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Another Low-Technology Estimate in Convex Geometry

GREG KUPERBERG

Abstract. We give a short argument that for some $C > 0$, every n-dimensional Banach ball K admits a 256-round subquotient of dimension at least $Cn/(\log n)$. This is a weak version of Milman's quotient of subspace theorem, which lacks the logarithmic factor.

Let V be a finite-dimensional vector space over $\mathbb{R}$ and let V^* denote the dual vector space. A *symmetric convex body* or *(Banach) ball* is a compact convex set with nonempty interior which is invariant under under $x \mapsto -x$. We define $K^\circ \subset V^*$, the *dual* of a ball $K \subset V$, by

$$K^\circ = \{y \in V^* \big| y(K) \subset [-1,1]\}.$$

A ball K is the unit ball of a unique Banach norm $\|\cdot\|_K$ defined by

$$\|v\|_K = \min\{t \big| v \in tK\}.$$

A ball K is an *ellipsoid* if $\|\cdot\|_K$ is an inner-product norm. Note that all ellipsoids are equivalent under the action of $\mathrm{GL}(V)$.

If V is not given with a volume form, then a volume such as Vol K for $K \subset V$ is undefined. However, some expressions such as (Vol K)(Vol K°) or (Vol K)/(Vol K') for $K, K' \subset V$ are well-defined, because they are independent of the choice of a volume form on V, or equivalently because they are invariant under $\mathrm{GL}(V)$ if a volume form is chosen.

An n-dimensional ball K is *r-semiround* [8] if it contains an ellipsoid E such that

$$(\text{Vol } K)/(\text{Vol } E) \le r^n.$$

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It is r-*round* if it contains an ellipsoid E such that $K \subseteq rE$. Santaló's inequality states that if K is an n-dimensional ball and E is an n-dimensional ellipsoid,

$$(\text{Vol } K)(\text{Vol } K^\circ) \leq (\text{Vol } E)(\text{Vol } E^\circ).$$

(Saint-Raymond [7], Ball [1], and Meyer and Pajor [4] have given elementary proofs of Santaló's inequality.) It follows that if K is r-round, then either K or K° is $\sqrt{r}$-semiround.

If K is a ball in a vector space V and W is a subspace, we define $W \cap K$ to be a *slice* of K and the image of K in V/W to be a *projection* of K; they are both balls. Following Milman [5], we define a *subquotient* of K to be a slice of a projection of K. Note that a slice of a projection is also a projection of a slice, so that we could also have called a subquotient a proslice. It follows that a subquotient of a subquotient is a subquotient. Note also that a slice of K is dual to a projection of K°, and therefore a subquotient of K is dual to a proslice (or a subquotient) of K°.

In this paper we prove the following theorem:

Theorem 1. *Suppose that K is a $(2^{k+1}n)$-dimensional ball which is $(2^{(3/2)^k} \cdot 4)$-semiround, with $k \geq 0$. Then K has a 256-round, n-dimensional subquotient.*

Corollary 2. *There exists a constant $C > 0$ such that every n-dimensional ball K admits a 256-round subquotient of dimension at least $Cn/(\log n)$.*

The corollary follows from the theorem of John that every n-dimensional ball is $(\sqrt{n})$-round.

The corollary is a weak version of a celebrated result of Milman [5; 6]:

Theorem 3 (Milman). *For every $C > 1$, there exists $D > 0$, and for every $D < 1$ there exists a C, such that every n-dimensional ball K admits a C-round subquotient of dimension at least Dn.*

However, the argument given here for Theorem 1 is simpler than any known proof of Theorem 3.

Theorem 3 has many consequences in the asymptotic theory of convex bodies, among them a dual of Santalo's inequality:

Theorem 4 (Bourgain, Milman). *There exists a $C > 0$ such that for every n and for every n-dimensional ball K,*

$$(\text{Vol } K)(\text{Vol } K^\circ) \geq C^n (\text{Vol } E)(\text{Vol } E^\circ).$$

Theorem 4 is an asymptotic version of Mahler's conjecture, which states that for fixed n, $(\text{Vol } K)(\text{Vol } K^\circ)$ is minimized for a cube. In a previous paper, the author [3] established a weak version of Theorem 4 also, namely that

$$\text{Vol }(K) \, \text{Vol }(K^\circ) \geq (\log_2 n)^{-n} \, \text{Vol }(E) \, \text{Vol }(E^\circ)$$

for $n \geq 4$. That result was the motivation for the present paper.

The author speculates that there are elementary arguments for both Theorems 3 and 4, which moreoever would establish reasonable values for the arbitrary constants in the statements of these theorems.

The Proof

The proof is a variation of a construction of Kashin [8]. For every k let Ω_k be the volume of the unit ball in $\mathbb{R}^k$; Ω_k is given by the formula

$$\frac{\pi^{k/2}}{\Gamma(\frac{k}{2}+1)}.$$

Let V be an n-dimensional vector space with a distinguished ellipsoid E, to be thought of as a round unit ball in V, so that V is isometric to standard $\mathbb{R}^n$ under $\|\cdot\|_E$. Give V the standard volume structure $d\vec{x}$ on $\mathbb{R}^n$. In particular, $\mathrm{Vol}\ E = \Omega_n$. Endow ∂E, the unit sphere, with the invariant measure μ with total weight 1. If K is some other ball in V, then

$$\mathrm{Vol}\ K = \Omega_n \int_{\partial E} \|x\|_K^{-n} d\mu$$

and, more generally,

$$\int_K \|x\|_E^k \, d\vec{x} = \frac{n\Omega_n}{n+k} \int_{\partial E} \|x\|_K^{-n-k} d\mu.$$

Let f be a continuous function on ∂E. Let $0 < d < n$ be an integer and consider the space of d-dimensional subspaces of V. This space has a unique probability measure invariant under rotational symmetry. If W is such a subspace chosen at random with respect to this measure, then for any continuous function f,

$$\int_{\partial E} f(x) d\mu = \mathrm{E}\left[\int_{\partial(E\cap W)} f(x) d\mu\right], \tag{1}$$

where μ denotes the invariant measure of total weight 1 on $E \cap W$ also. In particular, there must be some W for which the integral of f on the right side of equation (1) is less than or equal to that of the left side, which is the average value.

The theorem follows by induction from the case $k = 0$ and from the claim that if K is a $(2n)$-dimensional ball which is r-semiround, then K has an n-dimensional slice K'' such that either K'' or its dual is $(2r)^{2/3}$-semiround. In both cases, we assume that K is r-semiround and has dimension $2n$ and we proceed with a parallel analysis.

There exists an $(n+1)$-dimensional subspace V' of V such that:

$$\int_{\partial E'} \|x\|_K^{-2n} d\mu \leq \frac{\mathrm{Vol}\ K}{\mathrm{Vol}\ E} = r^{2n}, \tag{2}$$

where $E' = E \cap V'$. Let $K' = V' \cap K$. Then

$$\int_{\partial E'} \|x\|_K^{-2n} d\mu = \frac{2n}{(n-1)\Omega_{n+1}} \int_{K'} \|x\|_{E'}^{n-1}\, d\vec{x}. \tag{3}$$

Let p be a point in K' such that $s = \|p\|_E$ is maximized; in particular K' is s-round Let V'' be the subspace of V' perpendicular to p and define $K'' = V'' \cap K$ and $E'' = V'' \cap E$. The convex hull $S(K'')$ of $K'' \cup \{p, -p\}$ is a double cone with base K'' (or suspension of K''), and $S(K'') \subseteq K'$. We establish an estimate that shows that either s or Vol K'' is small. Let x_0 be a coordinate for V' given by distance from V''. Then

$$\begin{aligned}
\int_{K'} \|x\|_{E'}^{n-1}\, d\vec{x} &\geq \int_{S(K'')} \|x\|_{E'}^{n-1}\, d\vec{x} > \int_{S(K'')} |x_0|^{n-1}\, d\vec{x} \\
&= 2\int_0^s x_0^{n-1} \Big(\mathrm{Vol}\ \Big(1 - \frac{x_0}{s}\Big) K''\Big)\, dx_0 \\
&= 2(\mathrm{Vol}\ \ K'') s^n \int_0^1 t^{n-1}(1-t)^n\, dt \\
&= (\mathrm{Vol}\ \ K'') s^n \frac{2(n-1)!n!}{(2n)!}.
\end{aligned} \tag{4}$$

We combine equations (2), (3), and (4) with the inequality

$$\frac{\Omega_n 4n(n-1)!n!}{\Omega_{n+1}(n-1)(2n)!} = \frac{2\Gamma(\frac{n+3}{2})(n-2)!n!}{\sqrt{\pi}\Gamma(\frac{n+2}{2})(2n-1)!} > 4^{-n}.$$

(Proof: Let $f(n)$ be the left side. By Stirling's approximation, $f(n)4^n \to 2^{3/2}$ as $n \to \infty$. Since

$$\frac{f(n+2)}{f(n)} = \frac{1}{4}\frac{n^2+2n-3}{4n^2+8n+3} < \frac{1}{16},$$

the limit is approached from above.) The final result is that

$$\frac{\mathrm{Vol}\ \ K''}{\mathrm{Vol}\ \ E''} \leq (2r)^{2n} s^{-n}.$$

In the case $k = 0$, $r = 8$. Since $E'' \subseteq K''$, Vol $K'' \geq$ Vol E'', which implies that $s \leq 4r^2 = 256$. Since K'' is s-round, it is the desired subquotient of K.

If $k > 1$, then suppose first that $s \leq (2r)^{4/3}$. In this case K'' is $(2r)^{4/3}$-round, which implies by Santaló's inequality that either K'' or K''° is $(2r)^{2/3}$-semiround. On the other hand, if $s \geq (2r)^{4/3}$, then K'' is $(2r)^{2/3}$-semiround. In either case, the induction hypothesis is satisfied.

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Greg Kuperberg
Department of Mathematics
University of California
Davis, CA 95616
United States of America
greg@math.ucdavis.edu