$\mathrm{Hom}(H \otimes P, Q^* \otimes K) \hookrightarrow \mathrm{Hom}(H \otimes P, (Q \otimes J)^*) \cong \mathrm{Hom}((Q \otimes J) \otimes (H \otimes P), k)$$

are $H \bowtie J$-linear, where $H \bowtie J$ acts on the Hom spaces by conjugation. By taking $H \bowtie J$-invariants we obtain $\mathrm{Hom}(P, Q^*) \cong \mathrm{Hom}(Q \otimes P, k)$, which gives a natural identification between the double complexes. □

From the dual of the embedding $\varphi'_\cdot : X'_\cdot(\mathfrak{g}) \to X'_\cdot(J)$, a map $\psi^\cdot : Y^\cdot(K) \to Y^\cdot(\mathfrak{g}^*)$ of complexes in $\mathcal{C}$ is induced. Define a map of double complexes by

$$\alpha_0^{\cdot\cdot} = \mathrm{Hom}_H^K(\varphi_\cdot, \psi^\cdot) : B_0^{\cdot\cdot} \to C_0^{\cdot\cdot}.$$

Compose this with the isomorphism $A_0^{\cdot\cdot} \xrightarrow{\cong} B_0^{\cdot\cdot}$ given by Proposition 8.8 to obtain a map $A_0^{\cdot\cdot} \to C_0^{\cdot\cdot}$ of double complexes. As desired maps, we define κ_{1-n} : $H^n(\mathrm{Tot}\, A_0^{\cdot\cdot}) \to H^n(\mathrm{Tot}\, C_0^{\cdot\cdot})$ $(n = 0, 1)$ to be the induced maps between the cohomology groups. This proves Part 2. □

We see that the maps just defined are so as described by Proposition 6.5 if $(U\mathfrak{g})^{\circ}$ is replaced by $(U\mathfrak{g})^{\circ}_{\mathcal{I}}$.

To prove Parts 3 and 4 of Theorem 8.3, it suffices to show that the maps $H^n(\operatorname{Tot} B_0^{\cdot\cdot}) \to H^n(\operatorname{Tot} C_0^{\cdot\cdot})$ $(n = 0, 1)$ induced by $\alpha_0^{\cdot\cdot}$ are isomorphisms.

Form the double complexes

$$B^{\cdot\cdot} = \operatorname{Hom}(X_{\cdot}(H), Y^{\cdot}(K)),\ \ C^{\cdot\cdot} = \operatorname{Hom}(X_{\cdot}(\mathfrak{f}), Y^{\cdot}(\mathfrak{g}^*))$$

in $H \bowtie J$-Mod, which look as follows.

$$B^{\cdot\cdot} = \quad \begin{array}{ccccc} \vdots & & & & \\ \uparrow & & & & \\ \operatorname{Hom}(H\otimes H_+, K^{+\otimes 2}\otimes K) & \longrightarrow & \vdots\ \cdots & & \\ \uparrow & & \uparrow & & \\ \operatorname{Hom}(H\otimes H_+, K^{+}\otimes K) & \longrightarrow & \operatorname{Hom}(H\otimes H_+^{\otimes 2}, K^{+}\otimes K) & \longrightarrow & \cdots \end{array}$$

$$C^{\cdot\cdot} = \quad \begin{array}{ccccc} \vdots & & & & \\ \uparrow & & & & \\ \operatorname{Hom}(H\otimes \mathfrak{f}, (\wedge^2\mathfrak{g})^*\otimes K) & \longrightarrow & \vdots\ \cdots & & \\ \uparrow & & \uparrow & & \\ \operatorname{Hom}(H\otimes \mathfrak{f}, \mathfrak{g}^*\otimes K) & \longrightarrow & \operatorname{Hom}(H\otimes \wedge^2\mathfrak{f}, \mathfrak{g}^*\otimes K) & \longrightarrow & \cdots \end{array}$$

Their subcomplexes of $H \bowtie J$-invariants are precisely $B_0^{\cdot\cdot}$ and $C_0^{\cdot\cdot}$, respectively. The map

$$\alpha^{\cdot\cdot} = \operatorname{Hom}(\varphi_{\cdot}, \psi^{\cdot}) : B^{\cdot\cdot} \to C^{\cdot\cdot}$$

is restricted to $\alpha_0^{\cdot\cdot}$. Denote by

$$(B^{\cdot}, d^{\cdot}) = \operatorname{Tot} B^{\cdot\cdot},\ \ (C^{\cdot}, \partial^{\cdot}) = \operatorname{Tot} C^{\cdot\cdot},\ \ \alpha^{\cdot} : B^{\cdot} \to C^{\cdot}$$

the two total complexes and the total map of $\alpha^{\cdot\cdot}$. Then we have the following commutative diagram in $H \bowtie J$-Mod:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B^0 & \xrightarrow{d^0} & B^1 & \xrightarrow{d^1} & \operatorname{Im} d^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha^0} & & \downarrow{\scriptstyle \alpha^1} & & \downarrow{\scriptstyle \alpha^2} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^0 & \xrightarrow{\partial^0} & C^1 & \xrightarrow{\partial^1} & \operatorname{Im} \partial^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{8.11}$$

We suppose $H^1(\mathfrak{g}, K) = 0 = H^2(\mathfrak{g}, K)$, and claim the following.

Claim 8.12. $\alpha^0, \alpha^1, \alpha^2, 0, \ldots$ *give a homotopy equivalence between the complexes in* $H \bowtie J$-Mod.

If this is proved, we see by taking $H \bowtie J$-invariants that κ_0 and κ_1 are isomorphisms, which proves Part 4. Under the assumption $H^1(\mathfrak{g}, K) = 0$, Part 3 will be proved by similar argument with the diagram (8.11) replaced by the reduced one involving $d^0 : B^0 \to \operatorname{Im} d^0$, $\partial^0 : C^0 \to \operatorname{Im} \partial^0$ in its rows.

Since we have resolutions $0 \leftarrow H^+ \leftarrow X_\cdot(H)$ and $0 \to K_+ \to Y^\cdot(K)$ in $\mathcal{C}$, the first row in (8.11) gives a resolution of $\operatorname{Hom}(H^+, K_+)$ in $H \bowtie J$-Mod. Here the augmentations are given by the product $H \otimes H_+ \to H^+$, the coproduct $K_+ \to K^+ \otimes K$ and the map $e : \operatorname{Hom}(H^+, K_+) \to B^0$ induced by them. By Lemma 8.9 together with the assumption of cohomologies vanishing, one forms an exact sequence in $\mathcal{C}$ by splicing the first three terms $\mathfrak{g}^* \otimes K \to (\wedge^2 \mathfrak{g})^* \otimes K \to (\wedge^3 \mathfrak{g})^* \otimes K$ in $Y^\cdot(\mathfrak{g}^*)$ with the injection $0 \to K_+ \to \mathfrak{g}^* \otimes K$ induced by the coproduct of K. Since also the product $H \otimes \mathfrak{f} \to H^+$ makes $0 \leftarrow H^+ \leftarrow X_\cdot(\mathfrak{f})$ into a resolution in $\mathcal{C}$, the second row in (8.11) gives again a resolution of $\operatorname{Hom}(H^+, K_+)$ in $H \bowtie J$-Mod, whose augmentation $\eta : \operatorname{Hom}(H^+, K_+) \to C^0$ are induced from the last injection and the product. Clearly we have $\eta = \alpha^0 \circ e$.

To prove Claim 8.12, return to the general situation given after Remark 8.4, in which we are given a matched pair $(H, J, \triangleleft, \triangleright)$ of cocommutative Hopf algebras with continuous actions.

Definition 8.13 (cf. [M, Def. 6.8]). We define a category $\mathcal{D}$ as follows. An object in $\mathcal{D}$ is an $H \bowtie J$-module M, and so in particular an H- and J-module, such that

(a) M is a topological vector space with topological basis consisting of J-submodules,
(b) The action $J \otimes (M) \to M$ is continuous and
(c) The action $(H) \otimes M \to M$ is continuous.

A morphism in $\mathcal{D}$ is a continuous $H \bowtie J$-linear map.

One sees that $\mathcal{D}$ is a k-additive category. Let M be an object in $\mathcal{D}$, and suppose $N \subset M$ is an $H \bowtie J$-submodule. Then, N and M/N are objects in $\mathcal{D}$ respectively with the sub- and the quotient topologies (cf. [M, Prop. 6.9]), so that any morphism in $\mathcal{D}$ has kernel and cokernel. However, $\mathcal{D}$ is not abelian in general, since a monomorphism (an epimorphism) is not necessarily a (co)kernel.

Recall that the Singer pair $(H, K = J^\circ_{\mathcal{I}}, \rightharpoonup, \rho)$ arising from $(H, J, \triangleleft, \triangleright)$ defines the category $\mathcal{C}$; see Definition 2.3.

Lemma 8.14. *$\mathcal{C}$ is regarded as a full subcategory of $\mathcal{D}$ which consists of the discrete objects.*

This is shown similarly by the idea to prove Lemma 8.6.

For discrete vector spaces V and W, we regard $\operatorname{Hom}(V, W)$ as a topological vector space with topological basis consisting of $\operatorname{Hom}(V/V_\lambda, W)$, where V_λ ranges over all finite-dimensional subspaces of V. If M and N are in $\mathcal{C}$, we see that $\operatorname{Hom}(M, N)$ is an object in $\mathcal{D}$ with the conjugate action by $H \bowtie J$; cf. [M, Lemma 6.10 1)]. Thus we have a k-linear functor

$$\operatorname{Hom}(\ ,\ ) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D},$$

so that (8.11) is a commutative diagram in $\mathcal{D}$. In particular for the objects $\mathbb{F}(P)$, $\mathbb{G}(Q)$ in $\mathcal{C}$ given by (2.16), (2.18), $L := \operatorname{Hom}(\mathbb{F}(P), \mathbb{G}(Q))$ is an object in $\mathcal{D}$. This behaves like as an injective object as seen below.

Proposition 8.15. *Suppose that a morphism $f : M \to N$ in $\mathcal{D}$ is strict in the sense that the $H \bowtie J$-linear isomorphism $M/\operatorname{Ker} f \xrightarrow{\cong} \operatorname{Im} f$ induced from f is a homeomorphism. Then for any $g : M \to L$ in $\mathcal{D}$ with $g(\operatorname{Ker} f) = 0$, there exists $h : N \to L$ in $\mathcal{D}$ such that $g = h \circ f$.*

This is essentially the same as [M, Cor. 6.14].

Let us return to the diagram (8.11). Since one sees as in the proof of [M, Lemma 6.15] that the differentials d^0, ∂^0 and the augmentations e, η are strict, the familiar argument for uniqueness of injective resolution proves that $\alpha^0, \alpha^1, \alpha^2, 0, \dots$ give a homotopy equivalence between the complexes in $\mathcal{D}$, so in $H \bowtie J$-Mod. This proves Claim 8.12, and so Part 4 of Theorem 8.3. □

To prove Part 5 of the theorem, suppose that $I \cap \mathfrak{g} = 0$ for some $I \in \mathcal{I}$, or equivalently that the restriction $(U\mathfrak{g})^\circ_{\mathcal{I}} \to \mathfrak{g}^*$ is a surjection; see Remark 8.4. Then we construct as in Section 6 the symmetric monoidal functor $(A) \mapsto (\mathfrak{l}_A)$, $\mathcal{O}pext(U\mathfrak{f}, (U\mathfrak{g})^\circ_{\mathcal{I}}) \to \mathcal{O}pext(\mathfrak{f}, \mathfrak{g}^*)$. Since we see that the induced maps between the K_0, K_1 groups coincide with κ_0, κ_1, it follows that the functor is equivalent if (and only if) these maps are isomorphisms; see the proof of Theorem 6.2. This proves Part 5, and completes the proof of Theorem 8.3. □

Let us see that the theorem just proved implies Theorems 6.2 and 7.3. For this, it suffices to prove that the cohomologies H^1 and H^2 vanish when $\mathcal{I}$ is as in (2) or (3) in Example 8.2 (and $\mathfrak{g}$ is nilpotent in the latter case).

To see H^1 vanishes, note that in either case, $\mathcal{I}$ satisfies the following condition which is stronger than (i).

(i') For any $I_1, I_2 \in \mathcal{I}$, there exists $I \in \mathcal{I}$ such that $I \subset I_1 I_2$.

Then the desired result follows from the next proposition.

Proposition 8.16. *If $\mathcal{I}$ satisfies Condition (i'), then we have $H^1(\mathfrak{g}, (U\mathfrak{g})^\circ_{\mathcal{I}}) = 0$.*

Proof. Write $K = (U\mathfrak{g})^\circ_{\mathcal{I}}$. Recalling $H^1(\mathfrak{g}, K) = \operatorname{Ext}^1_{U\mathfrak{g}}(k, K)$ by definition, we will prove that any short exact sequence $0 \to K \to M \to k \to 0$ of $U\mathfrak{g}$-modules splits. Note that K is injective as a K-comodule. Then we have only to prove that the discrete $U\mathfrak{g}$-module M is *continuous* in the sense that the action

$U\mathfrak{g} \otimes (M) \to M$ is continuous, since then the short exact sequence is that of K-comodules and hence splits. Given $U\mathfrak{g}$-modules $N \subset M$, we will prove that, if N and M/N are continuous, then M is, too. We may suppose M is finitely generated. Then so is N as well as M/N, since $U\mathfrak{g}$ is noetherian. Since M/N is continuous, $I_2 M \subset N$ for some $I_2 \in \mathcal{I}$. Since N is continuous, $I_1 N = 0$ for some $I_1 \in \mathcal{I}$. Take $I \in \mathcal{I}$ such that $I \subset I_1 I_2$. Then, $IM \subset I_1 I_2 M \subset I_1 N = 0$, so that M is continuous. □

The H^2 vanishes since we have the following.

Theorem 8.17. (1) (Schneider [M, Thm. 5.2]) $H^2(\mathfrak{g}, (U\mathfrak{g})^\circ) = 0$.
(2) (Koszul [Kos, Thm. 6]) *If* $\mathfrak{g}$ *is nilpotent*, $H^n(\mathfrak{g}, (U\mathfrak{g})') = 0$ *for* $n > 0$.

We remark that $H^3(\mathfrak{g}, (U\mathfrak{g})^\circ) \neq 0$ if $\mathfrak{g}$ is semisimple; see [M, Remark 5.9].

Note 8.18. Suppose (H, K) is a Singer pair (Definition 2.2) with K finite-dimensional. Recall from Section 2 the double cosimplicial abelian group

$$\mathrm{Reg}_H^K(\Phi.(k), \Psi^{\cdot}(k)),$$

and let $D^{\cdot\cdot}$ denote the associated, normalized double complex. After this paper was accepted, I found explicit homotopy equivalences between the total complex $\mathrm{Tot}\, D^{\cdot\cdot}$ and the standard complex for computing the Sweedler cohomology $H^n(H \bowtie K^*, k)$. Obtained as a biproduct is a direct, homological proof of the generalized Kac exact sequence due to Schauenburg; see Remark 1.11 (3). The results are contained in my preprint "Cohomology and coquasi-bialgebra extensions associated to a matched pair of bialgebras".

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