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Hankel Determinants as Fredholm Determinants

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Abstract. Hankel determinants which occur in problems associated with orthogonal polynomials, integrable systems and random matrices are computed asymptotically for weights that are supported in an semi-infinite or infinite interval. The main idea is to turn the determinant computation into a random matrix "linear statistics" type problem where the Coulomb fluid approach can be applied.

1. Introduction

Let w be a weight function supported on L (a subset of $\mathbb{R}$) that has finite moments of all orders

$$\mu_n = \int_L x^n w(x)\,dx.$$

With $w(x)$ we associate the Hankel matrix (μ_{i+j}), where $i, j = 0, \dots, n-1$. The purpose of this paper is the determination of

$$D_n[w] := \det(\mu_{i+j})_{i,j=0}^{n-1}$$

for large n with suitable conditions on w. If L is a single interval, say $[-1, 1]$, then the asymptotic form of such determinants was computed by Szegö [1918] and later by Hirschmann [1966] for quite general w.

Our main result is as follows. Suppose we replace $w(x)$ by a function given in the form $w_0(x)U(x)$ where $w_0(x)$ is the weight $e^{-x}x^{\nu}$. Then for appropriate functions w, the determinants are given asymptotically as $n \to \infty$ by

$$D_n[w] = \exp\big(c_1 n^2 \log n + c_2 n^2 + c_3 n \log n + c_4 n + c_5 n^{1/2} + c_6 \log n + c_7 + o(1)\big) \quad (1)$$

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where

$$
\begin{aligned}
c_1 &= 1, \qquad c_2 = -3/2, \qquad c_3 = \nu, \qquad c_4 = -\nu + \log 2\pi,\\
c_5 &= \frac{2}{\pi}\int_0^\infty \log(U(x^2))\,dx, \qquad c_6 = \nu^2/2 - 1/6,\\
c_7 &= 4/3 \log G(1/2) + (1/3+\nu/2)\log\pi + (\nu/2 - 1/18)\log 2\\
&\qquad - \log G(1+\nu) - (\nu/2)\log U(0) + \frac{1}{2\pi^2}\int_0^\infty xS(x)^2\,dx,\\
S(x) &= \int_0^\infty \cos(xy)\log U(y^2)\,dy,
\end{aligned}
$$

and G is the Barnes function (see Section 2).

In Section 2 we establish an identity relating $D_n[w]$, $D_n[w_0]$, and a certain Fredholm determinant and a description of the Fredholm determinant from a "linear statistics" point of view. A computation of $D_n[w_0]$ is also included. Then in Section 3 the Coulomb fluid approach is used to compute the asymptotics of the Fredholm determinant. This, along with the computation of $D_n[w_0]$ allows us to give a heuristic, Coulomb fluid derivation of the formula. A rigorous proof based on operator theory techniques developed in [Basor 1997; Tracy and Widom 1994] will appear in a forthcoming paper. In Section 3 the Hermite case is also included.

2. Preliminaries

Let $p_i(x)$ be polynomials orthonormal with respect to $w_0(x)$ (the reference weight function) over L

$$\int_L p_i(x)p_j(x)w_0(x)\,dx = \delta_{i,j},$$

with strictly positive leading coefficients. For later convenience we also write $\phi_j = \sqrt{w_0}\,p_j$ as the orthonormal functions. Consider the determinant,

$$\det\left(\int_L p_i(x)p_j(x)w(x)\,dx\right)_{i,j=0}^{n-1}.$$

If $p_i(x) = \sum_{j=0}^{i} c_{ij}x^j$, then

$$
\begin{aligned}
\det\left(\int_L p_ip_j w\,dx\right)_{i,j=0}^{n-1} &= \det\left(\int_L \sum_{k=0}^{i}\sum_{l=0}^{j} c_{ik}c_{jl}x^{k+l}w\,dx\right)_{i,j=0}^{n-1}\\
&= \left(\prod_{i=0}^{n-1} c_{ii}\right)^2 \det\left(\int_L x^{j+k}w\,dx\right)_{j,k=0}^{n-1}\\
&= \left(\prod_{i=0}^{n-1} c_{ii}\right)^2 \det\left(\mu_{j+k}\right)_{j,k=0}^{n-1}. \qquad (2)
\end{aligned}
$$

If $w = w_0$ then the left side of (2) is 1. So,

$$\det\left(\int_L p_i p_j w \, dx\right)_{i,j=0}^{n-1} = \frac{\det\left(\mu_{i+j}\right)_{i,j=0}^{n-1}}{\det\left(\mu_{i+j}^0\right)_{i,j=0}^{n-1}} = \frac{D_n[w]}{D_n[w_0]}, \tag{3}$$

where

$$\mu_i^0 := \int_L x^i w_0 \, dx,$$

are the moments of the reference weight w_0, and

$$D_n[w_0] = \frac{1}{\left(\prod_{i=0}^{n-1} c_{ii}\right)^2}.$$

We now express the left side of (3) as a Fredholm determinant.

$$\begin{aligned}\det\left(\int_L p_i p_j w \, dx\right) &= \det\left(\int_L \phi_i \phi_j \left(1 - \left(1 - \frac{w}{w_0}\right)\right) dx\right) \\ &= \det(\delta_{i,j} - M_{i,j}),\end{aligned}$$

where

$$M_{i,j} := \int_L \phi_i \phi_j (1 - w/w_0) \, dx =: \int_L \phi_i \phi_j F \, dx = \int_L \phi_i \phi_j (1 - U) \, dx.$$

We have the standard expansion

$$-\log\det(\delta_{i,j} - M_{i,j}) = \sum_{p=1}^{\infty} \frac{\operatorname{tr} \mathbf{M}^p}{p},$$

where the matrix $\mathbf{M}$ has elements $M_{i,j}$. To compute $\operatorname{tr}\mathbf{M}^p$, we first look at the simpler case of $p = 3$. We see that $\operatorname{tr}\mathbf{M}^3$ equals

$$\begin{aligned}&\sum_{i,j,k=0}^{n-1} M_{i,j} M_{j,k} M_{k,i} \\ &= \int dX \sum_{i,j,k=0}^{n-1} \phi_i(x_1)\phi_j(x_1)F(x_1)\phi_j(x_2)\phi_k(x_2)F(x_2)\phi_k(x_3)\phi_i(x_3)F(x_3) \\ &= \int dX \, F(x_1)F(x_2)F(x_3) \sum_{i=0}^{n-1} \phi_i(x_3)\phi_i(x_1) \sum_{j=0}^{n-1} \phi_j(x_1)\phi_j(x_2) \sum_{k=0}^{n-1} \phi_k(x_2)\phi_k(x_3) \\ &= \int dX \, K_n(x_1, x_2)K_n(x_2, x_3)K_n(x_3, x_1)F(x_1)F(x_2)F(x_3),\end{aligned}$$

where $\int dX$ stands for $\int_L \cdots \int_L dx_1 \, dx_2 \, dx_3$ and where

$$K_n(x, y) := \sum_{i=0}^{n-1} \phi_i(x)\phi_i(y) = a_n \frac{\phi_n(x)\phi_{n-1}(y) - \phi_n(y)\phi_{n-1}(x)}{x - y}. \tag{4}$$

The last equality of (4) is the Christoffel–Darboux formula [Szegő 1975] and a_n are the off diagonal recurrence coefficients of p_i. The generalization to integer p is obvious and we find,

$$\operatorname{tr}\mathbf{M}^p = \operatorname{tr}(K_nF)^p,$$

where the operator K_nF has kernel $K_n(x,y)F(y)$. So

$$\det(\delta_{i,j} - M_{i,j}) = \det(I - K_nF), \tag{5}$$

and I in (5) has kernel $\delta(x-y)$. We now come to the linear statistics.

If x_i, for $i = 1, \ldots, n$, are random variables with the joint probability density function

$$p(x_1, \ldots, x_n) \propto \prod_{i=1}^{n} w_0(x_i) \prod_{1\le j,k\le n} |x_j - x_k|^2, \quad x_i \in L, \tag{6}$$

then $Q = \sum_{i=1}^{n} f(x_i)$ (the linear statistics) is also a random variable. Consider the generating function of Q, $\langle\exp(-Q)\rangle$, where

$$\langle(\ldots)\rangle := \frac{\int_L \cdots \int_L (\ldots) p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n}{\int_L \cdots \int_L p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n}.$$

Recall the Heine formula [Szegő 1975] for Hankel determinants:

$$\det(\mu_{i+j})_{i,j=0}^{n-1} = \frac{1}{n!} \int_L \cdots \int_L dx_1 \ldots dx_n \prod_{i=1}^{n} w(x_i) \prod_{1\le j,k\le n} |x_j - x_k|^2, \tag{7}$$

and the analogous one with μ_{i+j} replaced by μ^0_{i+j} and w replaced by w_0. If we write $w = \exp(-v)$, $w_0 = \exp(-v_0)$ and $v = v_0 + f$ then

$$D_n[w] = \det(\mu^0_{i+j})_{i,j=0}^{n-1}\langle\exp(-Q)\rangle = D_n[w_0]\langle\exp(-Q)\rangle. \tag{8}$$

In this notation $f(x) = -\log U(x)$. So our strategy is to choose a suitable w_0 for which there is exact result for $D_n[w_0]$, and compute $\langle\exp(-Q)\rangle$ as a Fredholm determinant for n large. In the next section we will use a heuristic method to give an indication how the results for $\langle\exp(-Q)\rangle$ for large n can be found. If we take $w_0(x) = x^\nu \exp(-x)$, $\nu > -1$ and $L = [0, \infty)$ then $p_i(x)$ are the orthonormal Laguerre polynomials. It is well known [Szegő 1975] that

$$c_{ii}^2 = \frac{1}{\Gamma(1+i+\nu)\Gamma(1+i)}.$$

So

$$D_n[w_0] = \prod_{i=0}^{n-1} \Gamma(1+i+\nu)\Gamma(1+i) = \frac{G(n+\nu+1)}{G(\nu+1)} \frac{G(n+1)}{G(1)}, \tag{9}$$

where the Barnes function G [Barnes 1900; Whittaker and Watson 1962] satisfies the functional equation $G(z+1) = \Gamma(z)G(z)$, with the initial condition $G(1) = 1$. The asymptotics of the Barnes function are computed in [Whittaker and Watson 1962] and since $G(1+a+n)$ is asymptotic to

$$n^{(n+a)^2/2-1/12} e^{-3/4n^2-an} (2\pi)^{(n+a)/2} G^{2/3}(1/2)\pi^{1/6} 2^{-1/36}$$

we can directly apply this formula with $a = 0$ and $a = \nu$ to obtain asymptotically

$$D_n[w_0] = \exp\left\{d_1 n^2 \log n + d_2 n^2 + d_3 n \log n + d_4 n + d_5 \log n + d_6 + o(1)\right\} \quad (10)$$

where

$$d_1 = 1, \quad d_2 = -3/2, \quad d_3 = \nu, \quad d_4 = -\nu + \log 2\pi, \quad d_5 = \nu^2/2 - 1/6,$$
$$d_6 = (4/3)\log G(1/2) + (1/3 + \nu/2)\log \pi + (\nu/2 - 1/18)\log 2 - \log G(1+\nu).$$

3. The Coulomb Fluid Method

For suitably chosen f, $\big\langle \exp(-Q)\big\rangle$ for large n was computed in [Chen and Lawrence 1998] starting from the Heine formula. An alternative and shorter derivation is given here. Now if Q is in some sense "small" then by expanding up to Q^2, we have, $\big\langle \exp(-Q)\big\rangle \approx 1 - \langle Q\rangle + \frac{1}{2}\langle Q^2\rangle$. This can be reproduced by expanding

$$\exp\left(-\langle Q\rangle - \frac{\big(\langle Q\rangle^2 - \langle Q^2\rangle\big)}{2}\right),$$

up to $\langle Q^2\rangle$ and $\langle Q\rangle^2$. With the introduction of the microscopic density $\varrho(x) := \sum_{i=1}^{n} \delta(x - x_i)$, one finds

$$\langle Q\rangle = \int_L f(x)\langle \varrho(x)\rangle\, dx,$$
$$\langle Q\rangle^2 - \langle Q^2\rangle = \int_L \int_L f(x)\left(\langle \varrho(x)\rangle\langle \varrho(y)\rangle - \langle \varrho(x)\varrho(y)\rangle\right) f(y)\, dx\, dy.$$

In the Coulomb fluid approach, expected to be valid for large n, we replace $\langle \varrho(x)\rangle$ by the equilibrium density $\sigma(x)$, which is supposed to be supported in a single interval (a, b). It is then a simple exercise to show that the correlation function

$$\big\langle \varrho(x)\big\rangle\big\langle \varrho(y)\big\rangle - \big\langle \varrho(x)\varrho(y)\big\rangle$$

is replaced by

$$\frac{1}{2\pi^2\sqrt{(b-x)(x-a)}}\frac{\partial}{\partial y}\left(\frac{\sqrt{(b-y)(y-a)}}{x-y}\right).$$

Therefore,

$$\big\langle \exp(-Q)\big\rangle \sim \exp(-S_1 - S_2), \quad (11)$$

where

$$S_1 = \frac{1}{4\pi^2}\int_a^b \int_a^b \frac{f(x)}{\sqrt{(b-x)(x-a)}}\frac{\partial}{\partial y}\left(\frac{\sqrt{(b-y)(y-a)}}{x-y}\right) f(y)\, dx\, dy,$$
$$S_2 = \int_a^b \sigma(x) f(x)\, dx.$$

The end points of the interval, a and b, are determined by the normalization condition $\int_a^b \sigma(x)\, dx = n$ and a supplementary condition [Chen and Lawrence 1998].

So the constants $c_i,\ i=1,\dots,6$ and part of c_7 are obtained from the asymptotic expansion of $D_n[w_0]$ while c_5 and the last two terms of c_7 are obtained from the large n behaviour of S_2 and S_1 respectively. For $w_0(x)=x^{\nu}\exp(-x),\ x\geq 0$ it is known that $a=0$, $b=4n+2\nu$ and

$$\sigma(x)=-\nu\delta_{+}(x)+\frac{1}{2\pi}\sqrt{\frac{b-x}{x}},\quad 0\leq x<b.$$

So,

$$\begin{aligned} S_2 &= -\frac{\nu}{2}f(0)+\frac{1}{2\pi}\int_0^b f(x)\sqrt{\frac{b-x}{x}}\,dx \\ &\to \frac{\nu}{2}\log U(0)-\frac{2n^{1/2}}{\pi}\int_0^{\infty}\log U(x^2)\,dx,\quad \text{as } n\to\infty. \end{aligned} \tag{12}$$

As $n\to\infty$, S_1 tends to

$$\frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty}\frac{f(x)}{\sqrt{x}}f(y)\frac{\partial}{\partial y}\frac{\sqrt{y}}{x-y}\,dx\,dy.$$

Changing the integration variables $x=s^2$, $y=t^2$ and noting

$$-\frac{1}{2}\int_{-\infty}^{\infty}|x|\exp[-ixt]\,dx=\frac{1}{t^2},$$

we find

$$\begin{aligned} S_1 &\to -\frac{1}{2\pi^2}\int_0^{\infty}x\left(\int_0^{\infty}\log U(s^2)\cos(xs)ds\right)^2 dx \\ &= -\frac{1}{2\pi^2}\int_0^{\infty}xS(x)^2\,dx. \end{aligned} \tag{13}$$

Therefore (1) follows from (10), (12) and (13).

We can also make use of the above information to determine the recurrence coefficients of the monic polynomials $P_j(x)$, orthogonal with respect to $x^{\nu}\exp(-tx)U(x)$. The parameter t is introduced for later convenience. The recurrence relations reads

$$xP_n(x)=P_{n+1}(x)+\alpha_n(t)P_n(x)+\beta_n(t)P_{n-1}(x).$$

From the basic properties of the Hankel determinant generated by the weight $w(x,t)=x^{\nu}\exp(-tx)U(x)$, one finds

$$\begin{aligned} \alpha_n(t) &= -\frac{d}{dt}\ln\frac{D_{n+1}(t)}{D_n(t)}, \\ \beta_n(t) &= \frac{D_{n+1}(t)D_{n-1}(t)}{\big(D_n(t)\big)^2}, \end{aligned}$$

where $D_n(t) = D_n[w(\cdot, t)]$. With the asymptotics, we find, as $n \to \infty$, that

$$\alpha_n(1) = 2n + \nu + 1 - \left(\frac{1}{\pi}\int_0^\infty \log U(x^2)\,dx\right)n^{-1/2} + o(1)$$
$$\beta_n(1) = n^2 + \nu n - \left(\frac{1}{2\pi}\int_0^\infty \log U(x^2)\,dx\right)n^{1/2} + o(1),$$

are the recurrence coefficients of those monic polynomials orthogonal with respect to $w(x) = x^\nu \exp(-x)U(x)$. As a further application of the asymptotic formula, we study the short noise generating function of an n-channel disordered conductor [Muttalib and Chen 1996] where the f of the linear statistics is

$$f(x) := M \ln\frac{x+z}{x+1}, \quad |z| = 1.$$

As $n \to \infty$, S_1 tends to

$$-M^2 \log\frac{\sqrt{z}+1}{2z^{1/4}},$$

while S_2 tends to

$$\frac{\nu M}{2}\log z + 2\sqrt{n}M(1-\sqrt{z}).$$

This generalises the results of [Muttalib and Chen 1996] to $\nu \neq 0$.

Now suppose w is supported in $(-\infty, \infty)$. We adopt the same strategy to determine the large n behaviour of the associated Hankel determinant:

$$D_n[w] = D_n[w_0]\frac{D_n[w]}{D_n[w_0]},$$

where the "reference" Hankel determinant is generated by the Hermite weight $w_0(x) = \exp(-x^2)$, where $x \in (-\infty, \infty)$. Now $a = -b = -\sqrt{2n}$ and $\sigma(x) = \frac{1}{\pi}\sqrt{x^2 - b^2}$. Thus, as $n \to \infty$,

$$\frac{D_n[u]}{D_n[w_0]} \sim \exp(-S_1 - S_2),$$

where

$$S_1 = -\frac{1}{8\pi^2}\int_{-\infty}^\infty |k|\hat{f}(-k)\hat{f}(k)\,dk$$
$$S_2 = \frac{\sqrt{2n}}{\pi}\int_{-\infty}^\infty f(x)\,dx. \tag{14}$$

Here $\hat{f}(k) = \int_{-\infty}^\infty \exp(ikx)f(x)\,dx$. Equations (14) are essentially those found by Kac and by Akhiezer [Akhiezer 1964]. Therefore the large n behaviour of $D_n[w]$ follows from

$$D_n[w_0] = (2\pi)^{n/2}2^{-n^2/2}G(n+1),$$

the asymptotics of the Barnes function, and Equation (14).

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