Coideal Subalgebras and
Quantum Symmetric Pairs

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ABSTRACT. Coideal subalgebras of the quantized enveloping algebra are surveyed, with selected proofs included. The first half of the paper studies generators, Harish-Chandra modules, and associated quantum homogeneous spaces. The second half discusses various well known quantum coideal subalgebras and the implications of the abstract theory on these examples. The focus is on the locally finite part of the quantized enveloping algebra, analogs of enveloping algebras of nilpotent Lie subalgebras, and coideals used to form quantum symmetric pairs. The last family of examples is explored in detail. Connections are made to the construction of quantum symmetric spaces.

The introduction of quantum groups in the early 1980's has had a tremendous influence on the theory of Hopf algebras. Indeed, quantum groups provide a source of new and interesting examples. We shall discuss the reverse impact: the theory of quantum groups uses the Hopf structure extensively. This special structure is often hidden in the classical setting, while it is prominent and fundamental for quantum analogs.

Let $\mathbf{g}$ be a semisimple Lie algebra and write G for the corresponding connected, simply connected algebraic group. There are two standard types of quantum groups associated to $\mathbf{g}$ and G. The first is the quantized enveloping algebra which is a quantum analog of the enveloping algebra of $\mathbf{g}$. The second is the quantized function algebra which is a quantum analog of the algebra of regular functions on G. We will be focusing on a particular aspect of the Hopf theory of both types of quantum groups: the study of (one-sided) coideal subalgebras.

One of the reasons coideal subalgebras are so important in the study of quantum groups is that quantum groups do not have "enough" Hopf subalgebras. This shortage of Hopf subalgebras is especially notable for quantized enveloping algebras. Consider a Lie subalgebra $\mathbf{t}$ of the Lie algebra $\mathbf{g}$. The enveloping algebra $U(\mathbf{t})$ of $\mathbf{t}$ is a Hopf subalgebra of $U(\mathbf{g})$. However, upon passage to the

quantum case, $U_q(\mathbf{t})$, even when it is defined, is often not isomorphic to a Hopf subalgebra of $U_q(\mathbf{g})$. In many cases, there are subalgebras of $U_q(\mathbf{g})$ which are not Hopf subalgebras but are still good quantum analogs of $U(\mathbf{t})$. Moreover, these subalgebras often turn out to be coideals. For example, let $\mathbf{g} = \mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$ be the triangular decomposition of $\mathbf{g}$. There is a natural subalgebra U^+ of $U_q(\mathbf{g})$ which is an analog of the subalgebra $U(\mathbf{n}^+)$ of $U(\mathbf{g})$. This subalgebra U^+ is a coideal but is not a Hopf subalgebra of the quantizing enveloping algebra. Of more interest to us is the fixed Lie subalgebra $\mathbf{g}^\theta$ corresponding to an involution θ of $\mathbf{g}$. In the classical case, the symmetric pair $\mathbf{g}^\theta, \mathbf{g}$ is used to form symmetric spaces. However, in the quantum case, $U_q(\mathbf{g}^\theta)$ does not usually embed inside of $U_q(\mathbf{g})$. Thus it was initially unclear how to develop the theory of quantum symmetric spaces. In [K], Koornwinder constructed two-sided coideal analogs of $\mathbf{g}^\theta$ in type A_1 and used them to produce quantum symmetric spaces. More families of coideal analogs were discovered in [N], [NS],[DN], and [L1]. In [L2], a uniform approach was developed in the maximally split case using one-sided coideal subalgebras of the quantized enveloping algebra. The one-sided coideal condition turned out to be critical in characterizing these quantum analogs of $U(\mathbf{g}^\theta)$.

Quantum symmetric spaces were first defined using the quantized function algebra. (See for example [KS, 11.6.3 and 11.6.4].) Koornwinder's work [K] inspired the development of a quantum symmetric space theory using analogs of $\mathbf{g}^\theta$ contained in $U_q(\mathbf{g})$. The axiomatic theory of quantum symmetric spaces (see [Di]) proceeded more rapidly than the discovery of a general way to construct examples. Indeed, Dijkhuizen [Di, end of Section 3] outlined the desirable properties that analogs of $\mathbf{g}^\theta$ contained in $U_q(\mathbf{g})$ should have in order to form "nice" quantum symmetric spaces. As in Koornwinder's work [K], one of the key properties is the coideal condition. Another crucial property of an analog is that its finite-dimensional spherical modules be characterized in a similar way to the characterization in the classical case. This is obtained in [L3] for the coideal subalgebras of [L2]. The proof uses quantum Harish-Chandra modules associated to quantum symmetric pairs. The coideal condition plays a prominent role in defining and developing the theory of quantum Harish-Chandra modules ([L3]).

This paper is based on a talk given at the MSRI Hopf Algebra Workshop. It offers a panorama of the use of coideal subalgebras in constructing quantum symmetric pairs, in forming quantum Harish-Chandra modules, and in producing quantum symmetric spaces. In the first half of the paper, we present topics in the general theory of quantum coideal subalgebras. Section 1 sets notation and presents some basic facts about coideal subalgebras inside arbitrary Hopf algebras. In Section 2, we define Harish-Chandra modules associated to quantum "reductive" pairs. We prove a basic result: every $U_q(\mathbf{g})$ module contains a large Harish-Chandra module associated to a quantum reductive pair. In Section 3, we discuss how coideal subalgebras of the quantized enveloping algebra can be used in the dual quantum function algebra setting. Connections are made to

the theory of quantum homogeneous spaces. Section 4 studies filtrations on the quantized enveloping algebra and their impact on coideal subalgebras. As a result, we obtain a nice description of the generators of a coideal subalgebra under mild restrictions.

The final three sections are devoted to specific coideal subalgebras of the quantized enveloping algebra. Section 5 discusses the locally finite part, $F(U)$, of $U_q(\mathbf{g})$. It is well known that the classical enveloping algebra $U(\mathbf{g})$ can be written as a direct sum of finite-dimensional ad $\mathbf{g}$ modules. This result plays an important role in understanding the structure of $U(\mathbf{g})$ and classifying its primitive ideals. Unfortunately, the quantized enveloping algebra contains infinite dimensional $U_q(\mathbf{g})$ modules with respect to the adjoint action. Thus it is often necessary to use the locally finite part, $F(U)$, which is the maximal subalgebra of $U_q(\mathbf{g})$ that can be written as a direct sum of finite-dimensional simple ad $U_q(\mathbf{g})$ modules. This algebra $F(U)$ is not a Hopf subalgebra of $U_q(\mathbf{g})$, but it is a coideal. The structure of this coideal subalgebra is briefly reviewed with some consideration for the implications of the results of Section 4. Certain quantum Harish-Chandra modules defined originally in [JL3] using $F(U)$ are elucidated in terms of the general approach presented in Section 2. Section 6 considers coideal subalgebra analogs of enveloping algebras of nilpotent and parabolic Lie subalgebras of $\mathbf{g}$. Much of the material in this section is based on [Ke]. The last part, Section 7, is devoted to the theory of quantum symmetric pairs. This material is largely drawn from [L2] and [L3]. However, since the papers appeared, we have found simpler approaches which are presented here with many proofs included. We show how to lift a maximally split involution θ of $\mathbf{g}$ to the quantum setting. Exploiting this lift, we define a coideal subalgebra B_θ of $U_q(\mathbf{g})$. As in [L2], B_θ is characterized as the "unique" maximal coideal subalgebra of $U_q(\mathbf{g}^\theta)$ which specializes to $U(\mathbf{g}^\theta)$ as q goes to 1. Using the results of Section 4, we give a new proof of this uniqueness theorem which does not involve the intricate specialization arguments found in [L2]. We also take the opportunity to make some corrections in the case work necessary to make the uniqueness tight. Results on the Harish-Chandra module and quantum symmetric space theory associated to these pairs are described.

1. Background and Notation

Let H be a Hopf algebra over a field k with comultiplication Δ, antipodal map σ, and counit ε. Given any $a \in H$, write $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$ using Sweedler notation. A vector subspace I of H is called a left coideal if

$$\Delta(I) \subset H \otimes I.$$

Similarly, I is called a right coideal if $\Delta(I) \subset I \otimes H$. In particular, a left (resp. right) coideal is a left (resp. right) H comodule contained in H. If I is both a left (resp. right) coideal and a subalgebra of H, then we simply say that I is a left (resp. right) coideal subalgebra. There is also a notion of two sided coideals but those are generally not considered here. We will usually choose to discuss left coideals and left coideal subalgebras; analogous results can be proved for the right-handed versions.

We first present two general results about coideals inside of an arbitrary Hopf algebra. First, assume that H contains the group algebra kG of a group G. Choose a vector space complement Y to kG in H. Let P be the projection map of H onto kG as vector spaces using the decomposition $H = kG \oplus Y$. Assume that H is a left kG comodule where the comodule structure comes from the comultiplication and the projection P. In particular, H is the direct sum of vector subspaces H_g where

$$(P \otimes \mathrm{Id})\Delta(H_g) \subset g \otimes H_g. \tag{1.1}$$

Given any left coideal I of H, set $I_g = I \cap H_g$.

LEMMA 1.1. *A left coideal I contained in H is equal to a direct sum of the vector spaces I_g. Thus I is a left kG comodule.*

PROOF. Write $a \in I$ as $a = \sum_{g \in G} a_g$ where each $a_g \in H_g$. The lemma follows if we show that each $a_g \in I$. By (1.1),

$$\Delta(a) \in \sum_{g \in G} g \otimes a_g + Y \otimes H.$$

The coideal property now ensures that each $a_g \in I$. $\square$

Every Hopf algebra H comes equipped with a (left) adjoint action. Using this adjoint action, H becomes an $(\mathrm{ad}\, H)$ module. In particular, given $a, b \in H$,

$$(\mathrm{ad}\, a)b = \sum a_{(1)} b \sigma(a_{(2)}). \tag{1.2}$$

In the quantum case, it is often interesting to consider ad-invariant coideals. The following result (which is basically [Jo, Lemma 1.3.5]) is particularly useful.

LEMMA 1.2. *Let I be a left coideal in H and let M be a Hopf subalgebra of H. Then $(\mathrm{ad}\, M)I$ is an $\mathrm{ad}\, M$ invariant left coideal of H.*

PROOF. First note that ([Jo, 1.1.10])

$$\Delta(\sigma(a)) = \sum \sigma(a_{(2)}) \otimes \sigma(a_{(1)}). \tag{1.3}$$

Hence

$$\Delta((\text{ad } a)b) = \Delta(\sum a_{(1)} b \sigma(a_{(2)})) = \sum (a_{(1)} b_{(1)} \sigma(a_{(4)})) \otimes (a_{(2)} b_{(2)} \sigma(a_{(3)})).$$

The result follows from the fact that $\Delta(a_{(2)}) = \sum a_{(2)} \otimes a_{(3)}$. $\square$

Before defining the quantized enveloping algebra, we recall some basic facts about semisimple Lie algebras. Denote the set of nonnegative integers by $\mathbb{N}$, the complex numbers by $\mathbb{C}$, and the real numbers by $\mathbb{R}$. Let $\mathbf{g}$ be a complex semisimple Lie algebra with triangular decomposition $\mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$. Write $h_1, \ldots, h_n$ for a basis of $\mathbf{h}^*$. Let Δ denote the root system of $\mathbf{g}$ and write Δ^+ for the set of positive roots. Recall that Δ is a subset of $\mathbf{h}^*$. Furthermore, $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) has a basis of root vectors $\{e_\beta | \beta \in \Delta^+\}$ (resp. $\{f_{-\beta} | \beta \in \Delta^+\}$). These root vectors are common eigenvectors, called weight vectors, for the adjoint action of $\mathbf{h}$ on $\mathbf{g}$. In particular, $(\text{ad } h_i)e_\beta = [h_i, e_\beta] = \beta(h_i)e_\beta$ and $(\text{ad } h_i)f_{-\beta} = [h_i, f_{-\beta}] = -\beta(h_i)f_{-\beta}$ for each $\beta \in \Delta^+$. We further assume that $\{e_\beta, f_{-\beta} | \beta \in \Delta^+\} \cup \{h_1, \ldots, h_n\}$ is a Chevalley basis for $\mathbf{g}$ ([H, Theorem 25.2]). Let $\pi = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots in Δ^+ and $(\ ,\ )$ denote the Cartan inner product on $\mathbf{h}^*$. Recall further that $(\ ,\ )$ is positive definite on the real vector space spanned by π. The set π is a basis for $\mathbf{h}^*$. Given $\alpha_i \in \pi$, we write e_i (resp. f_i) for e_{α_i} (resp. $f_{-\alpha_i}$). The Cartan matrix associated to the root system Δ is the matrix with entries $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. (The reader is referred to [H, Chapters II and III] for additional information on semisimple Lie algebras and root systems.)

Let q be an indeterminate and set $q_i = q^{(\alpha_i, \alpha_i)/2}$. Let $[m]_q$ denote the q number $(q^m - q^{-m})/(q - q^{-1})$ and $[m]_q!$ denote the q factorial $[m]_q[m-1]_q \cdots [1]_q$. The q binomial coefficients are defined by

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{[m]!_q}{[j]!_q[m-j]!_q}.$$

The quantized enveloping algebra $U = U_q(\mathbf{g})$ is generated by $x_1, \ldots, x_n, t_1^{\pm 1}, \ldots, t_n^{\pm 1}, y_1, \ldots, y_n$ over $\mathbb{C}(q)$ with the relations listed below (see for example [Jo, 3.2.9] or [DK, Section 1]).

(1.4) $x_i y_j - y_j x_i = \delta_{ij}(t_i - t_i^{-1})/(q_i - q_i^{-1})$ for each $1 \leq i \leq n$.
(1.5) The $t_1^{\pm 1}, \ldots, t_n^{\pm 1}$ generate a free abelian group T of rank n.
(1.6) $t_i x_j = q^{(\alpha_i, \alpha_j)} x_j t_i$ and $t_i y_j = q^{-(\alpha_i, \alpha_j)} y_j t_i$ for all $1 \leq i, j \leq n$.
(1.7) The quantum Serre relations:

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} x_i^{1-a_{ij}-m} x_j x_i^m = 0$$

and

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} y_i^{1-a_{ij}-m} y_j y_i^m = 0$$

for all $1 \le i, j \le n$ with $i \ne j$.

The algebra U is a Hopf algebra with comultiplication Δ, antipode σ, and counit ε defined on generators as follows.

(1.8) $\Delta(t) = t \otimes t$ $\varepsilon(t) = 1$ $\sigma(t) = t^{-1}$ for all t in T

(1.9) $\Delta(x_i) = x_i \otimes 1 + t_i \otimes x_i$ $\varepsilon(x_i) = 0$ $\sigma(x_i) = -t_i^{-1} x_i$

(1.10) $\Delta(y_i) = y_i \otimes t_i^{-1} + 1 \otimes y_i$ $\varepsilon(y_i) = 0$ $\sigma(y_i) = -y_i t_i$

for $1 \le i \le n$.

It is well known that the algebra U specializes to $U(\mathbf{g})$ as q goes to 1. This can be made more precise as follows. Set A equal to $\mathbb{C}[q, q^{-1}]_{(q-1)}$. Let $\hat{U}$ be the A subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}$, and $(t_i - 1)/(q - 1)$ for $1 \le i \le n$. Then $\hat{U} \otimes_A \mathbf{C}$ is isomorphic to $U(\mathbf{g})$. (See for example [L2, beginning of Section 2]). Given a subalgebra S of U, set $\hat{S} = S \cap \hat{U}$. We say that S specializes to the subalgebra $\bar{S}$ of $U(\mathbf{g})$ if the image of $\hat{S}$ in $\hat{U} \otimes_A \mathbf{C}$ is $\bar{S}$.

Set $Q(\pi)$ equal to the integral lattice generated by π. Let $Q^+(\pi)$ (resp. $Q^-(\pi)$) be the subset of $Q(\pi)$ consisting of nonnegative (resp. nonpositive) integer linear combinations of elements in π. The standard partial ordering on the root lattice $Q(\pi)$ is defined by $\lambda \ge \mu$ provided $\lambda - \mu$ is in $Q^+(\pi)$. Let $P^+(\pi)$ denote the set of dominant integral weights associated to π. In particular, $\lambda \in \mathbf{h}^*$ is an element of $P^+(\pi)$ if and only if $2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ is a nonnegative integer for all $1 \le i \le n$. There is an isomorphism τ of abelian groups from $Q(\pi)$ to T defined by $\tau(\alpha_i) = t_i$, for $1 \le i \le n$. Using this isomorphism, we can replace condition (1.6) with

$$\tau(\lambda) x_i \tau(\lambda)^{-1} = q^{(\lambda, \alpha_i)} x_i \quad \text{and} \quad \tau(\lambda) y_i \tau(\lambda)^{-1} = q^{-(\lambda, \alpha_i)} y_i \qquad (1.11)$$

for all $\tau(\lambda) \in T$ and $1 \le i \le n$.

Let M be a U module. A nonzero vector $v \in U$ has weight $\gamma \in \mathbf{h}^*$ provided that $\tau(\lambda) \cdot v = q^{(\lambda, \gamma)} v$ for all $\tau(\lambda) \in T$. Given a subspace $V \subset M$, the subspace of V spanned by the γ weight vectors is called the γ weight space of V and denoted by V_γ. Now U can be given the structure of a U module using the quantum adjoint action (1.2). Let v be an element of U. We say that v has weight γ provided that it is a γ weight vector in terms of this adjoint action. In particular, v has weight γ if $\tau(\lambda) v \tau(\lambda)^{-1} = q^{(\lambda, \gamma)} v$ for all $\tau(\lambda) \in T$.

Let G^+ be the subalgebra of U generated by $x_1 t_1^{-1}, \ldots, x_n t_n^{-1}$. Similarly, let U^- be the subalgebra of U generated by $y_1, \ldots, y_n$. Let U^o be the group algebra of T. It is well known that both U^- and G^+ are a direct sum of their weight spaces. The quantized enveloping algebra U admits a triangular decomposition. More precisely, there is an isomorphism of vector spaces using the multiplication

map ([R]):
$$U \cong U^- \otimes U^o \otimes G^+. \tag{1.12}$$
It follows that there is a direct sum decomposition
$$U = \bigoplus_{t \in T} U^- G^+ t. \tag{1.13}$$

Let G_+^+ (resp. U_+^-) denote the augmentation ideal of G^+ (resp. U^-) and set Y equal to the vector space $(U_+^- G^+ U^o + U^- G_+^+ U^o)$. The direct sum decomposition (1.13) implies that
$$U = U^o \oplus Y. \tag{1.14}$$
Using the definition of the comultiplication of U, it is straightforward to check that for any $b \in U^- G^+ t$
$$\Delta(b) \in t \otimes b + Y \otimes U.$$

Thus the projection of $\Delta(U)$ onto $U^o \otimes U$ makes U into a left U^o comodule with $U_t = U^- G^+ t$. Hence we have the following version of Lemma 1.1 for quantized enveloping algebras.

LEMMA 1.3. *Let I be left coideal of U. Then*
$$I = \bigoplus_{t \in T} (I \cap U^- G^+ t).$$

2. Harish-Chandra Modules

Consider a Lie subalgebra $\mathbf{k}$ of the semisimple Lie algebra $\mathbf{g}$. A Harish-Chandra module associated to the pair $\mathbf{g}, \mathbf{k}$ is a $\mathbf{g}$ module which can be written as a direct sum of finite-dimensional simple $\mathbf{k}$ modules. Harish-Chandra modules are an important tool in classical representation theory. This is especially true when $\mathbf{g}, \mathbf{k}$ is a symmetric pair. Harish-Chandra modules associated to symmetric pairs provide an algebraic approach to the representation theory of real reductive Lie groups.

There is a nice introduction to the theory of Harish-Chandra modules presented in [D, Chapter 9] from an algebraic point of view. The first section of [D, Chapter 9] only assumes that $\mathbf{k}$ is reductive in $\mathbf{g}$. A basic result which is used repeatedly in this chapter of [D] is the following.

THEOREM 2.1 [D, Proposition 1.7.9]. *The direct sum of all the finite-dimensional simple $\mathbf{k}$ modules inside a $\mathbf{g}$ module is a Harish-Chandra module for the pair $\mathbf{g}, \mathbf{k}$.*

This theorem allows one to find large Harish-Chandra modules inside of infinite-dimensional $\mathbf{g}$ modules. Its proof uses the fact that $\mathbf{k}$ is reductive in $\mathbf{g}$ and that $U(\mathbf{g})$ is a locally finite $\mathrm{ad}\, U(\mathbf{g})$ module.

In the quantum setting, $U_q(\mathbf{k})$ is not always a subalgebra of $U_q(\mathbf{g})$ when $\mathbf{k}$ is a Lie subalgebra of $\mathbf{g}$. However, one often finds a quantum analog of $U(\mathbf{k})$ which is a one-sided coideal subalgebra of $U_q(\mathbf{g})$. Thus any good theory of quantum

Harish-Chandra modules must work for pairs $U_q(\mathbf{g})$, I where I is a one-sided coideal subalgebra of $U_q(\mathbf{g})$. In order to begin such a theory, it is necessary to have an analog of Theorem 2.1. This presents two difficulties. The first is that U, in contrast to the classical situation, is not a locally finite $\operatorname{ad} U$ module. (We will return to this obstruction in Section 4.) The second is: what does it mean for a coideal subalgebra to be reductive in U?

In this section, we present a quantum version of Theorem 2.1 using the locally finite part $F(U)$ of U and a certain condition on coideal subalgebras which substitutes for reductivity. The material of this section is based on [L1, Section 4] and [L3, Section 3]. This result sets the stage for the development of a quantum Harish-Chandra module theory. Indeed, the author has checked that many of the results of [D, Section 9.1] and their proofs carry over to coideal subalgebras satisfying this quantum version of Theorem 2.1. Some properties of quantum principal series modules analogous to those in [D, Section 9.3] are proved in [L3, Section 6]. Spherical modules (see [D, 9.5.4]) have been classified in the quantum case (see Section 7, Theorem 7.7 and [L3, Section 4]). This is discussed further in Section 7.

Recall the definition of the adjoint action (1.2) and define

$$F(U) = \{v \in U \,|\, \dim(\operatorname{ad} U)v < \infty\}. \tag{2.1}$$

By [JL1, Corollary 2.3, Theorem 5.12, and Theorem 6.4], $F(U)$ is an algebra, it can be written as a direct sum of finite-dimensional simple U modules, and it is "large" in U. It is also true that $F(U)$ is a left coideal of U, a subject we will return to in Section 5.

Fix a left coideal subalgebra I of U. Note that

$$F(U)I = \{\sum f_i r_i \,|\, f_i \in F(U), r_i \in I\}$$

is also a subalgebra of U. This follows from the fact that $rf = \sum r_{(1)}\varepsilon(r_{(2)})f = \sum((\operatorname{ad} r_{(1)})f)r_{(2)}$ for any $r \in I$ and $f \in F(U)$. Since I is a left coideal, each $r_{(2)} \in I$. Furthermore the ad-invariance of $F(U)$ implies that $(\operatorname{ad} r_{(1)})f$ is in $F(U)$. We use $F(U)I$ to define Harish-Chandra modules.

DEFINITION 2.2. A *Harish-Chandra module* for the pair U, I is an $F(U)I$ module which is a direct sum of finite-dimensional simple I modules.

Let us take a closer look at the condition that $\mathbf{k}$ is reductive in $\mathbf{g}$. Reductivity means that $(\operatorname{ad} \mathbf{k})$ acts semisimply on $\mathbf{g}$. This assumption is enough to prove that $\mathbf{k}$ is itself reductive and that the center of $\mathbf{k}$ can be extended to a Cartan subalgebra of $\mathbf{g}$. It is unclear what the corresponding assumption in the quantum case, namely that $(\operatorname{ad} I)$ acts semisimply on $F(U)$, implies. It seems unlikely that this assumption alone will yield an analog of Theorem 2.1.

Of course, there would be no problem if I acted semisimply on all finite-dimensional I modules. When I turns out to be a Hopf subalgebra of U isomorphic to a quantized enveloping algebra of a semisimple Lie subalgebra of $\mathbf{g}$, this

is certainly true. However, complete reducibility does not hold in general for the large class of coideal subalgebras considered in Section 7. Thus we need a replacement for the notion of reductive in $\mathbf{g}$. This substitute is invariance under the action of a certain conjugate linear antiautomorphism of U.

Let κ denote the conjugate linear form of the quantum Chevalley antiautomorphism. In particular, let $U_{\mathbb{R}(q)}$ denote the $\mathbb{R}(q)$ subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}$, for $1 \leq i \leq n$. The antiautomorphism κ of $U_{\mathbb{R}(q)}$ is defined by $\kappa(x_i) = y_i t_i$, $\kappa(y_i) = t_i^{-1} x_i$ and $\kappa(t) = t$ for all $t \in T$. We then extend κ to U using conjugation. More precisely, given $a \in \mathbb{C}$, write $\bar{a}$ for the complex conjugate of a. Set $\bar{q} = q$. Then $\kappa(au) = \bar{a}\kappa(u)$ for all $u \in U_{\mathbb{R}(q)}$.

It is straightforward to check using (1.8), (1.9), and (1.10) that

$$\Delta(\kappa(b)) = (\kappa \otimes \kappa)\Delta(b) \tag{2.2}$$

for all $b \in U$. Moreover κ gives U the structure of a Hopf $*$ algebra where $* = \kappa$ ([CP, Section 4.1F]).

For the remainder of this section, we assume that I is a left coideal subalgebra such that $\kappa(I) = I$. Thus one can think of I as a $*$ subalgebra of U.

The field $\mathbb{R}(q)$ can be made into a real ordered field ([J, Section 11.1]) where the positive elements are defined as follows. Write a polynomial $f(q)$ in the form $(q-1)^s(f_m(q-1)^m + \cdots + f_1(q-1) + f_0)$ where each $f_i \in \mathbb{R}$ and $m, s \in \mathbb{N}$. Then $f(q)$ is positive if and only if $f_0 > 0$. An element $h \in \mathbb{R}(q)$ is positive if and only if h can be written as a quotient of positive polynomials. This induces a total order on $\mathbb{R}(q)$.

We next specify a class of "nice" finite-dimensional I modules.

DEFINITION 2.3. An I module W is called **unitary** if it admits a sesquilinear form S_W (i.e. linear in the first variable and conjugate linear in the second variable) such that

(i) $S_W(av, w) = S_W(v, \kappa(a)w)$ for all $a \in I$ and v, w in W
(ii) $S_W(v, v)$ is a positive element of $\mathbb{R}(q)$ for each nonzero vector $v \in W$
(iii) $S_W(v, w) = \overline{S_W(w, v)}$ for all v and w in W.

Let W be a finite-dimensional unitary I module. Choose a nonzero vector $v \in W$ such that $S_W(v, v) = 1$. Now suppose that $w \in W$ such that $S_W(w, v) = 0$. By Definition 2.3(iii), it follows that $S_W(v, w)$ also equals zero. Hence one can show using induction that W has an orthonormal basis with respect to S_W.

The following result and its corollary show that I has an extensive family of unitary modules, namely the finite-dimensional simple I submodules of any finite-dimensional simple U module.

THEOREM 2.4. *Every finite-dimensional unitary I module can be written as a direct sum of finite-dimensional simple unitary I modules.*

PROOF. Let W be a finite-dimensional unitary I module with sesquilinear form $S = S_W$ as in Definition 2.3. Let V be a finite-dimensional simple I submodule

inside of W. By Definition 2.3, the restriction of S to W again satisfies conditions (i), (ii), and (iii). Furthermore, Definition 2.3(i) implies that the orthogonal complement $W^\perp$ of W with respect to S is an I module. Hence $V \cong W \oplus W^\perp$, a direct sum of unitary I modules with smaller dimension. The proof follows by induction on $\dim V$. $\square$

COROLLARY 2.5. *Every finite-dimensional simple U module V is a Harish-Chandra module for the pair U, I.*

PROOF. Let V be a finite-dimensional simple U module. It is well known that finite-dimensional U modules are a direct sum of their weight spaces. Moreover, the weight space of maximal weight is one dimensional. Let v be a basis vector for this highest weight space and note that v generates V as a U module. The vector v is called a highest weight generating vector of V. Recall that U_+^- denotes the augmentation ideal of U^-. Note that $U_+^- v$ is the subspace of V spanned by those weight vectors whose weights are strictly less than that of v. Furthermore, V is the direct sum of $\mathbb{C}(q)v$ and $U_+^- v$.

Let φ be the projection of U onto U^o using the direct sum decomposition $U = U^o \oplus Y$ (1.14). Define a sesquilinear form S on V by $S(v,v) = 1$ and $S(av, bv) = S(v, \varphi(\kappa(a)b)v)$ for all $a, b \in U$. Since $\varphi(b) = 0$ for all b in U_+^-, it follows that $S(v, U_+^- v) = 0$.

Note that S satisfies Definition 2.3(i). As in say ([L1, Lemma 4.2]), S specializes to the classical positive definite Shapovalov form of [Ka, 11.5 and Theorem 11.7]. Thus ([L1, Lemma 4.2]) $S(w,w) \neq 0$ for any nonzero vector $w \in V$. It is straightforward to check that S restricts to a $\mathbb{R}(q)$ bilinear form on $U_{\mathbb{R}(q)}v$. Moreover, S restricted to $\hat{U}_{\mathbb{R}(q)}$ takes values in $\mathbb{R}[q, q^{-1}]_{(q-1)}$. Let w be an element in $\hat{U}_{\mathbb{R}(q)}v$. We can write $S(w,w) = f(q)$ with $f(q)$ in $\mathbb{R}[q, q^{-1}]_{(q-1)}$. Since the specialization of S is positive definite, we must have that $f(1) \geq 0$. It follows that $f(q) \geq 0$. This fact and the nondegeneracy property implies that S satisfies Definition 2.3(ii).

Recall the direct sum decomposition (1.14) of U. Note that $\kappa(Y) = Y$ and $\kappa(a) = a$ for all $a \in U^o \cap U_{\mathbb{R}(q)}$. Therefore $\varphi(\kappa(b)) = \varphi(b)$ for all $b \in U_{\mathbb{R}(q)}$. It follows that S is symmetric when restricted to $U_{\mathbb{R}(q)}v$. In particular, S satisfies Definition 2.3(iii). Thus V is a unitary I module. The result now follows from Theorem 2.4. $\square$

Note that we cannot expect $\Delta(I)$ to be a subset of $I \otimes I$. Hence the tensor product of two I modules does not necessarily admit an action of I via the comultiplication of U. However, since I is a left coideal, the tensor product $V \otimes W$ of a U module V with an I module W is an I module. In particular, $a(v \otimes w) = \sum a_{(1)}v \otimes a_{(2)}w$ for all $v \otimes w \in V \otimes W$ and $a \in I$. The next lemma shows that the notion of unitary behaves well with respect to the tensor product of a U module with an I module.

LEMMA 2.6. *Let V be a finite-dimensional U module and let W be a finite-dimensional unitary I module. Then $V \otimes W$ is a finite-dimensional unitary I module.*

PROOF. Let S_V (resp. S_W) denote the sesquilinear form on V (resp. W) satisfying the conditions of Definition 2.3. Define a sesquilinear form $S = S_{V \otimes W}$ on $V \otimes W$ by setting $S(a \otimes b, a' \otimes b') = S_V(a, a')S_W(b, b')$. It is easy to check Definition 2.3(iii) holds for S. Let $\{v_i\}$ be an orthonormal basis for V with respect to S_V and let $\{w_i\}$ be an orthonormal basis for W with respect to S_W. Then $S(\sum b_{ij}v_i \otimes w_j, \sum b_{ij}v_i \otimes w_j) = \sum b_{ij}\bar{b}_{ij}$. Thus Definition 2.3(ii) holds for S. Condition (2.2) on κ ensures that S satisfies Definition 2.3(i). In particular, for $c \in I$, we have $S(c(a \otimes b), a' \otimes b') = S(\sum c_{(1)}a \otimes c_{(2)}b, a' \otimes b') = S(a \otimes b, \sum \kappa(c_{(1)})a' \otimes \kappa(c_{(2)})b') = S(a \otimes b, \kappa(c)(a' \otimes b'))$. □

We now obtain a quantum analog of Theorem 2.1.

THEOREM 2.7. *The sum of all the finite-dimensional simple unitary I modules inside of the $F(U)I$ module M is a Harish-Chandra module for the pair U, I.*

PROOF. Assume that W is a finite-dimensional simple unitary I module contained in M. It suffices to show that the $F(U)I$ module generated by W is a direct sum of finite-dimensional simple unitary modules. Note that $F(U)IW = F(U)W = IF(U)W$ is an I module. The vector space $F(U) \otimes W$ is also an I module where the action is given by

$$a \cdot (f \otimes w) = \sum (\mathrm{ad}\, a_{(1)})f \otimes a_{(2)}w$$

for all $f \in F(U)$, $w \in W$, and $a \in I$. Furthermore, $F(U)W$ is a homomorphic image of the I module $F(U) \otimes W$. Recall that $F(U)$ is a direct sum of finite-dimensional simple $(\mathrm{ad}\, U)$ modules. By Corollary 2.5, each finite-dimensional simple $(\mathrm{ad}\, U)$ module is a unitary I module. Thus by Lemma 2.6, $F(U) \otimes W$, and hence $F(U)W$, splits into a direct sum of finite-dimensional simple unitary I modules. □

Let $\mathcal{H}_{\mathbb{R}}$ be the set of all Hopf algebra automorphisms of U which restrict to a Hopf algebra automorphism of $U_{\mathbb{R}(q)}$. Let $\Upsilon \in \mathcal{H}_{\mathbb{R}}$. Suppose that I is a left coideal subalgebra such that $\Upsilon^{-1}\kappa\Upsilon(I) = I$. Then the results of this section hold for I where we define unitary I modules using $\Upsilon^{-1}\kappa\Upsilon$ instead of κ.

3. The Dual Picture

In this section, we consider the connection between coideal subalgebras of U and coideal subalgebras inside the Hopf dual of U. The results presented in this section are well known and are related to the theory of quantum homogeneous spaces. A good reference for most of the material presented here and for other basic results about quantum homogeneous spaces is [KS, Chapter 11] (see also [Jo, 1.4.15]).

Let $R_q[G]$ denote the quantized function algebra of the connected, simply connected algebraic Lie group G with Lie algebra $\mathbf{g}$. (See [Jo, Section 9.1] for a precise definition.) Note that up to a finite group, $R_q[G]$ is the Hopf dual of U. Furthermore, $R_q[G]$ satisfies a Peter–Weyl theorem ([Jo, 9.1.1 and 1.4.13]). That is, there is an isomorphism of U bimodules

$$R_q[G] \cong \bigoplus_{\lambda \in P^+(\pi)} L(\lambda) \otimes L(\lambda)^*. \tag{3.1}$$

Here $L(\lambda)$ is the (left) finite-dimensional simple U module with highest weight λ contained in the set $P^+(\pi)$ of dominant integral weights. Moreover, $L(\lambda)^*$ is thought of as a right U module. Thus, the right U module action on $R_q[G]$ comes from the action of U on $L(\lambda)^*$, while the left action comes from the action of U on $L(\lambda)$.

Given a left coideal I of U and a (left) U module M, a (left) invariant is an $m \in M$ such that $am = \varepsilon(a)m$ for all $a \in I$. Write M_l^I for the collection of all left invariants in M. Equivalently, M_l^I is equal to the elements of M annihilated (on the left) by the augmentation ideal of I. Consider the special case where I is the quantum analog of the enveloping algebra of a Lie subalgebra of $\mathbf{g}$ corresponding to a subgroup H of G. Then $R_q[G]_l^I$ is often written as $R_q[G/H]$. In particular, $R_q[G/H]$ is thought of as the quantized function algebra on the quotient space G/H.

THEOREM 3.1. *For any left coideal I of U, $R_q[G]_l^I$ is a left coideal subalgebra of $R_q[G]$.*

PROOF. Let ϕ, ϕ' be elements of $R_q[G]_l^I$ and r an element of I. We first show that $\phi\phi'$ is also in $R_q[G]_l^I$. To see this, consider

$$r \cdot (\phi\phi') = \sum (r_{(1)} \cdot \phi)(r_{(2)} \cdot \phi')$$
$$= \sum (r_{(1)} \cdot \phi)\varepsilon(r_{(2)})\phi' = (r \cdot \phi)\phi' = \varepsilon(r)\phi\phi'.$$

We now check the coideal condition. One can show using the precise definition of the action of U on $R_q[G]$ and the coalgebra structure of $R_q[G]$ that

$$\Delta(r \cdot \phi) = (1 \otimes r)\Delta(\phi) = \sum \phi_{(1)} \otimes r \cdot \phi_{(2)}. \tag{3.2}$$

On the other hand,

$$\Delta(r \cdot \phi) = \Delta(\varepsilon(r)\phi) = \sum \phi_{(1)} \otimes \varepsilon(r)\phi_{(2)}. \tag{3.3}$$

Since we can choose the $\phi_{(1)}$ to be linearly independent, (3.2) and (3.3) force $r \cdot \phi_{(2)} = \varepsilon(r)\phi_{(2)}$. Thus each $\phi_{(2)} \in R_q[G]_l^I$. $\qquad\square$

In [KS, Chapter 11.6], a quantum homogeneous space associated to $R_q[G]$ is defined up to isomorphism as a one-sided coideal subalgebra of $R_q[G]$. (Note that quantum homogeneous spaces are actually quantum analogs of the algebra of regular functions on classical homogeneous spaces.) Thus the theorem above

shows there is a left quantum homogeneous space, $R_q[G]_I^I$, associated to each left coideal subalgebra I. Using the Peter–Weyl decomposition (3.1), we obtain the following nice description of $R_q[G]_l^I$.

$$R_q[G]_l^I \cong \bigoplus_{\lambda \in P^+(\pi)} L(\lambda)_l^I \otimes L(\lambda)^*. \qquad (3.4)$$

Now $R_q[G]_l^I$ is the set of left I invariants of $R_q[G]$. We may similarly define the set of right I invariants $R_q[G]_r^I$. Using the fact that the right action satisfies $\Delta(\phi \cdot r) = \Delta(\phi) \cdot (r \otimes 1)$, the same argument as in the proof of Theorem 3.1 shows that $R_q[G]_r^I$ is a right coideal subalgebra of $R_q[G]$. One may also study the set of bi-invariants $R_q[G]_{bi}^I = R_q[G]_l^I \cap R_q[G]_r^I$. As above, $R_q[G]_{bi}^I$ is a subalgebra of $R_q[G]^I$. However, it is not a coideal.

4. Generators and Filtrations

Consider the Hopf algebra $U(L)$, the universal enveloping algebra of a complex Lie algebra L. Since $U(L)$ is cocommutative, the one-sided coideal subalgebras of $U(L)$ are exactly the subbialgebras of $U(L)$. It is very easy to understand the coideal subalgebras of $U(L)$. Indeed,the next observation is well known. It follows from say [Mo, Theorem 5.6.5] and the fact that every subcoalgebra of $U(L)$ is connected ([Mo, Definition 5.1.5 and Lemma 5.1.9]). (Theorem 5.6.5 of [Mo] is stated for Hopf algebras, however, the proof also works for bialgebras.)

THEOREM 4.1.   *The set of (left) coideal subalgebras of $U(L)$ is the set of enveloping algebras $U(L')$ of Lie subalgebras L' of L.*

An immediate consequence of the above result is that any coideal subalgebra of $U(L)$ is generated by elements of the underlying Lie algebra L. We would like to obtain a similar result for coideal subalgebras of the quantized enveloping algebra. However, passing to the quantum case, the situation becomes more complicated. Indeed the coalgebra structure is not cocommutative for quantized enveloping algebras. So the set of coideal subalgebras of the quantized enveloping algebra is much larger than the set of subbialgebras. By analyzing and deepening Lemma 1.3 and studying the comultiplication of U, we are able to obtain detailed information about coideal subalgebras and their generators.

The next result is known as well. It describes the coideal subalgebras of a group algebra.

LEMMA 4.2.   *Let I be a (left) coideal subalgebra of the group algebra of the group G. Then $I \cap G$ is a semigroup and $I \cap kG$ is spanned by $I \cap G$ as a vector space.*

We introduce two subalgebras of U which are similar to U^- and G^+. Let U^+ be the subalgebra of U generated by $x_1, \ldots, x_n$ and G^- be the subalgebra of U generated by $y_1 t_1, \ldots, y_n t_n$. Once again, we have that U^+ and G^- are a direct

sum of their weight spaces. We may replace U^- by G^- and G^+ by U^+ to obtain the following version of the triangular decomposition.

$$U \cong G^- \otimes U^o \otimes U^+. \tag{4.1}$$

In this section, we show how to break up a coideal subalgebra into three parts corresponding to coideal subalgebras of G^-, U^o, and U^+ respectively. First, however, we obtain basic properties of coideal subalgebras of G^- and U^+.

Using the formulas for comultiplicaton (1.8), (1.9), and (1.10), it is straightforward to check that G^- and U^+ are left coideal subalgebras of U. Consider now an arbitrary coideal subalgebra J of U which is either a subset of G^- or U^+. Note that if J is also an $\mathrm{ad}\,T$ module, then J can be written as a direct sum of its weight spaces. We obtain a nice result on the generators of J analogous to Theorem 4.1 when J is an $\mathrm{ad}\,T$ module.

LEMMA 4.3. *Let J be an $\mathrm{ad}\,T$ submodule and a coideal subalgebra of G^- (resp. U^+). Then there exists a subset Δ' of Δ^+ and weight vectors $\tilde{f}_{-\gamma}$ of weight $-\gamma$, $\gamma \in \Delta'$ (resp. $\tilde{e}_\gamma$ of weight $\gamma \in \Delta'$) which generate J as an algebra. Moreover, each $\tilde{f}_{-\gamma}$ (resp. $\tilde{e}_\gamma$) specializes to the root vector $f_{-\gamma}$ (resp. e_γ) as q goes to 1.*

PROOF. Note that the weight spaces of G^- are finite-dimensional. Hence J has finite-dimensional weight spaces. Let $\bar{J}$ denote the specialization of J as q goes to 1. Consider a weight space J_μ of J. We have that $\hat{J}_\mu = \hat{U} \cap J_\mu = \hat{G}^- \cap J_\mu$. Also, G^- is a free A module and A is a principal ideal domain with unique maximal ideal generated by $(q-1)$. Hence one can find a basis for J_μ which is a subset of $\hat{J}_\mu$ and remains linearly independent modulo $(q-1)\hat{U}$. In particular, the specialization of this basis as q goes to 1 is a basis for $\bar{J}_\mu$. Hence the weight spaces of $\bar{J}$ have the same dimension as the weight spaces of J.

Note that the comultiplication of U specializes to the comultiplication of $U(\mathbf{g})$. Hence $\bar{J}$ is a coideal subalgebra of $U(\mathbf{n}^-)$. By Theorem 4.1, $\bar{J}$ is an enveloping algebra of a Lie subalgebra, say $\mathbf{a}$, of $\mathbf{n}^-$. Now $\bar{J}$ is a direct sum of its weight spaces. Hence there exists a subset Δ' of Δ^+ such that the set $\{f_{-\gamma} | \gamma \in \Delta'\}$ is a basis of $\mathbf{a}$. Thus for each $\gamma \in \Delta'$, we can find a vector $\tilde{f}_{-\gamma}$ of weight $-\gamma$ in J such the image of $\tilde{f}_{-\gamma}$ under specialization is $f_{-\gamma}$. Write $\Delta' = \{\gamma_1, \ldots, \gamma_m\}$. A standard argument shows that the set

$$\mathcal{B}_\eta = \{f_{-\gamma_1}^{i_1} \ldots f_{-\gamma_m}^{i_m} | i_j \in \mathbb{N} \text{ for } 1 \leq j \leq m \text{ and } i_1\gamma_1 + \ldots + i_m\gamma_m = \eta\}$$

is a basis for the $-\eta$ weight space of $U(\mathbf{a})$. Furthermore $\mathcal{B} = \cup_\eta \mathcal{B}_\eta$ is a basis for $U(\mathbf{a})$. Since the dimensions of the $-\eta$ weight spaces of $U(\mathbf{a})$ and J agree, the corresponding set $\tilde{\mathcal{B}}_\eta$ with $\tilde{f}$ playing the role of f is a basis of J_η. Thus the set $\tilde{\mathcal{B}} = \cup_\eta \tilde{\mathcal{B}}_\eta$ is a basis for J. It follows that the $\tilde{f}_{-\gamma}$, $\gamma \in \Delta'$ generate J as an algebra.

The same analysis applies to coideal subalgebras of U^+.   □

The direct sum decomposition (1.13) can be made finer using weight spaces. It is well known that the set of weights of G^+ and U^+ equals $Q^+(\pi)$ and the set of weights of G^- and U^- equals $Q^-(\pi)$. Thus there are direct sum decompositions

$$U = \oplus_{\lambda,\mu} U^-_{-\lambda} G^+_\mu U^o \text{ and } U = \oplus_{\lambda,\mu,t} U^-_{-\lambda} G^+_\mu t \qquad (4.2)$$

where λ and μ run over elements of $Q^+(\pi)$ and t runs over elements in T. Let $\pi_{\lambda,\mu}$ be the projection of U onto the subspace $U^-_{-\lambda} G^+_\mu U^o$. Write $[\lambda,\mu]$ for a typical element in $Q(\pi) \times Q(\pi)$ (so as to avoid confusion with the Cartan inner product).

Consider elements $c \in U^-_{-\lambda}$ and $d \in G^+_\mu$. It follows from the definition of the comultiplication map on the generators of U ((1.8), (1.9), and (1.10)) that

$$(\pi_{\lambda,\mu} \otimes \mathrm{Id}) \Delta(cd) = cd \otimes \tau(-\lambda - \mu), \qquad (4.3)$$

$$(\pi_{\lambda,0} \otimes \mathrm{Id}) \Delta(cd) = \sum c \otimes \tau(-\lambda) d, \qquad (4.4)$$

$$(\pi_{0,\mu} \otimes \mathrm{Id}) \Delta(cd) = \sum d \otimes c\tau(-\mu). \qquad (4.5)$$

In the next theorem, we consider coideal subalgebras of U which behave rather nicely in terms of the second decomposition in (4.2).

THEOREM 4.4. *Let I be a left coideal subalgebra of U such that*

$$I = \sum_{\lambda,\mu,t} (I \cap U^-_{-\lambda} G^+_\mu t) \qquad (4.6)$$

and $I \cap T$ is a group. Then $I \cap G^-$, $I \cap U^o$, and $I \cap U^+$ are $\mathrm{ad}\, T$ submodules and left coideal subalgebras of I. Moreover, the multiplication map induces an isomorphism

$$I \cong (I \cap G^-) \otimes (I \cap U^+) \otimes (I \cap U^o)$$

of vector spaces.

PROOF. Since I, G^-, U^o, and U^+ are all left coideal subalgebras, so are $I \cap G^-$, $I \cap U^o$, and $I \cap U^+$. Note that every element in $U^-_{-\lambda} G^+_\mu t$ is a weight vector of weight $-\lambda + \mu$. Thus I is a direct sum of its weight spaces and I is an $\mathrm{ad}\, T$ module. It follows that $I \cap G^-$, $I \cap U^o$, and $I \cap U^+$ are all $\mathrm{ad}\, T$ modules.

The triangular decomposition of U (4.1) ensures that the multiplication map induces an injection

$$(I \cap G^-) \otimes (I \cap U^+) \otimes (I \cap U^o) \to I$$

of vector spaces. We obtain an isomorphism by showing that each element of I is contained in $(I \cap G^-)(I \cap U^+)(I \cap U^o)$.

Recall the direct sum decomposition of I given in Lemma 1.3. Let b be an element of $I \cap U^-_{-\lambda} G^+_\mu t$ where $t \in T$. There exists $c_i \in U^-_{-\lambda}$ and $d_i \in G^+_\mu$ so that $b = \sum_i c_i d_i t$. We may further assume that $\{c_i\}$ and $\{d_i\}$ are each linearly independent sets. By (1.8) and (4.3), $(\pi_{\lambda,\mu} \otimes \mathrm{Id}) \Delta(b) = b \otimes \tau(-\lambda - \mu)t$. Hence

$\tau(-\lambda - \mu)t$ is an element of $I \cap T$. Since $I \cap T$ is a group, $\tau(\lambda + \mu)t^{-1}$ is also contained in $I \cap T$.

Equation (4.4) implies that

$$(\pi_{\lambda,0} \otimes \mathrm{Id})\Delta(b) = \sum_i c_i t \otimes \tau(-\lambda)d_i t.$$

Hence each $\tau(-\lambda)d_i t \in I$. Recall that d_i is a weight vector of weight μ in G^+. Thus, multiplying $\tau(-\lambda)d_i t$ by $\tau(\lambda + \mu)t^{-1}$ yields that $d_i \tau(\mu)$ is an element of $U_\mu^+ \cap I$. Similarly, (4.5) ensures that

$$(\pi_{0,\mu} \otimes \mathrm{Id})\Delta(b) = \sum_i d_i t \otimes c_i \tau(-\mu)t.$$

So $c_i \tau(-\mu)t \in I$ and hence $c_i \tau(-\mu)t\tau(\lambda+\mu)t^{-1} = c_i\tau(\lambda)$ is an element of $I \cap G^-$. Therefore,

$$b = \sum_i c_i d_i t = \sum_i q^{(-\lambda,\mu)}(c_i\tau(\lambda))(d_i\tau(\mu))\tau(-\lambda-\mu)t$$

$$\in (I \cap G^-)(I \cap U^+)(I \cap U^o). \qquad \square$$

Let I be a left coideal subalgebra of U such that $I \cap T$ is a group and I satisfies (4.6). Then Theorem 4.4 combined with Lemmas 4.2 and 4.3 imply that I is generated by $I \cap T$ and quantum analogs of root vectors in G^- and U^+. This description of the generators can be thought of as an analog of Theorem 4.1 for these special coideal subalgebras. Below, we generalize these results to other coideal subalgebras by introducing filtrations and associated gradings of U.

Filtration I. Define the filtration $\mathcal{F}$ on U using the degree function:

$$\deg x_i t_i^{-1} = \deg y_i = 1 \qquad \deg t_i = -1$$

for all $1 \le i \le n$. Write $\mathrm{gr}_{\mathcal{F}}U$ for the associated graded algebra of this filtration. This filtration is invariant under the adjoint action and used to understand the locally finite part of U (see [JL2, Section 2.2]). (It should be noted that the quantized enveloping algebra is defined in a different though equivalent manner in [JL2]. So the x_i (resp. t_i) in this paper corresponds to $x_i t_i$ (resp. t_i^2) in [JL2]. Furthermore the degree of an element as defined in [JL2] is twice the degree of the corresponding element given here.)

Given $\gamma = \sum_{\alpha_i \in \pi} m_i \alpha_i$, set $\mathrm{ht}(\gamma) = \sum_i m_i$. Note that any nonzero element of $U_{-\lambda}^- G_\mu^+$ has degree $\mathrm{ht}(\lambda + \mu)$. Let $x \in U$ and set $\mathrm{supp}(x) = \{[\lambda,\mu] | \pi_{\lambda,\mu}(x) \neq 0\}$. Further, for x an element of $U^- G^+ t$ for some $t \in T$, set

$$\mathrm{max}_{\mathrm{ht}}(x) = \{[\lambda,\mu] | [\lambda,\mu] \in \mathrm{supp}(x) \text{ and } \mathrm{ht}(\lambda+\mu) = \deg(x) - \deg(t)\}.$$

The next lemma connects the filtration $\mathcal{F}$ with the height function.

LEMMA 4.5. *Let I be a left coideal of U and let b be an element of $I \cap U^- G^+ t$ for some $t \in T$. Then*

$$b = \sum_{\{[\lambda,\mu]\,|\,[\lambda,\mu]\in\max_{\mathrm{ht}}(b)\}} \pi_{\lambda,\mu}(b) + \text{ lower degree terms.}$$

PROOF. The lemma follows from the fact that $\deg \pi_{\lambda,\mu}(b) = \deg b$ if and only if $[\lambda, \mu] \in \max_{\mathrm{ht}}(b)$. □

By induction on $\mathrm{ht}(\lambda + \mu)$ and the definition of the comultiplication (1.8), (1.9), and (1.10), we have the following:

$$\Delta(U^-_{-\lambda} G^+_\mu) \subset \sum_{\gamma+\beta=\lambda,\alpha+\xi=\mu} U^-_{-\gamma} G^+_\alpha \otimes U^-_{-\beta} G^+_\xi \tau(-\gamma - \alpha). \qquad (4.7)$$

Consider a subset S of $Q^+(\pi) \times Q^+(\pi)$. Set $|S|$ equal to the number of elements in S. We call S *transversal* if whenever both $[\lambda, \mu]$ and $[\lambda', \mu']$ are in S and $[\lambda, \mu] \neq [\lambda', \mu']$ then $\lambda \neq \lambda'$ and $\mu \neq \mu'$. Now assume that $b \in U^- G^+ t$ and that $\max_{\mathrm{ht}}(b)$ is transversal. Given $[\lambda, \mu] \in \max_{\mathrm{ht}}(b)$, find $c_i \in U^-_{-\lambda}$ and $d_i \in G^+_\mu$ such that $\pi_{\lambda,\mu}(b) = \sum_i c_i d_i t$. As in the proof of Theorem 4.4, we may further assume that $\{c_i\}$ and $\{d_i\}$ are each linearly independent sets. It follows from (4.7), (4.3), (4.4), and (4.5) that

$$(\pi_{\lambda,\mu} \otimes \mathrm{Id})\Delta(b) = \sum_i c_i d_i t \otimes \tau(-\lambda - \mu)t, \qquad (4.8)$$

$$(\pi_{\lambda,0} \otimes \mathrm{Id})\Delta(b) = \sum_i c_i t \otimes (\tau(-\lambda)d_i t + \text{terms of lower degree}) \qquad (4.9)$$

$$(\pi_{0,\mu} \otimes \mathrm{Id})\Delta(b) = \sum d_i t \otimes (c_i \tau(-\mu)t + \text{terms of lower degree}) \qquad (4.10)$$

A consequence of the next lemma is that any left coideal subalgebra which is also an $\mathrm{ad}\, T$ module has a basis $\mathcal{B}$ such that $\max_{\mathrm{ht}}(b)$ is transversal for each $b \in \mathcal{B}$. This in turn is used to generalize Theorem 4.4.

LEMMA 4.6. *Let $b \in U$ be a weight vector. Then $\max_{\mathrm{ht}}(b)$ is transveral.*

PROOF. Fix η and let b be an element in U of weight η. Note that $\pi_{\lambda,\mu}(b) \neq 0$ implies that $-\lambda + \mu = \eta$. Now assume that both $[\lambda, \mu]$ and $[\lambda', \mu']$ are in $\mathrm{supp}(b)$. Hence $-\lambda + \mu = -\lambda' + \mu'$. Thus $\lambda = \lambda'$ if and only if $\mu = \mu'$. In particular, $\mathrm{supp}(b)$ is transversal. The lemma now follows from the fact that $\max_{\mathrm{ht}}(b)$ is a subset of $\mathrm{supp}(b)$. □

Given a left coideal subalgebra I of U, set I^-_η equal to the subset of G^- such that $I \cap G^- \tau(\eta) = I^-_\eta \tau(\eta)$. Similarly, set I^+_η equal to the subset of U^+ such that $I \cap U^+ \tau(\eta) = I^+_\eta \tau(\eta)$. The following result can be thought of as an analog of Theorem 4.4 for coideal subalgebras which admit an $\mathrm{ad}\, T$ module structure.

THEOREM 4.7. *Let I be a left coideal subalgebra and $\mathrm{ad}\, T$ submodule of U. Then*

$$\mathrm{gr}_{\mathcal{F}} I \subset \sum_{\{\eta\,|\,\tau(\eta)\in I\cap T\}} \mathrm{gr}_{\mathcal{F}} I^-_\eta I^+_\eta \tau(\eta).$$

PROOF. Let b be a weight vector of I which is also contained in $I \cap U^- G^+ \tau(\beta)$ for some $\tau(\beta) \in T$. By Lemma 4.6, $\max_{\mathrm{ht}}(b)$ is transversal. We prove the theorem when $\max_{\mathrm{ht}}(b)$ contains exactly one element $[\lambda, \mu]$. The same argument works in general. We argue as in the proof of Theorem 4.4. Find $c_i \in U_{-\lambda}^-$ and $d_i \in G_\mu^+$ so that $\pi_{\lambda,\mu}(b) = \sum_i c_i d_i \tau(\beta)$. We may further assume that $\{c_i\}$ and $\{d_i\}$ are each linearly independent sets. By our assumption on $\max_{\mathrm{ht}}(b)$ and Lemma 4.5,

$$b = \sum_i c_i d_i \tau(\beta) + \text{ lower degree terms.}$$

Set $\eta = -\lambda - \mu + \beta$. By (4.8), $\tau(\eta)$ is in I. Now (4.9) implies that there exist elements $\tau(-\lambda) D_i \tau(\beta) \in I$ such that $D_i = d_i +$ lower degree terms and

$$(\pi_{\lambda,0} \otimes \mathrm{Id}) \Delta(b) = \sum_i c_i \otimes \tau(-\lambda) D_i \tau(\beta).$$

Note that (4.7) ensures that $D_i - d_i$ is an element of $U^- G^+ \tau(-\lambda + \beta)$. Also, d_i is in $G_\mu^+ \tau(-\lambda + \beta)$. Thus d_i has degree $\mathrm{ht}(\mu + \lambda - \beta)$. By Lemma 4.5, $[\xi, \gamma] \in \mathrm{supp}(D_i - d_i)$ implies that $\mathrm{ht}(\xi + \gamma) < \mathrm{ht}(\mu)$. Since ξ is in $Q^+(\pi)$, we also have $\mathrm{ht}(-\xi + \gamma) < \mathrm{ht}(\mu)$ and thus $-\xi + \gamma$ is not equal to μ. Therefore, for each $[\xi, \gamma] \in \mathrm{supp}(D_i - d_i)$, $\pi_{\xi,\gamma}(D_i - d_i)$ has weight $-\xi + \gamma$ which is different from the weight μ of d_i. Since I is an $\mathrm{ad}\, T$ module, it follows that the μ weight term of $\tau(-\lambda) D_i \tau(\beta)$, namely $\tau(-\lambda) d_i \tau(\beta)$, is contained in I. Hence $d_i \tau(\mu) \tau(\eta) \in I \cap U^+ \tau(\eta)$ and $d_i \tau(\mu) \in I_\eta^+$. A similar argument shows that $c_i \tau(\lambda) \in I_\eta^-$. Therefore

$$\mathrm{gr}_{\mathcal{F}} b = \mathrm{gr}_{\mathcal{F}} \sum_i c_i d_i \tau(\eta) = \mathrm{gr}_{\mathcal{F}} \sum_i q^{-(\lambda,\mu)} (c_i \tau(\lambda))(d_i \tau(\mu)) \tau(\eta)$$

is an element of $\mathrm{gr}_{\mathcal{F}} I_\eta^- I_\eta^+ \tau(\eta)$.   □

Filtration II. Order the set $\mathbb{N} \times \mathbb{N}$ lexicographically from left to right. Define a filtration on U by

$$\mathcal{G}_{m,n}(U) = \{u \in U \mid (\mathrm{ht}(\lambda), \mathrm{ht}(\mu)) \leq (m, n) \text{ for all } [\lambda, \mu] \in \mathrm{supp}(u)\}.$$

The associated graded algebra for this filtration is defined by setting

$$\mathrm{gr}_{\mathcal{G}}^{m,n}(U) = \mathcal{G}_{m,n}(U) \Big/ \sum_{(m',n') < (m,n)} \mathcal{G}_{m,n}(U)$$

and

$$\mathrm{gr}_{\mathcal{G}}(U) = \bigoplus_{m,n} \mathrm{gr}_{\mathcal{G}}^{m,n}(U).$$

Given a subset S of $Q^+(\pi) \times Q^+(\pi)$, set $\|S\|_1 = \max_{[\lambda,\mu] \in S} \{\mathrm{ht}(\lambda)\}$. We can define a bidegree: for x in U, we say that $\mathrm{bideg}(x) = (m, n)$ if (m, n) is the smallest element of $\mathbb{N} \times \mathbb{N}$ such that $x \in \mathcal{G}_{m,n}(U)$. Set $\max(x) = \{[\lambda, \mu] \mid [\lambda, \mu] \in$

$\mathrm{supp}(x)$ and $\mathrm{bideg}(x) = (\mathrm{ht}(\lambda), \mathrm{ht}(\mu))\}$. Now consider an element $b \in U^- G^+ t$ for some $t \in T$. The inclusion (4.7) implies the following variation of (4.3):

$$(\pi_{\lambda,\mu} \otimes \mathrm{Id})(b) = \pi_{\lambda,\mu}(b) \otimes t\tau(-\lambda - \mu) \text{ for all } [\lambda, \mu] \in \max(b). \qquad (4.11)$$

LEMMA 4.8. *Let I be a left coideal subalgebra such that $I \cap T$ is a group. Then I has a basis $\mathcal{B}$ such that for each $b \in \mathcal{B}$, $\max(b)$ is transversal.*

PROOF. Recall that I is a direct sum of the subspaces $I \cap U^- G^+ t$. Let $C = \{x \in I | \max(x) \text{ is transversal}\}$. It is enough to show that for each $t \in T$, every element of $I \cap U^- G^+ t$ is contained in the span of C. Consider $b \in I \cap U^- G^+ t$. We prove this result under the additional assumption that $\max(b)$ consists of exactly two elements $[\lambda, \mu]$ and $[\lambda, \mu']$. A similar argument works in the general case using induction on $|\max(b)|$ and $\|\max(b)\|_1$. Note that

$$b = \pi_{\lambda,\mu}(b) + \pi_{\lambda,\mu'}(b) + \text{ terms of lower bidegree}.$$

By (4.11), $t\tau(-\lambda - \mu)$ and $t\tau(-\lambda - \mu')$ are both elements of the group $I \cap T$. Hence $\tau(\mu - \mu')$ is contained in $I \cap T$. Consider the element

$$b' = \tau(\mu - \mu')b\tau(\mu - \mu')^{-1} = q^{(-\lambda+\mu,\mu-\mu')}\pi_{\lambda,\mu}(b) + q^{(-\lambda+\mu',\mu-\mu')}\pi_{\lambda,\mu'}(b)$$
$$+ \text{ terms of lower bidegree}.$$

Now $(\mu - \mu', \mu - \mu')$ is positive since $(\,,\,)$ is positive definite on $Q(\pi)$. Hence $q^{(-\lambda+\mu,\mu-\mu')} \neq q^{(-\lambda+\mu',\mu-\mu')}$. Taking linear combinations of b and b', it follows that there exists b_1 and b_2 in $U^- G^+ t \cap I$ such that $\{[\lambda, \mu]\} = \max(b_1)$ and $\{[\lambda, \mu']\} = \max(b_2)$. In particular, both $\max(b_1)$ and $\max(b_2)$ are transversal and b is a linear combination of b_1 and b_2. $\qquad \square$

Now assume that $b \in U^- G^+ t$ and that $\max(b)$ is transversal. We have versions of (4.4) and (4.5) similar to (4.9) and (4.10) in the discussion of the first filtration. Given $[\lambda, \mu] \in \max(b)$, find $c_i \in U^-_{-\lambda}$ and $d_i \in G^+_\mu$ so that $\pi_{\lambda,\mu}(b) = \sum_i c_i d_i t$ and that $\{c_i\}$ and $\{d_i\}$ are each linearly independent sets. It follows from (4.7), (4.4), and (4.5) that

$$(\pi_{\lambda,0} \otimes \mathrm{Id})\Delta(b) = \sum_i c_i t \otimes (\tau(-\lambda)d_i t + \text{terms of lower bidegree}) \quad (4.12)$$

$$(\pi_{0,\mu} \otimes \mathrm{Id})\Delta(b) = \sum d_i t \otimes (c_i \tau(-\mu)t + \text{terms of lower bidegree}). \quad (4.13)$$

The filtration $\mathcal{G}$ restricts to filtrations on the subalgebras G^+, U^- and U^o. Indeed, $U^o = \mathcal{G}_{0,0}(U)$ and so $\mathrm{gr}_\mathcal{G} U^o \cong U^o$ as algebras. Upon restriction to G^+, the filtration $\mathcal{G}$ becomes filtration by the degree function associated to the first filtration $\mathcal{F}$. The subalgebra of G^+ satisfies exactly the same relations as U^+. In particular, the only relations satisfied by the elements of G^+ are the quantum Serre relations (1.7) (see the discussion in Section 7 concerning (7.18)) which are homogeneous with respect to degree. Hence we have an algebra isomorphism $\mathrm{gr}_\mathcal{G} G^+ \cong G^+$. A similar argument shows that $\mathrm{gr}_\mathcal{G} U^- \cong U^-$. Since the elements

in T have bidegree $(0,0)$, we further have that $\mathrm{gr}_{\mathcal{G}}G^- \cong G^-$ and $\mathrm{gr}_{\mathcal{G}}U^+ \cong U^+$. For the rest of the paper, we will often identify $\mathrm{gr}_{\mathcal{G}}G^-$ with G^-, $\mathrm{gr}_{\mathcal{G}}U^+$ with U^+, and $\mathrm{gr}_{\mathcal{G}}U^o$ with U^o.

Now the images of x_i and y_j commute with each other inside the associated graded algebra of U with respect to $\mathcal{G}$. (See relation (1.4) of U.) It follows that the image of U^-U^+ in the associated graded algebra is isomorphic to the tensor product $U^- \otimes U^+$ as an algebra. If we replace U^- by G^-, the images of the elements x_i and y_jt_j do not commute. However, they do commute up to a power of q. Thus the image of G^-U^+ in the associated graded algebra can be thought of as a q form of the tensor product which we write as $G^- \otimes_q U^+$.

The group algebra U^o acts on weight vectors by $\tau(\lambda) \cdot a_\mu = \tau(\lambda)a_\mu\tau(\lambda)^{-1} = q^{(\lambda,\mu)}a_\mu$ for $a_\mu \in U_\mu$. Thus we obtain the following algebra isomorphism using a smash product construction:

$$\mathrm{gr}_{\mathcal{G}}(U) \cong U^o\#(G^- \otimes_q U^+). \qquad (4.14)$$

(Compare this with a similar result for a different filtration in [Jo, 7.4.7].)

THEOREM 4.9. *Let I be a left coideal subalgebra such that $I \cap T$ is a group. Then*

$$\mathrm{gr}_{\mathcal{G}}(I) \cong (I \cap U^o)\#((\mathrm{gr}_{\mathcal{G}}(I) \cap G^-) \otimes_q ((\mathrm{gr}_{\mathcal{G}}(I) \cap U^+)).$$

PROOF. The proof is a graded version of the proof of Theorem 4.4. Using Lemma 1.3 and Lemma 4.8, we can find a basis $\mathcal{B}$ of I such that $\mathcal{B} = \bigcup_t(\mathcal{B} \cap U^-G^+t)$ and $\max(b)$ is transversal for each $b \in \mathcal{B}$. By (4.14),

$$(I \cap U^o)\#((\mathrm{gr}_{\mathcal{G}}(I) \cap G)^- \otimes_q ((\mathrm{gr}_{\mathcal{G}}(I) \cap U^+))$$

is isomorphic to a subalgebra of $\mathrm{gr}_{\mathcal{G}}I$. To show this subalgebra is all of $\mathrm{gr}_{\mathcal{G}}I$ it is sufficient to show that each element of $\mathcal{B}$ is contained in $((\mathrm{gr}_{\mathcal{G}}(I) \cap G^-)((\mathrm{gr}_{\mathcal{G}}(I) \cap U^+))\mathrm{gr}_{\mathcal{G}}(I \cap U^o)$.

Let t be an element of T and let b be an element of $\mathcal{B} \cap U^-G^+t$. Choose $[\lambda, \mu] \in \max(b)$. There exists $c_i \in U_{-\lambda}^-$ and $d_i \in G_\mu^+$ so that $\pi_{\lambda,\mu}(b) = \sum_i c_id_it$ and the $\{c_i\}$ and $\{d_i\}$ are each linearly independent sets. Using (4.11), (4.12), and (4.13) and arguing as in the proofs of Theorem 4.4 and Theorem 4.7, I contains $\tau(-\lambda - \mu)t$ and $\tau(\lambda + \mu)t^{-1}$ and elements $\tilde{d}_i$ and $\tilde{c}_i$ such that

$$\tilde{d}_i = d_i\tau(\mu) + \text{ terms of lower bidegree}$$

and

$$\tilde{c}_i = c_i\tau(\lambda) + \text{ terms of lower bidegree}.$$

Note that $\mathrm{gr}_{\mathcal{G}}\tilde{d}_i \in \mathrm{gr}_{\mathcal{G}}(I) \cap U^+$ and $\mathrm{gr}_{\mathcal{G}}\tilde{c}_i \in \mathrm{gr}_{\mathcal{G}}(I) \cap G^-$. Set

$$b' = b - \sum_i q^{-(\lambda,\mu)}\tilde{c}_i\tilde{d}_i\tau(-\mu - \lambda)t.$$

Note that b' is in I. By construction, $\pi_{\lambda,\mu}(b') = 0$. Thus either $\max(b') = \max(b) - \{[\lambda,\mu]\}$ or the bidegree of b' is strictly smaller than the bidegree of b. The theorem now follows by induction on $|\max(b)|$ and the bidegree of b. $\square$

Consider a left coideal subalgebra I such that $I \cap T$ is a group. Given x in U, set $\operatorname{tip}(x) = \sum_{[\lambda,\mu] \in \max(x)} \pi_{\lambda,\mu}(x)$. The element $\operatorname{tip}(x)$ can be thought of as the highest bidegree term of x. Note that $\operatorname{gr}_{\mathcal{G}}(I) \cap \operatorname{gr}_{\mathcal{G}}(G^-)$ identifies with $\operatorname{tip}(I) \cap G^-$ under the isomorphism $G^- \cong \operatorname{gr}_{\mathcal{G}}(G^-)$. Thus $\operatorname{tip}(I) \cap G^-$ is a subalgebra of G^-. Consider the elements $\tilde{c}_i$ in the proof of Theorem 4.9. Note that each $\operatorname{tip}(\tilde{c}_i)$ is a weight vector. In particular, it follows implicitly from the proof of Theorem 4.9 that $\operatorname{tip}(I) \cap G^-$ is spanned by weight vectors and hence is an $\operatorname{ad} T$ module. One can further show using (4.7) that $\operatorname{tip}(I) \cap G^-$ is a left coideal of G^-. Thus $\operatorname{tip}(I) \cap G^-$ is a left coideal subalgebra and $\operatorname{ad} T$ submodule of G^-. Similarly, $\operatorname{tip}(I) \cap U^+$ is a left coideal subalgebra and $\operatorname{ad} T$ submodule of U^+. Thus combining Theorem 4.9 with Lemmas 4.2 and 4.3 yields the following.

COROLLARY 4.10. *Let I be a left coideal subalgebra of U such that $I \cap T$ is a subgroup of T. Then there exists subsets Δ' and Δ'' of Δ^+ such that I is generated by elements $c_{-\gamma}, \gamma \in \Delta'$; $d_\beta, \beta \in \Delta''$; and $I \cap T$. Moreover each $\operatorname{tip}(c_{-\gamma})$ (resp. $\operatorname{tip}(d_\beta)$) is a weight vector of weight $-\gamma$ (resp. β) which specializes to the root vector $f_{-\gamma}$ (resp. e_β) of $U(\mathbf{g})$.*

5. The Locally Finite Part of U

One of the most important coideal subalgebras contained in the quantized enveloping algebra is the locally finite part, $F(U)$, defined by (2.1). This subalgebra is studied extensively in [JL1] and [JL2] (see also [Jo]). Here we present some of the known results about this algebra by directly showing that $F(U)$ is a coideal subalgebra of U. We will see some of the implications of Section 4 on the structure of $F(U)$.

Recall that $F(U)$ is defined using the quantum adjoint action in Section 2. It is helpful to see how the generators of U act via the adjoint action. In particular

$$(\operatorname{ad} y_i)b = y_i b t_i - b y_i t_i \quad (\operatorname{ad} x_i)b = x_i b - t_i b t_i^{-1} x_i \quad (\operatorname{ad} t_i)b = t_i b t_i^{-1}$$

for all $1 \le i \le n$ and $b \in U$.

THEOREM 5.1. *$F(U)$ is a left coideal subalgebra of U.*

PROOF. Let $b \in F(U)$. A straightforward computation shows

$$\Delta((\operatorname{ad} x_i)b) = \sum x_i b_{(1)} \otimes b_{(2)} - \sum t_i b_{(1)} t_i^{-1} x_i \otimes t_i b_{(2)} t_i^{-1}$$
$$+ \sum t_i b_{(1)} \otimes (\operatorname{ad} x_i) b_{(2)} \quad (5.1)$$

for each $1 \le i \le n$.

We may write $\Delta(b)$ as a sum $\sum_{j=1}^{s} c_j \otimes b_j$ where the b_j, $1 \le j \le s$, are linearly independent weight vectors in U of weight λ_j respectively. Extend the standard

partial ordering on the integral lattice $Q(\pi)$ to a total archimedean ordering. (This can be done by embedding $Q^+(\pi)$ in the nonnegative real numbers.) We may further suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$. For sake of simplicity, assume that these inequalities are all strict. (A similar argument works in general.)

Let U_i be the subalgebra of U generated by $x_i, y_i, t_i^{\pm 1}$. Note that U_i is isomorphic to $U_q(\mathbf{sl}\, 2)$. We show below that each b_j generates a finite-dimensional $\mathrm{ad}\, U_i$ module for $1 \leq i \leq n$. By [JL1, Theorem 5.9], this forces each b_j to be an element of $F(U)$ (see also the proof of [JL1, Proposition 6.5]).

Suppose that $(\mathrm{ad}\, x_i)^m b = 0$. Using (5.1) and induction, we obtain

$$\Delta((\mathrm{ad}\, x_i)^m b) \in t_i^m c_1 \otimes (\mathrm{ad}\, x_i)^m b_1 + \sum_{\beta < \lambda_1 + m\alpha_i} U \otimes U_\beta.$$

Hence $(\mathrm{ad}\, x_i)^m b_1 = 0$. Choose r such that $(m-1)\alpha_i + \lambda_1 < (m+r)\alpha_i + \lambda_2$. We further have that

$$\Delta((\mathrm{ad}\, x_i)^{m+r} b) \in t_i^{m+r} c_2 \otimes (\mathrm{ad}\, x_i)^{m+r} b_2 + \sum_{\beta < \lambda_2 + (m+r)\alpha_i} U \otimes U_\beta.$$

Thus $(\mathrm{ad}\, x_i)^m b = 0$ also implies that $(\mathrm{ad}\, x_i)^{m+r} b_2 = 0$. By induction, it follows that there exists $M > 0$ such that $(\mathrm{ad}\, x_i)^M b_j = 0$ for all $1 \leq j \leq s$ and $1 \leq i \leq n$. One obtains a similar property for the action of each $\mathrm{ad}\, y_i$ on b. In particular, for each $1 \leq i \leq n$ and each $1 \leq j \leq s$, both $\mathrm{ad}\, y_i$ and $\mathrm{ad}\, x_i$ act nilpotently on b_j. Since b_j is a weight vector, it further follows that $\mathrm{ad}\, t_i$ acts semisimply on b_j. Thus b_j generates a finite-dimensional $\mathrm{ad}\, U_i$ module for all $1 \leq i \leq n$ and all $1 \leq j \leq s$. Therefore each $b_j \in F(U)$. $\qquad\square$

Set $T_F = T \cap F(U)$. It follows from Lemma 4.2 that the algebra generated by T_F is equal to the intersection of U^o with $F(U)$. By [JL1, 6.2], $\tau(\lambda) \in F(U)$ if and only if $(\mathrm{ad}\, x_i)$ and $(\mathrm{ad}\, y_i)$ act nilpotently on $\tau(\lambda)$ for $1 \leq i \leq n$. Furthermore, ([JL1, the proof of Lemma 6.1]) s is the least positive integer such that $(\mathrm{ad}\, x_i)^s \tau(\lambda) = 0$ and $(\mathrm{ad}\, y_i)^s \tau(\lambda) = 0$ if and only if

$$\frac{(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} = -s + 1.$$

Thus, $\tau(\lambda) \in F(U)$ if and only if $(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ is a nonpositive integer for all $1 \leq i \leq n$. In particular, $-\lambda/2$ is a dominant integral weight. So $\tau(\lambda)$ is in $F(U)$ if and only if λ is in $R(\pi) := Q(\pi) \cap -2P^+(\pi)$. (This is [JL1, Lemma 6.1. Note that the notation in [JL1] is different than in this paper. In particular, t_i here corresponds to t_i^2 in [JL1]. Thus divisibility by 4 in [JL1, Lemma 6.1] corresponds to divisibility by 2 in this paper.) For example, when $\mathbf{g}$ is $\mathbf{sl}\, 2$, then T_F is just the set

$$\{t^{-m} | m \in \mathbf{N}\}.$$

Note that this set is a semigroup but is not a group. This is true in general for T_F.

Recall $\mathcal{F}$, the first filtration discussed in Section 4. For each $\xi \in R(\pi)$, set K_ξ^- equal to the subspace of G^- such that $F(U) \cap G^- \tau(\xi) = K_\xi^- \tau(\xi)$. Similarly, set K_ξ^+ equal to the subspace of U^+ such that $F(U) \cap U^+ \tau(\xi) = K_\xi^+ \tau(\xi)$. It is shown in [JL2, Section 4.9, 4.10] that

$$\mathrm{gr}_\mathcal{F}(F(U)) = \oplus_{\xi \in R(\pi)} \mathrm{gr}_\mathcal{F}(K_\xi^- K_\xi^+ \tau(\xi)). \tag{5.2}$$

Note that the inclusion of the left hand side of (5.2) inside the right hand side is just Theorem 4.7 applied to $F(U)$. In particular, Theorem 4.7 gives a new proof of this inclusion. Moreover, Theorem 4.7 can be thought of as a generalization of this part of (5.2) to other left coideal subalgebras which admit an $\mathrm{ad}\, T$ module structure.

The analysis in [JL2, Section 4, see Section 4.10], shows that

$$\mathrm{gr}_\mathcal{F}(K_\xi^- K_\xi^+ \tau(\xi)) = (\mathrm{ad}\, U) \mathrm{gr}_\mathcal{F} \tau(\xi).$$

Moreover,

$$(\mathrm{ad}\, U) \mathrm{gr}_\mathcal{F} \tau(\xi) \cong (\mathrm{ad}\, U) \tau(\xi)$$

as $\mathrm{ad}\, U$ modules for each $\tau(\xi) \in T_F$. Thus one has the direct sum decomposition in the nongraded case [JL2, Corollary 4.11]):

$$F(U) = \oplus_{t \in T_F} (\mathrm{ad}\, U) t. \tag{5.3}$$

Now (5.3) implies that $F(U)$ is the $\mathrm{ad}\, U$ module generated by the algebra $F(U) \cap U^o$. The fact that $F(U)$ is a left coideal was originally proved using this fact and a weakened version of Lemma 1.2 ([JL3, Lemma 5.3]).

Note that $(\mathrm{ad}\, U)t$ is an *ad-invariant left coideal* of U for each $t \in T$. On the other hand, (4.11) guarantees that any left coideal of U contains an element of T. Thus by (5.3) the minimal ad-invariant left coideals of U contained in $F(U)$ are exactly the vector subspaces $(\mathrm{ad}\, U)t$ where $t \in T_F$. This argument and result is due to [HS, Theorem 3.9] where it is actually proved in the dual setting. The description of the minimal ad-invariant left coideals is, in turn, a crucial step in the classification of bicovariant differential calculi on the quantized function algebra $R_q[G]$.

The algebra $F(U)$ can be localized by the normal elements T_F to obtain the larger coideal subalgebra $\mathbf{F} = F(U)T_F^{-1}$. Now $\mathbf{F} \cap T$ is just the subgroup generated by T_F. It is straightforward to show that $\mathbf{F}$ is generated by $x_i, y_i t_i$, and $\mathbf{F} \cap T$ for $1 \leq i \leq n$. In particular, given i, there exists some $t \in T_F$ such that $(\mathrm{ad}\, x_i)t$ is a nonzero multiple of $x_i t$. Hence $x_i t \in F(U)$ and $x_i \in \mathbf{F}$. A similar argument shows that $y_i t_i \in \mathbf{F}$ for all $1 \leq i \leq n$. Thus $\mathbf{F}$ contains $\mathbf{F} \cap T$, x_i, and $y_i t_i$, for $1 \leq i \leq n$. In the notation of Corollary 4.10, we get that $\Delta' = \Delta'' = \Delta^+$, and, moreover, $\mathbf{F} \cap T$, x_i, and $y_i t_i$, $1 \leq i \leq n$, generate $\mathbf{F}$. It further follows that G^- and U^+ are subalgebras of $\mathbf{F}$ and that $G^- U^+ t$ is a subset of $\mathbf{F}$ for each $t \in \mathbf{F} \cap T$. Recall the notation of Theorem 4.7. Note that $\mathbf{F}_\eta^- = G^- \subset \mathbf{F}$ and

$\mathbf{F}_\eta^+ = U^+ \subset \mathbf{F}$ for all $\eta \in Q(\pi)$. Hence Theorem 4.7 implies the following direct sum decomposition of $\mathbf{F}$:

$$\mathbf{F} = \oplus_{t \in \mathbf{F} \cap T} G^- U^+ t.$$

Since $\mathbf{F} \cap T$ is a subgroup of finite index in T, we see that $\mathbf{F}$, and hence $F(U)$, is "large" in U (For a stronger version of this, see [JL1, Theorem 6.4]).

A particular type of quantum Harish-Chandra module, defined differently (and earlier) than those of Section 2, was introduced in [JL3] in order to classify the primitive ideals of U. These modules were originally specified as a subcategory of the $F(U)$ bimodules with a "compatible" $\operatorname{ad} U$ action (see [JL3, 5.4] or [Jo, 8.2.3 and 8.4.1]). In [JL3, 5.4], an $F(U)$ bimodule M has a compatible $\operatorname{ad} U$ module structure provided that

$$\sum ((\operatorname{ad} a)(b \cdot m \cdot c) = \sum (\operatorname{ad} a_{(1)})b \cdot (\operatorname{ad} a_{(2)})m \cdot (\operatorname{ad} a_{(3)})c \qquad (5.5)$$

and

$$(\operatorname{ad} t)m \cdot t = t \cdot m \qquad (5.6)$$

for all $a \in U$, b and c in $F(U)$, $m \in M$, and $t \in F(U) \cap T$. A different definition of compatible is given in [Jo, 8.2.3]. In particular, the $\operatorname{ad} U$ action must satisfy the following condition in [Jo, 8.2.3]:

$$\sum ((\operatorname{ad} a_{(1)})m) \cdot a_{(2)} = a \cdot m \qquad (5.7)$$

for all $a \in F(U)$ and $m \in M$. Note that (5.6) follows from (5.7) by setting $a = t$.

The purpose of introducing the compatibility conditions (5.5) and (5.6) was to study the specific Harish-Chandra module category $\mathcal{H}_\chi$ associated to a dominant regular weight Λ defined in [JL3, Section 5.7]. By [JL3, 5.12] and [Jo, 8.4.11], this category is the same as the one described in [Jo, 8.4.1] using condition (5.7). Hence this category consists of modules with an $F(U)$ bimodule structure and $(\operatorname{ad} U)$ module action which satisfy both (5.5) and (5.7). In this paper, we say that $F(U)$ has a compatible $\operatorname{ad} U$ module action if both (5.5) and (5.7) hold. We show here that $F(U)$ bimodules with a compatible $\operatorname{ad} U$ module action fit exactly into the framework of Section 2.

Let U^{op} denote the Hopf algebra with underlying vector space U, the opposite multiplication, the same comultiplication and counit as U, and with antipode σ^{-1} ([Jo, 1.1.12]). Note that the algebra $U \otimes U^{\mathrm{op}}$ can be made into a Hopf algebra with comultiplication $\Delta(a \otimes b) = (\operatorname{Id} \otimes \mathbf{tw} \otimes \operatorname{Id})(\Delta \otimes \Delta)(a \otimes b)$ where $\mathbf{tw}$ denotes the twist map sending $a \otimes b$ to $b \otimes a$. The other Hopf operations can be defined similarly. Observe that $U \otimes U^{\mathrm{op}}$ is isomorphic to $U_q(\mathbf{g} \oplus \mathbf{g}^*)$ as a Hopf algebra. There is an algebra embedding ψ of U into $U \otimes U^{\mathrm{op}}$ which sends an element u to $\sum u_{(1)} \otimes \sigma(u_{(2)})$. The image of U in $U \otimes U^{\mathrm{op}}$ under ψ is not a Hopf subalgebra of $U \otimes U^{\mathrm{op}}$. However, by the next lemma it is a coideal subalgebra.

LEMMA 5.2. *The algebra $\psi(U)$ is a left coideal of $U \otimes U^{\mathrm{op}}$.*

Proof. By (1.3),

$$\Delta(\sum u_{(1)} \otimes \sigma(u_{(2)})) = \sum (u_{(1)} \otimes \sigma(u_{(4)})) \otimes (u_{(2)} \otimes \sigma(u_{(3)})).$$

Thus $\psi(U)$ is a left coideal since $\Delta(u_{(2)}) = \sum u_{(2)} \otimes u_{(3)}$. $\square$

Let $F(U \otimes U^{\mathrm{op}})$ denote the locally finite part of $U \otimes U^{\mathrm{op}}$. We show that $F(U)$ modules with compatible $\mathrm{ad}\, U$ module action are $F(U \otimes U^{\mathrm{op}})\psi(U)$ modules. The next lemma relates $F(U \otimes U^{\mathrm{op}})$ to the locally finite part $F(U)$ of U.

Lemma 5.3. $F(U \otimes U^{\mathrm{op}}) = F(U) \otimes F(U)^{\mathrm{op}}$

Proof.   Let $\mathrm{ad}^{\mathrm{op}}$ denote the (left) adjoint action of U^{op}. With sufficient care to indentification of elements in U and U^{op}, one checks using (1.3) that $(\mathrm{ad}^{\mathrm{op}}\sigma(a))b = (\mathrm{ad}\, a)b$. Thus $F(U^{\mathrm{op}}) = F(U)^{\mathrm{op}}$ as algebras. $\square$

Recall that since $\psi(U)$ is a left coideal and $F(U \otimes U^{\mathrm{op}})$ is an $\mathrm{ad}\, U \otimes U^{\mathrm{op}}$ module, we have that $F(U \otimes U^{\mathrm{op}})\psi(U) = \psi(U)F(U \otimes U^{\mathrm{op}})$. The next lemma shows that $F(U)$ is a free as a left $\psi(U)$ module.

Lemma 5.4. *The multiplication map induces an isomorphism ϕ of vector spaces*

$$\psi(U) \otimes (1 \otimes F(U)^{\mathrm{op}}) \to \psi(U)F(U \otimes U^{\mathrm{op}}).$$

Proof. Let $a \in F(U)$. Note that

$$
\begin{aligned}
a \otimes 1 &= \sum a_{(1)}\varepsilon(a_{(2)}) \otimes 1 = \sum a_{(1)} \otimes \varepsilon(a_{(2)}) \\
&= \sum a_{(1)} \otimes a_{(3)}\sigma(a_{(2)}) = \sum \psi(a_{(1)})(1 \otimes a_{(2)}) \quad\quad (5.8)
\end{aligned}
$$

for all $a \in U$. It follows that

$$\psi(U)F(U \otimes U^{\mathrm{op}}) = \psi(U)(1 \otimes F(U)^{\mathrm{op}}).$$

This proves that ϕ is surjective.

Suppose that $\sum_i \psi(c_i)(1 \otimes b_i) = 0$ where the set $\{b_i\}$ is a linearly independent subset of $F(U)^{\mathrm{op}}$. We argue that each $\psi(c_i) = 0$. This in turn implies that ϕ is injective.

There is a version of Lemma 1.3 for right coideal subalgebras. In particular, $U = \oplus_t G^- U^+ t$ and each $G^- U^+ t$ is a right coideal of U. We may write $c_i = \sum_t c_{it}$ where each $c_{it} \in G^- U^+ t$. It follows that $\psi(c_{it})(1 \otimes b_i) \in G^- U^+ t \otimes U$ for each i and $t \in T$. Thus $\sum_i \psi(c_{it})(1 \otimes b_i) = 0$. This allows us to reduce to the case where there exists $t \in T$ such that c_i is in $G^- U^+ t$ for all i.

Recall the notation of Section 4, Filtration II. Let (M, N) be the maximum value of the set of bidegrees of the c_i. Reordering if necessary, we may assume that c_1 has bidegree (M, N). Choose $[\lambda, \mu] \in \max(c_1)$. By say (4.11), we have

$$(\pi_{\lambda,\mu} \otimes \mathrm{Id}) \sum_i \psi(c_i)(1 \otimes b_i) = \sum_i \pi_{\lambda,\mu}(c_i) \otimes \sigma(t)b_i.$$

Note that $\sigma(t) = t^{-1}$. Since the set $\{b_i\}$ is linearly independent, the set $\{\sigma(t)b_i\}$ is also linearly independent. Hence $\pi_{\lambda,\mu}(c_i) = 0$ for each i. The choice of $[\lambda, \mu]$ now forces $c_i = 0$ for each i. $\square$

The next result shows that $F(U)$ modules with compatible $\operatorname{ad} U$ modules are just $F(U \otimes U^{\mathrm{op}})\psi(U)$ modules.

THEOREM 5.5. *The set of $F(U)$ bimodules with compatible $\operatorname{ad} U$ module action can be identified with the set of $F(U \otimes U^{\mathrm{op}})\psi(U)$ modules.*

PROOF. Let M be a $F(U \otimes U^{\mathrm{op}})\psi(U)$ module. Note that M is a $F(U)$ bimodule in a natural way. In particular, set $a \cdot m \cdot b = (a \otimes b)m$ for all $a, b \in F(U)$ and $m \in M$. We define an action of $\operatorname{ad} U$ on M by setting $(\operatorname{ad} c)m = \psi(c)m$ for all $c \in U$. By (5.8), it follows that this $\operatorname{ad} U$ action satisfies (5.7). A straightforward computation shows that this action satisfies (5.5) as well.

Now let M be an $F(U)$ bimodule with compatible $\operatorname{ad} U$ module action. Make M into a $F(U \otimes U^{\mathrm{op}})\psi(U)$ module by setting

$$(a \otimes b)m = a \cdot m \cdot b \quad \text{and} \quad (\psi(c))m = (\operatorname{ad} c)m \tag{5.9}$$

for all $a \otimes b \in F(U \otimes U^{\mathrm{op}})$, $c \in U$, and $m \in M$.

One checks that $\psi(c)(1 \otimes b) = \sum(1 \otimes (\operatorname{ad} c_{(2)})b)\psi(c_{(1)})$. By (5.5), $(\operatorname{ad} c)(m \cdot b) = (\operatorname{ad} c_{(1)} \cdot m) \cdot ((\operatorname{ad} c_{(2)})b)$. Hence the action of $\psi(c)$ on $(1 \otimes b)m$ described in (5.9) agrees with the action of $(1 \otimes (\operatorname{ad} c_{(2)})b)$ on $\psi(c_{(1)})m$. Therefore, to show that the action in (5.9) is well defined it is sufficient to show that the action of an element $x \in F(U \otimes U^{\mathrm{op}})\psi(U)$ on M is independent of the way x is written as a sum of terms of the form bu where $b \in F(U \otimes U^{\mathrm{op}})$ and $u \in \psi(U)$.

The compatibility condition (5.7) ensures that

$$(\operatorname{ad} c)(a \cdot m \cdot b) = \sum(\operatorname{ad} ca_{(1)})(m \cdot ba_{(2)})$$

for all $a \otimes b \in F(U \otimes U^{\mathrm{op}})$, $c \in U$, and $m \in M$. Thus using (5.9) formally, we see that

$$\psi(c)((a \otimes b)m) = \psi(c)((a \cdot m \cdot b)) = \sum \psi(ca_{(1)})(m \cdot ba_{(2)})$$
$$= \sum \psi(ca_{(1)})((1 \otimes ba_{(2)})m).$$

In particular the action of $\psi(c)(a \otimes b)$ agrees with the action of $\sum \psi(ca_{(1)})(1 \otimes ba_{(2)})$ on M. By Lemma 5.4, every element in $F(U \otimes U^{\mathrm{op}})\psi(U)$ can be expressed uniquely in the form $\sum_i \psi(a_i)(1 \otimes b_i)$ where $\{b_i\}$ is a basis of $F(U)^{\mathrm{op}}$. The theorem now follows. $\square$

One can apply the results of Section 2 to the study of Harish-Chandra modules for the pair $U \otimes U^{\mathrm{op}}$, $\psi(U)$. Identify the algebra $U \otimes U^{\mathrm{op}}$ with $U_q(\mathbf{g} \oplus \mathbf{g}^*)$. Let $\tilde{\kappa}$ denote the conjugate linear Chevalley antiautomorphism of Section 2 associated here to the quantized enveloping algebra $U_q(\mathbf{g} \oplus \mathbf{g}^*)$. One can find a Hopf algebra automorphism $\Upsilon \in \mathcal{H}_{\mathbb{R}}$ such that $\Upsilon(\psi(U))$ is invariant under $\tilde{\kappa}$. Thus the results

in Section 2 apply here. However, the main results of Section 2, such as Theorem 2.7, can be proved easily in this case since $\psi(U)$ is isomorphic as an algebra to $U_q(\mathbf{g})$. Thus it acts completely reducibly on all finite-dimensional $\psi(U)$ modules. Furthermore, one checks that all finite-dimensional $\psi(U)$ modules are unitary using the fact that this is true for $U_q(\mathbf{g})$.

For an example of a Harish-Chandra module associated to the pair $U \otimes U^{\mathrm{op}}$, $\psi(U)$, consider two left U modules M and N. Define the U bimodule $\mathrm{Hom}(M, N)$ by $(a \cdot f \cdot b)(m) = af(bm)$. As explained in [JL3, 5.4] and [Jo, 8.2.3], $\mathrm{Hom}(M, N)$ has a compatible $(\mathrm{ad}\, U)$ module structure in the sense of (5.5) and (5.7) given by $(\mathrm{ad}\, a)f = \sum a_{(1)} \cdot f \cdot \sigma(a_{(2)})$. Thus from the above Theorem 5.5, we see that $\mathrm{Hom}(M, N)$ is a $(F(U) \otimes F(U)^{\mathrm{op}})\psi(U)$ module. By Theorem 2.7, the sum of all finite-dimensional $\mathrm{ad}\, U$ modules $F(M, N)$ inside of $\mathrm{Hom}(M, N)$ is a Harish-Chandra module for the pair $U \otimes U^{\mathrm{op}}$, $\psi(U)$.

In [JL3, Theorem 5.13] (see also [Jo, Chapter 8]), the theory of Harish-Chandra modules associated to the pair $U \otimes U^{\mathrm{op}}$, $\psi(U)$ is used to prove an equivalence of categories between certain Harish-Chandra modules and various category $\mathcal{O}$ modules. This is critical in obtaining the quantum version of Duflo's theorem: every primitive ideal of U is the annihilator of a highest weight simple module ([JL3, Corollary 6.4] or [Jo, 8.4.17]).

6. Nilpotent and Parabolic Coideal Subalgebras

Yet another left coideal subalgebra of U is G^-, an obvious quantum analog of $U(\mathbf{n}^-)$. In this section, we consider coideal subalgebras of G^- which correspond to classical enveloping algebras of Lie subalgebras of $\mathbf{n}^-$ and related Lie subalgebras of $\mathbf{g}$. Most of the results presented here are from [Ke].

Let π' be a subset of the simple roots π of $\mathbf{g}$. There are a number of Lie subalgebras of $\mathbf{g}$ which can be associated to π'. The most obvious is the semisimple Lie subalgebra $\mathbf{m}$ of $\mathbf{g}$ generated by the e_i, f_i, h_i, for those i with $\alpha_i \in \pi'$. Since the simple roots π' associated to the root system of $\mathbf{m}$ are contained in the simple roots π of $\mathbf{g}$, the entire picture can be lifted to the quantum setting. In particular, $U_q(\mathbf{g})$ contains a Hopf subalgebra $\mathcal{M}$ isomorphic to $U_q(\mathbf{m})$ and generated by the x_i, y_i, t_i, t_i^{-1} for the same i. Set $\mathcal{M}^- = \mathcal{M} \cap G^-$ and $\mathcal{M}^+ = \mathcal{M} \cap U^+$.

Let Δ' denote the set of positive roots associated to the simple roots π'. The vector space $\mathbf{n}_{\pi'}^-$ spanned by the root vectors $f_{-\gamma}$, γ in $\Delta^+ - \Delta'$, is a second Lie subalgebra of $\mathbf{n}^-$. Let $\mathbf{m}^-$ denote the Lie subalgebra of $\mathbf{m}$ generated by the f_i for $\alpha_i \in \pi'$. Then

$$\mathbf{n}^- = \mathbf{n}_{\pi'}^- \oplus \mathbf{m}^-.$$

Thus the multiplication map defines a vector space isomorphism:

$$U(\mathbf{n}^-) \cong U(\mathbf{n}_{\pi'}^-) \otimes U(\mathbf{m}^-). \tag{6.1}$$

We shall see that the algebra $U(\mathbf{n}_{\pi'}^-)$ can be lifted to the quantum setting using a coideal subalgebra.

Let $G_{\pi-\pi'}^-$ be the subalgebra of G^- generated by the y_it_i such that α_i is in $\pi - \pi'$. Note that $G_{\pi-\pi'}^-$ is a left coideal subalgebra of G^-. Now $G_{\pi-\pi'}^-$ is generated by weight vectors and in particular, $(\operatorname{ad}T)G_{\pi-\pi'}^- = G_{\pi-\pi'}^-$. Also, $(\operatorname{ad}x_i)y_jt_j = 0$ for all $i \neq j$. Thus $(\operatorname{ad}(\mathcal{M}^+T))G_{\pi-\pi'}^- \subset G_{\pi-\pi'}^-$. Recall that $\mathcal{M}$ is equal to the quantized enveloping $U_q(\mathbf{m})$. Hence the triangular decomposition (1.12) implies that $\mathcal{M}T = \mathcal{M}^-\mathcal{M}^+T$. Hence $(\operatorname{ad}\mathcal{M}^-)G_{\pi-\pi'}^-$ equals $(\operatorname{ad}\mathcal{M})G_{\pi-\pi'}^-$. By Lemma 1.2, $(\operatorname{ad}\mathcal{M}^-)G_{\pi-\pi'}^-$ is a left coideal. Let $N_{\pi'}^-$ be the subalgebra of G^- generated by $(\operatorname{ad}\mathcal{M}^-)G_{\pi-\pi'}^-$. It is a left coideal subalgebra since $(\operatorname{ad}\mathcal{M}^-)G_{\pi-\pi'}^-$ is a left coideal.

By [Ke], one has a quantum analog of (6.1). Namely there is an isomorphism of vector spaces

$$G^- \cong N_{\pi'}^- \otimes \mathcal{M}^-. \tag{6.2}$$

Kébé actually proves a stronger result with this as a consequence, namely, G^- is isomorphic to the smash product of $N_{\pi'}^-$ and $\mathcal{M}^-$.

By construction, $N_{\pi'}^-$ is generated by weight vectors and hence is a direct sum of its weight spaces. By Lemma 4.3, we can find a subset Δ_1 of Δ^+ such that $N_{\pi'}^-$ is generated by weight vectors $\tilde{f}_{-\gamma}$ of weight $\gamma \in \Delta_1$ which specialize to root vectors in $U(\mathbf{n}^+)$. By [L3, proof of Proposition 2.2], Δ_1 consists of those positive roots which are not linear combinations of roots in π'. In particular, $N_{\pi'}^-$ specializes to $U(\mathbf{n}_{\pi'}^-)$ as q goes to 1 ([L3, proof of Proposition 2.2]). Thus the left coideal subalgebra $N_{\pi'}^-$ is a natural choice of quantum analog of $U(\mathbf{n}_{\pi'}^-)$ inside of $U(\mathbf{g})$.

It is instructive to look at the generators of $N_{\pi'}^-$. Let I be a tuple $(i_1, \ldots, i_r)$ of (arbitrary) length r and suppose that α_{i_s} is in $\pi - \pi'$ for $1 \leq s \leq r$. By the argument in [L3, Proposition 2.2], the algebra $N_{\pi'}^-$ is generated by elements of the form

$$Y_{I,j} = (\operatorname{ad}y_{i_1} \cdots y_{i_r})y_jt_j$$

where $\alpha_j \notin \pi'$.

Now each $Y_{I,j}$ is an element of the subcoideal $(\operatorname{ad}\mathcal{M}^-)y_jt_j$ of $N_{\pi'}^-$ as well as an element of G^-. Hence

$$(\operatorname{Id}\otimes\pi_{0,0})\Delta(Y_{I,j}) = Y_{I,j} \otimes 1.$$

Thus

$$\Delta(Y_{I,j}) = Y_{I,j} \otimes 1 + \sum Y_i \otimes Y_i'$$

where Y_i is in U and Y_i' is in $(\operatorname{ad}\mathcal{M}^-)y_jt_j$. We can actually say more about the Y_i. First recall that $Y_{I,j}$ is in G^-. Set $\lambda = \alpha_{i_1} + \cdots + \alpha_{i_r} + \alpha_j$ and note that the weight of $Y_{I,j}$ is $-\lambda$. Set $\mu = 0$. We may apply (4.7) to $Y_{I,j}\tau(-\lambda)$ using this λ and μ. By (4.7) and weight space considerations, each Y_i is in $\mathcal{M}\cap G^-U^o$. Furthermore, (4.7) implies that each $Y_i \in U^-\tau(\lambda)$. Since $\tau(\lambda) \in \mathcal{M}t_j$, it follows

that each Y_i is an element of $(\mathcal{M} \cap U^- U^o)t_j$. In particular, we get that (see [AJS, Proposition C.5])

$$\Delta(Y_{I,j}) \in Y_{I,j} \otimes 1 + (\mathcal{M} \cap U^- U^o)t_j \otimes (\mathrm{ad}\, \mathcal{M}^-)(y_j t_j). \qquad (6.3)$$

The elements $Y_{I,j}$ also satisfy a uniqueness property. In particular, by [L2, Proposition 4.1], if Y is an element of G^- of weight $-\lambda$ such that

$$\Delta(Y) \in Y \otimes 1 + (\mathcal{M} \cap U^- U^o)t_j \otimes (\mathrm{ad}\, \mathcal{M}^-)(y_j t_j)$$

then Y is a nonzero scalar multiple of $Y_{I,j}$. This uniqueness property will be used in the uniqueness result Theorem 7.5 concerning quantum symmetric pairs.

Let $\mathbf{n}_{\pi'}^+$ be the Lie subalgebra of $\mathbf{n}^+$ spanned by the root vectors e_γ, where γ runs over $\Delta^+ - \Delta'$. One can similarly define left coideal subalgebras $N_{\pi'}^+$ of U^+ which are analogs of $U(\mathbf{n}_{\pi'}^+)$. These can be constructed directly using the same methods described above for $N_{\pi'}^-$.

Of course, one could take the perspective of right coideal subalgebras instead of left coideal subalgebras. This will be useful in the next section. For example, right coideal analogs of $U(\mathbf{n}_{\pi'}^+)$ are subalgebras of G^+ defined using the right adjoint action,

$$(\mathrm{ad}_r a)b = \sum \sigma(a_{(1)}) b a_{(2)} \qquad (6.4)$$

for all a and b in U. Let $G_{\pi-\pi'}^+$ be the subalgebra of G^+ generated by the $x_i t_i^{-1}$ for all i such that $\alpha_i \in \pi - \pi'$. Then the subalgebra $N_{\pi',r}^+$ generated by $(\mathrm{ad}_r \mathcal{M}^+) G_{\pi-\pi'}^+$ is a right coideal subalgebra of G^+ and an analog of $U(\mathbf{n}_{\pi'}^+)$. The algebra $N_{\pi',r}^+$ is generated by elements of the form

$$X_{I,j} = (\mathrm{ad}_r x_{i_1} \cdots x_{i_r}) x_j t_j^{-1}$$

where each $\alpha_{i_s} \in \pi'$ and $\alpha_j \notin \pi'$. Moreover the comultiplication of these elements is similar to that of the $Y_{I,j}$, e.g.,

$$\Delta(X_{I,j}) \in 1 \otimes X_{I,j} + (\mathrm{ad}_r \mathcal{M}^+)(x_j t_j^{-1}) \otimes (\mathcal{M} \cap G^+ U^o)t_j^{-1}. \qquad (6.5)$$

Using $N_{\pi'}^-$, $N_{\pi'}^+$, and $\mathcal{M}^-$, one can construct what are called generalized Verma modules. Let $\mathcal{P}$ be the subalgebra of U generated by $\mathcal{M}$, U^o, and $N_{\pi'}^+$. Note that $\mathcal{P}$ is a left coideal subalgebra since it is generated by left coideal subalgebras. It is an analog of the enveloping algebra of the parabolic Lie subalgebra $(\mathbf{m}+\mathbf{h}) \oplus \mathbf{n}_{\pi'}^+$. Using (6.2), one obtains an isomorphism of vector spaces via the multiplication map

$$U \cong N_{\pi'}^- \otimes \mathcal{P}. \qquad (6.6)$$

Let W be a finite-dimensional simple $\mathcal{M}$ module. Extend the action of $\mathcal{M}$ on W to U^o by insisting that the highest weight generating vector of W is a weight vector of say weight Λ with respect to the action of T. Extend further the action on W to $N_{\pi'}^+$ by insisting that the augmentation ideal of $N_{\pi'}^+$ acts as zero on all vectors in W. These extensions make W into a $\mathcal{P}$ module. The generalized Verma module $M_{\pi'}(\Lambda)$ is defined to be $U \otimes_{\mathcal{P}} W$. In particular, elements of U

act by left multiplication and $pu \otimes w = \sum (\operatorname{ad} p_{(1)})u \otimes p_{(2)}w$ for all $p \in \mathcal{P}$, $u \in U$, and $w \in W$. As a left $N_{\pi'}^{-}$ module, $U \otimes_{\mathcal{P}} W \cong N_{\pi'}^{-} \otimes W$. Furthermore, the action of $\mathcal{M}$ on $\mathbb{N}_{\pi'}^{-}$ is both locally finite and semisimple. Hence the generalized Verma module $M_{\pi'}(\Lambda)$ is a Harish-Chandra module for the pair U, $\mathcal{M}$.

Using the coideal subalgebras discussed in this section, one can form quantized homogenous spaces as in Section 3. For example, the homogeneous space associated to G^{-}, $R_q[G/N] = R_q[G]_l^{G^{-}}$ is studied in [Jo, Chapter 9] where it is used to obtain the complete description of the prime and primitive spectra of the quantized function algebra $R_q[G]$.

7. Quantum Symmetric Pairs

We turn now to the theory of quantum symmetric pairs. First, we present the construction and characterization of the coideal subalgebras used to form such pairs. The results are drawn from [L2] and [L3], but the methods in this paper are often simpler. The involutions used to construct these algebras are given in a concrete fashion here. The relations for the coideal subalgebras as algebras are also presented more explicitly. Moreover, using the results of Section 4, we give a new, less intricate, proof of the uniqueness characterization for the subalgebras used to form quantum symmetric pairs (see Theorem 7.5 below.) The Harish-Chandra module and symmetric space theory associated to these pairs is also described with the aid of Sections 2 and 3.

A symmetric pair is defined for each Lie algebra involution (equivalently, a Lie algebra automorphism of order 2) of $\mathbf{g}$. More precisely, let θ be a Lie algebra involution of $\mathbf{g}$. Write $\mathbf{g}^{\theta}$ for the Lie subalgebra of $\mathbf{g}$ consisting of elements fixed by θ. The pair $\mathbf{g}, \mathbf{g}^{\theta}$ is a classical symmetric pair. A classification of involutions and classical symmetric pairs up to isomorphism can be found in [He1, Chapter 10, Sections 2, 5, and 6] and [OV, Section 4.1.4].

Let $\mathbf{p} = \{v \in \mathbf{g} | \theta(v) = -v\}$. A commutative Lie subalgebra of $\mathbf{g}$ which is reductive in $\mathbf{g}$ and is equal to its centralizer in $\mathbf{p}$ is called a Cartan subspace of $\mathbf{p}$ (see [D, 1.13.5].) A Cartan subalgebra $\mathbf{h}'$ of $\mathbf{g}$ is called maximally split ([V, Section 0.4.1]) with respect to θ provided that $\mathbf{h}' \cap \mathbf{p}$ is a Cartan subspace of $\mathbf{p}$. By [D, 1.13.6, 1.13.7], $\mathbf{p}$ contains Cartan subspaces and moreover each Cartan subspace can be extended to a Cartan subalgebra of $\mathbf{g}$.

Recall that we have already specified a Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. Let θ be an involution of $\mathbf{g}$ such that $\mathbf{h}$ is maximally split with respect to θ. Let $\mathcal{L}$ be the set of Lie algebra automorphisms ψ of $\mathbf{g}$ such that $\psi(\mathbf{p} \cap \mathbf{h})$ is a subset of $\mathbf{h}$. If $\psi \in \mathcal{L}$ then $\mathbf{h}$ is also maximally split with respect to the involution $\psi\theta\psi^{-1}$. By [D, 1.13.7 and 1.13.8], one can replace θ by $\psi\theta\psi^{-1}$ for some $\psi \in \mathcal{L}$ so that θ also satisfies the following conditions:

$$\theta(\mathbf{h}) = \mathbf{h}; \tag{7.1}$$

$$\text{if } \theta(h_i) = h_i \text{ then } \theta(e_i) = e_i \text{ and } \theta(f_i) = f_i; \tag{7.2}$$

if $\theta(h_i) \neq h_i$ then $\theta(e_i)$ is a nonzero root vector in $\mathbf{n}^-$
$$\text{and } \theta(f_i) \text{ is a nonzero root vector in } \mathbf{n}^+. \quad (7.3)$$

By [D, 1.13.8], θ also induces an automorphism Θ of the root system Δ.

if $\theta(h_i) \neq h_i$ then $\theta(e_i)$ is a nonzero root vector in $\mathbf{n}^-$

Now consider an arbitrary involution θ' of $\mathbf{g}$. One can find a Lie algebra automorphism Υ of $\mathbf{g}$ so that $\mathbf{h}$ is maximally split with respect to the involution $\Upsilon\theta'\Upsilon^{-1}$. In the quantum case, we do not have as much flexibility in "moving" involutions around using an automorphism of U. In particular, there is only one choice of quantum Cartan subalgebra, since the only invertible elements of U are the nonzero scalars and the elements of T. Hence any automorphism of U restricts to an automorphism of T. Thus the relationship between an involution of $\mathbf{g}$ and the particular Cartan subalgebra $\mathbf{h}$ is important in lifting the involution to the quantum case. In this section, we call an involution θ of $\mathbf{g}$ a maximally split involution if $\mathbf{h}$ is maximally split with respect to θ and θ satisfies (7.1), (7.2), and (7.3). (Similar terminology was introduced in [Di, Section 5].) We discuss lifts of maximally split involutions and the associated quantum symmetric pairs. There are also a few scattered results on quantum symmetric pairs when the involution is not maximally split. The reader is referred to [G] and [BF] for more information.

For the remainder of this section, let θ be a maximally split involution with respect to the fixed Cartan subalgebra $\mathbf{h}$. Consider the Cartan subspace $\mathbf{a} = \mathbf{p} \cap \mathbf{h}$ of $\mathbf{p}$. Since $\mathbf{a}$ is subset of $\mathbf{h}$, the action of $\operatorname{ad}\mathbf{a}$ on $\mathbf{g}$ is semisimple. Given $\lambda \in \mathbf{a}^*$, set

$$\mathbf{g}_\lambda = \{x \in \mathbf{g} | (\operatorname{ad}a)x = \lambda(a)x \text{ for all } a \in \mathbf{a}\}.$$

Let

$$\Sigma = \{\lambda \in \mathbf{a}^* | \mathbf{g}_\lambda \neq 0\}.$$

We can write $\mathbf{g} = \oplus_{\lambda \in \Sigma}\mathbf{g}_\lambda$. Furthermore, by [OV, Theorem 3.4.2], Σ is an abstract root system called the restricted root system associated to θ (or more precisely, to $\mathbf{g}, \mathbf{a}$.) A classification of restricted root systems associated to involutions can be found in [Kn, Chapter VI, Section 11] (see also [He1, Chapter X, Section F under Exercises and Further Results]). Note that an abstract root system is slightly more general than an ordinary root system (often called a reduced root system) described in [H, Chapter III]. Good references for abstract root systems are [Kn, Chapter II, Section 5] and [OV, Chapter 3, Section 1.1]. The abstract root systems have been classified as the set of reduced root systems and one additional nonreduced family referred to as type BC ([Kn, Chapter II, Section 8]).

Before discussing the quantum case, we further describe the action of θ on the generators of $\mathbf{g}$. Set $\Delta_\Theta = \{\alpha \in \Delta | \Theta(\alpha) = \alpha\}$ where Θ is the associated root system automorphism. This is the root system for the semisimple Lie subalgebra $\mathbf{m}$ of $\mathbf{g}$ generated by the e_i, f_i, h_i with $\theta(h_i) = h_i$. Write $\mathbf{m} = \mathbf{m}^- \oplus \mathbf{m}^o \oplus \mathbf{m}^+$ for the obvious triangular decomposition of $\mathbf{m}$. Set $\pi_\Theta = \Delta_\Theta \cap \pi$. Note that π_Θ

is a set of positive simple roots for the root system Δ_Θ. Write $Q(\pi_\Theta)$ for the lattice of integral linear combinations of the simple roots in π_Θ. Let $Q^+(\pi_\Theta)$ be the set of nonnegative integral linear combinations of the elements in π_Θ.

Note that $\pi_\Theta = \Theta(\pi) \cap \pi$. Also, $\Theta(-\alpha_i) \in \Delta^+$ for all $\alpha_i \notin \pi_\Theta$ by (7.3). It follows that

$$\Theta(-\alpha_i) \in \sum_{\alpha_j \notin \pi_\Theta} \mathbf{N}\alpha_j + Q^+(\pi_\Theta) \tag{7.4}$$

for each $\alpha_i \notin \pi_\Theta$. Since Θ is a root system automorphism, every element of Δ can be written as an integral linear combination of roots in $\{\Theta(\alpha_i)|\alpha_i \in \pi\}$ where either all the coefficients are positive or all the coefficients are negative. Hence each $\alpha_i \notin \pi_\Theta$ can be written as a linear combination of elements in $\{\Theta(\alpha_i)|\alpha_i \notin \pi_\Theta\} \cup \pi_\Theta$ with just negative integers as coefficients. Observation (7.4) thus implies that there exists a permutation p on the set $\{i \,|\, \alpha_i \in \pi - \pi_\Theta\}$ such that for each $\alpha_i \in \pi - \pi_\Theta$,

$$\Theta(-\alpha_i) - \alpha_{p(i)} \in Q^+(\pi_\Theta). \tag{7.5}$$

Choose a maximal subset π^* of $\pi - \pi_\Theta$ such that if $j = p(j)$ then $\alpha_j \in \pi^*$ and if $j \neq p(j)$, then exactly one of the pair $\alpha_j, \alpha_{p(j)}$ is in π^*. Consider i such that $\alpha_i \in \pi^*$. The root vector $e_{p(i)}$ associated to the simple root $\alpha_{p(i)}$ satisfies $(\mathrm{ad}\, f_j)e_{p(i)} = [f_j, e_{p(i)}] = 0$ for all $\alpha_j \in \pi_\Theta$. Thus $e_{p(i)}$ is a lowest weight vector for the action of $\mathrm{ad}\,\mathbf{m}^-$. Let V be the corresponding simple $\mathrm{ad}\,\mathbf{m}$ module generated by $e_{p(i)}$. By (7.3), $\theta(f_i)$ is a root vector in $\mathbf{n}^+$. Furthermore (7.5) implies that the weight of this root vector is $\alpha_{p(i)}$ plus some element in $Q^+(\pi_\Theta)$. Thus $\theta(f_i)$ can be written as a bracket $[a_1[a_2, \ldots, [a_{s-1}, a_s]\ldots]$ where exactly one of the a_j equals $e_{p(i)}$ and the others are elements of $\mathbf{m}^+$. Using the Jacobi identity, we see that $\theta(f_i)$ is an element of $(\mathrm{ad}\,\mathbf{m}^+)e_{p(i)}$. In particular $\theta(f_i)$ is an element of V. Furthermore, since elements of $\mathbf{m}^+$ commute with f_i and thus with $\theta(f_i)$, we see that $\theta(f_i)$ must be a highest weight vector of V. Thus we can find a sequence of elements $\alpha_{i_1}, \ldots, \alpha_{i_r}$ in π_Θ and a sequence of positive integers $m_1, \ldots, m_r$ such that (up to a slight adjustment of θ)

$$\theta(f_i) = (\mathrm{ad}\, e_{i_1}^{(m_1)} \cdots e_{i_r}^{(m_r)})e_{p(i)}. \tag{7.6}$$

Here $e_j^{(m)} = e_j^m/m!$. We may further assume that both the sequence of roots and the sequence of integers are chosen so that each $(\mathrm{ad}\, e_{i_s}^{(m_s)} \cdots e_{i_r}^{(m_r)})e_{p(i)}$, $1 \leq s \leq r$, is an extreme vector of V. (In particular, $(\mathrm{ad}\, e_{i_s}^{(m_s)} \cdots e_{i_r}^{(m_r)})e_{p(i)}$ is a highest weight vector for the action of $\mathrm{ad}\, e_{i_s}$ and $(\mathrm{ad}\, e_{i_{s-1}}^{(m_{s-1})} \cdots e_{i_r}^{(m_r)})e_{p(i)}$ is a lowest weight vector for the action of $\mathrm{ad}\, f_{i_s}$.) Suppose that the sequence $\alpha_{j_1}, \ldots, \alpha_{j_s}$ of elements in π_Θ and the positive integers $n_1, \ldots, n_s$ also satisfy this condition on extreme vectors and that $\sum_k m_k \alpha_{i_k} = \sum_k n_k \alpha_{j_k}$. By [Ve], $(\mathrm{ad}\, e_{i_1}^{(m_1)} \cdots e_{i_r}^{(m_r)})e_{p(i)} = (\mathrm{ad}\, e_{j_1}^{(n_1)} \cdots e_{j_s}^{(n_s)})e_{p(i)}$. Thus (7.6) is independent of the choice of such sequences.

Using lowest weight vectors instead of highest weight vectors, we obtain

$$\theta(e_{p(i)}) = (\operatorname{ad} f_{i_r}^{(m_r)} \cdots f_{i_1}^{(m_1)}) f_i \tag{7.7}$$

up to a nonzero scalar. A straightforward **sl** 2 computation shows that

$$(\operatorname{ad} e_{i_1}^{(m_1)} \cdots e_{i_r}^{(m_r)})[(\operatorname{ad} f_{i_r}^{(m_r)} \cdots f_{i_1}^{(m_1)}) f_i] = f_i$$

and

$$(\operatorname{ad} f_{i_r}^{(m_r)} \cdots f_{i_1}^{(m_1)})[(\operatorname{ad} e_{i_1}^{(m_1)} \cdots e_{i_r}^{(m_r)}) e_{p(i)}] = e_{p(i)}. \tag{7.8}$$

Since θ^2 is the identity, the scalar in (7.7) must be 1.

Set $m(i) = m_1 + \ldots + m_r$. Now $[\theta(e_i), \theta(f_i)] = \theta(h_i)$ is an element of **h** by (7.1). Furthermore, by (7.2) and (7.3), $\theta(h_i)$ must be the coroot $h_{\Theta(\alpha_i)}$ associated to the root $\Theta(\alpha_i)$. The description of the Chevalley basis for **g** given in [H, Proposition 25.2 and Theorem 25.2] ensures that both $\theta(e_i)$ and $\theta(f_i)$ are Chevalley basis vectors up to a sign. Furthermore, by [H, Proposition 25.2(b)] and (7.6),we must have

$$\theta(e_i) = (-1)^{m(i)} (\operatorname{ad} f_{i_1}^{(m_1)} \cdots f_{i_r}^{(m_r)}) f_{p(i)}.$$

Similarly, by [H, Proposition 25.2(b)] and (7.7)

$$\theta(f_{p(i)}) = (-1)^{m(i)} (\operatorname{ad} e_{i_r}^{(m_r)} \cdots e_{i_1}^{(m_1)}) e_i.$$

Note that when $p(i) = i$, we have

$$(\operatorname{ad} e_{i_r}^{(m_r)} \cdots e_{i_1}^{(m_1)}) e_i = (\operatorname{ad} e_{i_1}^{(m_1)} \cdots e_{i_r}^{(m_r)}) e_i.$$

Hence $m(i)$ is even in this case.

The above analysis allows us to better describe the root space automorphism Θ. Let W' denote the Weyl group associated to the root system Δ_Θ of **m** considered as a subgroup of the Weyl group of Δ. Let w_o denote the longest element of W'. Note that w_o is a product of reflections in W' but can also be considered as an element of W. Let d be the diagram automorphism on π_Θ such that $d = -w_o$ when restricted to π_Θ. Note that d induces a permutation on the set $\{i | \alpha_i \in \pi_\Theta\}$ which we also denote by d. In particular, given $\alpha_i \in \pi_\Theta$, $d(\alpha_i) = \alpha_{d(i)}$. Extend d to a function on π, and thus to Δ, by setting $d(\alpha_i) = \alpha_{p(i)}$ for $\alpha_i \notin \pi_\Theta$. It follows that $\Theta = -w_o d$. Note that this forces d to be a diagram automorphism of the larger root system Δ.

Before lifting θ to the quantum case, we recall and introduce more notation. The right adjoint action is defined by (6.4). This action on the generators of U is given by:

$$(\operatorname{ad}_r y_i) b = b y_i - y_i t_i b t_i^{-1} \quad (\operatorname{ad}_r x_i) b = t_i^{-1} b x_i - t_i^{-1} x_i b \quad (\operatorname{ad}_r t_i) b = t_i^{-1} b t_i$$

for $1 \le i \le n$. Recall the definitions of $[m]_q$ and q_i used to define the quantized enveloping algebra ((1.4)-(1.7)). The divided powers of x_i and y_i are defined by $x_i^{(m)} = x_i^m / [m]_{q_i}!$ and $y_i^{(m)} = y_i^m / [m]_{q_i}!$. (Note that these are quantum analogs

of the divided power $e_i^{(m)}$.) Let $\mathfrak{M}$ denote the subalgebra of U generated by the corresponding elements x_i, y_i, t_i, t_i^{-1} where $\theta(h_i) = h_i$. Note that $\mathfrak{M}$ is just a copy of the quantized enveloping algebra $U_q(\mathbf{m})$ so this notation is consistent with that of Section 6. Let ι be the $\mathbb{C}$ algebra automorphism of U fixing $x_i t_i^{-1}$ and $t_i y_i$ for $1 \leq i \leq n$, sending t to t^{-1} for all $t \in T$ and q to q^{-1}. Recall the sequences $\alpha_{i_1}, \ldots, \alpha_{i_r}$ and $m_1, \ldots, m_r$ used in (7.6) and (7.7). (As in the classical case, using [Lu, Proposition 39.3.7], the description of $\tilde{\theta}(y_i)$ in (7.12) below is independent of the choice of such sequences.)

In the next theorem, we lift θ to a $\mathbb{C}$ algebra automorphism of U. This is in the spirit of [L2, Theorem 3.1]. The main difference here is that we do not insist that $\tilde{\theta}$ is a $\mathbb{C}$ algebra involution on all of U.

THEOREM 7.1. *There exists a $\mathbb{C}$ algebra automorphism $\tilde{\theta}$ on U such that*:

$$\tilde{\theta}(x_i) = x_i \text{ and } \tilde{\theta}(y_i) = y_i \text{ for all } \alpha_i \in \pi_\Theta. \tag{7.9}$$

$$\tilde{\theta}(\tau(\lambda)) = \tau(\Theta(-\lambda)) \text{ for all } \tau(\lambda) \in T. \tag{7.10}$$

$$\tilde{\theta}(q) = q^{-1}. \tag{7.11}$$

For $\alpha_i \in \pi^*$, we have $\tilde{\theta}(y_i) = [(\mathrm{ad}_r x_{i_1}^{(m_1)} \cdots x_{i_r}^{(m_r)}) t_{p(i)}^{-1} x_{p(i)}]$ and

$$\tilde{\theta}(y_{p(i)}) = (-1)^{m(i)} [(\mathrm{ad}_r x_{i_r}^{(m_r)} \cdots x_{i_1}^{(m_1)}) t_i^{-1} x_i]. \tag{7.12}$$

Furthermore, $\tilde{\theta}^2$ is the identity when restricted to $\mathfrak{M}$ and to T. Finally, $\tilde{\theta}$ specializes to θ as q goes to 1.

PROOF. To show that $\tilde{\theta}$ extends to a $\mathbb{C}$ algebra automorphism of U, we relate it to Lusztig's automorphisms. Let T_{w_o} be Lusztig's automorphism associated to w_o, the longest element of W'. We follow the notation of [DK, Section 1.6]. Fix $\alpha_i \in \pi_\Theta$. Recall that $-w_o(\alpha_i) = \alpha_{d(i)}$. By [DK, Section 1.6 and Proposition 1.6], T_{w_o} sends y_i to a nonzero scalar multiple of $x_{d(i)} t_{d(i)}^{-1}$, sends x_i to a nonzero scalar multiple of $y_{d(i)} t_{d(i)}$, and sends t_i to $t_{d(i)}^{-1}$. Furthermore, one checks using [DK, Remark 1.6] that for each $\alpha_i \notin \pi_\Theta$ the composition

$$(\iota \circ T_{w_o})(t_{p(i)}^{-1} x_{p(i)}) = u_i [(\mathrm{ad}_r x_{i_1}^{(m_1)} \cdots x_{i_r}^{(m_r)}) t_{p(i)}^{-1} x_{p(i)}]$$

for some nonzero scalar u_i.

Define a function $\tilde{\theta}$ on the generators of U using (7.9), (7.10), (7.11), (7.12), and setting

$$\tilde{\theta}(x_i) = u_i^{-1} (\iota \circ T_{w_o})(y_{p(i)} t_{p(i)}) \text{ and } \tilde{\theta}(x_{p(i)}) = (-1)^{m(i)} u_{p(i)}^{-1} (\iota \circ T_{w_o})(y_i t_i)$$

for each $\alpha_i \in \pi^*$. It is clear from (7.9) and (7.10) that $\tilde{\theta}$ extends to a $\mathbf{C}$ algebra automorphism on both $\mathfrak{M}$ and T. Now $\tilde{\theta}^2$ is clearly the identity on $\mathfrak{M}$. Since Θ is an involution on the root system of $\mathbf{g}$, condition (7.10) ensures that $\tilde{\theta}$ also restricts to an involution on the group T.

We check that $\tilde{\theta}$ extends to a $\mathbb{C}$ algebra automorphism of U. In particular, $\tilde{\theta}(y_i)\tilde{\theta}(x_i) - \tilde{\theta}(x_i)\tilde{\theta}(y_i) = (\iota \circ T_{w_o})(y_{p(i)} x_{p(i)} - x_{p(i)} y_{p(i)}) = \tilde{\theta}(y_i x_i - x_i y_i)$ for

$\alpha_i \notin \pi_\Theta$. Furthermore, for $\alpha_i \in \pi_\Theta$, $(\iota \circ T_{w_o})(t_{d(i)}x_{d(i)}) = y_i = \tilde{\theta}(y_i)$ up to some nonzero scalar. Hence the $\tilde{\theta}(y_i), 1 \leq i \leq n$ satisfy the quantum Serre relations (1.7). Similarly, $(\iota \circ T_{w_o})(y_it_i) = x_{d(i)} = \tilde{\theta}(x_{d(i)})$ up to a nonzero scalar when $\alpha_i \in \pi_\Theta$. It follows that the $\tilde{\theta}(x_i)$ for $1 \leq i \leq n$ satisfy the quantum Serre relations (1.7). Moreover, $\tilde{\theta}$ preserves the relations between the x_i and the y_j for $1 \leq i, j \leq n$. Thus $\tilde{\theta}$ extends to a $\mathbb{C}$ algebra automorphism $\tilde{\theta}$ of U.

Now consider an element b in $\mathcal{M} \cup T \cup \{y_i | 1 \leq i \leq n\}$ and write $\bar{b}$ for its specialization as q goes to 1. Note that the specialization of $\tilde{\theta}(b)$ is just $\theta(\bar{b})$. This is enough to force $\tilde{\theta}$ to specialize to θ. □

We are now ready to introduce the quantum analog of $U(\mathbf{g}^\theta)$. Set

$$T_\Theta = \{\tau(\lambda) | \Theta(\lambda) = \lambda\},$$

a subgroup of T. Let $B = B_{\tilde{\theta}}$ be the subalgebra of U generated by $\mathcal{M}$, T_Θ, and the elements

$$B_i = y_it_i + \tilde{\theta}(y_i)t_i$$

for $\alpha_i \notin \pi_\Theta$. The next result shows that B is a coideal subalgebra of U. This fact combined with the results of Sections 1 and 4 is used below to describe the relations satisfied by these generators. As a consequence, we show below that B specializes to $U(\mathbf{g}^\theta)$ as q goes to 1.

THEOREM 7.2. *B is a left coideal subalgebra of U.*

PROOF. We need to check that

$$\Delta(b) \in U \otimes B \qquad (7.13)$$

for all $b \in B$. Since Δ is an algebra homomorphism from U to $U \otimes U$, it is sufficient to check (7.13) for a set of generators of B. Now B is generated by the elements B_i, for $\alpha_i \notin \pi_\Theta$, and two Hopf algebras: $\mathcal{M}$ and the group algebra generated by T_Θ. In particular, each $b \in \mathcal{M}$ and each $b \in T_\Theta$ satisfies (7.13). Hence it is sufficient to check (7.13) holds for the remaining generators, namely when $b = B_i$ for $\alpha_i \notin \pi_\Theta$.

Set $\mathcal{M}^+ = U^+ \cap \mathcal{M}$. Note that $t_it_{p(i)}^{-1}$ is in T_Θ for all i with $\alpha_i \notin \pi_\Theta$. Thus by (6.5) and the definition of $\tilde{\theta}$, the element $\tilde{\theta}(y_i)t_i$ satisfies the following nice property with respect to the comultiplication of U:

$$\Delta(\tilde{\theta}(y_i)t_i) \in t_i \otimes \tilde{\theta}(y_i)t_i + U \otimes (\mathcal{M} \cap G^+U^o)t_{p(i)}^{-1}t_i$$
$$\subset t_i \otimes \tilde{\theta}(y_i)t_i + U \otimes \mathcal{M}^+T_\Theta. \qquad (7.14)$$

This combined with the formula for $\Delta(y_it_i)$ (see (1.8) and (1.10)) yields

$$\Delta(B_i) \in t_i \otimes B_i + U \otimes \mathcal{M}^+T_\Theta \subset U \otimes B \qquad (7.15)$$

and the theorem follows. □

We turn now to understanding the relations satisfied by the generators of B. The elements B_i have already been defined when $\alpha_i \notin \pi_\Theta$. Set $B_i = y_i t_i$ for $\alpha_i \in \pi_\Theta$. Given a tuple $I = (i_1, \ldots, i_r)$, set $|I| = r$, $\mathrm{wt}(I) = \alpha_{i_1} + \ldots + \alpha_{i_r}$, $B_I = B_{i_1} \cdots B_{i_r}$, and $Y_I = y_{i_1} t_{i_1} \cdots y_{i_r} t_{i_r}$.

Recall ((1.4) and (1.11)) that $x_i y_j t_j = q^{(-\alpha_i, \alpha_j)} y_j t_j x_i$ whenever $j \neq i$. By Theorem 7.1, $\tilde{\theta}(x_i) = x_i$ whenever $\alpha_i \in \pi_\Theta$. Furthermore, $\tilde{\theta}(y_j)$ and y_j have the same weight with respect to the adjoint action of T_Θ. Hence

$$x_i B_j = q^{(-\alpha_i, \alpha_j)} B_j x_i \text{ and } \tau(\lambda) B_j = q^{(\lambda, -\alpha_j)} B_j \tau(\lambda) \qquad (7.16)$$

for all $\alpha_i \in \pi_\Theta$ with $\alpha_j \notin \pi_\Theta$, and $\tau(\lambda) \in T_\Theta$. It follows that

$$B = \sum_I B_I \mathcal{M}^+ T_\Theta. \qquad (7.17)$$

Let $\mathcal{J}$ be a set such that $\{Y_J | J \in \mathcal{J}\}$ is a basis for G^-. Note that $B_J = Y_J +$ (terms of higher weight) for each tuple J. The triangular decomposition (4.1) of U implies that the subspaces $\{Y_J \mathcal{M}^+ T_\Theta | J \in \mathcal{J}\}$, and hence the subspaces $\{B_J \mathcal{M}^+ T_\Theta | J \in \mathcal{J}\}$, are linearly independent.

Let F_{ij} be the function in two variables X_1 and X_2 defined by

$$F_{ij}(X_1, X_2) = \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} X_1^{1-a_{ij}-m} X_2 X_1^m.$$

The quantum Serre relations (1.7) are the set of equations $F_{ij}(y_i, y_j) = 0$ for $i \neq j$. A straightforward computation shows that if $(\lambda_i, \alpha_j) = (\lambda_j, \alpha_i)$ then $F_{ij}(y_i \tau(\lambda_i), y_j \tau(\lambda_j)) = 0$. Hence

$$F_{ij}(y_i t_i, y_j t_j) = 0. \qquad (7.18)$$

It follows that the generators $y_i t_i$ of G^- satisfy the same relations as the generators of U^-. Furthermore, since $(\Theta(-\alpha_i), \alpha_j) = (\Theta(-\alpha_j), \alpha_i)$, we have

$$F_{ij}(\tilde{\theta}(y_i) t_i, \tilde{\theta}(y_j) t_j) = 0. \qquad (7.19)$$

We show below that the B_i for $1 \leq i \leq n$ satisfy relations which come from the quantum Serre relations on G^-. First, we consider the evaluation of the function F_{ij} at B_i, B_j in a few special cases.

If both α_i and α_j are in π_Θ, then $F_{ij}(B_i, B_j) = F_{ij}(y_i t_i, y_j t_j)$. Similarly, if $\alpha_i \in \pi_\Theta$ and $\alpha_j \notin \pi_\Theta$, then $F_{ij}(B_i, B_j) = F_{ij}(y_i t_i, y_j t_j) + F_{ij}(\tilde{\theta}(y_i) t_i, \tilde{\theta}(y_j) t_j)$. Hence (7.18) and (7.19) imply that

$$\text{if } \alpha_i \in \pi_\Theta \text{ then } F_{ij}(B_i, B_j) = 0. \qquad (7.20)$$

Now suppose that i and j are chosen such that $\pi_{0,0}(Y_{ij})$ is nonzero. It follows that Y_{ij} must have a zero weight summand. Checking the possibilities for the

quantum Serre relations, we must have $a_{ij} = 0$ and $\Theta(\alpha_i) = -\alpha_j$. In particular, $B_i = y_i t_i + q_i^{-2} x_j t_j^{-1} t_i$ and $B_j = y_j t_j + q_i^{-2} x_i t_i^{-1} t_j$. A straightforward computation shows that

$$a_{ij} = 0 \text{ and } \Theta(\alpha_i) = -\alpha_j \text{ imply } F_{ij}(B_i, B_j) = B_i B_j - B_j B_i = \frac{t_i^{-1} t_j - t_j^{-1} t_i}{q_i - q_i^{-1}}. \tag{7.21}$$

Given $\lambda \in Q(\pi)$, let P_λ be the projection of B onto $U^- G^+ \tau(\lambda)$ with respect to the direct sum decomposition of Lemma 1.3 applied to the coideal B. The next lemma provides more detailed information about $F_{ij}(B_i, B_j)$.

LEMMA 7.3. *Let $Y_{ij} = F_{ij}(B_i, B_j)$ for $i \neq j$ and $\lambda_{ij} = (1 - a_{ij})\alpha_i + \alpha_j$. If $(\pi_{\beta,\gamma} \circ P_{\lambda_{ij}})(Y_{ij}) \neq 0$ then $[\beta, \gamma] \neq 0$, $\tau(\lambda_{ij} - \beta) \notin T_\Theta$, and $\tau(\lambda_{ij} - \gamma) \notin T_\Theta$.*

PROOF. Set $P_{ij} = P_{\lambda_{ij}}$. Suppose that $(\pi_{0,0} \circ P_{ij})(Y_{ij}) \neq 0$. It follows that $\pi_{0,0}(Y_{ij}) \neq 0$. Hence $a_{ij} = 0$ and $\Theta(\alpha_i) = -\alpha_j$. Now $\lambda_{ij} = \alpha_i + \alpha_j$ in this case. By (7.21) $P_{ij}(Y_{ij}) = P_{ij}(t_i^{-1} t_j - t_j^{-1} t_i) = 0$. Therefore, $\pi_{0,0}(Y_{ij}) = 0$ for all choices of i and j.

By (7.20), we may assume that α_i is not in π_Θ. Assume that β and γ are chosen so that $\pi_{\beta,\gamma}(Y_{ij}) \neq 0$. Note that Y_{ij} can be written as a sum of monomials in $2 - a_{ij}$ terms where $1 - a_{ij}$ of those terms are from the set $\{y_i t_i, \tilde{\theta}(y_i) t_i\}$ and the other term is from the set $\{y_j t_j, \tilde{\theta}(y_j) t_j\}$. It follows that $\gamma = s_1 \alpha_{p(i)} + s_2 \alpha_{p(j)} + \eta$ for some $\eta \in Q^+(\pi_\Theta)$ and nonnegative integers s_1 and s_2 such that $s_1 \leq 1 - a_{ij}$ and $s_2 \leq 1$. Set $\gamma' = s_1 \alpha_i + s_2 \alpha_j$ and note that $\tau(\gamma - \gamma')$ is in T_Θ. The above description of the monomials which add to Y_{ij} further implies that $\gamma' + \beta \leq \lambda_{ij}$. Moreover, by (7.18), $0 \leq \beta < \lambda_{ij}$ and by (7.19), $0 \leq \gamma' < \lambda_{ij}$. Now β and γ' are both linear combinations of α_i and α_j. Thus the lemma follows if neither α_i nor α_j are elements of $Q^+(\pi_\Theta)$. In the case when $\alpha_j \in \pi_\Theta$, (7.19) further implies that that $0 \leq \gamma' < \lambda_{ij} - \alpha_j$ and $0 \leq \beta < \lambda_{ij} - \alpha_i$. The lemma thus follows in this case as well. $\square$

The next result gives a description of the generators and relations of B.

THEOREM 7.4. *Let $\tilde{B}$ be the algebra freely generated over $\mathcal{M}^+ T_\Theta$ by the elements $\tilde{B}_i$, $1 \leq i \leq n$. Then there exist elements $c_J^{ij} \in \mathcal{M}^+ T_\Theta$ such that $B \cong \tilde{B}/L$ where L is the ideal generated by the following elements:*

(i) $\tau(\lambda)\tilde{B}_i \tau(-\lambda) - q^{-(\lambda,\alpha_i)}\tilde{B}_i$ *for all $\tau(\lambda) \in T_\Theta$ and $\alpha_i \notin \pi_\Theta$.*
(ii) $t_j^{-1} x_j \tilde{B}_i - \tilde{B}_i t_j^{-1} x_j - \delta_{ij}(t_j - t_j^{-1})/(q_j - q_j^{-1})$ *for all $\alpha_j \in \pi_\Theta$ and $1 \leq i \leq n$.*
(iii)

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{q_i} \tilde{B}_i^{1-a_{ij}-m} \tilde{B}_j \tilde{B}_i^m - \sum_{\{J \in \mathcal{J} \mid \mathrm{wt}(J) < (1-a_{ij})\alpha_i + \alpha_j\}} \tilde{B}_J c_J^{ij}$$

for each $i \neq j$, $1 \leq i, j \leq n$.

PROOF. Relations (i) and (ii) follow from (7.16) and (1.4). We now show that the B_i, $1 \leq i \leq n$, satisfy the relations described in (iii). Fix a quantum Serre relation $Y = F_{ij}(B_i, B_j)$ for given α_i, α_j with $i \neq j$. Set $\lambda = (1 - a_{ij})\alpha_i + \alpha_j$ and $Z = P_\lambda(Y)$. By (4.7), $((P_\lambda \circ \pi_{0,0}) \otimes \mathrm{Id})\Delta(Y) = (\pi_{0,0} \otimes \mathrm{Id})\Delta(Z)$. Moreover, (4.7) ensures that

$$(\pi_{0,0} \otimes \mathrm{Id})\Delta(Z) = \tau(\lambda) \otimes Z.$$

By (7.15) and (7.17), we have

$$\Delta(Y) \in \tau(\lambda) \otimes Y + \sum_{\{J| \ \mathrm{wt}(J)<\lambda\}} U \otimes B_J \mathcal{M}^+ T_\Theta. \tag{7.22}$$

Now if J has weight less than λ, one checks from (1.7) that there is no quantum Serre relation of weight greater than or equal to $-\lambda$. Hence if $\mathrm{wt}(J) < \lambda$ then J is an element of the set $\mathcal{J}$. Now, (7.22) implies that

$$((P_\lambda \circ \pi_{0,0}) \otimes \mathrm{Id})\Delta(Y) \in \tau(\lambda) \otimes (Y + \sum_{\{J\in\mathcal{J}| \ wt(J)<\lambda\}} B_J \mathcal{M}^+ T_\Theta)$$

Thus we can find $X \in \sum_{\{J\in\mathcal{J}|\mathrm{wt}(J)<\lambda\}} B_J \mathcal{M}^+ T_\Theta$ such that $Y + X = Z$. We obtain a relation of the form described in (iii) by proving $Z = 0$.

Recall the notation of Section 4, Filtration II. Assume that Z is nonzero and hence $\max(Z)$ is nonempty. Choose $[\beta,\gamma] \in \max(Z)$. It follows that $\pi_{\beta,\gamma}(Z) \neq 0$. By Lemma 7.3, $[\beta,\gamma] \neq [0,0]$, and neither $\tau(\lambda - \beta)$ nor $\tau(\lambda - \gamma)$ is an element of T_Θ. Write $(\pi_{\beta,0} \otimes \mathrm{Id})(\Delta(Z)) = \sum v_i \otimes u_i$ where the $v_i \in U_\beta^- \tau(\lambda)$ and the $u_i \in G^+ T$. We may assume that the v_i are linearly independent elements of $U_{-\beta}^- \tau(\lambda)$. Note that at least one of the u_i has a (nonzero) summand of weight γ in $G_\gamma^+ \tau(\lambda - \beta)$. By (7.19), the maximality of $[\beta,\gamma]$, and the fact that $\beta \neq 0$, each u_i is in $G^+ T \cap \sum_{\{J\in\mathcal{J}|\mathrm{wt}(J)<\lambda\}} B_J \mathcal{M}^+ T_\Theta$. This intersection is just $\mathcal{M}^+ T_\Theta$. Hence $\tau(\lambda - \beta) \in T_\Theta$, a contradiction. This forces $\beta = 0$. It follows that $(\pi_{0,\gamma} \otimes \mathrm{Id})(\Delta(Z)) \in U \otimes \tau(\lambda - \gamma)$. Again $\tau(\lambda - \gamma)$ must be in T_Θ. This contradiction forces $\max(Z)$ to be empty. In particular, $Z = 0$.

We have shown that B is isomorphic to a homomorphic image of $\tilde{B}/L$. A consequence of relations (i), (ii), and (iii) is that $\mathcal{M}^+ T_\Theta \tilde{B}_I \subset \sum_{J\in\mathcal{J}} \tilde{B}_J \mathcal{M}^+ T_\Theta + L$ for each tuple I. Thus

$$\tilde{B}/L = \bigoplus_{J\in\mathcal{J}} (\tilde{B}_J \mathcal{M}^+ T_\Theta + L).$$

Since the elements B_i in B satisfy the relations (i), (ii), (iii), we also have the following direct sum decomposition:

$$B = \bigoplus_{J\in\mathcal{J}} (B_J \mathcal{M}^+ T_\Theta).$$

Therefore $B \cong \tilde{B}/L$. □   □

Note that (7.20) and (7.21) both provide examples of the relations described in Theorem 7.4 (iii). We illustrate how to compute the c_J^{ij} in a more complicated example. Consider the case where $\Theta(\alpha_i) = -\alpha_i$, $\Theta(\alpha_j) = -\alpha_j$ and $a_{ij} = -1$. So $B_i = y_i t_i + q_i^{-2} x_i$ and $B_j = y_j t_j + q_j^{-2} x_j$ and $Y = B_i^2 B_j - (q_i + q_i^{-1}) B_i B_j B_i + B_j B_i^2$. Thus by (1.9) and (1.10),

$$\Delta(B_r) = B_r \otimes 1 + t_r \otimes B_r$$

for $r = i, j$. It follows that

$$\Delta(Y) = t_i^2 t_j \otimes Y + (B_i^2 t_j - (q_i + q_i^{-1}) B_i t_j B_i + t_j B_i^2) \otimes B_j + W \otimes B$$

for some W which satisfies $\pi_{0,0}(W) = 0$. A straightforward computation using the relations of U shows that $P_\lambda \circ \pi_{0,0}((B_i^2 t_j - (q_i + q_i^{-1}) B_i t_j B_i + t_j B_i^2) = -q_i^{-1} t_i^2 t_j$. Thus

$$0 = (P_\lambda \circ \pi_{0,0}) \otimes \mathrm{Id}(\Delta(Y)) = t_i^2 t_j \otimes Y - q_i^{-1} t_i^2 t_j \otimes B_j.$$

It follows that

$$B_i^2 B_j - (q_i + q_i^{-1}) B_i B_j B_i + B_j B_i^2 = -q_i^2 B_j.$$

(This relation is also computed in [L1, Lemma 2.2 (2.2)]. The generators for U and B are somewhat different in [L1]. In particular, when $\alpha_i = -\Theta(\alpha_i)$, B_i in [L1] is equal to $y_i t_i + x_i$ in the notation of this paper. Thus using a Hopf algebra automorphism of U, the B_i in [L1] corresponds to q_i^{-1} times the B_i defined in this paper. This explains the difference in coefficient of B_j found in the two papers.) Note that a similar argument shows that $c_J^{ij} = 0$ whenever $-\alpha_i \neq \alpha_{p(i)}$ and $-\alpha_i \neq \alpha_{p(j)}$. The c_J^{ij} are computed in [L1, Lemma 2.2] for the cases when $a_{ij} \geq -2$ and $\Theta(-\alpha_i) = \alpha_{p(i)}$.

Note that the generators of B specialize to the generators of $U(\mathbf{g}^\theta)$ as q goes to 1. Thus the specialization of $B_J \mathcal{M}^+ T_\Theta$ is contained in $U(\mathbf{g}^\theta)$. Moreover the set of spaces $\{B_J \mathcal{M}^+ T_\Theta, \ J \in \mathcal{J}\}$ remain linearly independent after specialization. As q goes to 1, since these spaces span B, we conclude that B specializes to $U(\mathbf{g}^\theta)$.

The algebra B also satisfies a maximality condition. Indeed, suppose that C is a subalgebra of U containing B and that C also specializes to $U(\mathbf{g}^\theta)$. Then by [L2, Theorem 4.9], $C = B$. The proof in [L2] uses a quantum version of the Iwasawa decomposition. The result also follows directly from Theorem 7.4. The idea is as follows. Recall the notation of Section 6. Set $N_\Theta^+ = N_{\pi_\Theta}^+$. By (6.6) (interchanging the roles of $N_{\pi'}^+$ with $N_{\pi'}^-$), we have

$$U = \sum_{J \in \mathcal{J}} Y_J \mathcal{M}^+ T N_\Theta^+.$$

By induction on $|J|$ (as in [L2, Lemma 4.3]), one can show that U is spanned by the spaces B; Bt, $t \notin T_\Theta$; and $BT(N_\Theta^+)_+$ where $(N_\Theta^+)_+$ is the augmentation

ideal of N_Θ^+. Let X be in C. Subtracting an element of B if necessary, we may assume that X is a linear combination of elements in $B(t-1)/(q-1)$ for $t \notin T_\Theta$, and $BT(N_\Theta^+)_+$. Assume that X is nonzero. Rescale X by a power of $(q-1)$ so that it is an element of $\hat{C} - (q-1)\hat{C}$. It follows that X does not specialize to an element of $U(\mathbf{g}^\theta)$. This contradiction forces $X = 0$ and thus $B = C$.

We have shown that the algebra B satisfies the following properties.

B is a left coideal in U.   (7.23)

B specializes to $U(\mathbf{g}^\theta)$.   (7.24)

If $B \subset C$ and C is a subalgebra of U specializing to $U(\mathbf{g}^\theta)$ then $B = C$. (7.25)

We now turn to characterizing all subalgebras of U which satisfy (7.23), (7.24), and (7.25). First, we present two variations which satisfy these conditions as well.

Variation 1. For sake of simplicity, we assume first that $\mathbf{g}$ is simple. Recall the permutation p used in (7.5). Suppose that there exists an $r \in \{1, 2, \ldots, n\}$ such that $\alpha_r \notin \pi_\Theta$ and $p(r) \neq r$. Assume further that $(\alpha_r, \Theta(\alpha_r)) \neq 0$. Recall the Cartan subspace $\mathbf{a} = \{x \in \mathbf{h} | \theta(x) = -x\}$ and the restricted root system Σ associated to θ introduced at the beginning of this section. Let β be the restricted root corresponding to e_r. Note that β is just the restriction of $\alpha_r \in \mathbf{h}^*$ to $\mathbf{a}^*$. Furthermore, $(\mathrm{ad}\, a)[e_r, \Theta(f_r)] = 2\beta(a)[e_r, \Theta(f_r)]$ for all $a \in \mathbf{a}$. In particular, the restricted root system Σ contains both β and 2β. Thus Σ is nonreduced and hence must be of type BC. One can choose the positive roots of Σ so that each α_j restricted to $\mathbf{a}^*$ is either zero or a simple positive root in Σ. Furthermore, α_r and α_j restrict to the same root if and only if $j = r$ or $j = p(r)$. It follows from [Kn, Chapter II, Section 8] that there is exactly one positive simple root in Σ such that twice this root is also in Σ. Hence r and $p(r)$ are the only values of j such that $(\alpha_j, \Theta(\alpha_j)) \neq 0$.

Let c be an element in $A = \mathbb{C}[q, q^{-1}]_{(q-1)}$ that specializes to 1 as q goes to 1. Define the $\mathbb{C}$ algebra automorphism $\tilde{\theta}_c$ of U by requiring that

$$\tilde{\theta}_c(y_r) = c^{-1}\tilde{\theta}(y_r), \quad \tilde{\theta}_c(x_r) = c\tilde{\theta}(x_r),$$

and that $\tilde{\theta}_c$ agrees with $\tilde{\theta}$ on all other generators of U. Note that $\tilde{\theta}_c$ is also a $\mathbb{C}$ algebra automorphism of U which specializes to θ and restricts to $\tilde{\theta}$ on $\mathcal{M}T_\Theta$. Define $B_{\tilde{\theta}_c}$ in the same way as $B_{\tilde{\theta}}$ using $\tilde{\theta}_c$ instead of $\tilde{\theta}$. Thus $B_{\tilde{\theta}_c}$ is generated by $\mathcal{M}$, T_Θ, and elements $B_i^c = y_i t_i + \tilde{\theta}_c(y_i)t_i$ for $\alpha_i \notin \pi_\Theta$. Moreover, $B_i^c = B_i$ for $i \neq r$. Since $\tilde{\theta}_c(y_i)$ is a scalar multiple of $\tilde{\theta}(y_i)$ for all i, the proof of Theorem 7.2 also works for $B_{\tilde{\theta}_c}$. Hence $B_{\tilde{\theta}_c}$ is a left coideal subalgebra of U. Consider a quantum Serre relation $F_{ij}(y_i, y_j)$ where either i or j equals r. Note that if $\Theta(\alpha_i) = -\alpha_j$ then, by the assumptions on r, $\{i, j\} = \{r, p(r)\}$ and $(\alpha_i, \alpha_j) \neq 0$. Thus as in the proof of Lemma 7.3, $(\pi_{0,0} \circ P_{ij})(F_{ij}(B_i^c, B_j^c)) \neq 0$ whenever $i \neq j$. Hence the arguments for $B_{\tilde{\theta}}$ used to prove Lemma 7.3 and Theorem 7.4 work for $B_{\tilde{\theta}_c}$ as well. In particular, $B_{\tilde{\theta}_c}$ satisfies conditions (7.23), (7.24), and (7.25).

Note that $B_{\tilde{\theta}_c}$ is not isomorphic to $B_{\tilde{\theta}}$ via a Hopf algebra automorphism of U for $c \neq 1$. It appears unlikely in general that two such algebras are isomorphic using just an algebra isomorphism. It should be noted that the existence of this one parameter family of analogs is implicit in the proof of [L2, Theorem 5.8]. However, it was mistakenly concluded in the paragraph directly preceding [L2, Theorem 5.8] that all the analogs of Variation 1 were isomorphic to $B_{\tilde{\theta}}$ via a Hopf algebra automorphism.

In the general semisimple case, the one parameter c is replaced by a multiparameter $\mathbf{c}$. In particular, each parameter corresponds to a pair of roots $\alpha_{i_j}, \alpha_{p(i_j)}$ such that $(\alpha_{i_j}, \Theta(\alpha_{i_j})) \neq 0$. The automorphism $\tilde{\theta}_{\mathbf{c}}$ is defined in a similar fashion to $\tilde{\theta}_c$. Let $[\Theta]$ be the set of automorphisms of the form $\tilde{\theta}_{\mathbf{c}}$. Following the convention in [L3], we refer to $B_{\tilde{\theta}'}$, $\tilde{\theta}' \in [\Theta]$ as a standard analog of $U(\mathbf{g}^\theta)$.

Variation 2. Let $\mathcal{S}_1$ be the subset of $\pi - \pi_\Theta$ consisting of those roots α_i such that $\Theta(\alpha_i) = -\alpha_i$. Let $\mathcal{S}$ be the subset of $\mathcal{S}_1$ such that if $\alpha_i \in \mathcal{S}$ and $\alpha_j \in \mathcal{S}_1$ then $2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ is even. Let $\mathbf{S}$ be the set of n tuples $\mathbf{s} = (s_1, \ldots, s_n)$ such that each s_i is in $A = \mathbf{C}[q, q^{-1}]_{(q-1)}$ and $s_i \neq 0$ implies $\alpha_i \in \mathcal{S}$. Given $\hat{\theta} \in [\Theta]$, let $B_{\hat{\theta}, \mathbf{s}}$ be the subalgebra of U generated by $T_\Theta, \mathcal{M}$, the B_i for $\alpha_i \in \pi - \mathcal{S}$, and the $B_{i,\mathbf{s}}$ defined by

$$B_{i,\mathbf{s}} = y_i t_i + q^{-(\alpha_i, \alpha_i)} x_i + s_i t_i$$

for $\alpha_i \in \mathcal{S}$. In particular, when the entries of $\mathbf{s}$ are all zero, $B_{i,\mathbf{s}}$ is just equal to B_i.

Note that

$$\Delta(B_{i,\mathbf{s}}) = t_i \otimes B_i + (y_i t_i + q^{-(\alpha_i, \alpha_i)} x_i) \otimes 1.$$

Thus by the same arguments as in Theorem 7.2 , $B_{\hat{\theta}, \mathbf{s}}$ is a left coideal subalgebra of U. Note that if $\tau(\lambda) \in T_\Theta$ and $\alpha_i \in \mathcal{S}$ then $(\lambda, \alpha_i) = 0$. It follows that $a B_{i,\mathbf{s}} = B_{i,\mathbf{s}} a$ for all $\alpha_i \in \mathcal{S}$ and $a \in \mathcal{M} T_\Theta$. Recall the notation of Lemma 7.3. To show that $B_{\hat{\theta}, \mathbf{s}}$ satisfies the conditions (7.23), (7.24), and (7.25), it suffices to check for all i, j, $i \neq j$, that $(\pi_{0,0} \circ P_{ij})(F_{ij}(B_{i,\mathbf{s}}, B_{j,\mathbf{s}})) = 0$. A lengthy but routine computation shows that this holds exactly when the n tuple $\mathbf{s}$ is in $\mathbf{S}$.

Following the convention in [L3], the $B_{\hat{\theta}, \mathbf{s}}$, $\hat{\theta} \in [\Theta]$ are called nonstandard analogs of $U(\mathbf{g}^\theta)$. A nonstandard analog $B_{\hat{\theta}, \mathbf{s}}$ is not isomorphic to a standard analog using a Hopf algebra automorphism of U. However, ([L2, Lemma 5.7]) $B_{\hat{\theta}, \mathbf{s}}$ is isomorphic as an algebra to $B_{\hat{\theta}}$.

Nonstandard analogs were first observed (to the suprise of the author) in [L2, Section 5]. In [L3, Section 2], nonstandard analogs were claimed to exist when $\mathbf{S}$ is defined using the larger set $\mathcal{S}_1$ instead of $\mathcal{S}$. (See in particular the definition of $\mathbf{S}$ given following [L3, (2.11) and Theorem 2.1].) Our analysis in Variation 2 corrects this point.

We are now ready to show that the only possible subalgebras of U which satisfy (7.23), (7.24), and (7.25) are our standard and nonstandard analogs associated

to an automorphism in $[\Theta]$. In particular, we give a new proof of [L2, Theorem 5.8] using the approach and results of Section 4. Note that when the restricted roots Σ associated to the involution θ do not contain a component of type BC, then all the analogs described below are isomorphic to each other as algebras. This is precisely what Theorem 5.8 in [L2] states. On the other hand, by the discussion of Variation 1, if Σ contains m components of type BC, then there is an m parameter family of analogs up to algebra isomorphism.

THEOREM 7.5. *A subalgebra B of U satisfies* (7.23), (7.24), *and* (7.25) *if and only if B is isomorphic as an algebra to $B_{\hat{\theta}}$ for some $\hat{\theta} \in [\Theta]$. In particular, B is isomorphic to a standard or nonstandard analog of $U(\mathbf{g}^{\theta})$ corresponding to an element $\hat{\theta}$ in $[\Theta]$ and an element $\mathbf{s}$ in $\mathcal{S}$ via a Hopf algebra automorphism of U.*

PROOF. We use the notation of the second filtration introduced in Section 4. Let B be a subalgebra of U which satisfies (7.23), (7.24), and (7.25). The proof of this theorem has three steps:

(i) $B \cap T = T_{\Theta}$
(ii) $B \cap U^{+} = \mathcal{M}^{+}$
(iii) $\mathrm{gr}_{\mathcal{G}} B \cap G^{-} = G^{-}$.

More precisely, we first prove that $B \cap T$ is a subgroup of T_{Θ} and $B \cap U^{o}U^{+}$ is a coideal subalgebra of $\mathcal{M}^{+}T_{\Theta}$. We then use the second filtration introduced in Section 4 to analyze $\mathrm{gr}_{\mathcal{G}} B \cap G^{-}$ and thus prove (iii). This information is then used to show that $B \cap U^{+}U^{o}$ specializes to $U(\mathbf{g}^{\theta}) \cap U(\mathbf{n}^{+} + \mathbf{h})$. Next we obtain (i) and (ii). The last part of the proof takes a closer look at the generators of B whose tip is in G^{-} and show they are of the desired form. The details follow.

Consider the set $B \cap T$. By (7.24), $B \cap T$ is a subset of T_{Θ}. Hence $B \cap T = B \cap T_{\Theta}$. Note that any element of B can be written as a direct sum of weight vectors with respect to $B \cap T$. Hence by (7.25), we may assume that $B \cap T_{\Theta}$ is a group. Condition (7.23) and Lemma 4.2 ensure that $B \cap U^{o}$ is the group algebra generated by $B \cap T_{\Theta}$. Since T_{Θ} is free abelian of finite rank, $B \cap T_{\Theta}$ is free abelian of rank at most the rank of T_{Θ}.

Consider the coideal subalgebra $B \cap U^{+}U^{o}$ of B. We show that $B \cap U^{+}U^{o}$ is a subalgebra of $\mathcal{M}^{+}T_{\Theta}$. By Lemma 1.3, $B \cap U^{o}U^{+}$ is a direct sum of the vector spaces $B \cap G^{+}\tau(\mu)$, where $\tau(\mu) \in T$. Suppose that $c \in B \cap G^{+}\tau(\mu)$. Choose γ maximal with respect to the standard partial ordering on $Q^{+}(\pi)$ so that $\pi_{0,\gamma}(c) \neq 0$ and $\gamma \in Q^{+}(\pi_{\Theta})$. Then by (4.7),

$$\pi_{0,\gamma}(c) \in G_{\gamma}^{+}\tau(\mu) \otimes Y$$

where $Y \in \tau(\mu - \gamma) + \sum_{\gamma' > \gamma} G_{\gamma'-\gamma}^{+}\tau(\mu - \gamma)$. Since B is a coideal, Y is an element of B. Rescaling if necessary, we may assume that Y is in $\hat{B} - (q-1)\hat{B}$. Hence Y specializes to a nonzero element in $U(\mathbf{g}^{\theta})$. The choice of γ implies that $\gamma' - \gamma \notin Q^{+}(\pi_{\Theta})$ for all γ' which appear in the definition of Y. Hence,

$Y \in \tau(\mu - \gamma) + (q-1)\sum_{\gamma' > \gamma} \hat{G}^+_{\gamma'-\gamma}\tau(\mu - \gamma)$. But then $(q-1)^{-1}(Y-1)$ is also in $\hat{B}$ and thus specializes to an element of $U(\mathbf{g}^\theta)$. This forces $\tau(\mu - \gamma)$, and thus $\tau(\mu)$, to be in T_Θ. Now consider λ maximal such that $\pi_{0,\lambda}(c) \neq 0$. Then by (4.7), $\tau(\mu - \lambda) \in T_\Theta$. Hence $\lambda \in Q^+(\pi_\Theta)$. Note that if $\lambda' \in Q^+(\pi)$ and $\lambda' < \lambda$ then λ' is also in $Q^+(\pi_\Theta)$. It follows that c is a sum of weight vectors with weights in $Q^+(\pi_\Theta)$. In particular, $c \in \mathcal{M}^+ T_\Theta$ and $B \cap U^o U^+$ is a subalgebra of $\mathcal{M}^+ T_\Theta$.

We next analyze the part of B whose top degree terms are in G^-. To do this, we introduce the left B module B/N where N is the left ideal $B(B \cap (U^+ U^o)_+)$ of B. (Here $(U^+ U^o)_+$ is equal to the augmentation ideal of $U^+ U^o$.) The filtration $\mathcal{G}$ on B induces a filtration which we also denote by $\mathcal{G}$ on B/N which makes $\mathrm{gr}_\mathcal{G} B/N$ into a $\mathrm{gr}_\mathcal{G} B$ module. By Theorem 4.9, the only important contributions to this graded module occur in bidegree $(m,0)$ for $m \geq 0$. In particular, $\mathrm{gr}_\mathcal{G} B/N$ is spanned by elements $b+N$ where $b \in B$ and $\mathrm{tip}(b) \in G^-$. Note that the subspace of G^- of elements of bidegree less than or equal to $(m,0)$ is finite dimensional. Thus the filtration on B/N is a finite discrete filtration. Moreover, $\mathrm{gr}_\mathcal{G} B$ is finitely generated by the image of the generators of B described in Corollary 4.10. Hence we have equality of Gelfand Kirillov dimension: $\mathrm{GKdim}\, \mathrm{gr}_\mathcal{G}(B/N) = \mathrm{GKdim}\, B/N$ ([KL, Prop. 6.6]). Now $\mathrm{gr}_\mathcal{G} B/N$ identifies with $\mathrm{gr}_\mathcal{G}(B) \cap G^-$ as a left $\mathrm{gr}_\mathcal{G}(B) \cap G^-$ module. It is straightforward to check that the GK dimension of $\mathrm{gr}_\mathcal{G} B/N$ as a $\mathrm{gr}_\mathcal{G} B$ module is equal to the GK dimension of $\mathrm{gr}_\mathcal{G} B/N$ as a $\mathrm{gr}_\mathcal{G}(B) \cap G^-$ module. Hence, the form of the generators of B given in Corollary 4.10 implies that $\mathrm{GKdim}\, \mathrm{gr}_\mathcal{G} B/N \leq \dim \mathbf{n}^-$.

Let $\mathbf{r}$ be a Lie subalgebra of $\mathbf{g}^\theta$. A standard argument similar to the argument in the previous paragraph yields that the $U(\mathbf{g}^\theta)$ module $U(\mathbf{g}^\theta)/(U(\mathbf{g}^\theta)\mathbf{r})$ has GK dimension equal to $\dim \mathbf{g}^\theta - \dim \mathbf{r}$. (This follows for example from [D, Proposition 2.2.7].) Consider the $\hat{B}$ module $\hat{B}/\hat{N}$. Write $\bar{N}$ for the specialization of N at $q = 1$. By Theorem 4.1, $B \cap (U^+ U^o)$ specializes to the enveloping algebra of a Lie subalgebra, say $\mathbf{s}$, of $\mathbf{g}^\theta$. Note that $\bar{N} = U(\mathbf{g}^\theta)\mathbf{s}$. Now $B \cap U^+ U^o \subset \mathcal{M}^+ T_\Theta$. Hence $\mathbf{s}$ is a Lie subalgebra of $\mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h})$. The map which sends each $b + \hat{N}$ in $\hat{B}/\hat{N}$ to $\bar{b} + \bar{N}$ in $U(\mathbf{g}^\theta)/\bar{N}$ allows us to specialize the left $\hat{B}$ module $\hat{B}/\hat{N}$ to the $U(\mathbf{g}^\theta)$ module $U(\mathbf{g}^\theta)/\bar{N}$ at $q = 1$. We can choose generating sets for B and B/N which specialize to generating sets of $U(\mathbf{g}^\theta)$ and $U(\mathbf{g}^\theta)/\bar{N}$ respectively. Hence $\mathrm{GKdim}\, B/N \geq \mathrm{GKdim}\, U(\mathbf{g}^\theta)/\bar{N}$. Note that

$$\mathrm{GKdim}\, U(\mathbf{g}^\theta)/\bar{N} = \dim \mathbf{g}^\theta - \dim \mathbf{s} \geq \dim \mathbf{g}^\theta - \dim(\mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h})) = \dim \mathbf{n}^-.$$

By the previous paragraph, this inequality is an equality. Hence

$$\mathrm{GKdim}\, U(\mathbf{g}^\theta)/\bar{N} = \mathrm{GKdim}\, \mathbf{n}^- = \mathrm{GKdim}\, G^-.$$

Moreover $\dim \mathbf{s} = \dim(\mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h}))$. Since $\mathbf{s}$ is a subalgebra of $\mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h})$, it follows that $\mathbf{s} = \mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h})$. Thus $B \cap U^+ U^o$ specializes to $U(\mathbf{m}^+ + (\mathbf{g}^\theta \cap \mathbf{h}))$.

Recall the set Δ' of Corollary 4.10 used to define the generators of B whose top degree term is in G^-. The description of the generators of B in Corollary 4.10 implies that $\mathrm{GKdim}\, B/N$ is equal to the number of elements in Δ'. Since the

number of elements in Δ^+ is just the dimension of $\mathbf{n}^-$, it follows that $\Delta' = \Delta^+$. Hence by Corollary 4.10, B contains elements $y_i t_i + b_i$, $1 \leq i \leq n$, where b_i is in $U^+ U^o$. It follows that $\mathrm{tip}(B) \cap G^- = G^-$. This proves (iii).

Let N' be the left ideal of $B \cap U^+ U^o$ generated by the augmentation ideal of $B \cap U^o$. We can analyze the left $B \cap U^+ U^o$ module $(B \cap U^+ U^o)/N'$ in a similar fashion to the analysis of B/N. It follows that $B \cap U^+ U^o = B \cap \mathcal{M}^+ T_\Theta$ contains elements $x_i + c_i \in \hat{B}$ for each $\alpha_i \in \pi_\Theta$. Furthermore, $c_i \in U^o$ and $B \cap U^o$ specializes to $U(\mathbf{g}^\theta \cap \mathbf{h})$. Now $B \cap U^o$ is just the group algebra generated by $B \cap T_\Theta$. Therefore, rank $B \cap T_\Theta =$ rank T_Θ. Hence we can find generators of T_Θ such that a power of each generator lies in B. This in turn implies that B can be written as a direct sum of T_Θ weight spaces. By the maximality condition (7.25) of B, we obtain $B \cap T_\Theta = T_\Theta$. This completes the proof of step (i).

Since $T_\Theta \subset B$, any element in $U^+ U^o \cap B = \mathcal{M}^+ T_\Theta \cap B$ is a sum of T_Θ weight vectors contained in B. Thus $x_i + c_i \in B$ implies $x_i \in B$. In particular B contains x_i for all $\alpha_i \in \pi_\Theta$. Hence $B \cap U^+ U^o = \mathcal{M}^+ T_\Theta$ and (ii) follows.

Fix i and consider again the element $y_i t_i + b_i$ in B where $b_i \in U^+ U^o$. Replacing b_i by another element in $U^+ U^o$ if necessary, we may assume that $y_i t_i + b_i$ is a weight vector for the action of T_Θ. By Lemma 1.3, we may further assume that $b_i \in G^+ t_i$. First consider the case when $\alpha_i \in \pi_\Theta$. Choose β maximal with respect to the standard partial ordering on $Q(\pi)$ such that $\pi_{0,\beta}(b_i) \neq 0$. By (4.7), $(\pi_{0,\beta} \otimes \mathrm{Id}) \Delta(y_i t_i + b_i)$ is a nonzero element of $G_\beta^+ t_i \otimes \tau(-\beta) t_i$. Hence $\tau(-\beta) t_i \in T_\Theta$ and $\beta \in Q^+(\pi_\Theta)$. If $0 < \gamma < \beta$, then γ is also in $Q^+(\pi_\Theta)$. Thus $\mathrm{supp}(b_i)$ is a subset of $\{0\} \times Q^+(\pi_\Theta)$. This forces b_i to be an element of $\mathcal{M}^+ T_\Theta$ and so $y_i t_i \in B$.

Now assume that $\alpha_i \notin \pi_\Theta$. Choose β such that $[0, \beta] \in \max(b)$. Then by (4.11), $(\pi_{0,\beta} \otimes \mathrm{Id}) \Delta(y_i t_i + b_i)$ is a nonzero element of $G_\beta^+ t_i \otimes \tau(-\beta) t_i$. In particular, $\tau(-\beta) t_i = \tau(-\beta + \alpha_i)$ is in T_Θ. Since $\beta \in Q^+(\pi)$, it follows that $\beta \in \alpha_i + Q^+(\pi_\Theta)$ or $\beta \in \alpha_{p(i)} + Q^+(\pi_\Theta)$. However, β must also be of the same T_Θ weight as $-\alpha_i$. The only possibility is $\beta = \Theta(-\alpha_i)$. By the uniqueness property of the $Y_{I,j}$ and $X_{I,j}$ discussed in Section 6 (see (6.3) and the following discussion), the β weight term is a scalar multiple of $\tilde{\theta}(y_i) t_i$. Indeed this is necessary in order for $\Delta(y_i t_i + b_i)$ to be an element of $U \otimes B$. Therefore $b_i = c\tilde{\theta}(y_i) t_i + d t_i$ for some scalar c and element $d \in G^+$ of bidegree less than $\mathrm{bideg}(\tilde{\theta}(y_i) t_i)$. By (7.23),

$$\Delta(y_i t_i + b_i) \in t_i \otimes (y_i t_i + b_i) + U \otimes B.$$

By (1.8), (1.10), and (7.14), it follows that

$$\Delta(d t_i) \in t_i \otimes d t_i + U \otimes \mathcal{M}^+ T_\Theta.$$

Since $\alpha_i \notin \pi_\Theta$, this forces $d t_i$ to be a scalar multiple of t_i. Hence, up to a Hopf algebra automorphism of U, the only possibility for B is one of the standard or nonstandard analogs of $U(\mathbf{g}^\theta)$. $\qquad\square$

Let us return for now to our first analog $B_{\tilde{\theta}}$. Recall the definition of the anti-automorphism κ. One checks that $\kappa((\mathrm{ad}_r x_j)b) = -((\mathrm{ad}_r y_j)\kappa(b))$ for any $b \in U$ and $1 \leq j \leq n$. Recall that $m(i) = m_1 + \cdots + m_r$. Hence

$$\kappa[(\mathrm{ad}_r x_{i_1}^{(m_1)} \cdots x_{i_r}^{(m_r)})t_{p(i)}^{-1} x_{p(i)}] = (-1)^{m(i)}(\mathrm{ad}_r y_{i_1}^{(m_1)} \cdots y_{i_r}^{(m_r)})y_{p(i)}.$$

A straightforward $U_q(\mathbf{sl}\,2)$ computation as in the classical case (see (7.8)) yields

$$(\mathrm{ad}_r y_{i_1}^{(m_1)} \cdots y_{i_r}^{(m_r)})(\mathrm{ad}_r x_{i_r}^{(m_r)} \cdots x_{i_1}^{(m_1)})t_i^{-1} x_i = t_i^{-1} x_i.$$

Set $y_j \cdot bt_{p(i)} = bt_{p(i)} q^{(\alpha_{p(i)}, \alpha_j)} y_j - y_j t_j bt_{p(i)} t_j^{-1}$ for any $b \in U$ and $1 \leq i, j \leq n$. Note that $y_j \cdot bt_{p(i)} = ((\mathrm{ad}_r y_j)b)t_{p(i)}$. Recall the definition of π^* immediately following (7.5). We have

$$(-1)^{m(i)}[y_{i_1}^{(m_1)} \cdots y_{i_r}^{(m_r)} \cdot B_{p(i)}]t_{p(i)}^{-1} t_i$$
$$= ([(-1)^{m(i)}(\mathrm{ad}_r y_{i_1}^{(m_1)} \cdots y_{i_r}^{(m_r)})y_{p(i)}]t_i + t_i^{-1} x_i t_i$$
$$= \kappa(\tilde{\theta}(y_i)t_i) + q^{-(\alpha_i, \alpha_i)}\kappa(y_i t_i)$$

is an element of $B_{\tilde{\theta}}$ for each $\alpha_i \in \pi^*$. A similar argument shows that

$$\kappa(\tilde{\theta}(y_{p(i)})t_{p(i)}) + q^{-(\alpha_i, \alpha_i)}\kappa(y_{p(i)} t_{p(i)})$$

is also in $B_{\tilde{\theta}}$ for each $\alpha_i \in \pi^*$. Thus one can find a Hopf algebra automorphism Υ in $\mathcal{H}_{\mathbb{R}}$ such that Υ restricts to the identity on $\mathcal{M}$ and T_{Θ} and $\Upsilon(B)$ contains $\kappa(\Upsilon(B_i))$ for each $\alpha_i \notin \pi_{\Theta}$. Furthermore, one can show that $\Upsilon(B)$ is generated by $\mathcal{M}$, T_{Θ}, and the $\kappa(\Upsilon(B_i))$, $\alpha_i \notin \pi_{\Theta}$. Now $\kappa(\Upsilon(\mathcal{M})) = \mathcal{M}$ and $\kappa(\Upsilon(T_{\Theta})) = T_{\Theta}$. It follows that $\kappa(\Upsilon(B)) = \Upsilon(B)$. Hence the results of Section 2 hold for B.

The same argument works for analogs of Variations 1 and 2 provided that all entries of the tuples involved are from $\mathbb{R}(q)$. In particular, let $[\Theta]_r$ be the set $\{\theta_{\mathbf{b}} |$ all entries of $\mathbf{b}$ are in $\mathbb{R}(q)\}$. We refer to analogs of the form $B_{\theta_{\mathbf{b}}, \mathbf{s}}$ for $\theta_{\mathbf{b}} \in [\Theta]_r$ and all entries of $\mathbf{s}$ are in $\mathbb{R}(q)$ as real analogs of $U(\mathbf{g}^{\theta})$. Given $\theta_{\mathbf{b}} \in [\Theta]_r$, one can find $\Upsilon \in \mathcal{H}_{\mathbb{R}}$ such that $\Upsilon^{-1} \kappa \Upsilon(B_{\theta_{\mathbf{b}}}) = B_{\theta_{\mathbf{b}}}$. Furthermore, for any $\mathbf{s}$ such that all of its entries are in $\mathbb{R}(q)$, we also have that $\Upsilon^{-1} \kappa \Upsilon(B_{\theta_{\mathbf{b}}, \mathbf{s}}) = B_{\theta_{\mathbf{b}}, \mathbf{s}}$. Hence, we may apply results of Section 2 to all real analogs of $U(\mathbf{g}^{\Theta})$.

Consider a real analog B of $U(\mathbf{g}^{\theta})$. Given a U module M, set $X(M)$ equal to the sum of all the finite-dimensional unitary B submodules of M. The next result on basic Harish-Chandra modules associated to the pair U, B follows from Section 2.

THEOREM 7.6. *Let B be a real analog of $U(\mathbf{g}^{\theta})$ and let M be a U module. Then any finite-dimensional U module is a B unitary module and a Harish-Chandra module for the pair U, B. Furthermore both $F(U)$ and $X(M)$ are Harish-Chandra modules for the pair U, B.*

We continue the assumption that B is a real analog of $U(\mathbf{g}^{\theta})$. Using the approach of Section 3, we can define the quantum homogeneous space associated to B. The left invariants $R_q[G]_l^B$ are often referred to as $R_q[G/K]$ (or $\mathcal{A}_q[G/K]$) in the

literature (see for example [NS,(2.5)]). Here K can be thought of merely as a symbol or as the complexification of the compact Lie group in G with Lie algebra $\mathbf{g}^\theta$. Thus the homogeneous space G/K is a symmetric space. The notation $R_q[G/K]$ suggests that the right B invariants of $R_q[G]$ is the quantum analog of the ring of regular functions on G/K. In [L3], it is shown that B is a "good" analog of $U(\mathbf{g}^\theta)$ for constructing quantum symmetric spaces in the sense of [Di, end of Section 3]. In particular, $R_q[G/K]$ has the same left U module structure as its classical counterpart (see Theorem 7.8 below). We summarize this and related results here. A good survey on how to construct quantum symmetric spaces which includes a description of the classical situation is [Di]. For further information about classical symmetric spaces, the reader is referred to [He1] and [He2].

A finite-dimensional U module V is called a spherical module for B if the space of invariants V^B has dimension 1. Recall the notion of Cartan subspace and restricted root system introduced at the beginning of this section. Let $\mathbf{a}$ be the Cartan subspace $\{x \in \mathbf{h} | \theta(x) = -x\}$ and let Σ be the associated restricted root system. Let P_Θ^+ be the subset of $P^+(\pi)$ containing those λ such that

(i) $(\lambda, \beta) = 0$ for all $\beta \in Q(\pi)$ such that $\Theta(\beta) = \beta$;
(ii) the restriction $\tilde{\lambda}$ of λ to $\mathbf{a}^*$ satisfies $(\tilde{\lambda}, \beta)/(\beta, \beta)$ is an integer for every restricted root β.

The set P_Θ^+ is exactly the set of dominant integral weights such that the corresponding finite-dimensional simple $\mathbf{g}$ module is spherical ([Kn, Theorem 8.49]). By [L3, Theorem 4.2 and Theorem 4.3] we have the same classification in the quantum case.

THEOREM 7.7. *Let $L(\lambda)$ be a finite-dimensional U module with highest weight λ up to some possible roots of unity. Then*

$$\dim L(\lambda)^B \le 1. \tag{i}$$

Moreover,

$$\dim L(\lambda)^B = 1 \text{ if and only if } \lambda \in P_\Theta^+. \tag{ii}$$

SKETCH OF PROOF. (See [L3] for full details.) Let v_λ denote the highest weight generating vector of $L(\lambda)$. Recall that for each $y \in G^-$ there exists a $b \in B$ such that $b = y+$ higher weight terms. Now $L(\lambda)$ is spanned by weight vectors of the form yv_λ where $y \in G^-$. Hence $\dim L(\lambda)/B^+v_\lambda \le 1$ where B^+ is the augmentation ideal of B. Statement (i) follows from the fact that $B^+v_\lambda \cap L(\lambda)^B$ is empty. A careful analysis using the form of the generators of B further shows that $v_\lambda \in B^+v_\lambda$ if and only if $\lambda \notin P_\Theta^+$. This in turn implies (ii). The argument turns out to be much more delicate when B is a nonstandard analog. $\square$

Theorem 7.7, the Peter–Weyl decomposition of $R_q[G]$, (3.1), and (3.4) imply the following characterization of $R_q[G]_l^B$ as a right U module.

THEOREM 7.8. *There is an isomorphism of right U modules*

$$R_q[G]_l^B \cong \bigoplus_{\lambda \in P_\Theta^+} L(\lambda)^*$$

There is an analogous statement for the right B invariants of $R_q[G]$. One can also describe the B bi-invariants in a nice way. Identifying $L(\lambda)$ with a subspace of $R_q[G]_l^B$, set $\mathcal{H}(\lambda) = R_q[G]_{bi}^B \cap L(\lambda)$. Note that $\mathcal{H}(\lambda)$ is a trivial left and right B module. Moreover, by Theorem 7.7, $\mathcal{H}(\lambda)$ is one-dimensional if $\lambda \in P_\Theta^+$ and zero otherwise. The following direct sum decomposition into trivial one-dimensional B bimodules is thus an immediate consequence of Theorem 7.8.

$$R_q[G]_{bi}^B \cong \bigoplus_{\lambda \in P_\Theta^+} \mathcal{H}(\lambda).$$

Let $\mathcal{A}$ be the subgroup of T consisting of those elements $\tau(\lambda)$ such that $\Theta(\lambda) = -\lambda$. Thus $\mathcal{A}$ can be thought of as a quantum version of $\mathbf{a}$. Let W_Θ denote the Weyl group associated to the restricted root system Σ. Since $\Sigma \subset \mathbf{a}^*$, $\mathbf{a}$ and hence $\mathcal{A}$ inherit an action of W_Θ. The author has recently shown that, $R_q[G]_{bi}^B$ is commutative and moreover is isomorphic to $\mathbb{C}(q)[\mathcal{A}]^{W_\Theta}$. Thus, the $\mathcal{H}(\lambda)$ are natural choices of quantum zonal spherical functions (see [Di, the discussion concerning (3.4)]). In special cases, these quantum zonal spherical functions have been determined to be Macdonald polynomials or other q hypergeometric series (see for example [K], [N], [DN], [NS]). Preliminary work by the author suggests that this should be true in general.

It should be noted that these papers use analogs of $U(\mathbf{g}^\theta)$ whose definition differs from the definition of the $B_{\tilde{\theta}}$ and its variations found in this paper. In [NM], one-sided coideal subalgebras are used. By [L2, Section 6], using Theorem 7.5, these are shown to be examples of the analogs presented here. In other papers, two-sided coideals analogs of $\mathbf{g}^\theta$ are used. The specialization of these two-sided coideals generate a much larger subalgebra than $U(\mathbf{g}^\theta)$. The important object in these papers, used to define quantum symmetric spaces, is the left ideal generated by these two-sided coideals analogs of $\mathbf{g}^\theta$. It seems likely that these left ideals can be shown to be generated by the augmentation ideal of one of the analogs presented here. This is certainly true for the left coideals studied in [K] and also for those in [N] (combine [N, Section 2.4] with [L2, Section 6]).

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