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Curvature of Nonlocal Markov Generators

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Abstract. Bakry's curvature-dimension condition will be extended to certain nonlocal Markov generators. In particular this gives rise to a possible notion of curvature for graphs.

1. Definition of Curvature

Let (Ω, μ) be a probability space and L a self-adjoint negative but not necessarily bounded operator on $L_2(\mu)$ given by

$$Lf(x) := \int (f(y) - f(x))K(x,y)\,\mu(dy) \tag{1}$$

where K is a non negative symmetric kernel. Obviously L remains unchanged if we change K on the diagonal. By $P_t = e^{tL}$ we denote the continuous contraction semigroup on $L_2(\mu)$ with generator L. We will assume that P_t is ergodic and that there exists an algebra $\mathcal{A} \subseteq \bigcap_n \operatorname{dom} L^n$ of bounded functions which is a form core of L. Then the Beurling–Deny condition implies that P_t is a symmetric Markov semigroup, i.e., P_t preserves positivity and extends to a continuous contraction semigroup on $L_p(\mu)$ for all $1 \le p < \infty$. We will also assume that $\mathcal{A}$ is stable under P_t. On $\mathcal{A} \times \mathcal{A}$ define

$$\Gamma(f,g) := \tfrac{1}{2}(L(fg) - fLg - gLf),$$
$$\Gamma_2(f,g) := \tfrac{1}{2}(L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)).$$

Following D. Bakry and M. Emery [BE, B] we define the curvature of L at the point $x \in \Omega$ by

$$R(L)(x) := \sup\bigl\{r \in \mathbb{R} : \Gamma_2(f,f)(x) \ge r\Gamma(f,f)(x) \text{ for all } f \in \mathcal{A}\bigr\},$$

and say that the curvature of L is bounded from below by R if $R(L)(x) \ge R$ for all $x \in \Omega$, i.e., $\Gamma_2(f,f) \ge R\Gamma(f,f)$ for all $f \in \mathcal{A}$. By the definition of R it is clear that $R(\lambda L) = \lambda R(L)$ for any $\lambda > 0$. Let us say a a word about the motivation for this definition. Assume L is the Laplacian on a Riemannian manifold, then

$\Gamma(f,f) = \|\mathrm{grad}\, f\|^2$ and $\Gamma_2(f,f) = \mathrm{Ric}\,(\mathrm{grad}\, f, \mathrm{grad}\, f) + \|Hf\|^2$, where Ric denotes the Ricci curvature and Hf the Hessian of f. Thus $R(L)$ coincides with the biggest lower bound for the Ricci curvature.

Given Γ we can define a metric d_Γ on Ω by

$$d_\Gamma(x,y) := \sup\big\{|f(x)-f(y)| : \Gamma(f,f) \le 1\big\}.$$

If L is the Laplacian on a Riemannian manifold this is just the metric induced by the Riemannian metric.

From now on we will assume that for all $y \in \Omega$ the function $x \mapsto \sqrt{K(x,y)}$ belongs to the algebra $\mathcal{A}$. In case L is given by (1) we obtain, by putting $\nabla_y f(x) := f(y) - f(x)$ and $d(x) := \int K(x,y)\,\mu(dy)$,

$$\Gamma(f,g)(x) = \tfrac{1}{2}\int \nabla_y f(x) \nabla_y g(x) K(x,y)\,\mu(dy),$$

$$\begin{aligned}\Gamma_2(f,f)(x) = \tfrac{1}{4}\int \nabla_y f(x)^2 \Big(\int K(x,z)K(y,z)\,\mu(dz) + K(x,y)(3d(y)-d(x))\Big)\,\mu(dy)\\ - \tfrac{1}{2}\iint \nabla_y f(x)\nabla_z f(x) K(x,y)(2K(y,z) - K(x,z))\,\mu(dy)\,\mu(dz)\end{aligned}$$

For simplicity of notation let us write $\langle f\rangle$ for the mean $\int f\,d\mu$ and $\langle f,g\rangle := \langle fg\rangle$. Suppose that the curvature of L is bounded from below by $R>0$, then

$$\langle (Lf)^2\rangle = \langle \Gamma_2(f,f)\rangle \ge R\langle \Gamma(f,f)\rangle,$$

and by Proposition 6.3 in [B] this is equivalent to the spectral gap inequality $\langle f^2\rangle - \langle f\rangle^2 \le R^{-1}\langle \Gamma(f,f)\rangle$. Thus $R \le \lambda_1$, where λ_1 is the spectral gap of $-L$.

Now we are going to check that Ledoux's proof [L1] of the concentration of measure phenomenon on compact Riemannian manifolds still works in the above setting.

1. (Bakry) If $f \in \mathcal{A}$ and if the curvature of L is bounded from below by $R > 0$, then by differentiation of the function $F(s) := P_s\Gamma(P_{t-s}f, P_{t-s}f)$ it is easy to see that $F' \ge 2RF$ and hence, for all f satisfying $\Gamma(f,f) \le 1$,

$$\Gamma(P_t f, P_t f) \le e^{-2Rt} P_t \Gamma(f,f) \le e^{-2Rt}. \tag{2}$$

2. For $f \in \mathcal{A}$ and $\lambda \ge 0$ we have

$$\langle \Gamma(f, e^{\lambda f})\rangle \le \lambda \langle e^{\lambda f}, \Gamma(f,f)\rangle. \tag{3}$$

This follows from the elementary inequality $(e^y - e^x)/(y-x) \le \frac{1}{2}(e^y + e^x)$.

3. (Ledoux) For $\lambda > 0$, $f \in \mathcal{A}$ such that $\Gamma(f,f) \le 1$ and $\langle f\rangle = 0$ define $F(t) := \langle e^{\lambda P_t f}\rangle$, then

$$\begin{aligned}-F'(t) &= -\lambda\langle LP_t f, e^{\lambda P_t f}\rangle = \lambda\langle \Gamma(P_t f, e^{\lambda P_t f})\rangle\\ &\le \lambda\langle \lambda e^{P_t f}, \Gamma(P_t f, P_t f)\rangle \le \lambda^2 e^{-2Rt}\langle e^{\lambda P_t f}\rangle = \lambda^2 e^{-2Rt} F(t),\end{aligned}$$

where we first used (3) and then (2). Thus $(\log F)'(t) \le -\lambda^2 e^{-2Rt}$ and since $F(\infty) = 1$ we conclude that $F(0) \le e^{\lambda^2/2R}$, which implies the deviation inequality

$$\mu(f > \varepsilon) \le e^{-\frac{1}{2}R\varepsilon^2}. \tag{4}$$

If Ω is a finite graph with counting measure μ we define

$$Lf(x) := \sum_{y\sim x} (f(y) - f(x)).$$

$y \sim x$ meaning that y and x are connected by an edge. Suppose that f is 1-Lipschitz with respect to the graph distance, then $\Gamma(f,f)(x) \le d(x)/2$, where $d(x)$ is the degree of the vertex x. Also, in this case $\Gamma_2(f,f)(x)$ only depends on points whose graph distance to x is at most 2. In this respect the curvature is a local quantity.

Suppose L_1 and L_2 are generators of type (1) on $L_2(\Omega_1, \mu_1)$ and $L_2(\Omega_1, \mu_1)$ respectively. Let P_t^1 and P_t^2 be the corresponding contraction semigroups on $L_2(\Omega_1, \mu_1)$ and $L_2(\Omega_1, \mu_1)$. Then $L := L_1 \otimes 1 + 1 \otimes L_2$ is the generator of $P_t(f \otimes g) := P_t^1 f \otimes P_t^2 g$ and

$$\Gamma^L(f\otimes g, f\otimes g) = f^2 \otimes \Gamma^{L_2}(g,g) + \Gamma^{L_1}(f,f)\otimes g^2,$$
$$\Gamma_2^L(f\otimes g, f\otimes g) = f^2 \otimes \Gamma_2^{L_2}(g,g) + \Gamma_2^{L_1}(f,f)\otimes g^2 + 2\Gamma^{L_1}(f,f)\otimes \Gamma^{L_2}(g,g).$$

For simplicity we assume $L_1 = \cdots = L_n$; we will also drop the superscripts. By induction we obtain, for $F = \bigotimes_{j=1}^n f_j$,

$$\Gamma(F,F) = \sum_j f_1^2 \otimes\cdots\otimes f_{j-1}^2 \otimes \Gamma(f_j,f_j)\otimes f_{j+1}^2\otimes\cdots\otimes f_n^2$$
$$\begin{aligned}\Gamma_2(F,F) = &\sum_j f_1^2 \otimes\cdots\otimes f_{j-1}^2 \otimes \Gamma_2(f_j,f_j)\otimes f_{j+1}^2\otimes\cdots\otimes f_n^2 \\ &+2\sum_{i<j} f_1^2\otimes\cdots\otimes f_{i-1}^2\otimes\Gamma(f_i,f_i)\otimes f_{i+1}^2\otimes\cdots\otimes f_{j-1}^2 \\ &\qquad\qquad\otimes\Gamma(f_j,f_j)\otimes f_{j+1}^2\otimes\cdots\otimes f_n^2.\end{aligned}$$

Let $x = (x_1,\dots,x_n) \in \Omega^n$; put $\widehat{x}_j = (x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)$ and define $F_{\widehat{x}_j} : \Omega \to \mathbb{R}$ by $F_{\widehat{x}_j}(x_j) = F(x)$, then the terms involving Γ in the second sum can be written as

$$\iint (F_{\widehat{x}_i}(y_i) - F_{\widehat{x}_i}(x_i))^2 (F_{\widehat{x}_j}(y_j) - F_{\widehat{x}_j}(x_j))^2 K(x_i,y_i)K(x_j,y_j)\,\mu(dy_i)\,\mu(dy_j).$$

For $x \in \Omega^n$ and $y \in \Omega$, we define $\nabla_y^j F(x) := F_{\widehat{x}_j}(y) - F_{\widehat{x}_j}(x_j)$; then the preceding expression equals

$$\iint \left(\nabla_{y_i}^i \nabla_{y_j}^j F(x)\right)^2 K(x_i,y_i)K(x_j,y_j)\,\mu(dy_i)\,\mu(dy_j).$$

Hence, for all $F \in \bigotimes_{j=1}^n \mathcal{A}$,

$$\Gamma(F,F)(x) = \sum_j \Gamma(F_{\widehat{x}_j}, F_{\widehat{x}_j})(x_j),$$
$$\Gamma_2(F,F)(x) = \sum_j \Gamma_2(F_{\widehat{x}_j}, F_{\widehat{x}_j})(x_j) + 2\sum_{i<j} \iint \left(\nabla^i_{y_i}\nabla^j_{y_j} F(x)\right)^2 K(x_i,y_i)K(x_j,y_j)\,\mu(dy_i)\,\mu(dy_j).$$

We thus have the following analogue to manifolds:

Theorem 1.1. *Suppose the curvatures of $L_1,\dots,L_n$ are bounded from below by $R_1,\dots,R_n$. Then the curvature of*

$$L_1 \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes L_n.$$

is bounded from below by $\min_j R_j$.

Finally let us note a somewhat more convenient formula for Γ_2: For each $y \in \Omega$ define $X_y : \mathcal{A} \to \mathcal{A}$ by $X_y f(x) = \sqrt{K(x,y)}(f(y)-f(x))$.

Proposition 1.2. *For all $f,g \in \mathcal{A}$ we have*

$$\Gamma(f,g) = \tfrac{1}{2}\int X_y f X_y g\,\mu(dy),$$
$$\Gamma_2(f,f) = \tfrac{1}{2}\int \Gamma(X_y f, X_y f) + X_y f[L,X_y]f\,\mu(dy),$$

where $[L,X]$ denotes the commutator $LX - XL$.

Proof. The first formula is just the definition of X_y. As for the second we note that

$$\tfrac{1}{2}L\Gamma(f,f) = \tfrac{1}{4}\int L(X_y f)^2\,\mu(dy) = \tfrac{1}{2}\int \Gamma(X_y f, X_y f) + X_y f L X_y f\,\mu(dy)$$

and thus the formula follows by the definition of Γ_2. □

2. Curvature of Graphs

Let us consider the trivial example $K(x,y)=1$. In this case

$$\Gamma(f,f)(x) = \tfrac{1}{2}\int \nabla_y f(x)^2\,\mu(dy),$$
$$\Gamma_2(f,f)(x) = \tfrac{1}{4}\Big(3\int \nabla_y f(x)^2\,\mu(dy) - 2\Big(\int \nabla_y f(x)\,\mu(dy)\Big)^2\Big).$$

Choosing $R = \frac{1}{2}$ the inequality $\Gamma_2(f,f) \ge R\Gamma(f,f)$ is thus equivalent to

$$\int \nabla_y f(x)^2\,\mu(dy) \ge \Big(\int \nabla_y f(x)^2\,\mu(dy)\Big)^2,$$

i.e., the curvature of L is bounded from below by $\frac{1}{2}$. If Ω is a complete graph of order n then we obtain a slightly larger lower bound for the curvature: $R = 1/2+1/n$. In this case the deviation inequality can be obtained much more easily: following M. Ledoux [L2] and using the elementary inequality $(e^x - e^y)/(x-y) \leq \frac{1}{2}(e^x + e^y)$, we get, for all $f \in \mathcal{A}$,

$$\begin{aligned} \langle fe^f\rangle - \langle e^f\rangle \log\langle e^f\rangle &\leq \tfrac{1}{2} \iint (f(x) - f(y))(e^{f(x)} - e^{f(y)})\, \mu(dx)\, \mu(dy) \\ &\leq \tfrac{1}{4} \iint (f(x) - f(y))^2(e^{f(x)} + e^{f(y)})\, \mu(dx)\, \mu(dy) \\ &= \langle e^f, \Gamma(f,f)\rangle, \end{aligned} \tag{5}$$

which implies (see [L2]) that $\mu(f - \langle f\rangle > \varepsilon) \leq e^{-\varepsilon^2/4}$ provided $\Gamma(f,f) \leq 1$. The latter condition implies that f is bounded: if $\langle f\rangle = 0$, then $|f| \leq \sqrt{2 - \langle f^2\rangle}$. Ledoux's point is not this deviation inequality but rather the fact that (5) tensorizes easily. In our context this is reflected by the fact that if the curvature of L is bounded from below by R, then so is the curvature of $L \otimes 1 + 1 \otimes L$. In the particular case of the cube $\Omega = \{-1, +1\}$ and the normalized Haar measure μ_1 we get by 4 and Theorem 1.1:

COROLLARY 2.1. *Let $f : \Omega^N \to \mathbb{R}$ be a 1-Lipschitz function with respect to the graph distance. If $\langle f\rangle = 0$, then $\mu_N(f > \varepsilon) \leq e^{-2\varepsilon^2/N}$, where μ_N is the product probability.*

PROOF. Since $\int(\nabla_y f(x))^2\, \mu_1(dy) \leq 1/2$ we get $\Gamma(f,f) \leq N/4$. □

More generally:

COROLLARY 2.2. *Let Ω be a complete graph of order n with normalized counting measure μ_1 and Ω^N the product graph with the product measure μ_N. Suppose $f : \Omega^N \to \mathbb{R}$ is a 1-Lipschitz function with respect to the graph distance such that $\langle f\rangle = 0$, then*

$$\mu_N(f > \varepsilon) \leq \exp\Big(-\frac{n+2}{2N(n-1)}\varepsilon^2\Big).$$

Now suppose $\Omega = \{0, 1, \ldots, n-1\}$ is a finite graph of order n. Any function $f : \Omega \to \mathbb{R}$ can be thought of as a vector $f = (f_0, \ldots, f_{n-1}) \in \mathbb{R}^n$. By μ we denote the counting measure on Ω and by μ_0 the normalized counting measure. For any function $f : \Omega \to \mathbb{R}$ we will also write $\langle f\rangle$ for the mean of f with respect to μ_0. Define $(Lf)_j = \sum_i (f_i - f_j)K_{i,j}$, where $K_{i,j}$ is 1 if and only if $i \sim j$ and put $x_i := f_i - f_0$ and $d_i := \sum_l K_{i,l}$, the degree of the vertex i. Then we obtain

$$\Gamma(f,f)_0 = \tfrac{1}{2} \sum_{i=1}^{n-1} x_i^2 K_{i,0}$$

and

$$\Gamma_2(f,f)_0 = \tfrac{1}{4}\sum_{i=1}^{n-1} x_i^2\Big(\sum_l K_{l,0}K_{i,l} + K_{i,0}(3d_i - d_0 + 2)\Big) \\ - \sum_{1\le i<j\le n-1} x_i x_j (K_{i,j}(K_{i,0}+K_{j,0}) - K_{i,0}K_{j,0}).$$

Define symmetric matrices $A = (a_{i,j})$ and $G = (g_{i,j})$, $1 \le i,j \le n-1$ by

$$a_{i,j} = \begin{cases} \frac{1}{2}(\sum_l K_{l,0}K_{l,i}) + \frac{1}{2}K_{i,0}(3d_i - d_0 + 2) & \text{if } i = j,\\ -K_{i,j}(K_{i,0}+K_{j,0}) + K_{i,0}K_{0,j} & \text{if } i \ne j,\end{cases}$$

and $g_{i,i} = K_{i,0}$ and 0 off the diagonal. Then the curvature R_0 at 0 is bounded from below by

$$\sup\{r \in \mathbb{R} :\ A - rG \ge 0\}.$$

In this case we conclude by (4) that for all 1-Lipschitz functions $f : \Omega \to \mathbb{R}$

$$\mu_0(f - \langle f\rangle > \varepsilon) \le e^{-(R/d)\varepsilon^2} \tag{6}$$

where $R = \inf R_i$ and $d = \max d_i$.

The off-diagonal entries of A can take on the values 0, 1 or -1 only:

$$a_{i,j} = \begin{cases} -1 & \text{if } i \sim j \text{ and } (i \sim 0 \text{ or } j \sim 0),\\ \ \ 1 & \text{if } i \not\sim j \text{ and } j \sim 0 \text{ and } i \sim 0,\\ \ \ 0 & \text{otherwise.}\end{cases}$$

For $\varepsilon > 0$ let $B_\varepsilon(i)$ be the ball $\{j \in \Omega : d(i,j) \le \varepsilon\}$. Let $I = B_1(0) \setminus \{0\}$ be the set of vertices, which are connected with 0 and put $J = B_2(0) \setminus B_1(0)$. Then, for $i,j \in I$ with $i \ne j$, we get

$$a_{i,i} = \tfrac{1}{2}(c_{i,0}^{(3)} + 3d_i - d_0 + 2) \quad \text{and} \quad a_{i,j} = \begin{cases} -1 & \text{if } i \sim j,\\ +1 & \text{if } i \not\sim j,\end{cases}$$

where $c_{i,0}^{(3)}$ is the number of 3-cycles containing both 0 and i. If $i \ne j$, $i,j \in J$, then

$$a_{i,i} = \tfrac{1}{2}p_{i,0}^{(2)} \quad \text{and} \quad a_{i,j} = 0$$

where $p_{i,0}^{(2)}$ is the number of paths of length 2 joining 0 and i. Finally, if $i \in I$ and $j \in J$, then

$$a_{i,j} = \begin{cases} -1 & \text{if } i \sim j,\\ \ \ 0 & \text{if } i \not\sim j.\end{cases}$$

By restricting the vertices to $I \times I$, $J \times J$, $I \times J$ and $J \times I$, we get four submatrices A_{II}, A_{JJ}, A_{IJ} and $A_{JI} = A_{IJ}^t$:

$$A = \begin{pmatrix} A_{II} & A_{IJ}\\ A_{JI} & A_{JJ}\end{pmatrix}.$$

Thus the smallest eigenvalue of A_{II} is a lower bound for the curvature at 0. Since $c_{i,0}^{(3)} \le d_i$, we conclude that

$$R_0 \le \inf\{2d_j - d_0/2 + 1 : j \sim 0\}.$$

Thus the lower bound of the curvature cannot be positive if there exist two connected points i and j such that $d_i \ge 4d_j + 2$. Another more or less obvious upper bound for R involves the diameter D of Ω:

$$D := \sup\big\{|f_i - f_j| : i, j \in \Omega, f \in \mathrm{Lip}_1(\Omega)\big\}.$$

Since for all $\varepsilon > 0$ there exists $f \in \mathrm{Lip}_1(\Omega)$ such that $\mu_0(|f - c| > (D/2 - \varepsilon)) \ge 1/n$ for all $c \in [0, D]$, we obtain, by (6), $R \le 4d \log(2n)/D^2$.

Now we turn to a more homogeneous situation: we will assume that for all $i, j \in \Omega$ there exists an isomorphism $h_{i,j}$ from $B_2(i)$ onto $B_2(j)$ such that $h_{i,j}(i) = j$. If follows that each vertex has the same degree d and a lower bound R for the curvature at any point is also a lower bound for the curvature of L. Therefore we will call these graphs, graphs of constant curvature. This situation in particular occurs if there is an underlying group structure that determines the graph: Let $I = \{g_1, \dots, g_d\} \subseteq G \setminus \{e\}$ be a symmetric subset of a finite group G with neutral element e. Suppose further that $B_1(e) = I \cup \{e\}$ generates G, i.e., $\bigcup_n B_1(e)^n = G$. Two points $x, y \in G$ are connected if there exists a $g_j \in I$ such that $y = g_j x$. Obviously the map $h_{x,y} : B_2(x) \to B_2(y)$ defined by $h_{x,y}(z) := zx^{-1}y$ is an isomorphism.

PROPOSITION 2.3. *Let ∇_j be the operator $\nabla_j f(x) := f(g_j x) - f(x)$. Then the following statements are equivalent.*

1. *The graph distance is a bi-invariant metric.*
2. *For all $g, h \in I$, $ghg^{-1} \in I$.*
3. *For all j the operator ∇_j commutes with L.*

The Ricci curvature of a Lie group with bi-invariant Riemannian metric is always non negative. The following proposition is the analogue of this fact for finite discrete groups.

PROPOSITION 2.4. *Let G be a finite group, $I = \{g_1, \dots, g_d\}$ a symmetric subset of $G \setminus \{e\}$ such that condition 2 of Proposition 2.3 holds. Then R is a lower bound for the curvature of G if and only if, for all $f : G \to \mathbb{R}$,*

$$\sum_{j,k} (\nabla_j \nabla_k f)^2 \ge 2R \sum_j (\nabla_j f)^2.$$

In particular the curvature of such groups is always non negative.

PROOF. By Proposition 2.3 the commutators vanish and thus the assertion follows from Proposition 1.2. ☐

A bi-invariant metric on a finite group need not necessarily evolve from a graph structure: The Hilbert–Schmidt metric

$$d_{HS}(\pi_1,\pi_2) := \frac{1}{2}\sqrt{(\pi_1-\pi_2)(\pi_1-\pi_2)^*}$$

is a bi-invariant metric on the symmetric group Π_n and $B_1(e)\setminus\{e\}$ is the set of all transpositions, but the metric d determined by $B_1(e)\setminus\{e\}$ is different from d_{HS}.

It's very likely that the curvature of (Π_n, d) is bounded from below by 2. However, this is too small to recover Maurey's deviation inequality for Π_n; see [M] or [MS].

For $n \geq 2$ let $\Omega = \{e_1, e_2, \dots, e_n, -e_n, \dots, -e_1\}$ be the set of extreme points of the unit ball of ℓ_1^n. Two points are connected if they are connected by a 1 dimensional face of the unit ball. In this case A is a $(2n-1)\times(2n-1)$ matrix whose diagonal is given by $\{3(n-1), \dots, 3(n-1), (n-1)\}$. The off diagonal entries $a_{i,j}$ are 1 if $i+j = 2n-1$ and -1 otherwise. The curvature of this graph is bounded from below by n.

The curvature of the icosahedron is bounded from below by $(11-3\sqrt{5})/2$.

The curvature of the dodecahedron is bounded from below by 0.

For $n \geq 5$ the curvature of the additive group $\mathbb{Z}_n$ with $I = \{1, n-1\}$ is bounded from below by 0.

Let $(\{1,\dots,n\}, d)$ be a finite metric space. Putting $K(i,j) := 1/d(i,j)^2$, then $d_\Gamma = d$. Thus the curvature can be defined for any finite metric space.

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