Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

Random Points in Isotropic Convex Sets

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Abstract. Let K be a symmetric convex body of volume 1 whose inertia tensor is isotropic, i.e., for some constant L we have $\int_K \langle x, y\rangle^2\, dx = L^2|y|^2$ for all y. It is shown that if m is about $n(\log n)^3$ then with high probability, this tensor can be approximately realised by an average over m independent random points chosen in K,

$$\frac{1}{m}\sum_{i=1}^{m}\langle x_i, y\rangle^2.$$

Our aim is to prove the following fact:

Proposition. *Let $K \subset \mathbb{R}^n$ be a convex centrally symmetric body of volume* 1, *in isotropic position*, *i.e.*,

$$\int_K \langle x, e_i\rangle\langle x, e_j\rangle\, dx = L^2\delta_{ij} \quad \text{where } L = L_K (\gtrsim 1). \tag{1}$$

Fix $\delta > 0$ and choose m random points $x_1, \dots, x_m \in K$, where

$$m > C(\delta) n\, (\log n)^3. \tag{2}$$

Then, *with probability $> 1-\delta$*,

$$(1-\delta)L^2 < \frac{1}{m}\sum_{i=1}^{m}\bigl|\langle x_i, y\rangle\bigr|^2 < (1+\delta)L^2 \tag{3}$$

for all $y \in S^{n-1} = [|y| = 1]$.

We first use the following probabilistic estimate:

Lemma 1. *Let $f_1, \dots, f_m$ be independent copies of a random variable f satisfying*

$$\int f^2 = 1, \tag{4}$$

$$\|f\|_{\psi_1} < C \quad (\text{where } \psi_1(t) = e^t), \tag{5}$$

$$\|f\|_\infty < B. \tag{6}$$

(Here and in the sequel we use c and C to denote positive constants, not necessarily the same each time.) *Let* $\varepsilon > 0$ *and assume* $B > 1/\varepsilon$ *say. Then*

$$\operatorname{mes}\left[(1-\varepsilon)m < \sum_{i=1}^{m} f_i^2 < (1+\varepsilon)m\right] > 1 - e^{-c\frac{\varepsilon}{B}m}. \tag{7}$$

Proof (standard). For real λ (to be specified),

$$\int e^{\lambda(\sum_{i=1}^{m}(f_i^2-1))} = \left(\int e^{\lambda(f^2-1)}\right)^m. \tag{8}$$

By (4)

$$\int e^{\lambda(f^2-1)} = 1 + \sum_{j\geq 2} \frac{1}{j!}\,\lambda^j \int (f^2-1)^j. \tag{9}$$

From (5) and (6),

$$\int (1+|f|)^{2j} < \min\bigl((Cj)^{2j},\, (1+B)^j (Cj)^j\bigr). \tag{10}$$

for each j. Hence, substituting (10) in (9),

$$\int e^{\lambda(f^2-1)} < 1 + \sum_{j\geq 2} (C\lambda)^j (j \wedge B)^j < 1 + C\lambda^2 \tag{11}$$

provided

$$\lambda < \frac{c}{B}\,. \tag{12}$$

for an appropriate c. Thus (8) $< (1+C\lambda^2)^m < e^{C\lambda^2 m}$ and from this fact and Tchebychev's inequality

$$\operatorname{mes}\left[\left|\frac{1}{m}\sum_{i=1}^{m}(f_i^2-1)\right| > \varepsilon\right] < e^{-\lambda m\varepsilon} e^{C\lambda^2 m} < e^{-c\frac{\varepsilon}{B}m} \tag{13}$$

for appropriate λ satisfying (12) (and since $1/\varepsilon < B$). □

Recall the important fact (following from the Brunn–Minkowski inequality) that, for K convex with $\operatorname{Vol} K = 1$, there is equivalence

$$\|\langle y, x\rangle\|_{L^{\psi_1}(K,dx)} \sim \|\langle y, x\rangle\|_{L^2(K,dx)} \tag{14}$$

(with an absolute constant). Hence, in our situation

$$\|\langle y, x\rangle\|_{L^{\psi_1}(K,dx)} < CL \quad \text{if} \quad |y| = \|y\|_2 \leq 1. \tag{15}$$

It follows that

$$\operatorname{mes}\bigl[x \in K \bigm| |x| > \lambda L\sqrt{n}\,\bigr] < e^{-C\lambda} \quad \text{for } \lambda > 1. \tag{16}$$

The next estimate may be refined significantly in terms of an estimate on the ℓ^2-operator norm (see remark at the end) but for our purposes the following cruder form is sufficient.

LEMMA 2. *Let K be as above and $x_1, \dots, x_m$ random points in K. Then, with probability $> 1-\delta$,*

$$\Big|\sum_{i\in E} x_i\Big| < C(\delta)\, L\, \log n\big(|E|^{1/2}\, n^{1/2} + |E|\big) \tag{17}$$

holds for all subsets $E \subset \{1, \dots, m\}$.

PROOF. Write

$$\Big|\sum_{i\in E} x_i\Big|^2 = \sum_{i\in E} |x_i|^2 + 2 \sum_{\substack{i\neq j\\ i,j\in E}} \langle x_i, x_j\rangle. \tag{18}$$

From (15), we may clearly assume

$$|x_i| < CL\log n\sqrt{n} \quad \text{for all} \quad i = 1, \dots, m.$$

Hence the first term of (18) may be assumed bounded by $C\,L^2(\log n)^2 n\,|E|$.

To estimate the second term of (18), we use a standard decoupling trick.

We can find subsets E_1, E_2 of E satisfying $E_1 \cap E_2 = \varnothing$, $|E_1| \geq |E_2|$, and

$$\sum_{i\neq j, i,j\in E} \langle x_i, x_j\rangle \leq 4 \sum_{i\in E_1} \Big|\Big\langle x_i, \sum_{j\in E_2} x_j\Big\rangle\Big|. \tag{19}$$

Hence we are reduced to bounding expressions of the form (19).

Rewrite

$$\sum_{i\in E_1} \Big|\Big\langle x_i, \sum_{j\in E_2} x_j\Big\rangle\Big| = \Big|\sum_{j\in E_2} x_j\Big| \sum_{i\in E_1} |\langle x_i, y_{E_2}(x)\rangle| \tag{20}$$

where

$$y_{E_2}(x) = \frac{\sum_{j\in E_2} x_j}{\Big|\sum_{j\in E_2} x_j\Big|}\ ; \quad \text{thus} \ \ |y_{E_2}| = 1. \tag{21}$$

Observe that the system $(x_i)_{i\in E_1}$ is independent of y_{E_2}, since $E_1 \cap E_2 = \varnothing$. Fix size scales $|E_1| \sim m_1$, $|E_2| \sim m_2$, $m \geq m_1 \geq m_2 \geq 1$.

Thus for fixed $m_1 > m_2$, (E_1, E_2) run over at most m^{Cm_1} pairs of subsets of $\{1, \dots, m\}$. For given $y, |y| = 1$, (15) easily implies that

$$\int e^{\frac{C}{L}\sum_{i\in E_1} |\langle x_i, y\rangle|} \prod_{i\in E_1} dx_i < 2^{|E_1|}; \tag{22}$$

hence, for $\mu > C$,

$$\operatorname{mes}\Big[(x_i)_{1\leq i\leq m} \in K^m \ \Big|\ \sum_{i\in E_1} |\langle x_i, y_{E_2}(x)\rangle| > \mu\, L\, |E_1|\Big] < e^{-c\mu|E_1|}. \tag{23}$$

Consequently, from (20) and the preceding, we may write

$$\sum_{i\in E_1}\left|\left\langle x_i, \sum_{j\in E_2} x_j\right\rangle\right| < \left|\sum_{j\in E_2} x_j\right| \mu\, L\, |E_1| \tag{24}$$

for all $|E_1|\sim m_1$, $|E_2|\sim m_2$, $E_1\cap E_2=\varnothing$ provided

$$m^{Cm_1}\, e^{-\mu m_1} < 2^{-m_1}; \quad \text{thus } \mu\sim\log m\sim\log n. \tag{25}$$

Thus, letting $\mu\sim\log n$, (24) may be assumed valid for all $E_1, E_2\subset\{1,\dots,m\}$ with $E_1\cap E_2=\varnothing$.

Substituting (19), (24) in (18) thus yields, for all $E\subset\{1,\dots,m\}$,

$$\left|\sum_{i\in E} x_i\right|^2 \le CL^2(\log n)^2\, n\, |E| + CL(\log n)|E| \max_{E_2\subset E}\left|\sum_{j\in E_2} x_j\right| \tag{26}$$

and (17) immediately follows. □

Proposition. *Fix $\delta>0$ and choose random points $x_1,\dots,x_m\in K$, with $m>C(\delta)n(\log n)^3$. Then with probability $>1-\delta$*

$$(1-\delta)L^2 < \frac{1}{m}\sum_{i=1}^m |\langle x_i, y\rangle|^2 < (1+\delta)L^2 \quad \textit{for all} \quad y\in S^{n-1}. \tag{27}$$

Proof. Restrict y to a $\frac{\delta}{10}$-dense set $\mathcal{F}_\delta$ in the unit sphere S^{n-1}, $\#\mathcal{F}_\delta < \left(\frac{C}{\delta}\right)^n$. Fix $y\in\mathcal{F}$ and define

$$f = f^y(x) = \begin{cases} \frac{1}{L}|\langle x,y\rangle| & \text{if } \ |\langle x,y\rangle| < C_1(\log n)\, L, \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

(with C_1 to be specified).

Thus

$$1-\int f^2 = \frac{1}{L^2}\int_{K\cap|\langle x,y\rangle|>C_1\,(\log n)\,L]} |\langle x,y\rangle|^2 dx < e^{-c_1\log n}. \tag{29}$$

Applying Lemma 1 with $B=C_1\log n, \varepsilon=\frac{\delta}{10}$, it follows that for a random choice $x_1,\dots,x_m$ of points in K, with probability $>1-e^{-c(\varepsilon/\log n)m}$,

$$\int f^2 - \varepsilon < \frac{1}{m}\sum_{i=1}^m f^y(x_i)^2 < \left(\int f^2+\varepsilon\right); \tag{30}$$

hence, by (28) and (29),

$$\left|1-\frac{1}{L^2 m}\sum\big\{\langle x_i,y\rangle^2 \mid |\langle x_i,y\rangle| < C_1 L\, \log n\big\}\right| < \varepsilon + \left(1-\int f^2\right) < 2\varepsilon. \tag{31}$$

Letting

$$\left(\tfrac{C}{\delta}\right)^n e^{-c(\varepsilon/\log n)m} \ll 1, \quad \text{i.e., } m\gtrsim \frac{1}{\varepsilon}\log\frac{1}{\delta}(\log n)n, \tag{32}$$

we may then assume (31) for all $y\in\mathcal{F}_\delta$.

On the other hand, from Lemma 2, a random choice $\{x_i \mid i = 1, \ldots, m\}$ of m points in K will also with probability $> 1-\delta$ satisfy (17) for all $E \subset \{1, \ldots, m\}$. This permits to estimate $\# E_\beta$, where for given y satisfying $|y| = 1$,

$$E_\beta = E_\beta(y) = \big\{ i = 1, \ldots, m \mid |\langle x_i, y\rangle| > \beta \big\}, \quad \beta > C_1 (\log n)\, L. \tag{33}$$

Indeed, it follows from (17) that

$$\tfrac{1}{2}\beta |E_\beta| < C\, L\, \log n \left(|E_\beta|^{1/2} n^{1/2} + |E_\beta| \right) \tag{34}$$

hence

$$|E_\beta| < C\, \frac{L^2 (\log n)^2 n}{\beta^2} \tag{35}$$

from the choice of β. Consequently

$$\begin{aligned} \frac{1}{L^2 m} \sum \big\{ \langle x_i, y\rangle^2 \bigm| |\langle x_i, y\rangle| \geq C_1\, L \log n \big\} &< \frac{1}{L^2 m} \sum_{\substack{n > \beta > C_1\, L \log n \\ \beta \text{ dyadic}}} \beta^2 |E_\beta| \\ &< C(\delta)\, (\log n)^3 \frac{n}{m} < \frac{\delta}{10} \end{aligned} \tag{36}$$

by the choice of m.

Finally, combining (36) and (31), it follows that for all $y \in F_\delta$

$$\left| 1 - \frac{1}{L^2 m} \sum_{i=1}^m \langle x_i, y\rangle^2 \right| < 2\varepsilon + \frac{\delta}{10} < \frac{\delta}{3} \tag{37}$$

and therefore also (27). □

Remark. By refining a bit the method of proof of Lemma 2, one may obtain the following result: Let $x_1, \ldots, x_n$ be a choice of n independent vectors in $\mathbb{R}^n$ according to a probability measure μ on $\mathbb{R}^n$ satisfying

$$\|\langle x, y\rangle\|_{L^{\psi_1}(\mu(dx))} < \frac{1}{\sqrt{n}} \quad \text{for all } y \in S^{n-1}. \tag{38}$$

Then, with probability $> 1 - \delta$, one gets for the matrix $(x_1, \ldots, x_n)$ the bound

$$\|(x_1, \ldots, x_n)\|_{B(\ell_n^2)} < C(\delta) \left(\int \left(\max_{1 \leq i \leq n} |x_i| \right) d\mu + 1 \right). \tag{39}$$

This is the same estimate as one would get assuming an L^{ψ_2}-bound

$$\|\langle x, y\rangle\|_{L^{\psi_2}(\mu(dx))} < \frac{1}{\sqrt{n}} \quad \text{for } y \in S^{n-1} \tag{40}$$

instead of (38).

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