Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

An Extremal Property of the Regular Simplex

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Abstract. If C is a convex body in $\mathbb{R}^n$ such that the ellipsoid of minimal volume containing C—the Löwner ellipsoid—is the euclidean ball B_2^n, then the mean width of C is no smaller than the mean width of a regular simplex inscribed in B_2^n.

1. Introduction and Notation

Suppose that C is a convex body in $\mathbb{R}^n$ such that 0 is an interior point of C, then the mean width $w(C)$ is defined by

$$\begin{aligned} w(C) :&= \int_{S^{n-1}} \Bigl(\sup_{y\in C}\langle x,y\rangle - \inf_{y\in C}\langle x,y\rangle \Bigr)\,\sigma(dx) \\ &= 2\int_{S^{n-1}} \sup_{y\in C} |\langle x,y\rangle|\,\sigma(dx) = 2c_n \int_{\mathbb{R}^n} \sup_{y\in C} |\langle x,y\rangle|\,\gamma_n(dx) \end{aligned}$$

where c_n is a constant depending only on the dimension, σ the normalized Haar measure on the sphere S^{n-1} and γ_n the n-dimensional standard gaussian measure. Denoting by C^* the polar of C with respect to 0 and by $\|.\|_C$ the gauge of C, we obtain the well known formula

$$w(C) = 2c_n \int_{\mathbb{R}^n} \|x\|_{C^*}\ \gamma_n(dx) =: 2c_n \ell(C^*).$$

The euclidean ball B_2^n is the Löwner ellipsoid of C if and only if B_n^2 is the John ellipsoid of C^* i.e., the ellipsoid of maximal volume contained in C^*. Hence, in order to prove that the regular simplex has minimal mean width, it is enough to prove that for all convex bodies K whose John ellipsoid is the euclidean ball, we necessarily have $\ell(K) \geq \ell(T)$, i.e., the ℓ-norm of K is bounded from below by the ℓ-norm of the regular simplex T.

The proof of this inequality will follows closely Keith Ball's proof in [B1], where it is shown that for any convex body K there exists an affine image $\widetilde{K}$ of K for which the isoperimetric quotient $\mathrm{Vol}_{n-1}(\partial\widetilde{K})/\mathrm{Vol}_n(\widetilde{K})^{\frac{n-1}{n}}$ is no larger than the isoperimetric quotient of a regular simplex. Franck Barthe [B] proved a reversed inequality: among convex bodies whose Löwner ellipsoid is the euclidean ball the regular simplex has maximal ℓ-norm.

2. The Proof

The first ingredient of the proof is a well-known theorem of F. John [J]:

THEOREM 2.1. *Let K be a convex body in $\mathbb{R}^n$. Then the euclidean ball B_n^2 is the John ellipsoid of K if and only if there exist unit vectors $u_j \in \partial K$, $1 \le j \le m$ and positive numbers c_j such that*

(i) $\sum_{j=1}^m c_j u_j \otimes u_j = id_{\mathbb{R}^n}$ *and*
(ii) $\sum_{j=1}^m c_j u_j = 0$.

The second is an inequality due to Brascamp and Lieb [BL]. We state this inequality in its normalized form, as it was introduced by Ball in [B2].

THEOREM 2.2. *Let u_j, $1 \le j \le m$, be a sequence of unit vectors in $\mathbb{R}^n$ and c_j positive numbers such that $\sum_{j=1}^m c_j u_j \otimes u_j = id_{\mathbb{R}^n}$. Then, for all nonnegative integrable functions $f_j : \mathbb{R} \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\langle x, u_j\rangle)^{c_j}\,dx \le \prod_{j=1}^m \Big(\int f_j\Big)^{c_j}.$$

Equality holds if, for example, the f_j's are identical gaussians or the u_j's form an orthonormal basis.

By John's theorem there exist unit vectors $u_j \in \partial K$ and positive numbers c_j such that

$$\sum_{j=1}^m c_j u_j \otimes u_j = id_{\mathbb{R}^n} \quad \text{and} \quad \sum_{j=1}^m c_j u_j = 0.$$

Putting $v_j := \Big(\sqrt{\frac{n}{n+1}}\,u_j, -\frac{1}{\sqrt{n+1}}\Big) \in \mathbb{R}^{n+1}$ and $d_j = \frac{n+1}{n} c_j$ it is easily checked that

$$\sum_{j=1}^m d_j v_j \otimes v_j = id_{\mathbb{R}^{n+1}} \quad \text{and} \quad \sum_{j=1}^m d_j v_j = -\sqrt{n+1}\,\mathrm{Pr}_{n+1} \tag{1}$$

The first identity implies $\sum d_j \langle z, v_j\rangle^2 = \|z\|_2^2$ and $\sum d_j = n+1$.

For $\alpha \in \mathbb{R}$ let μ be the measure on $\mathbb{R}$ with density

$$\tfrac{1}{\sqrt{2\pi}} \exp(\alpha t\sqrt{n+1} - t^2/2).$$

Then by (1) we obtain

$$\begin{aligned}\gamma_n\otimes\mu\Big(\bigcap\,[v_j\le 0]\Big) &= \iint\prod_{j=1}^m I_{(-\infty,0]}(\langle z,v_j\rangle)\,\gamma_n(dx)\,e^{\alpha t\sqrt{n+1}}\,\gamma_1(dt)\\ &= \int_{\mathbb{R}^{n+1}} I_{(-\infty,0]}(\langle z,v_j\rangle)^{d_j}\exp\Big(-\tfrac12\sum_{j=1}^m d_j\langle z,v_j\rangle^2\Big)\\ &\qquad\times\Big(\tfrac{1}{\sqrt{2\pi}}\Big)^{n+1}\exp\Big(-\alpha\sum_{j=1}^m d_j\langle z,v_j\rangle\Big)\,dz\end{aligned}$$

Putting $f(s)=\frac{1}{\sqrt{2\pi}}e^{-s^2/2-\alpha s}I_{(-\infty,0]}(s)$ we conclude by the Brascamp–Lieb inequality that

$$\begin{aligned}\gamma_n\otimes\mu\Big(\bigcap\,[v_j\le 0]\Big) &= \int\prod_{j=1}^m f(\langle z,v_j\rangle)^{d_j}\,dz\\ &\le\Big(\int f(s)\,ds\Big)^{\sum d_j}=\Big(\int f(s)\,ds\Big)^{n+1},\end{aligned}$$

and equality holds if the vectors v_j form an orthonormal basis in $\mathbb{R}^{n+1}$ i.e., if the vectors u_j span a regular simplex. Thus, denoting by u_j^0, $1\le j\le n+1$, the contact points of a regular simplex T and the euclidean ball and by v_j^0 the corresponding unit vectors in $\mathbb{R}^{n+1}$, the above inequality states that

$$\gamma_n\otimes\mu\Big(\bigcap\,[v_j\le 0]\Big)\le\gamma_n\otimes\mu\Big(\bigcap\,[v_j^0\le 0]\Big). \tag{2}$$

On the other hand

$$\bigcap\,[v_j\le 0]=\Big\{z=(x,t)\in\mathbb{R}^n\times\mathbb{R}: t\ge 0,\ x\in\tfrac{t}{\sqrt{n}}\widetilde{K}\Big\},$$

where $\widetilde{K}:=\bigcap\,[u_j\le 1]\supseteq K$. Hence we get, by Fubini's theorem,

$$\gamma_n\otimes\mu\Big(\bigcap\,[v_j\le 0]\Big)=\tfrac{1}{\sqrt{2\pi}}\int_0^\infty\gamma_n\Big(\tfrac{t}{\sqrt{n}}\widetilde{K}\Big)\,e^{\alpha t\sqrt{n+1}-t^2/2}\,dt.$$

Now, since $K\subseteq\widetilde{K}$, this implies by (2),

$$\tfrac{1}{\sqrt{2\pi}}\int_0^\infty\gamma_n\Big(\tfrac{t}{\sqrt{n}}K\Big)\,e^{\lambda t-t^2/2}\,dt\le\tfrac{1}{\sqrt{2\pi}}\int_0^\infty\gamma_n\Big(\tfrac{t}{\sqrt{n}}T\Big)\,e^{\lambda t-t^2/2}\,dt,$$

and therefore

$$\tfrac{1}{\sqrt{2\pi}}\int_0^\infty\gamma_n\Big(\|.\|_K>\tfrac{t}{\sqrt{n}}\Big)\,e^{\lambda t-t^2/2}\,dt\ge\tfrac{1}{\sqrt{2\pi}}\int_0^\infty\gamma_n\Big(\|.\|_T>\tfrac{t}{\sqrt{n}}\Big)\,e^{\lambda t-t^2/2}\,dt.$$

Multiplying both sides by $e^{-\lambda^2/2}$ and integrating over $\lambda\in\mathbb{R}$ we obtain, by Fubini's theorem,

$$\int_0^\infty\gamma_n\Big(\|.\|_K>\tfrac{t}{\sqrt{n}}\Big)\,dt\ge\int_0^\infty\gamma_n\Big(\|.\|_T>\tfrac{t}{\sqrt{n}}\Big)\,dt$$

from which we readily deduce that $\ell(K) \geq \ell(T)$. More generally we get, for each non negative function φ,

$$\int_0^\infty \gamma_n\left(\|\cdot\|_K > \tfrac{t}{\sqrt{n}}\right) \int_{\mathbb{R}} \varphi(t-x)e^{-x^2/2}\,dx\,dt \geq \int_0^\infty \gamma_n\left(\|\cdot\|_T > \tfrac{t}{\sqrt{n}}\right) \int_{\mathbb{R}} \varphi(t-x)e^{-x^2/2}\,dx\,dt.$$

REMARK. If we restrict the problem to convex and symmetric bodies, then we get an inequality for the distribution function (see [SS]): For all convex symmetric bodies B in $\mathbb{R}^n$ whose John ellipsoid is the euclidean ball we have, for all $t > 0$,

$$\gamma_n(\|\cdot\|_B > t) \geq \gamma_n(\|\cdot\|_\infty > t).$$

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