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Liftings of Kernels Shift-Invariant in Scattering Systems

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To Daniel J. Goldstein, without whom this story could not have unfolded.

Abstract. The Generalized Bochner Theorem (GBT) provides both integral representations and extensions of forms and kernels invariant under the shift operator. Even in the simplest setting of trigonometric polynomials, it allows a unified approach encompassing the Nehari approximation theorem and the Helson–Szegő and Helson–Sarason prediction theorems. It also gives results on weighted Lebesgue spaces that had been out of reach of classical methods.

The GBT's lifting approach is valid in abstract algebraic and hilbertian scattering systems, with one or several evolution groups (not necessarily commuting), and integral representations of Toeplitz extensions of Hankel forms are obtained in many such systems. These integral representations lead to applications to harmonic analysis in product spaces, such as the polydisk, and in symplectic spaces. In a different direction, a noncommutative extension of the GBT is given for kernels defined in terms of completely positive maps.

Introduction

The study of the generalized Toeplitz kernels and forms started as an attempt to apply Kreĭn's moment theory methods to the Hilbert transform. In particular, a generalization of the classical Herglotz–Bochner theorem, the GBT, yields a characterization of the pairs of measures for which the Hilbert transform operator is continuous in the corresponding weighted L^2 spaces. Yet the GBT, unlike the Bochner theorem, provides not only integral representations of bounded forms, but invariant extensions of them without norm increase. The GBT, therefore, is

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closely related with the far-reaching lifting theory of Sz.-Nagy and Foiaş: their lifting theorem for intertwining contractions can be obtained as a corollary of its abstract generalization to scattering systems (see Section 2 below). Moreover, immediate applications to harmonic analysis follow through appropriate integral representations of the extended forms.

This paper is a self-contained exposition of research, joint with Mischa Cotlar, extending over two decades. (The abbreviation C-S is used in references to our papers.) Shortly after initiating this program, we collaborated with Rodrigo Arocena, who later carried on this lifting approach in his significant work in a somewhat different direction. Developments parallel to ours occurred during the same period in the work of other schools, some of which is represented in this volume. In many cases the relation between these other developments and ours still remains obscure, and should be investigated further.

This exposition concentrates on the basic lifting results, presents sketches of their proofs, and outlines some applications. It traces the historical development of our approach, starting with the concrete example of kernels on the integers, followed by the abstraction of that example in the setting of scattering systems. For background material, the reader can consult [Nikol'skiĭ 1986].

Section 1 centers on the GBT, introducing it as a result on integral representation of positive generalized Toeplitz kernels, and then as an extension property for bounded Hankel forms. The Helson–Szegő theorem for the Hilbert transform and the Nehari theorem for Hankel operators are given as corollaries, as well as our results on boundedness for those operators acting in two different weighted spaces. It is observed that the GBT holds for matrix- and operator-valued kernels, with essentially the same proof as presented here.

In Section 2, algebraic and hilbertian scattering structures appear as the natural settings for bounded Hankel forms. Thus the lifting theorems 2.1 and 2.2 emerge as a natural extension of the GBT. A constructive proof is sketched for them, based on the Wold–Kolmogorov decomposition. Satisfactory integral representations of the Hankel forms are obtained in the special cases of Adamyan–Arov and Lax–Phillips scattering systems. Section 2 also includes a lifting theorem for forms defined in general semi-invariant subspaces, such as the internal state space of a hilbertian scattering system, under a condition of "essential invariance." The Sarason representation theorem for contractions commuting with a compression of the shift, and the Sz.-Nagy–Foiaş lifting theorem for intertwining contractions are given as corollaries. Section 2 ends with a conditional lifting theorem from which follows the abstract Adamyan–Arov–Kreĭn (AAK) theorem for singular numbers of Hankel operators, and some of its applications.

In Section 3 the Lifting Theorem is extended to bounded Hankel forms in pairs of scattering systems with several evolution groups (not necessarily commuting). This result is one of the most significant features of our program. Although a physical interpretation of this setting is not yet clear, it allows many applications to function spaces in several variables. In particular, it yields a noncommutative Nehari theorem for forms acting in the space of Hilbert–Schmidt operators.

Section 4 shows another sense in which noncommutativity can usefully be introduced into our theory: the GBT can be extended from positive definite numerical-valued kernels to completely positive kernels whose values are operator-valued sesquilinear forms on a C^*-algebra. A Nehari theorem for sequences defined in a C^*-algebra is given as an application. The results of this section are being published here for the first time.

This whole paper is focused on liftings in discrete scattering systems, where the evolution group is a unitary representation of $\mathbb{Z}$. Alternatively, continuous scattering systems can be considered, where the evolution group is a unitary representation of $\mathbb{R}$. The lifting theorems of Section 2, as well as those of Section 3 for several evolution groups, also hold in the continuous case [C-S 1988; 1990a; 1994b]. The continuous version of the theory is invoked in the present paper only at the end of Section 3, when we consider operators in the symplectic plane.

The object of this paper is to show that invariant (Hankel) forms, bounded with respect to quadratic invariant norms, have invariant (Toeplitz) liftings. Two significant extensions of the theory — left out of this exposition for the sake of brevity — treat the cases when those norms are invariant but not quadratic (e.g., L^p norms for $p \neq 2$), or quadratic but not invariant (e.g., Sobolev norms). For the first case, see [C-S 1989a], where the pertinent previous papers are summarized, and [C-S 1990b], where the Hilbert transform in weighted $L^p(\mathbb{T}^2)$ is studied. For the second, see [C-S 1991], where the problem is related to unitary extensions in Kreĭn spaces and to scattering systems with evolution operators unitary with respect to an indefinite metric.

Other significant extensions not presented here are the (local) nonlinear theorems of [C-S 1989b], and the study of stationary, harmonizable and generalized stationary processes in scattering systems [C-S 1984/85; 1988; 1989b].

1. The General Bochner Theorem and Some of Its Applications

Positive definite functions admit integral representation as Fourier transforms of positive measures. This classical result, fertile in applications, is the starting point of our exposition.

In the simplest case, a *positive definite* sequence, that is, a sequence $s : \mathbb{Z} \to \mathbb{C}$ satisfying

$$\sum_{m,n} s(m-n)\,\lambda(m)\,\overline{\lambda(n)} \geq 0 \quad \text{for all } \lambda : \mathbb{Z} \to \mathbb{C} \text{ finitely supported,} \tag{1–1}$$

is characterized as the Fourier coefficient sequence of a finite positive measure μ defined on $\mathbb{T}$:

$$s(n) = \hat{\mu}(n) := \int_{\mathbb{T}} e^{-int}\, d\mu \quad \text{for all } n \in \mathbb{Z}. \tag{1–2}$$

Note that we do not require strict inequality in (1–1).

Since (1–1) stands for the positive definiteness of the kernel $K\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$, defined by $K(m,n) := s(m-n)$, and thus satisfying the *Toeplitz condition*

$$K(m+1,\, n+1) = K(m,n) \quad \text{for all } n \in \mathbb{Z}, \tag{1–3}$$

the result can be stated as

Theorem 1.1 (Herglotz–Bochner). *A kernel $K : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ is positive definite and Toeplitz if and only if there exists a uniquely determined finite measure $\mu \ge 0$ on $\mathbb{T}$ such that*

$$K(m,n) = \hat{\mu}(m-n) \quad \textit{for all } m,n \in \mathbb{Z}. \tag{1–4}$$

In this section, this integral representation of positive kernels is extended to a class that includes the Toeplitz kernels and more. In this process a lifting property appears, having as corollaries classical theorems such as those of Helson–Szegő and Nehari — seemingly unrelated to the Herglotz–Bochner theorem — as well as new results.

Examples of numerical positive definite Toeplitz kernels are the autocorrelation kernels of stationary discrete (Hilbert space valued) stochastic processes $X : \mathbb{Z} \to H$, for H a Hilbert space, with autocorrelation $K(m,n) = \langle X(m), X(n)\rangle$. Not only is this kernel given by the Fourier coefficients of a positive measure μ (Herglotz–Bochner), but the process itself can be identified with the Fourier sequence of an orthogonally scattered bounded H-valued measure ν (Bochner–Khinchine):

$$X(n) = \hat{\nu}(n) \quad \text{for all } n \in \mathbb{Z}. \tag{1–5}$$

Among the simplest nonstationary processes (i.e., those whose kernels are not Toeplitz) are the harmonizable and the generalized stationary processes. Harmonizable processes admit a weakened version of representation (1–5) in terms of bounded (but not necessarily orthogonally scattered) ν. Alternatively, their kernels can be represented by an integration formally similar to (1–4), but involving not measures on $\mathbb{T}$ but bimeasures on $\mathbb{T}\times\mathbb{T}$. On the other hand, generalized stationary processes (those stationary except for one point) admit both a representation (1–4) in terms of positive numerical matrix-valued vector measures and a representation (1–5) in terms of pairs of bounded (mutually orthogonally scattered) vector-valued vector measures. This follows from Theorem 1.2 below.

Definition. A kernel $K : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ is a *generalized Toeplitz kernel* (GTK) if

$$K(m{+}1,\, n{+}1) = K(m,n) \quad \text{except possibly when } m=-1 \text{ or } n=-1. \tag{1–6}$$

The autocorrelation kernel of a generalized stationary process is a GTK. The analog of the Herglotz–Bochner theorem for these kernels is the following result, whose proof will be sketched later in the section.

THEOREM 1.2 (THE GENERALIZED BOCHNER THEOREM [C-S 1979]). *A kernel* $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ *is positive definite and a GTK if and only if there exists a positive* 2×2 *matrix* $\mu = (\mu_{ij})$ *of (complex) measures defined on* $\mathbb{T}$ *such that*

$$K(m,n) = \hat{\mu}_{ij}(m-n) \quad \text{for } (m,n) \in \mathbb{Z}_i \times \mathbb{Z}_j, \quad i,j = 1,2, \tag{1–7}$$

where $\mathbb{Z}_1 = \{k \in \mathbb{Z} : k \geq 0\}$ *and* $\mathbb{Z}_2 = \{k \in \mathbb{Z} : k < 0\}$.

For $\mu = (\mu_{ij})$ a 2×2 matrix of complex measures on $\mathbb{T}$, saying that μ is positive (denoted $\mu \geq 0$) is equivalent to saying that

$$\begin{aligned} &\mu_{11} \geq 0, \quad \mu_{22} \geq 0, \quad \mu_{21} = \overline{\mu_{12}}, \quad \text{and} \\ &|\mu_{12}(D)|^2 \leq \mu_{11}(D)\mu_{22}(D) \quad \text{for every Borel set } D \text{ in } \mathbb{T}. \end{aligned} \tag{1–8}$$

Since every Toeplitz kernel is obviously a GTK, the GBT (Theorem 1.2) includes the Herglotz–Bochner representation of the kernel, but here μ is *not* unique. This fact is at the heart of what makes the GBT not only a result on integral representation but on extension of forms. To show this it is helpful to rewrite the result as follows.

Let $\mathcal{P}$ be the set of trigonometric polynomials on $\mathbb{T}$. That is, $\mathcal{P}$ consists of finite sums of the form $f = \sum_n c_n e^{int}$. Given a hermitian kernel K, a sesquilinear form $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ can be defined by setting

$$B(e^{imt}, e^{int}) = K(m,n) \quad \text{for all } m, n \in \mathbb{Z},$$

and extending by linearity.

A form $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ is positive (that is, $B(f,f) \geq 0$ for all $f \in \mathcal{P}$) if and only if the corresponding kernel K is positive definite. If B is positive we write $B \geq 0$.

A kernel K is Toeplitz if and only if the corresponding form B is invariant under the shift operator $S : f \mapsto e^{it} f$, that is,

$$B(Sf, Sg) = B(f,g) \quad \text{for all } (f,g) \in \mathcal{P} \times \mathcal{P}. \tag{1–9}$$

The forms B satisfying (1–9) are called *Toeplitz* or S*-invariant in* $\mathcal{P} \times \mathcal{P}$.

In this setting, the Herglotz–Bochner theorem translates to: B *is positive and* S*-invariant in* $\mathcal{P} \times \mathcal{P}$ *if and only if there exists* $\mu \geq 0$ *such that*

$$B(f,g) = \int f \bar{g} \, d\mu \quad \text{for all } f, g \in \mathcal{P}. \tag{1–10}$$

Setting $\mathcal{P}_1 = \{f \in \mathcal{P} : f \text{ analytic}\}$ and $\mathcal{P}_2 = \{f \in \mathcal{P} : f \text{ antianalytic}\}$, we have $\mathcal{P} = \mathcal{P}_1 \dotplus \mathcal{P}_2$, and the domain of the form B splits into the four pieces

$\mathcal{P}_i \times \mathcal{P}_j$, for $i, j = 1, 2$. A weaker concept than S-invariance in $\mathcal{P} \times \mathcal{P}$ as in (1–9) is S-invariance in each $\mathcal{P}_i \times \mathcal{P}_j$; the form B has this property if

$$B(Sf, Sg) = B(f, g) \quad \text{for all } (f, g) \in (\mathcal{P}_1 \times \mathcal{P}_1) \cup (S^{-1}\mathcal{P}_2 \times S^{-1}\mathcal{P}_2) \cup (P_1 \times S^{-1}\mathcal{P}_2).$$

Then the GBT asserts that B *is positive in* $\mathcal{P} \times \mathcal{P}$ *and* S*-invariant in each* $\mathcal{P}_i \times \mathcal{P}_j$ *if and only if there exists* $\mu = (\mu_{ij}) \geq 0$ *such that*

$$B(f_1 + f_2,\, g_1 + g_2) = \sum_{i,j=1,2} \int f_i \bar{g}_j \, d\mu_{ij} \quad \text{for all } f_1, g_1 \in \mathcal{P}_1,\ f_2, g_2 \in \mathcal{P}_2. \tag{1–11}$$

Sketch of the proof of the GBT (for either forms or kernels). $B \geq 0$ in $\mathcal{P} \times \mathcal{P}$ (or K positive definite) gives rise to a (possibly degenerate) scalar product in $\mathcal{P}$. Under the usual procedure, there is a Hilbert space H and a linear operator $J : \mathcal{P} \to H$ such that $J\mathcal{P}$ is dense in H and $B(f, g) = \langle Jf, Jg \rangle$ for $f, g \in \mathcal{P}$. Consider the operator V defined by $V(Jf) := J(Sf)$ for $f \in \mathcal{P}$. If B were S-invariant in $\mathcal{P} \times \mathcal{P}$ (or K Toeplitz), V would extend to a unitary operator in H. As B is S-invariant only in each $\mathcal{P}_i \times \mathcal{P}_j$ (K is a GTK), V extends to an isometry in H, with domain $J\mathcal{P}_1 + J(S^{-1}\mathcal{P}_2)$ and range $J(S\mathcal{P}_1) + J(\mathcal{P}_2)$. As is well known, such an isometry extends to a unitary operator U in a larger Hilbert space $\mathcal{H} \supset H$. The cyclic pair $\xi_1 = J(1)$, $\xi_2 = J(e^{-it})$ in $\mathcal{H}$ gives

$$B(e^{imt}, e^{int}) = K(m, n) = \langle U^m U^{(i-1)} \xi_i, U^n U^{(j-1)} \xi_j \rangle$$

if $m \in \mathbb{Z}_i, n \in \mathbb{Z}_j$, for $i, j = 1, 2$. Now define $\mu = (\mu_{ij})$ by

$$\mu_{ij}(D) = \langle E(D)\xi_i,\, \xi_j \rangle \quad \text{for any Borel set } D \text{ in } \mathbb{T},$$

where E is the spectral measure of U. This gives the representation (1–7), (1–11). □

This proof holds also for operator-valued kernels $K : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(N)$, where N is a Hilbert space. Then $B : \mathcal{P}(N) \times \mathcal{P}(N) \to \mathbb{C}$ is the corresponding form, for $\mathcal{P}(N)$ the set of vector-valued trigonometric polynomials, $f = \sum_n \xi_n e^{int}$ a finite sum with $\xi_n \in N$, and $B(f, g) = \sum_{m,n} \langle K(m, n)\xi_m,\, \xi_n \rangle$.

Observe here that if B is S-invariant in $\mathcal{P} \times \mathcal{P}$ we have

$$B \geq 0 \iff \mu \geq 0,$$

whereas if B is S-invariant in each $\mathcal{P}_i \times \mathcal{P}_j$ we have

$$B \geq 0 \iff \sum_{i,j} \int f_i \bar{f}_j \, d\mu_{ij} \geq 0 \text{ only for } f_1 \in \mathcal{P}_1 \text{ and } f_2 \in \mathcal{P}_2, \tag{1–12}$$

which is far less than

$$\mu \geq 0 \iff \sum_{i,j} \int f_i \bar{f}_j \, d\mu_{ij} \geq 0 \text{ for all } f_1, f_2 \in \mathcal{P}, \tag{1–13}$$

To unveil what this discrepancy means, we look at the restrictions of $B \geq 0$ to $\mathcal{P}_i \times \mathcal{P}_j$. The restrictions $B_1 = B|(\mathcal{P}_1 \times \mathcal{P}_1)$ and $B_2 = B|(\mathcal{P}_2 \times \mathcal{P}_2)$ are also ≥ 0, and thus define (possibly degenerate) scalar products; in contrast, $B_0 = B|(\mathcal{P}_1 \times \mathcal{P}_2)$ is not positive, but it is bounded in the sense that

$$|B_0(f_1, f_2)| \leq B_1(f_1, f_1)^{1/2} B_2(f_2, f_2)^{1/2} \quad \text{for all } f_1 \in \mathcal{P}_1,\ f_2 \in \mathcal{P}_2. \tag{1–14}$$

Conversely, if $B_0 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$, $B_1 : \mathcal{P}_1 \times \mathcal{P}_1 \to \mathbb{C}$, and $B_2 : \mathcal{P}_2 \times \mathcal{P}_2 \to \mathbb{C}$ satisfy (1–14), the form $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ coinciding with each of them in the respective domain is positive.

If $B \geq 0$ is S-invariant in $\mathcal{P}_i \times \mathcal{P}_j$, by the GBT, the positive forms B_1 and B_2 are given by the positive measures μ_{11} and μ_{22}, while

$$B_0(f_1, f_2) = \int f_1 \bar{f}_2 \, d\mu_{12} \qquad \text{only for } f_1 \in \mathcal{P}_1,\ f_2 \in \mathcal{P}_2.$$

Furthermore, by (1–14) B_0 is bounded on $L^2(\mu_{11}) \times L^2(\mu_{22})$, with $\|B_0\| \leq 1$. But the complex measure μ_{12}, which is bounded by μ_{11} and μ_{22} in the sense of (1–8), defines a form B' in all of $\mathcal{P} \times \mathcal{P}$:

$$B'(f_1, f_2) := \int f_1 \bar{f}_2 \, d\mu_{12} \quad \text{for all } f_1, f_2 \in \mathcal{P}. \tag{1–15}$$

Thus the GBT not only gives an integral representation of B, but extends B_0 to all of $\mathcal{P} \times \mathcal{P}$, without increasing its norm!

Corollary 1.3 (Extension property for bounded S-invariant forms in $\mathcal{P}_1 \times \mathcal{P}_2$). *Given two positive measures μ_{11} and μ_{22} and a form $B_0 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ satisfying*

$$B_0(Sf_1, f_2) = B_0(f_1, S^{-1} f_2) \tag{1–16}$$

and

$$|B_0(f_1, f_2)| \leq \|f_1\|_{L^2(\mu_{11})} \|f_2\|_{L^2(\mu_{22})}, \tag{1–17}$$

there exists an S-invariant form $B' : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ such that $B'|(\mathcal{P}_1 \times \mathcal{P}_2) = B_0$ and $\|B'\| = \|B_0\|$.

Furthermore, *B' has the integral representation* (1–15) *in terms of a complex measure μ_{12} satisfying inequality* (1–8) *with respect to the given μ_{11} and μ_{22}.*

The value of the GBT as an *extension* result is highlighted when the forms are already given by an integral representation through measures. Such is the case for the first application of the GBT [C-S 1979], still the most striking, since, as we shall show, it provides a direct link between the lifting theory of Sz.-Nagy and Foiaş and the continuity of the Hilbert transform in weighted spaces.

Let H be the *Hilbert transform operator*, defined in $\mathcal{P} = \mathcal{P}_1 \dot{+} \mathcal{P}_2$ by

$$H(f_1 + f_2) = -i f_1 + i f_2 \quad \text{for } f_1 \in \mathcal{P}_1,\ f_2 \in \mathcal{P}_2. \tag{1–18}$$

The problem now is to characterize the positive measures μ and ν on $\mathbb{T}$ for which the *weighted norm inequality*

$$\int |Hf|^2 \, d\mu \leq M^2 \int |f|^2 \, d\nu \tag{1–19}$$

holds for all $f \in \mathcal{P}$. By (1–18), this can be rewritten as

$$\int |f_1 - f_2|^2 \, d\mu \le M^2 \int |f_1 + f_2|^2 \, d\nu \quad \text{for } f_1 \in \mathcal{P}_1,\ f_2 \in \mathcal{P}_2,$$

or, equivalently, as

$$\sum_{i,j=1,2} \int f_i \bar{f}_j \, d\rho_{ij} \ge 0 \quad \text{for } f_1 \in \mathcal{P}_1,\ f_2 \in \mathcal{P}_2, \tag{1–20}$$

where

$$\rho_{11} = \rho_{22} = M^2\nu - \mu, \quad \rho_{12} = \rho_{21} = M^2\nu + \mu \tag{1–21}$$

are positive measures on $\mathbb{T}$.

Define $B : \mathcal{P} \times \mathcal{P} \to \mathbb{C}$ by

$$B(f,g) = B(f_1 + f_2,\, g_1 + g_2) = \sum_{i,j=1,2} \int f_i \bar{g}_j \, d\rho_{ij}.$$

Then B is S-invariant in $\mathcal{P}_i \times \mathcal{P}_j$ by definition. Furthermore, if the ρ_{ij}'s are related with μ, ν, M via (1–21), condition (1–19) is equivalent to $B \ge 0$. Then, by the GBT, there exist measures μ_{ij}, for $i, j = 1, 2$, satisfying (1–8) and such that $\hat{\rho}_{11}(n) = \hat{\mu}_{11}(n)$, $\hat{\rho}_{22}(n) = \hat{\mu}_{22}(n)$ for all $n \in \mathbb{Z}$, while $\hat{\rho}_{12}(n) = \hat{\mu}_{12}(n)$ only for $n < 0$. By the uniqueness of the Fourier transform and the theorem of F. and M. Riesz for analytic measures, this is equivalent to

$$\mu_{11} = \mu_{22} = M^2\nu - \mu \text{ and } \mu_{12} = M^2\nu + \mu - h \text{ with } h \in H^1(\mathbb{T}). \tag{1–22}$$

Therefore, (1–8) implies this result:

THEOREM 1.4 (HELSON–SZEGŐ THEOREM FOR TWO MEASURES [C-S 1979]). *Given $\mu \ge 0$ and $\nu \ge 0$, the Hilbert transform H is a bounded operator from $L^2(\nu)$ to $L^2(\mu)$ with norm M if and only if there exists $h \in H^1(\mathbb{T})$ such that*

$$\left| (M^2\nu + \mu)(D) - \int_D h \, dt \right| \le (M^2\nu - \mu)(D) \quad \textit{for all Borel sets } D \subset \mathbb{T}. \tag{1–23}$$

In particular, μ is an absolutely continuous measure: $d\mu = w\, dt$ for some $w \in L^1$ satisfying $w \ge 0$.

In the case $\mu = \nu$, we have:

COROLLARY 1.5 ([C-S 1979]; see also [C-S 1983]). *For weights $\omega \in L^1(\mathbb{T})$ satisfying $\omega \ge 0$, the following conditions are equivalent:*

(i) *The Hilbert transform H is a bounded operator in $L^2(\omega)$ with norm M.*
(ii) *There is a positive constant M and an $h \in H^1(\mathbb{T})$ for which*

$$\left| (M^2+1) w(t) - h(t) \right| \le (M^2 - 1)\, \omega(t) \quad \textit{a.e. in } \mathbb{T}.$$

(iii) *There is some $h \in H^1(\mathbb{T})$ such that, for appropriate constants c, C, ε,*

$$\operatorname{Re} h(t) \ge c\omega(t), \quad |h(t)| \le C\omega(t), \quad \left| \arg h(t) \right| \le \pi/2 - \varepsilon, \quad \textit{a.e. in } \mathbb{T}.$$

(iv) *There exist real-valued bounded functions* u, v *such that*, *for appropriate constants* C *and* ε, *we have* $\|u\|_\infty \le C$, $\|v\|_\infty \le \pi/2 - \varepsilon$, *and* $\omega = \exp(u + Hv)$.

(v) *There exists a real-valued function* w *such that*, *for appropriate constants* c, C, M', *we have* $c\omega(t) \le w(t) \le C\omega(t)$ *and*

$$|Hw(t)| \le M' w(t) \quad \text{a.e. in } \mathbb{T}.$$

The constants in (i)–(v) *are related*; *in particular*, *the same* M *can be taken in* (i) *and* (ii).

Remark. The equivalence of (iv) with (i) for *some* M is the Helson–Szegő theorem [Helson and Szegő 1960]. The sufficiency of condition (v) had already appeared in work of Gaposhkin in the late 1950's, as noted in [Helson and Szegő 1960]; its direct equivalence with (i) is the most significant sharpening of the Helson–Szegő theorem provided by the GBT. In fact, it opened the way for the characterization of weights for which the Hilbert transform is bounded in $L^p(\omega)$, for $1 < p < \infty$, equivalent to the A_p condition (see [C-S 1983], and also [C-S 1990b] for further results in product spaces).

Remark. In the same context, another corollary of the GBT is the Helson–Sarason theorem on past and future [Helson and Sarason 1967], obtained by replacing $\mathcal{P}_1$ by $e^{ikt}\mathcal{P}_1$ for a positive integer k [Arocena, Cotlar, and Sadosky 1981]. The flexibility for making this change is an essential feature of this approach, as will be seen in Section 2.

On the other hand, the *integral representation* provided by the GBT for the extension of the S-invariant form $B_0 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ bounded in $L^2(\mu_1) \times L^2(\mu_2)$ is essential for obtaining another significant corollary, the Nehari theorem for Hankel operators.

Let H^2 be the closure of $\mathcal{P}_1$ in the norm of L^2, and H^2_-, that of $\mathcal{P}_2$. A linear operator $\Gamma : H^2 \to H^2_-$ is *Hankel* if

$$\Gamma S = (1 - P) S \Gamma \quad \text{for } P : L^2 \to H^2 \text{ the orthoprojector.} \tag{1–24}$$

An example of a bounded Hankel operator is given by a bounded *symbol* φ, i.e.,

$$\Gamma_\varphi : f \mapsto (1 - P)\varphi f, \quad \text{where } \varphi \in L^\infty(\mathbb{T}). \tag{1–25}$$

In fact,

$$\bigl|\langle \Gamma_\varphi f, g\rangle\bigr| = \left|\int \varphi f \bar{g}\, dt\right| \le \|\varphi\|_\infty \, \|f\|_2 \, \|g\|_2 \quad \text{for all } f \in \mathcal{P}_1,\ g \in \mathcal{P}_2,$$

and $\|\Gamma_\varphi\| \le \|\varphi\|_\infty$. Hankel operators may have many symbols, and

$$\Gamma_\varphi = \Gamma_\psi \iff \varphi - \psi = h \quad \text{for } h \text{ an analytic function.} \tag{1–26}$$

Set $B_\varphi(f, g) = \langle \Gamma_\varphi f, g\rangle$. Then $B_\varphi : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ is S-invariant and bounded, with $\|B_\varphi\| = \|\Gamma_\varphi\|$. Conversely, if $B : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ is bounded in $L^2 \times L^2$ and

S-invariant, the associated bounded operator Γ such that $\langle \Gamma f, g\rangle = B(f,g)$ is Hankel. This justifies calling the S-invariant forms in $\mathcal{P}_1 \times \mathcal{P}_2$ *Hankel forms.*

The Nehari theorem shows that all bounded Hankel operators are as those in the example above. This important result, key to the solution of many interpolation and moment problems, and essential in modern H^∞-control theory, follows immediate from Corollary 1.3.

THEOREM 1.6 [Nehari 1957]. *Let $\Gamma : H^2 \to H^2_-$ be a Hankel operator. Then Γ is bounded if and only if there exists $\varphi \in L^\infty$ such that $\Gamma = \Gamma_\varphi$ and $\|\varphi\|_\infty = \|\Gamma\|$.*

PROOF. Let $B_0 : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ be the bounded Hankel form corresponding to Γ, and consider the absolutely continuous measures $d\mu_{11} = d\mu_{22} = \|\Gamma\|\, dt$. From Corollary 1.3, there exists a complex measure μ_{12} such that

$$\langle \Gamma f, g\rangle = \int f\bar{g}\, d\mu_{12} \quad \text{for all } f \in \mathcal{P}_1, g \in \mathcal{P}_2,$$

satisfying

$$|\mu_{12}(D)| \le \|\Gamma\|\, |D| \quad \text{for all Borel sets } D \subset \mathbb{T}.$$

Therefore, $d\mu_{12} = \varphi\, dt$, with $\|\varphi\|_\infty \le \|\Gamma\|$, while $\Gamma = \Gamma_\varphi$. □

Thus, Corollary 1.3 is the *Nehari theorem for Hankel operators in weighted Hardy spaces*, $\Gamma : H^2(\mu_{11}) \to H^2_-(\mu_{22})$, where $H^2(\mu_{11})$ is the closure of $\mathcal{P}_1$ in the norm of $L^2(\mu_{11})$, while $H^2_-(\mu_{22})$ is the closure of $\mathcal{P}_2$ in the norm of $L^2(\mu_{22})$.

In [C-S 1993b] it was shown that when $L^2(\mu_{22}) = H^2(\mu_{22})$ every finite-rank Hankel operator is zero, while if $L^2(\mu_{22}) \ne H^2(\mu_{22})$ such operators admit a Kronecker-type representation of their symbols, in terms of the reproducing kernel of $H^2(\mu_{22})$.

In the case of absolutely continuous measures, we have the following characterization of the symbols of Hankel operators in two different weighted spaces.

COROLLARY 1.7 (NEHARI THEOREM IN TWO WEIGHTED SPACES [C-S 1993b]). *Let $\Gamma : H^2(w_1) \to H^2_-(w_2)$ be a Hankel operator, where w_1, w_2 are weights. The following conditions are equivalent*:

(a) $\|\Gamma\| = \|\Gamma\|_{H^2(w_1)\to H^2_-(w_2)} = 1$.

(b) *There exists $\varphi \in L^\infty$ such that $\|\varphi\|_\infty = 1$ and $\varphi\sqrt{w_1/w_2}$ is a symbol of Γ, i.e., $\Gamma f = P_2\varphi\sqrt{w_1/w_2} f$ for all $f \in H^2(w_1)$, where $P_2 : L^2(w_2) \to H^2_-(w_2)$ is the orthoprojector.*

(c) $\Gamma 1$ *is the unique symbol in $H^2_-(w_2)$.*

(d) *If ψ is a symbol of Γ, then $\psi = \Gamma 1 - h/w_2$, for some analytic function h.*

To summarize, even in the most elementary setting of trigonometric polynomials in $\mathbb{T}$, the GBT

- unifies the solution of problems from different areas,
- sharpens known results, and

- solves *new* problems, with the *same proofs* with which it solves old ones, while those new problems were not approachable with the classical methods.

The extensions and integral representations given by the GBT are valid (with essentially the same proofs) for *vector-valued* functions (i.e., for *operator-valued* kernels), as noted above. Thus, results such as the Nehari–Page theorem for Hankel operators $\Gamma : H^2(\mathcal{H}) \to H^2_-(\mathcal{H})$, for $\mathcal{H}$ a Hibert space [Page 1970], are part of the theory.

2. Lifting Theorems in Scattering Systems and Integral Representations

Results analogous to those in Section 1 are valid in abstract settings, provided they have an underlying "scattering structure." The Lax–Phillips scattering theory considers systems defined in a Hilbert space H, with outgoing and incoming spaces being closed subspaces of H, and evolutions given by one-parameter groups of unitary operators in $\mathcal{L}(H)$. In classical mechanics, however, one sometimes deals with groups of linear isomorphisms in other vector spaces, thus it is natural to consider also algebraic scattering systems, defined as follows.

A (discrete) *algebraic scattering system* $[V; W^+, W^-; \sigma]$ consists of a vector space V, two subspaces W^+, W^- of V, and a linear isomorphism $\sigma : V \to V$ such that the discrete group $\{\sigma^n : n \in \mathbb{Z}\}$ satisfies the scattering property

$$\sigma^n W^+ \subset W^+ \quad \text{and} \quad \sigma^{-n} W^- \subset W^- \quad \text{for all } n \geq 0. \tag{2–1}$$

W^+ and W^- are called, respectively, the *outgoing* and the *incoming* spaces of the system.

The trigonometric polynomials of Section 1 (with scalar or vector-valued coefficients) are an example of such a system for $V = \mathcal{P}$, $W^+ = \mathcal{P}_1$, $W^- = \mathcal{P}_2$, and $\sigma = S$. A more general example is a function system $[V(E); W^+(E), W^-(E); T]$, defined by an arbitrary set E, two subsets $E_1 \subset E$ and $E_2 \subset E$, and a bijection $T : E \to E$ such that $TE_1 \subset E_1$, $T^{-1}E_2 \subset E_2$, with

$$\begin{aligned} V(E) &:= \{f : E \to \mathbb{C} \text{ finitely supported}\}, \\ W^+(E) &:= \{f \in V : \operatorname{supp} f \subset E_1\}, \\ W^-(E) &:= \{f \in V : \operatorname{supp} f \subset E_2\}, \\ Tf(x) &:= f(Tx), \end{aligned}$$

for all $f \in V$ and $x \in E$. Setting $E = \mathbb{Z}$, $E_1 = \mathbb{Z}_1$, $E_2 = \mathbb{Z}_2$, and $T : n \mapsto n + 1$, we get back to the previous example by identifying the trigonometric polynomials with the finite sequences of their Fourier coefficients. In this example we have

$$E_1 \cup E_2 = E, \quad E_1 \cap E_2 = \varnothing. \tag{2–2}$$

In general, neither of the equalities in (2–2) need hold. But if both hold for some E, E_1, E_2, then for every kernel $K : E \times E \to \mathbb{C}$ there exists a unique sesquilinear form $B = B_K : V(E) \times V(E) \to \mathbb{C}$ satisfying $B(1_x, 1_y) = K(x, y)$, the correspondence $K \mapsto B$ is bijective, and statements concerning kernels translate to statements on forms and vice versa.

If V is a Hilbert space, the outgoing and incoming spaces are closed subspaces of it, and σ is a unitary operator acting in V, then $[V; W^+, W^-; \sigma]$ is called a *hilbertian scattering system*. This name is justified by the fact that, under the additional conditions

$$\bigcap_{n \geq 0} \sigma^n W^+ = \{0\} = \bigcap_{n \geq 0} \sigma^{-n} W^- \tag{2–3}$$

and

$$V = V^1 \vee V^2, \quad \text{where } V^1 := \bigvee_{n \in \mathbb{Z}} \sigma^n W^+, \ V^2 := \bigvee_{n \in \mathbb{Z}} \sigma^n W^-, \tag{2–4}$$

$[V; W^+, W^-, \sigma]$ is an *Adamyan–Arov* (A-A) *scattering system*, with evolution group $\{\sigma^n : n \in \mathbb{Z}\}$. If, furthermore,

$$W^+ \perp W^- \tag{2–5}$$

and

$$V = V^1 = V^2, \tag{2–6}$$

then $[V; W^+, W^-; \sigma]$ is a *Lax–Phillips* (L-P) *scattering system*. Condition (2–3) implies that the trajectory $\{\sigma^n f : n \in \mathbb{Z}\}$ of every $f \in W^+$ contains some element of the complement of W^+, and similarly for the trajectory of every $f \in W^-$, while (2–4) means that V is spanned by the trajectories.

Let $[V; W^+, W^-; \sigma]$ be any scattering system. By analogy with the trigonometric case, a sesquilinear form $B : V \times V \to \mathbb{C}$ is called *Toeplitz* in the system if

$$B(\sigma f, \sigma g) = B(f, g) \quad \text{for all } f, g \in V, \tag{2–7}$$

while a form $B_0 : W^+ \times W^- \to \mathbb{C}$ is called *Hankel* in it if

$$B_0(\sigma f, g) = B_0(f, \sigma^{-1} g) \quad \text{for all } f \in W^+, \ g \in W^-. \tag{2–8}$$

This is the analog to Corollary 1.3 in algebraic scattering systems:

Theorem 2.1 (Liftings of Hankel forms bounded with respect to Toeplitz forms in algebraic scattering systems [C-S 1987]). *Given an algebraic scattering system $[V; W^+, W^-; \sigma]$ and two positive Toeplitz forms B_1 and B_2 in it, for every Hankel form $B_0 : W^+ \times W^- \to \mathbb{C}$ bounded by B_1 and B_2 in the sense that*

$$|B_0(f, g)|^2 \leq B_1(f, f) B_2(g, g) \quad \textit{for all } f \in W^+, \ g \in W^-, \tag{2–9}$$

there exists a Toeplitz lifting $B : V \times V \to \mathbb{C}$ such that $B|(W^+ \times W^-) = B_0$ and

$$|B(f, g)|^2 \leq B_1(f, f) B_2(g, g) \quad \textit{for all } f, g \in V. \tag{2–10}$$

A short simple proof of this result is sketched in [C-S 1987]; see also [C-S 1990a] for details. Alternatively, Theorem 2.1 can be seen as a special case of Theorem 2.2 for hilbertian scattering systems, as follows.

Observe that, given an algebraic scattering system $[V; W^+, W^-; \sigma]$, a positive Toeplitz form $B_1 : V \times V \to \mathbb{C}$ can provide an inner product for a hilbertian system $[H_1; W_1^+, W_1^-; \sigma_1]$, where H_1 is the Hilbert space in which V can be identified as a dense subspace, W_1^+ and W_1^- are the closures in H_1 of W^+ and W^-, and σ extends to $\sigma_1 \in \mathcal{L}(H_1)$ as a unitary operator.

Thus, given an algebraic scattering system $[V; W^+, W^-; \sigma]$ and two positive Toeplitz forms B_1 and B_2, for any Hankel form B_0 satisfying (2–9), we can consider $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ as bounded in $H_1 \times H_2$, where, for $i = 1, 2$, H_i, W_i^+, and W_i^- are the closures of V, W^+, and W^-, respectively, in the norm induced by B_i.

A form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ bounded in $H_1 \times H_2$ is called *bounded in the pair of scattering systems* $[H_1; W_1^+, W_1^-; \sigma_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2]$.

In the next theorem, the hilbertian scattering systems satisfy only the defining condition (2–1), which is sufficient to insure the existence of Toeplitz liftings, and even to provide some form of integral representation for them. The case of A–A systems will be treated separately because the functional realization of these systems [Adamyan and Arov 1966] permits simplified integral representations.

THEOREM 2.2 (LIFTING THEOREM FOR HANKEL FORMS BOUNDED IN A PAIR OF HILBERTIAN SCATTERING SYSTEMS [C-S 1988; 1993a]). *Given two hilbertian scattering systems* $[H_1; W_1^+, W_1^-; \sigma_1]$ *and* $[H_2; W_2^+, W_2^-; \sigma_2]$, *every Hankel form* $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$, *bounded in the pair has a Toeplitz lifting* $B : H_1 \times H_2 \to \mathbb{C}$, $B|(W_1^+ \times W_2^-) = B_0$, *such that* $\|B\| = \|B_0\|$.

SKETCH OF PROOF. Since $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ is bounded in $H_1 \times H_2$, the hermitian form

$$\langle (f_1, f_2), (g_1, g_2) \rangle := \langle f_1, g_1 \rangle_{H_1} + \langle f_2, g_2 \rangle_{H_2} + B_0(f_1, g_2) + \overline{B_0(g_1, f_2)} \qquad (2\text{–}11)$$

is positive and gives a pre-Hilbert structure to $W_1^+ \times W_2^-$, where $\sigma : (f_1, f_2) \mapsto (\sigma_1 f_1, \sigma_2 f_2)$ is an isometry, with domain $W_1^+ \times \sigma_2^{-1} W_2^-$ and range $\sigma_1 W_1^+ \times W_2^-$. Then there is a unitary operator $T \in \mathcal{L}(N)$ in a larger Hilbert space $N \supset W_1^+ \times W_2^-$, such that $T = \sigma$ on its domain and $T^{-1} = \sigma^{-1}$ on the range. Identifying W_1^+ with $W_1^+ \times \{0\}$ and W_2^- with $\{0\} \times W_2^-$, we can consider W_1^+ and W_2^- as subspaces of N. Set

$$R_1 = W_1^+ \ominus \sigma_1 W_1^+ \quad \text{and} \quad R_2 = W_2^- \ominus \sigma_2^{-1} W_2^-. \qquad (2\text{–}12)$$

As a consequence of the Wold–Kolmogorov decomposition, H_1 and H_2 can be expressed with respect to σ_1 and σ_2, respectively, as

$$H_1 = \bigoplus_{n \in \mathbb{Z}} \sigma_1^n R_1 \oplus H_1^0 \oplus H_1^1, \qquad H_2 = \bigoplus_{n \in \mathbb{Z}} \sigma_2^{-n} R_2 \oplus H_2^0 \oplus H_2^1, \qquad (2\text{–}13)$$

where $H_1^1 = \bigcap_{n\ge 0} \sigma_1^n W_1^+$ and $H_2^1 = \bigcap_{n\ge 0} \sigma_2^{-n} W_2^-$, so that

$$\sigma_1 H_1^0 = H_1^0, \quad \sigma_1 H_1^1 = H_1^1, \quad \sigma_2^{-1} H_2^0 = H_2^0, \quad \sigma_2^{-1} H_2^1 = H_2^1,$$

and every $f_i \in H_i$ can be written as

$$f_i = \bigoplus_{n\in\mathbb{Z}} \sigma_i^{\pm n} f_{n,i} + f_i^0 + f_i^1 \text{ with } f_{n,i} \in R_i,\ f_i^0 \in H_i^0,\ f_i^1 \in H_i^1, \qquad \text{for } i = 1, 2. \tag{2–14}$$

For f_i as in (2–14), set

$$[f_1] := \bigoplus_n T^n f_{n,1} + f_1^1, \qquad [f_2] := \bigoplus_n T^{-n} f_{n,2} + f_2^1,$$

and, for $i = 1, 2$, set $J_i : H_i \to N$ by $J_i f_i = [f_i]$.

Observing that $\bigl\|[f_i]\bigr\|_N \le \|f_i\|_{H_i}$, for $i = 1, 2$, define a scalar product in $H_1 \times H_2$ by

$$\bigl\langle (f_1, f_2),\, (g_1, g_2)\bigr\rangle := \langle f_1, g_1\rangle_{H_1} + \langle f_2, g_2\rangle_{H_2} + \bigl\langle [f_1], [g_2]\bigr\rangle_N + \overline{\bigl\langle [g_1], [f_2]\bigr\rangle_N} \tag{2–15}$$

and, as before, call H the Hilbert space obtained from it. Hence, H contains $H_1 \equiv H_1 \times \{0\}$ and $H_2 \equiv \{0\} \times H_2$ as subspaces, and $U \in \mathcal{L}(H)$ is the unitary operator such that $U(f_1, f_2) = (\sigma_1 f_1, \sigma_2 f_2)$ for $(f_1, f_2) \in H_1 \times H_2$.

Define $B : H_1 \times H_2 \to \mathbb{C}$ by

$$B(f_1, f_2) = \langle f_1, f_2\rangle_H \quad \text{for all } f_1 \in H_1,\ f_2 \in H_2. \tag{2–16}$$

Then $\|B\| \le 1$, and it is not difficult to check that B is Toeplitz since

$$\bigl\langle [\sigma_1 f_1], [\sigma_2 f_2]\bigr\rangle_N = \bigl\langle [f_1], [f_2]\bigr\rangle_N, \quad \text{for all } f_1 \in H_1,\ f_2 \in H_2. \tag{2–17}$$

Furthermore, for any $f_1 \in W_1^+$ and $f_2 \in W_2^-$, we have $[f_1] = f_1$ and $[f_2] = f_2$. Therefore $B|(W_1^+ \times W_2^-) = B_0$, since

$$B(f_1, f_2) = \langle f_1, f_2\rangle_H = \bigl\langle [f_1], [f_2]\bigr\rangle_N = \langle f_1, f_2\rangle_N = B_0(f_1, f_2).$$

This completes the proof that B is the desired lifting of B_0. □

Remark. The spaces $R_1 = W_1^+ \ominus \sigma_1 W_1^+$ and $R_2 = W_2^- \ominus \sigma_2^{-1} W_2^-$ play in this proof the role played in the proof of the GBT (Section 1) by $\mathcal{P}_1 \ominus S\mathcal{P}_1 = \{c1\}$ and $\mathcal{P}_2 \ominus S^{-1}\mathcal{P}_2 = \{ce^{-it}\}$, where $J(1)$ and $J(e^{-it})$ were a cyclic pair. Here R_1 and R_2 need not be one-dimensional, but they are still cyclic sets.

The essence of the proof above is that there are two metrics defined in $W_1^+ \times W_2^- \subset H_1 \times H_2$, one by B_0 in (2–11), and the other induced in $H_1 \times H_2$ by (2–15). The maps $J_1 : H_1 \to N$ and $J_2 : H_2 \to N$ allow the transference of the metric in $W_1^+ \times W_2^-$ to the whole of $H_1 \times H_2$, providing the lifting.

From the proof of Theorem 2.2 it follows that, if E is the spectral measure of the unitary operator $T \in \mathcal{L}(N)$, for each pair of elements $(f_1, f_2) \in H_1 \times H_2$

there is a numerical measure μ_{f_1,f_2} defined on $\mathbb{T}$ by its values on the Borel sets $D \subset \mathbb{T}$:

$$\mu_{f_1,f_2}(D) := \big\langle E(D)J_1 f_1,\, J_2 f_2 \big\rangle_N. \tag{2–18}$$

On the pair $(f_1, f_2) \in H_1 \times H_2$, equation (2–14) gives the representation of the lifting $B : H_1 \times H_2 \to \mathbb{C}$ as

$$B(f_1,f_2) = \sum_{m,n} \int e^{i(m-n)t}\, d\mu_{f_{m,1},f_{n,2}} + \sum_n \int e^{int}\, d\mu_{f_1^1,f_{n,2}} + \sum_m \int e^{imt}\, d\mu_{f_{m,1},f_2^1} + \int d\mu_{f_1^1,f_2^1}. \tag{2–19}$$

In the particular case of forms acting in A-A scattering systems (including the Lax–Phillips type), the number of measures necessary for the integral representation can be substantially reduced (compare [C-S 1986; 1988]), which is not surprising since, for starts, $H_1^1 = H_2^1 = \{0\}$ in that situation.

A more economical integral representation of the lifting $B : H_1 \times H_2 \to \mathbb{C}$ in A-A scattering systems can be given by just *one* operator-valued measure, as done in [C-S 1988]. This is obtained through the functional realization of the systems given by Adamyan and Arov [1966]. For a scattering system $[H; W^+, W^-; \sigma]$ satisfying (2–3) and (2–4), their realization provides a scattering function $s : \mathbb{T} \to \mathcal{L}(R_1, R_2)$, where $R_1 = W^+ \ominus \sigma W^+$ and $R_2 = W^- \ominus \sigma^{-1} W^-$, with $\|s(t)\| \le 1$ for all $t \in \mathbb{T}$, and an isometric mapping $\mathcal{F}$ of H onto $L^2(R_2) \oplus L^2_\Delta(R_1)$ for which $\mathcal{F}(\sigma f)(t) = e^{it}\mathcal{F}(f)(t)$ for all $f \in H$ and $t \in \mathbb{T}$. Here $L^2(R_i) = L^2(\mathbb{T}; R_i)$, for $i = 1, 2$, and $L^2_\Delta(R_1)$ is the closure of the space

$$\{\phi \in L^2(R_1) : \phi = \Delta h \text{ for } h \in L^2(R_1)\},$$

where $\Delta(t) = (I_{R_1} - s^*(t)s(t))^{1/2}$.

Furthermore, there are two isometries, $j_1 : W^+ \to H^2(R_1)$ and $j_2 : W^- \to H^2_-(R_2)$, such that, for $(f_1, f_2) \in W^+ \times W^-$, we have

$$\begin{aligned} \mathcal{F}(f_2) &= j_2 f_2 \oplus \{0\}, \\ \mathcal{F}(f_1) &= s j_1 f_1 \oplus \Delta j_1 f_1, \\ j_1(\sigma f_1)(t) &= e^{it}(j_1 f_1)(t), \\ j_2(\sigma^{-1} f_2)(t) &= e^{-it}(j_2 f_2)(t). \end{aligned}$$

Let us consider this in the simpler case of a pair of scattering systems consisting of two copies of the same $[H; W^+, W^-; \sigma]$. Then $B_0 : W^+ \times W^- \to \mathbb{C}$ is a Hankel form bounded in $[H; W^+, W^-; \sigma]$, an A-A scattering system. Now, through the functional realization of the system, the Toeplitz extension $B : H \times H \to \mathbb{C}$ of B_0 can be written explicitly in all of $H \times H$ as

$$B(\sigma^m f_1, \sigma^n f_2) = \int \langle d\mu\, e^{imt}\varphi_1, e^{int}\varphi_2\rangle \quad \text{for all } m, n \in \mathbb{Z}, \tag{2–20}$$

for $\varphi_1 = j_1 f_1$, $\varphi_2 = j_2 f_2$, $f_1 \in W^+$, $f_2 \in W^-$, where μ is an $\mathcal{L}(R_1, R_2)$-valued measure on $\mathbb{T}$. Since $\mathcal{F}$ respects scalar products, we have

$$\begin{aligned} B_0(f_1, f_2) &= \big\langle (f_1, 0),\, (0, f_2)\big\rangle_H = \big\langle \mathcal{F}(f_1, 0),\, \mathcal{F}(0, f_2)\big\rangle_{L^2(R_2)\oplus L^2_\Delta(R_1)} \\ &= \int_{\mathbb{T}} \big\langle s(t)\varphi_1(t), \varphi_2(t)\big\rangle\, dt, \end{aligned}$$

so that $d\mu = s(t)\,dt$ is given by the scattering function of the system. In the case of a Lax–Phillips scattering system, s coincides with the Heisenberg scattering function as defined in [Lax and Phillips 1967]. Details are given in [C-S 1988].

Remark. The role of the operator-valued scattering function in the integral representation of the bounded Hankel form is played, in the trigonometric case, by the BMO functions. In fact, for $\phi \in L^2(\mathbb{T})$, a real-valued function, $B_\phi : \mathcal{P}_1 \times \mathcal{P}_2 \to \mathbb{C}$ is defined by $B_\phi(f, g) = \int f\bar{g}\phi$. Then B_ϕ is bounded in $L^2 \times L^2$ if and only if $B_\phi = B_\varphi$ for some $\varphi \in L^\infty$ with $\|\varphi\|_\infty \le \|B_\phi\|$ and $\phi - \varphi = h$, an analytic function. Then $\phi = \operatorname{Re}\phi = \operatorname{Re}\varphi + \operatorname{Re} h = \operatorname{Re}\varphi - H(\operatorname{Im} h) = \operatorname{Re}\varphi + H(\operatorname{Im}\varphi) \in L^\infty + HL^\infty \equiv \mathrm{BMO}(\mathbb{T})$.

Liftings of forms defined on general semi-invariant spaces. Whereas in the trigonometric example $V = \mathcal{P} = \mathcal{P}_1 \dot{+} \mathcal{P}_2 = W^+ + W^-$, in the general case of a hilbertian scattering system $[H; W^+, W^-; \sigma]$ in which $W^+ \perp W^-$, the subspace

$$W = H \ominus (W^+ \oplus W^-) \tag{2–21}$$

will be nontrivial, and will deserve study in itself. In major applications, it is normal to call this subspace the *internal state space*.

A subspace $V \subset H$ is called *semi-invariant* relative to a unitary $\sigma \in \mathcal{L}(H)$ if there exist V_1 and $V_2 \subset V_1$, both invariant under σ, such that

$$V = V_1 \ominus V_2,$$

or, equivalently, if there exist V_1' and $V_2' \subset V_1'$, both invariant under σ^{-1}, such that

$$V = V_1' \ominus V_2'. \tag{2–22}$$

Observe that in the setting of $[H; W^+, W^-; \sigma]$ the internal state space W defined in (2–21) is semi-invariant. For that matter, so are W^+, W^-, and their orthogonal complements.

The forms $B_0 : W_1 \times W_2 \to \mathbb{C}$ defined in a pair of (semi-invariant) internal state spaces are of special interest. Given a pair of hilbertian scattering systems $[H_1; W_1^+, W_1^-; \sigma_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2]$, the condition under which the lifting for bounded forms $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ was obtained is that of *invariance* with respect to σ_1 and σ_2, that is,

$$B_0(\sigma_1, f_1, f_2) = B_0(f_1, \sigma_2^{-1} f_2) \quad \text{for } f_1 \in W_1^+,\ f_2 \in W_2^-. \tag{2–23}$$

This condition makes sense since, by the scattering property (2–1), the domain $W_1^+ \times W_2^-$ is invariant under $(\sigma_1, \sigma_2^{-1})$.

This need not be true for $B_0 : W_1 \times W_2 \to \mathbb{C}$, where $W_i = H_i \ominus (W_i^+ \oplus W_i^-)$, for $i = 1, 2$, are only semi-invariant subspaces.

Thus we consider an invariance condition for $B_0 : W_1 \times W_2 \to \mathbb{C}$ that, although weaker than the Hankel condition (2–23), is still sufficient for the existence of invariant liftings in $H_1 \times H_2$.

A form $B_0 : W_1 \times W_2 \to \mathbb{C}$ is called *essentially invariant* in a pair of scattering systems if

$$B_0(P_1\sigma_1 f_1, f_2) = B_0(f_1, P_2\sigma_2^{-1} f_2) \quad \text{for all } f_1 \in W_1,\ f_2 \in W_2, \tag{2–24}$$

where $P_1 : H_1 \to W_1$ and $P_2 : H_2 \to W_2$ are the orthoprojectors. In the special case when $W_1^+ = \{0\} = W_2^-$, we have $\sigma_1 W_1 \subset W_1$ and $\sigma_2^{-1} W_2 \subset W_2$, and the condition (2–24) reduces to $B_0 : W_1 \times W_2 \to \mathbb{C}$ being Hankel. Conversely, for $[H_i; W_i^+, W_i^-; \sigma_i]$ and $W_i = H_i \ominus (W_i^+ \oplus W_i^-)$, setting

$$V_1 = W_1 \oplus W_1^+ \quad \text{and} \quad V_2 = W_2 \oplus W_2^-,$$

it is easy to see that

$$\sigma_1 V_1 \subset V_1 \quad \text{and} \quad \sigma_2^{-1} V_2 \subset V_2. \tag{2–25}$$

Let a bounded essentially invariant form $B_0 : W_1 \times W_2 \to \mathbb{C}$ be given. If we define $B_0^{\#} : V_1 \times V_2 \to \mathbb{C}$ by

$$B_0^{\#}(f_1 + f_1^+,\ f_2 + f_2^-) := B_0(f_1, f_2), \tag{2–26}$$

it is not difficult to check that $\|B_0^{\#}\| = \|B_0\|$, and that $B_0^{\#}$ is Hankel in $V_1 \times V_2$.

From (2–26) and Theorem 2.2 we obtain:

Corollary 2.3 (Liftings for bounded essentially Hankel forms defined in semi-invariant subspaces in scattering systems [C-S 1993a]). *Consider two scattering systems* $[H_1; W_1^+, W_1^-; \sigma_1]$ *and* $[H_2; W_2^+, W_2^-; \sigma_2]$ *with internal spaces* $W_i = H_i \ominus (W_i^+ \oplus W_i^-)$, *for* $i = 1, 2$, *and a bounded form* $B_0 : W_1 \times W_2 \to \mathbb{C}$, *essentially Hankel in the sense of* (2–24). *Then*:

(a) *There exists a Toeplitz form* $B : H_1 \times H_2 \to \mathbb{C}$ *such that* $B|(W_1 \times W_2) = B_0$ *and* $\|B\| = \|B_0\|$.

(b) *Furthermore*, $B = 0$ *on* $W_1^+ \times W_2^-$, $W_1 \times W_2^-$, *and* $W_1^+ \times W_2$.

Corollary 2.3 has important applications. For example, we now use both parts of it to provide a simple proof of Sarason's interpolation theorem, without relying on Beurling's characterization of the invariant subspaces of $H^2(\mathbb{T})$.

Theorem 2.4 [Sarason 1967]. *Let* $W \subset H^2(\mathbb{T})$ *be a subspace invariant under the shift*, S, *let* $K = H^2 \ominus W$ *be the model space, and let* $T = P_K S|K$ *be the compression of* S *to* K. *For each contraction* $A \in \mathcal{L}(K)$ *commuting with* T, *there exists a bounded holomorphic function* a *satisfying* $\|a\|_\infty \le 1$ *and*

$$Af = P_K(af) \quad \textit{for all } f \in K. \tag{2–27}$$

Proof. In Corollary 2.3, take $H_1 = H_2 = L^2(\mathbb{T})$, $W_1 = W_2 = K$, $W_1^+ = W_2^+ = W$, $W_1^- = W_2^- = L^2 \ominus H^2 = H^2_-$, $\sigma_1 = \sigma_2 = S$, and define $B_0 : W_1 \times W_2 \to \mathbb{C}$ by

$$B_0(f,g) := \langle Af, g\rangle \quad \text{for all } f,g \in K. \tag{2–28}$$

Then $\|A\| \le 1$ is equivalent to $\|B_0\| \le 1$, and $AT = TA$ is equivalent to B_0 being essentially Hankel as in (2–24). Then, by Corollary 2.3, there is a bounded Toeplitz lifting $B : L^2 \times L^2 \to \mathbb{C}$ of B_0, with $\|B\| \le 1$. The restriction to trigonometric polynomials of the Toeplitz lifting B is given by $B(f,g) = L(f\bar g)$, for L a linear functional. Since $|B(f,g)| \le \|f\|_2\,\|g\|_2$, we have $|L(f)| \le \|f\|_1$, and there exists $a \in L^\infty$ with $\|a\|_\infty \le 1$ and such that $B(f,g) = \int af\bar g\,dt$. From $B = B_0$ in $K \times K$ follows the representation (2–27), while from $B = 0$ in $(K \oplus W) \times H^2_- = H^2 \times H^2_-$ it follows that $a1 \perp H^2_-$, which means $a \in H^2 \cap L^\infty = H^\infty$. □

The most important result for which Corollary 2.3 gives an immediate proof is the Lifting Theorem for intertwining contractions of Sz.-Nagy and Foiaş:

Theorem 2.5 [Sz.-Nagy and Foiaş 1967]. *Let $T_1 \in \mathcal{L}(H_1)$ and $T_2 \in \mathcal{L}(H_2)$ be two contractions, with strong unitary dilations $U_1 \in \mathcal{L}(\mathcal{H}_1)$ and $U_2 \in \mathcal{L}(\mathcal{H}_2)$. If $X : H_1 \to H_2$ is a contraction intertwining T_1 and T_2, that is, such that $XT_1 = T_2X$, then there exists a contraction $Y : \mathcal{H}_1 \to \mathcal{H}_2$ such that Y intertwines U_1, U_2 that is, $YU_1 = U_2Y$, and*

$$X = P_{H_2} Y | H_1, \tag{2–29}$$

where $P_{H_2} : \mathcal{H}_2 \to H_2$ is the orthoprojector.

Proof. By Sarason's Lemma [Sarason 1965], for $i = 1, 2$, the fact that U_i is a strong dilation of T_i means that $\mathcal{H}_i = H_i \oplus H_i^+ \oplus H_i^-$, where $U_i H_i^+ \subset H_i^+$ and $U_i^{-1} H_i^- \subset H_i^-$, and $T_i = P_{H_i} U_i | H_i$. Defining $B_0 : H_1 \times H_2 \to \mathbb{C}$ by

$$B_0(f_1, f_2) = \langle Xf_1, f_2\rangle_{H_2} \quad \text{for all } f_1 \in H_1,\ f_2 \in H_2,$$

the intertwining condition for X is equivalent to B_0 being essentially Hankel in U_1, U_2. From $B : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$, the lifting of B_0 in Corollary 2.3, define $Y : \mathcal{H}_1 \to \mathcal{H}_2$ by

$$\langle Yf_1, f_2\rangle_{\mathcal{H}_2} := B(f_1, f_2) \quad \text{for all } f_1 \in \mathcal{H}_1,\ f_2 \in \mathcal{H}_2.$$

Then $\|B\| \le 1$ is equivalent to $\|Y\| \le 1$, and the property that B is Toeplitz for U_1, U_2 is equivalent to $YU_1 = U_2Y$. Finally, $B = B_0$ in $H_1 \times H_2$ is equivalent to (2–29). □

Conditional liftings. To every form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ bounded in a pair of hilbertian scattering systems we associate a bounded operator $\Gamma : W_1^+ \to W_2^-$, by setting $\langle \Gamma f, g \rangle = B_0(f,g)$. Then $\|\Gamma\| = \|B\|$. If the form B_0 is Hankel, the operator satisfies the Hankel condition

$$\Gamma\sigma_1 = P_-\sigma_2\Gamma \quad \text{for } P_- : H_2 \to W_2^- \text{ the orthoprojector.} \tag{2–30}$$

The singular numbers s_n of B_0, for $n \geq 0$, are

$$s_n(B_0) := s_n(\Gamma) = \inf\bigl\{\|\Gamma - T_n\| : \operatorname{rank} T_n \leq n\bigr\}. \tag{2–31}$$

A form $B^{(n)} : H_1 \times H_2 \to \mathbb{C}$ is called an *n-conditional lifting* of B_0 if there exists a subspace $M_n \subset W_1^+$ of codimension at most n such that

$$B^{(n)}|(M_n \times W_2^-) = B_0 \quad \text{and} \quad \|B^{(n)}\| \leq s_n(B_0). \tag{2–32}$$

Theorem 2.6 (Existence of n-conditional liftings for Hankel forms bounded in hilbertian scattering systems [C-S 1993b]). *Given a Hankel form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$, bounded in a pair of hilbertian scattering systems $[H_1; W_1^+, W_1^-; \sigma_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2]$, there exists for every integer $n \geq 0$ a Toeplitz n-conditional lifting of B_0.*

Sketch of proof. Given n, let $s_n = s_n(B_0)$, and let Γ be the bounded Hankel operator associated with B_0. Considering the set

$$K = \{f \in W_1^+ : \|\Gamma f\| \leq s_n \|f\|\},$$

it is easy to check that $\sigma_1 K \subset K$. By a particular case of [Treil' 1985, Theorem 2], there exists a subspace $M_n \subset W_1^+$ such that $\operatorname{codim} M_n \leq n$, $M_n \subset K$, and $\sigma_1 M_n \subset M_n$. Since $\sigma_1 M_n \subset M_n$ and $\sigma_2^{-1} W_2^- \subset W_2^-$, and since $M_n \subset K$ implies that the restriction $B_0|(M_n \times W_2^-)$ has norm bounded by s_n, we can apply Theorem 2.2 to the Hankel form $B_0|(M_n \times W_2^-)$, bounded in the systems $[H_1; M_n, W_1^-; \sigma_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2]$, to obtain a Toeplitz form $B^{(n)} : H_1 \times H_2 \to \mathbb{C}$ such that $B^{(n)}|(M_n \times W_2^-) = B_0$ and $\|B^{(n)}\| \leq s_n$. □

Corollary 2.7 (Abstract Adamyan–Arov–Kreĭn theorem [C-S 1993b]). *Given a pair of hilbertian scattering systems $[H_1; W_1^+, W_1^-; \sigma_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2]$, and a bounded Hankel operator $\Gamma : W_1^+ \to W_2^-$, there exists for each integer $n \geq 0$ a Hankel operator Γ_n of finite rank at most n and such that*

$$\|\Gamma - \Gamma_n\| = s_n(\Gamma). \tag{2–33}$$

Proof. For $B^{(n)}$ as in Theorem 2.6, let $\tilde{\Gamma} : W_1^+ \to W_2^-$ be the operator associated to the form $B^{(n)}|(W_1^+ \times W_2^-)$. Setting $\Gamma_n := \Gamma - \tilde{\Gamma}$, we have $\|\Gamma - \Gamma_n\| = \|\tilde{\Gamma}\| \leq s_n = s_n(\Gamma)$. Furthermore, Γ_n is Hankel, and since by definition it vanishes on M_n, its rank is at most n. □

In the case when, for $i = 1, 2$, $H_i = L^2(\mathbb{T}; \mu_i)$ for μ_i a positive measure on $\mathbb{T}$, while $W_1^+ = H^2(\mathbb{T}; \mu_1)$ and $W_2^- = H^2_-(\mathbb{T}; \mu_2)$, we have the following consequence of Theorem 2.6 together with Corollary 1.7:

Corollary 2.8 (Weighted AAK theorem [C-S 1993b]). *Given a bounded Hankel form $B : H^2(\mathbb{T};\mu_1) \times H^2_-(\mathbb{T};\mu_2) \to \mathbb{C}$, where μ_1 and μ_2 are positive measures on $\mathbb{T}$, there exist for every integer $n \geq 0$ a complex measure μ on $\mathbb{T}$ and a subspace $M_n \subset H^2(\mathbb{T};\mu_1)$, of codimension at most n, such that*

$$B(f,g) = \int f\bar{g}\,d\mu \quad \text{for all } f \in M_n, g \in H^2_-(\mathbb{T};\mu_2), \tag{2–34}$$

while

$$s_n(B) \geq \sup_D \frac{|\mu(D)|}{\mu_1(D)^{1/2}\mu_2(D)^{1/2}},$$

where the supremum is taken over all Borel sets $D \subset \mathbb{T}$.

Remark. When both μ_1 and μ_2 are the Lebesgue measure on $\mathbb{T}$, Corollary 2.8 is the classical AAK theorem [Adamyan, Arov and Kreĭn 1971]. If, moreover, $n = 0$, we recover the classical Nehari theorem [1957]. Furthermore, if μ_2 is a deterministic measure, i.e., if $L^2(\mu_2) = H^2(\mu_2)$, then every Hankel form of finite rank in $H^2(\mu_1) \times H^2_-(\mu_2)$ is zero, while in the opposite case such a form can be represented in terms of the reproducing kernel of $H^2(\mu_2)$ [C-S 1993b].

3. Lifting of Forms Invariant with Respect to Several Evolution Groups. Some Applications to Analysis in Product Spaces

The lifting theorems of the preceding section extend to forms invariant in scattering systems having several evolution groups. In order to avoid notational complications, here we present only the case of two evolutions, but all concepts and results are general.

For simplicity, we consider only hilbertian scattering systems of the form $[H; W^+, W^-; \sigma, \tau]$, where both $\sigma \in \mathcal{L}(H)$ and $\tau \in \mathcal{L}(H)$ are unitary operators satisfying

$$\sigma\tau = e^{ia}\tau\sigma \text{ for some } a \in \mathbb{R}, \tag{3–1}$$

and such that W^+ is invariant with respect to both σ and τ, while W^- is invariant with respect to both σ^{-1} and τ^{-1}. Compare (2–1).

We are now concerned with forms invariant with respect to both evolution groups $\{\sigma^n : n \in \mathbb{Z}\}$ and $\{\tau^n : n \in \mathbb{Z}\}$. More precisely, given a pair of scattering systems $[H_1; W_1^+, W_1^-; \sigma_1, \tau_1]$ and $[H_2; W_2^+, W_2^-; \sigma_2, \tau_2]$ as described above, a form $B : H_1 \times H_2 \to \mathbb{C}$ is *Toeplitz* in the pair if

$$B(\sigma_1 f_1, \sigma_2 f_2) = B(f_1, f_2) = B(\tau_1 f_1, \tau_2 f_2) \quad \text{for all } f_1 \in H_1,\ f_2 \in H_2. \tag{3–2}$$

A form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ is *Hankel* in the pair if

$$B_0(\sigma_1 f_1, f_2) = B_0(f_1, \sigma_2^{-1} f_2) \quad \text{and} \quad B_0(\tau_1 f_1, f_2) = B_0(f_1, \tau_2^{-1} f_2) \quad \text{for all } f_1 \in W_1^+,\ f_2 \in W_2^-. \tag{3–3}$$

Theorem 2.2 applied to a Hankel form in a scattering system with two evolutions σ and τ provides two liftings of the form, one invariant with respect to $\{\sigma^n\}$ and the other invariant with respect to $\{\tau^n\}$, but it does not provide liftings invariant with respect to both groups.

A full extension of Theorem 2.2 providing lifting with respect to all evolutions of the scattering systems cannot hold in general. This follows from the relation between Theorem 2.2 and the Sz.-Nagy–Foiaş Lifting Theorem (compare Section 2, as well as [C-S 1987; 1993a]), since the latter does not have a full extension to two pairs of intertwining operators. What we can obtain is partial liftings, in the following sense.

Given a Hankel form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$, two Toeplitz forms $B' : H_1 \times H_2 \to \mathbb{C}$ and $B'' : H_1 \times H_2 \to \mathbb{C}$ form a *Toeplitz lifting pair* (B', B'') for B_0 if

$$\|B'\| \le \|B_0\|, \quad \|B''\| \le \|B_0\|, \tag{3–4}$$

$$B'|(W_1^+ \times W_2^\sigma) = B_0, \text{ and } B''|(W_1^+ \times W_2^\tau) = B_0, \tag{3–5}$$

where

$$\begin{aligned} W_2^\sigma &= \{f \in W_2^- : \sigma_2^k f \in W_2^- \quad \text{for all } k \in \mathbb{Z}\}, \\ W_2^\tau &= \{f \in W_2^- : \tau_2^k f \in W_2^- \quad \text{for all } k \in \mathbb{Z}\}. \end{aligned} \tag{3–6}$$

The following result provides Toeplitz lifting pairs for bounded Hankel forms in a pair of scattering systems with two evolutions. Its proof does not follow directly from a repeated use of Theorem 2.2, where a system with two evolutions $[H; W^+, W^-; \sigma, \tau]$ is taken as two single-evolution systems $[H; W^+, W^-; \sigma]$ and $[H; W^+, W^-; \tau]$, but requires an argument based on Banach limits.

THEOREM 3.1 [C-S 1990a; 1993a]. *Given a pair of scattering systems with two evolutions*, $[H_1; W_1^+, W_1^-; \sigma_1, \tau_1]$ *and* $[H_2; W_2^+, W_2^-; \sigma_2, \tau_2]$, *every Hankel form* $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ *bounded in the pair has a Toeplitz lifting pair* (B', B'') *in the sense of* (3–4)–(3–6).

REMARK. Theorem 3.1 holds equally when W_2^σ, W_2^τ are replaced by W_1^σ, W_1^τ, or by W_1^σ, W_2^τ, or by W_2^σ, W_1^τ, defined similarly to (3–6).

SKETCH OF PROOF. Consider B_0 as a Hankel form in the pair $[H_i; W_i^+, W_i^-; \sigma_i]$, $i = 1, 2$. Then Theorem 2.2 gives a lifting $B^\sigma : H_1 \times H_2 \to \mathbb{C}$ satisfying $\|B^\sigma\| = \|B_0\|$, $B^\sigma|(W_1^+ \times W_2^-) = B_0$, and $B^\sigma(\sigma_1 f_1, \sigma_2 f_2) = B^\sigma(f_1, f_2)$ for all $f_1 \in H_1$, $f_2 \in H_2$. Similarly, Theorem 2.2 gives another lifting $B^\tau : H_1 \times H_2 \to \mathbb{C}$ satisfying $\|B^\tau\| = \|B_0\|$, $B^\tau|(W_1^+ \times W_2^-) = B_0$, and $B^\tau(\tau_1 f_1, \tau_2 f_2) = B^\tau(f_1, f_2)$ for all $f_1 \in H_1$, $f_2 \in H_2$.

For a fixed pair $(f_1, f_2) \in H_1 \times H_2$ and any positive integer k, we have

$$|B^\tau(\sigma_1^k f_1, \sigma_2^k f_2)| \le \|B^\tau\| \, \|\sigma_1^k f_1\|_{H_1} \, \|\sigma_2^k f_2\|_{H_2} = \|B^\tau\| \, \|f_1\|_{H_1} \, \|f_2\|_{H_2}, \tag{3–7}$$

and $\{B^\tau(\sigma_1^k f_1, \sigma_2^k f_2)\}$ is a bounded numerical sequence, for which there is a Banach–Mazur limit. Thus, define $B' : H_1 \times H_2 \to \mathbb{C}$ by

$$B'(f_1, f_2) := \mathrm{LIM}_k \, B^\tau(\sigma_1^k f_1, \sigma_2^k f_2) \quad \text{for all } f_1 \in H_1, \ f_2 \in H_2. \tag{3–8}$$

The form B' is sesquilinear by the properties of the Banach–Mazur limits, and $\|B'\| \le \|B^\tau\| = \|B_0\|$ by (3–7). Also, for $f_1 \in H_1, f_2 \in H_2$, we have

$$B'(\sigma_1 f_1, \sigma_2 f_2) = \mathrm{LIM}_k\, B^\tau(\sigma_1^{k+1} f_1, \sigma_2^{k+1} f_2) = \mathrm{LIM}_k\, B^\tau(\sigma_1^k f_1, \sigma_2^k f_2) = B'(f_1, f_2).$$

Furthermore, (3–1), the sesquilinearity of B^τ, and its invariance with respect to τ_1, τ_2 yield

$$\begin{aligned} B'(\tau_1 f_1, \tau_2 f_1) &= \mathrm{LIM}_k\, B^\tau(\sigma_1^k \tau_1 f_1, \sigma_2^k \tau_2 f_2) = \mathrm{LIM}_k\, B^\tau(\tau_1 \sigma_1^k f_1, \tau_2 \sigma_2^k f_2) \\ &= \mathrm{LIM}_k\, B^\tau(\sigma_1^k f_1, \sigma_2^k f_2) = B'(f_1, f_2). \end{aligned}$$

Finally, since $B^\tau = B_0$ in $W_1^+ \times W_2^-$, and since $(f_1, f_2) \in W_1^+ \times W_2^\sigma$ implies $(\sigma_1^k f_1, \sigma_2^k f_2) \in W_1^+ \times W_2^-$ for all $k \ge 0$, we have

$$B^\tau(\sigma_1^k f_1, \sigma_2^k f_2) = B_0(\sigma_1^k f_1, \sigma_2^k f_2) = B_0(f_1, f_2) \quad \text{for all } k \ge 0,$$

and $B' = B_0$ in $W_1^+ \times W_2^\sigma$.

Defining the form $B'' : H_1 \times H_2 \to \mathbb{C}$ by

$$B''(f_1, f_2) = \mathrm{LIM}_k\, B^\sigma(\tau_1^k f_1, \tau_2^k f_2) \quad \text{for all } f_1 \in H_1, f_2 \in H_2,$$

we obtain the lifting pair (B', B'') for B_0. □

Observe that while the lifting B for a Hankel form in a pair of scattering systems with one evolution group determines the form B_0 in the whole of its domain $W_1^+ \times W_2^-$, the lifting pair (B', B'') in the case of systems with two evolution groups determines B_0 only in the two subspaces $W_1^+ \times W_2^\sigma$ and $W_1^+ \times W_2^\tau$ of the domain. Thus, the value of Theorem 3.1 in applications is determined by the relation between these subspaces. When, for instance,

$$W_2^- = W_2^\sigma + W_2^\tau \tag{3–9}$$

holds, in the sense that for each $f \in W_2^-$ there are $g \in W_2^\sigma$ and $h \in W_2^\tau$ such that $f = g + h$, the lifting pair determines B_0. This is the case in many examples in analysis, as shown below. In particular, Theorem 3.1 provides as corollaries multidimensional analogs of the one-dimensional Helson–Szegő and Nehari–Adamyan–Arov–Kreĭn theorems that were given in Section 1 as corollaries of the GBT, as well as related results in symplectic spaces and the Heisenberg group [C-S 1990a].

Applications to some classical operators in the two-dimensional torus. In what follows, fix $H_1 = H_2 = L^2(\mathbb{T}^2)$,

$$W_1^+ = H^2(\mathbb{T}^2) = \{f \in L^2 : \hat f(m, n) = 0 \text{ for } m < 0 \text{ or } n < 0\},$$

and $W_2^- = L^2 \ominus H^2 = H^{2\perp}$. Let $\sigma = S_x$ and $\tau = S_y$ be the shifts in each variable of $\mathbb{T}^2$, that is, $S_x : f(x, y) \mapsto e^{ix} f(x, y)$ and $S_y : f(x, y) \mapsto e^{iy} f(x, y)$. Then

$$\begin{aligned} W_2^\tau &= H^2_{-x} = \{f \in L^2 : \hat f(m, n) = 0 \text{ for } m \ge 0\}, \\ W_2^\sigma &= H^2_{-y} = \{f \in L^2 : \hat f(m, n) = 0 \text{ for } n \ge 0\}. \end{aligned}$$

Therefore (3–9) holds in this case, since $H^{2\perp} = H^2_{-x} + H^2_{-y}$.

REMARK. If, with the same H_1, H_2, W_1^+ and the same two shifts, we define $W_2^- = \{f \in L^2 : \hat{f}(m,n) = 0 \text{ for } m \geq 0\}$, then $W_2^\sigma = \{0\}$, while $W_2^\tau = W_2^-$. This example has applications that will not be explored here. On the other hand, choosing $W_2^- = \overline{H^2} = \{f \in L^2 : \hat{f}(m,n) = 0 \text{ for } m \geq 0 \text{ or } n \geq 0\}$, we have $W_2^\sigma = W_2^\tau = \{0\}$, and no lifting is obtained.

In this setting, since the Bochner theorem is valid in $\mathbb{T}^d$, for any $d \geq 1$, every positive Toeplitz form $B : L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2) \to \mathbb{C}$ can be represented in terms of a positive measure μ on $\mathbb{T}^2$, by $B(f,g) = \int f\bar{g}\,d\mu$.

By Theorem 3.1, any bounded Hankel form $B_0 : H^2 \times H^{2\perp} \to \mathbb{C}$ has a lifting pair of Toeplitz forms B' and B''. If μ' and μ'' are the measures on $\mathbb{T}^2$ corresponding to B' and B'', respectively, B_0 has an integral representation

$$B_0(f,g) = \begin{cases} B'(f,g) = \int f\bar{g}\,d\mu' & \text{for } f \in H^2,\ g \in H^2_{-y}, \\ B''(f,g) = \int f\bar{g}\,d\mu'' & \text{for } f \in H^2,\ g \in H^2_{-x}, \end{cases} \tag{3–10}$$

which implies $\hat{\mu}'(m,n) = \hat{\mu}''(m,n)$ for $m < 0$ and $n < 0$. Since both forms B' and B'' are bounded in $L^2 \times L^2$, (3–10) implies that

$$d\mu' = \varphi_1\,dx\,dy,\ d\mu'' = \varphi_2\,dx\,dy, \text{ with } \|\varphi_1\|_\infty \leq \|B_0\|,\ \|\varphi_2\|_\infty \leq \|B_0\|. \tag{3–11}$$

As in the one-dimensional case, to the bounded form B_0 is associated a bounded operator $\Gamma : H^2 \to H^{2\perp}$, called a *big Hankel operator*, which satisfies

$$\Gamma S_x = (1-P)S_x\Gamma,\ \Gamma S_y = (1-P)S_y\Gamma, \text{ for } P : L^2 \to H^2 \text{ orthoprojector.} \tag{3–12}$$

Again, a function $\phi \in L^2(\mathbb{T}^2)$ is called a *symbol* of Γ if $\Gamma = \Gamma_\phi$ for $\langle \Gamma_\phi f, g\rangle = \int f\bar{g}\phi$.

COROLLARY 3.2 (NEHARI THEOREM FOR BIG HANKEL OPERATORS IN THE TORUS [C-S 1993b]). *Let $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ be a big Hankel operator. Then $\|\Gamma\| \leq 1$ if and only if there exist two bounded functions $\varphi_1, \varphi_2 \in L^\infty(\mathbb{T}^2)$ satisfying $\|\varphi_1\|_\infty \leq 1$, $\|\varphi_2\|_\infty \leq 1$, $\hat{\varphi}_1(m,n) = \hat{\varphi}_2(m,n)$ for $m < 0$ and $n < 0$, and*

$$\langle \Gamma f, g\rangle = \begin{cases} \int f\bar{g}\varphi_1 & \text{for } f \in H^2,\ g \in H^2_{-y}, \\ \int f\bar{g}\varphi_2 & \text{for } f \in H^2,\ g \in H^2_{-x}. \end{cases}$$

Hence, we fail to assign a bounded symbol to Γ, but we get a "pair of partial symbols" φ_1, φ_2.

In [C-S 1994a] we introduced the space BMOr of functions $\phi \in L^2(\mathbb{T}^2)$ that can be expressed as

$$\phi = \varphi_1 + h_x = \varphi_2 + h_y = \varphi_0 + h^\perp \tag{3–13}$$

for some $\varphi_1, \varphi_2, \varphi_0 \in L^\infty(\mathbb{T}^2)$ and some $h_x \in H^2_x$, $h_y \in H^2_y$, and $h^\perp \in H^{2\perp}$. BMOr is a normed space under

$$\|\phi\|_{\mathrm{BMOr}} := \inf\big\{\max\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty, \|\varphi_0\|_\infty\} : \text{all decompositions (3–13)}\big\}.$$

Then Corollary 3.2 can be rewritten as: A big Hankel operator Γ is bounded if and only if there exists $\phi \in$ BMOr such that $\Gamma = \Gamma_\phi$ and

$$\|\phi\|_{\mathrm{BMOr}} \leq \|\Gamma\| \leq \sqrt{2}\,\|\phi\|_{\mathrm{BMOr}}.$$

The notion of BMOr and this formulation of the Nehari theorem in $\mathbb{T}^2$ have been fully explored in [C-S 1996].

Here we want only to underline that the Lifting Theorem 3.1 is the appropriate tool for obtaining a new description of the bounded Hankel operators in several-dimensional spaces, which presents important differences with the classical one-dimensional theory [C-S 1993b; 1994a; 1996]. To give the flavor of these fundamental differences, we state two results, omitting the proofs, which are based on the Lifting Theorems 3.1 and 2.6 and their corollaries.

THEOREM 3.3 [C-S 1993b]. *Given a bounded big Hankel operator* $\Gamma : H^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$, *its singular numbers* $s_n(\Gamma)$ *satisfy*

$$s_n(\Gamma) \geq 2^{-1/2}\|\Gamma\| \quad \textit{for all } n \geq 0.$$

An immediate consequence of this result is that there are no nonzero Hankel operators in $\mathbb{T}^2$ either of finite rank or compact; thus no AAK theory of approximation, in the sense of [Adamyan, Arov and Kreĭn 1971], can be developed. (For a substitute approach based on so-called "sigma numbers," see [C-S 1996].)

The structure of the space BMOr provides the next result:

THEOREM 3.4 [C-S 1996]. *There are bounded big Hankel operators in* $\mathbb{T}^2$ *without bounded symbols.*

Obviously, the big Hankel operators defined by bounded symbols are themselves bounded. The surprising result of Theorem 3.4 leaves open the question: Given the bounded big Hankel operator Γ_φ defined by $\varphi \in L^\infty(\mathbb{T}^2)$, is there $\psi \in L^\infty(\mathbb{T}^2)$ such that $\Gamma_\psi = \Gamma_\varphi$ and $\|\psi\|_\infty$ is equivalent to $\|\Gamma_\varphi\|$? This question was posed in lectures and in [C-S 1996], with the suggestion that a positive answer was unlikely. Recently, two ingenious constructive negative answers have been given independently by Ferguson [1997] and by Bakonyi and Timotin [1997]. This inequivalence establishes the essential role of BMOr.

Another interesting consequence of the Lifting Theorem 3.1 is the following characterization of the weights ω for which the product Hilbert transform $H = H_x H_y$ is continuous in $L^2(\omega)$.

THEOREM 3.5 (HELSON–SZEGŐ THEOREM IN $\mathbb{T}^2$ [C-S 1990b]). *For a weight* $0 \leq \omega \in L^1(\mathbb{T}^2)$ *the following conditions are equivalent* (*with related constants*):

(i) *The product Hilbert transform* $H = H_x H_y$ *is continuous in* $L^2(\omega)$ *with norm* M.

(ii) *There exist real-valued functions* $u_1, u_2, v_1, v_2 \in L^\infty(\mathbb{T}^2)$ *and constants* C, ε *such that* $\|u_i\|_\infty \leq C$ *and* $\|v_i\|_\infty \leq \pi/2 - \varepsilon$ *for* $i = 1, 2$, *and*

$$\log \omega = u_1 + H_x v_1 = u_2 + H_y v_2.$$

(iii) *There exist real-valued functions w_1 and w_2 and constants C, c, M', such that $c\omega \le w_i \le C\omega$ for $i = 1, 2$, and*

$$|H_x w_1| \le M' w_1, |H_y w_2| \le M' w_2 \ a.e. \ in \ \mathbb{T}^2.$$

For the details of the proof see [C-S 1990b], where a general result is also given for two weights, and for H acting in $L^p(\omega)$, $p \ne 2$.

It is noteworthy that condition (ii) in Theorem 3.5 is equivalent to $\varphi = \log w \in$ bmo, the proper subspace of product $\mathrm{BMO}(\mathbb{T}^2)$ consisting of functions of *bounded mean oscillation on rectangles*. For the relations and properties of $\mathrm{bmo}(\mathbb{T}^2) \subsetneq \mathrm{BMOr}(\mathbb{T}^2) \subsetneq \mathrm{BMO}(\mathbb{T}^2)$, see [C-S 1996].

The applications above followed from Theorem 3.1 through the integral representations of the lifting pairs provided by the classical Bochner theorem in $\mathbb{T}^2$. Applications to operators acting on the symplectic plane come from the continuous analog of Theorem 3.1 as follows (see [C-S 1990a]).

Consider $(\mathbb{C}, [\ ,\ ])$, the symplectic plane under the symplectic form $[z_1, z_2] = -\mathrm{Im}\, z_1 \overline{z_2}$, for $z_1, z_2 \in \mathbb{C}$. A unitary representation of the symplectic plane on a Hilbert space H is a function $z \mapsto W(z)$ assigning to each point z a unitary operator $W(z) \in \mathcal{L}(H)$, with $W(0) = I$ and satisfying the Weyl–Segal relation

$$W(z_1)W(z_2) = \exp(\pi_i [z_1, z_2]) W(z_1 + z_2).$$

All irreducible representations are unitarily equivalent to the Schrödinger representation on $H = L^2(\mathbb{R})$, defined by $z \mapsto w(z)$, with

$$(w(\zeta)\varphi)(z) = \exp(-\pi i [z, \zeta])\, \varphi(z + \zeta) \quad \text{for } \zeta = (s, t), z = (x, y) \in \mathbb{C}.$$

For every $f \in L^1(\mathbb{R}^2)$ its *Weyl transform* is the bounded operator in $L^2(\mathbb{R})$ defined by

$$W(f) := \int_{\mathbb{R}^2} f(x, y)\, w(-x, -y)\, dx\, dy. \tag{3–14}$$

The Weyl transform establishes an isometric isomorphism between $L^2(\mathbb{R}^2)$ and $\mathcal{L}^2 = \mathcal{L}^2(L^2(\mathbb{R}))$, the space of Hilbert–Schmidt operators, which is a Hilbert space with scalar product $\langle A_1, A_2 \rangle = \mathrm{tr}\, A_2^* A_1$, for $A_1, A_2 \in \mathcal{L}^2$. Moreover, the product of operators corresponds to the twisted convolution in the symplectic plane,

$$W(f)W(g) = W(f \natural g) \quad \text{for } f, g \in L^2(\mathbb{R}^2),$$

where

$$f \natural g(z) = \int f(z)\, g(z - \zeta) \exp(i\pi [z, \zeta])\, d\zeta.$$

Under the Weyl isomorphism the regular representation $\{W(z) : z \in \mathbb{C}\}$ of the symplectic plane in $L^2(\mathbb{R}^2)$ passes into the unitary representation $\{w(z) : z \in \mathbb{C}\}$ of the symplectic plane in $\mathcal{L}^2(L^2(\mathbb{R}))$.

Set $\Delta = \{z = x + iy \in \mathbb{C} : x \geq 0, y \geq 0\}$. For $H = \mathcal{L}^2(L^2(\mathbb{R}))$, considering the subspaces

$$\begin{aligned} W^+ &= \{A \in \mathcal{L}^2 : A = W(f),\ \operatorname{supp} f \subset \Delta\}, \\ W^- &= \{A \in \mathcal{L}^2 : A = W(f), \operatorname{supp} f \subset \Delta^c\}, \end{aligned} \tag{3–15}$$

and defining the evolution groups $\{\sigma_t\}$ and $\{\tau_t\}$ by

$$\sigma_t A = w(t + i0)A, \quad \tau_t A = w(0 + it)A, \quad \text{for all } A \in \mathcal{L}^2,\ t \in \mathbb{R}, \tag{3–16}$$

we obtain a *continuous scattering system* $[H; W^+, W^-; \sigma_t, \tau_t, t \in \mathbb{R}]$ since (3–1) is satisfied (although σ_s and τ_t do not commute, we have $\sigma_s \tau_t = \exp(i\pi st)\tau_t \sigma_s$, for all $t, s \in \mathbb{R}$).

A sesquilinear form $B : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{C}$ is Toeplitz in this scattering system if

$$B(\sigma_t A_1, \sigma_t A_2) = B(A_1, A_2) = B(\tau_t A_1, \tau_t A_2) \quad \text{for all } t \in \mathbb{R} \text{ and all } A_1, A_2 \in \mathcal{L}^2. \tag{3–17}$$

In $\mathcal{L}^2$ *measures* are replaced by their quantized analogs, *states* or trace class operators. A trace class operator $S \in \mathcal{L}^1(L^2(\mathbb{R}))$ satisfying $S \geq 0$ defines a form $B_S : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{C}$ by

$$B_S(A_1, A_2) = \operatorname{tr} S A_2^* A_1. \tag{3–18}$$

This definition keeps its sense when S is a bounded operator. The form B_S is Toeplitz and, for S bounded, B_S is continuous in the $\mathcal{L}^2$-topology, while for S trace class, B_S is bounded in the $\mathcal{L}^\infty$-topology of compact operators.

The following result of N. Wallach (compare [C-S 1990a]), later extended in [C-S 1990c], provides the representation for Toeplitz forms in this setting:

Theorem 3.6 (Bochner theorem for the unitary representation of the symplectic plane). *Given a Toeplitz form $B : \mathcal{L}^2 \times \mathcal{L}^2 \to \mathbb{C}$, continuous in $\mathcal{L}^2$, there exists a bounded operator S in $L^2(\mathbb{R})$ such that*

$$B(A_1, A_2) = B_S(A_1, A_2) = \operatorname{tr} S A_2^* A_1 \quad \textit{for all } A_1, A_2 \in \mathcal{L}^2. \tag{3–19}$$

Furthermore, if $B \geq 0$, then $S \geq 0$, and if B is continuous in $\mathcal{L}^\infty$, then S is a trace class operator.

Since the norm in $\mathcal{L}^2$ is given by $\|A\|_{\mathcal{L}^2} = \operatorname{tr} A^* A$, the expression $\operatorname{tr} S A^* A =: \|A\|_{\mathcal{L}^2(S)}$, for a bounded operator $S \geq 0$, can be considered as a "weighted" norm in $\mathcal{L}^2(S)$.

The continuous version of the Lifting Theorem 3.1 and the Representation Theorem 3.6 together imply:

Corollary 3.7 (Quantized Nehari theorem in weighted $\mathcal{L}^2$). *Consider a pair of quantized scattering systems $[\mathcal{L}^2(S_i); W_i^+, W_i^-; \sigma_t, \tau_t, t \in \mathbb{R}]$, for $i = 1, 2$, where the $S_i \geq 0$ are bounded operators, the W_i^+, W_i^- are as in* (3–15), *and*

σ_t, τ_t *as in* (3–16). *A Hankel form* $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ *in the pair is bounded*, $\|B_0\| \le 1$, *if and only if there exist two bounded operators* S' *and* S'' *satisfying*

$$\max\bigl\{|\operatorname{tr} S'A_2^*A_1|, |\operatorname{tr} S''A_2^*A_1|\bigr\} \le \|A_1\|_{\mathcal{L}^2(S_1)}\|A_2\|_{\mathcal{L}^2(S_2)}, \tag{3–20}$$

for all $A_1 \in \mathcal{L}^2(S_1)$, $A_2 \in \mathcal{L}^2(S_2)$, *and representing* B_0 *in the sense that*

$$B_0(A_1, A_2) = \begin{cases} \operatorname{tr} S'A_2^*A_1 & \textit{for } A_1 \in W_1^+,\ A_2 \in W_2^\sigma, \\ \operatorname{tr} S''A_2^*A_1 & \textit{for } A_1 \in W_1^+,\ A_2 \in W_2^\tau, \end{cases} \tag{3–21}$$

where the spaces

$$W_2^\sigma = \{A \in W_2^- : \sigma_t A \in W_2^- \textit{ for all } t \in \mathbb{R}\},$$
$$W_2^\tau = \{A \in W_2^- : \tau_t A \in W_2^- \textit{ for all } t \in \mathbb{R}\}$$

satisfy $W_2^\sigma + W_2^\tau = W_2^-$.

For $S \in \mathcal{L}^1(L^2(\mathbb{R}))$ a trace class operator, and for $s \in L^2(\mathbb{C})$, the equation $(W^{-1}S)(z) = s(z)$ in $\operatorname{Im} z > 0$ (or in $\operatorname{Re} z > 0$) is equivalent to

$$\operatorname{tr} SW(g)^*W(f) = \int s(z)(f \natural g^*)(z)\,dz = s(f \natural g^*)$$
$$\text{for } (W(f), W(g)) \in W^+ \times W^\sigma \text{ (or } W^+ \times W^\tau). \tag{3–22}$$

For S a bounded operator, $W^{-1}S$ is not defined, but the expression (3–22) still makes sense.

THEOREM 3.8 (NEHARI THEOREM FOR OPERATORS IN $L^2(\mathbb{R})$ [C-S 1990a]). *For a given linear functional* s *in* $L^2(\mathbb{R}^2)$, *the following conditions are equivalent*:

(i) *There exist two bounded operators* S' *and* S'' *in* $L^2(\mathbb{R})$, *of norm at most* 1, *satisfying*

$$W^{-1}S' = s \textit{ in } \operatorname{Im} z > 0, \quad W^{-1}S'' = s \textit{ in } \operatorname{Re} z > 0$$

in the sense of (3–22).

(ii) *We have* $\bigl|s(f \natural g^*)\bigr| \le \|f\|_{L^2}\,\|g\|_{L^2}$ *whenever* $\operatorname{supp} f \subset \Delta$, *and* $\operatorname{supp} g \subset \Delta^c$. (Recall that Δ is the closed first quadrant of the complex plane.)

For other results in the symplectic plane and in the dual of the Heisenberg group, see [C-S 1990a; 1990c; 1994b].

4. Lifting Theorem for Completely Positive Definite Kernels

Section 3 concluded with a noncommutative application of the lifting theorem in scattering systems with several evolution groups. This section deals with an entirely different way in which noncommutative objects can usefully be brought into the scope of our lifting theory.

The idea here is to replace the commutative algebra $C(X)$ of continuous functions by a general C^*-algebra, and the positive linear forms on $C(X)$ (i.e., measures), by completely positive maps on the C^*-algebra.

The framework and some results from [C-S 1994c] on lifting and integral representation of kernels defined through completely positive maps are summarized below, concluding with a Nehari theorem for sequences in C^*-algebras. These results are being published here for the first time.

First, let us look again at the commutative set-up to be generalized.

If X is a group and H is a Hilbert space, every function $f : X \to \mathcal{L}(H)$ gives rise to a kernel $K : X \times X \to \mathcal{L}(H)$ defined by $K(x, y) = f(y^{-1}x)$, and K is *positive definite* if and only if f is positive definite, i.e., if

$$\sum_{j,k} \langle f(x_k^{-1}x_j)\xi_j, \, \xi_k\rangle \geq 0 \tag{4–1}$$

for all finite sets $\{x_j\} \subset X$, $\{\xi_j\} \subset H$.

A kernel K defined by a function is invariant under the group action $\sigma_z : x \mapsto zx$, that is, $K(\sigma_z x, \sigma_z y) = K(x, y)$ for all $z, x, y \in X$. Conversely, every invariant kernel is defined by a function, as above.

How does this translate when functions are replaced by maps in a C^*-algebra?

If $\mathcal{A} \subset \mathcal{L}(H)$ is a unital C^*-algebra of operators, every linear map $\varphi : \mathcal{A} \to \mathcal{L}(H)$ gives rise to a kernel $K : \mathcal{A} \times \mathcal{A} \to \mathcal{L}(H)$ defined by $K(A, B) = \varphi(B^*A)$, and K is positive definite if and only if φ is *completely positive*, i.e., if

$$\sum_{j,k} \langle \varphi(A_k^* A_j)\xi_j, \, \xi_k\rangle \geq 0 \tag{4–2}$$

for all finite sets $\{A_j\} \subset \mathcal{A}$, $\{\xi_j\} \subset \mathcal{H}$.

In this setting, a kernel defined by a map is invariant under every unitary operator U, that is, $K(UA, UB) = K(A, B)$ for all $U, A, B \in \mathcal{A}$, but the converse need not hold.

In his study of quantum probability problems, Holevo [1988] combined the notions of positive definiteness and complete positivity by considering kernels Φ defined by a function f whose values $f(x)$, for $x \in X$, are linear maps, $\mathcal{A} \to \mathcal{L}(H)$. The kernel Φ is defined by $\Phi(x, y) = f(y^{-1}x)$, and is positive definite if and only if each $f(y^{-1}x)$ is completely positive, i.e., if

$$\sum_{j,k} \langle f(x_k^{-1}x_j)(A_k^* A_j)\xi_j, \, \xi_k\rangle \geq 0 \tag{4–3}$$

for all finite sets $\{x_j\} \subset X$, $\{A_j\} \subset \mathcal{A}$, $\{\xi_j\} \subset H$.

When the group $X = \{e\}$ consists of a single point, Φ is positive definite if and only if $\Phi(e)$ is completely positive. When $\mathcal{A} = \{cI\}$ is one-dimensional, Φ is positive definite if and only if f is positive definite. Observe that these kernels are invariant both under the group action σ_z, for $z \in X$, and under the left multiplication by unitary $U \in \mathcal{A}$. In the simplest case, when $X = \mathbb{Z}$, $H \cong \mathbb{C}$, and $\mathcal{A} \cong \mathbb{C}$, the map $\sigma = \sigma_1$ is the shift in $\mathbb{Z}$, $\sigma_1(n) = n+1$ for all $n \in \mathbb{Z}$, and Φ has domain in $\mathbb{Z} \times \mathbb{Z}$ and numerical values. These are the kernels of Section 1, and it is natural to seek the generalization of the GBT for general kernels Φ under suitable invariance conditions.

For this purpose we consider a setting somewhat more general than that of Holevo. Instead of assuming that the values $\Phi(x, y)$ are already defined as invariant under unitary operators, by $\Phi(x, y)(A, B) = \phi(x, y)(B^*A)$, we assume $\Phi(x, y)(A, B)$ to be any sesquilinear form in $\mathcal{A} \times \mathcal{A}$ with values in $\mathcal{L}(H)$, on which we impose invariance conditions under certain groups of automorphisms of the algebra $\mathcal{A}$. Furthermore, we assume X to be any set, and not necessarily a group. This more general setting for the kernels is chosen for two reasons. In the first place, positive definite kernels $\Phi : X \times X \to \mathcal{L}(\mathcal{A} \times \mathcal{A}, \mathcal{L}(H))$ can be considered as Hilbert space reproducing kernels. Secondly, in order to obtain invariant liftings, it is necessary to restrict the invariance condition with respect to $\mathcal{A}$ imposed to the kernels. Among the more general kernels having as values sesquilinear forms in $\mathcal{A} \times \mathcal{A}$ to $\mathcal{L}(H)$, those satisfying

$$\Phi(x, y)(A, B) = \Phi(x, y)(B^*A, I) \quad \text{for all } A, B \in \mathcal{A} \text{ and } x, y \in X, \tag{4–4}$$

are called of *Holevo type*.

For X a set, and $\mathcal{A} \subset \mathcal{L}(H)$ a unital C^*-algebra of operators, a kernel $\Phi : X \times X \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A}, \mathcal{L}(H))$ is positive definite if and only if

$$\sum_{jk} \langle \Phi(x_j, x_k)(A_j, A_k)\xi_j,\, \xi_k\rangle \geq 0 \tag{4–5}$$

for all finite sets $\{x_j\} \subset X$, $\{A_j\} \subset \mathcal{A}$, $\{\xi_j\} \subset H$. In what follows, in abuse of language, such a Φ is referred to as *completely positive definite*.

Fix a bijection $\sigma : X \to X$. The kernel Φ is called *σ-invariant* or *Toeplitz* if

$$\Phi(\sigma x, \sigma y) = \Phi(x, y) \quad \text{for all } x, y \in X. \tag{4–6}$$

For two fixed subsets X_1, X_2 satisfying $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \varnothing$, and

$$\sigma X_1 \subset X_1, \quad \sigma^{-1} X_2 \subset X_2, \tag{4–7}$$

a kernel $\Phi_0 : X_1 \times X_2 \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A}, \mathcal{L}(H))$ is *σ-invariant* in $X_1 \times X_2$ or *Hankel* if

$$\Phi_0(\sigma x, y) = \Phi_0(x, \sigma^{-1} y) \quad \text{for all } x \in X_1,\ y \in X_2. \tag{4–8}$$

Just as in Section 1, to give three kernels Φ_1, Φ_2, Φ_0, where $\Phi_1 : X_1 \times X_1 \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A},\, \mathcal{L}(H))$ and $\Phi_2 : X_2 \times X_2 \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A},\, \mathcal{L}(H))$ are Toeplitz and $\Phi_0 : X_1 \times X_2 \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A},\, \mathcal{L}(H))$ is Hankel, is the same as to give a GTK $\Phi : X \times X \to \mathcal{L}(\mathcal{A}{\times}\mathcal{A},\, \mathcal{L}(H))$ such that $\Phi|(X_1{\times}X_1) = \Phi_1$, $\Phi|(X_2{\times}X_2) = \Phi_2$, and $\Phi|(X_1{\times}X_2) = \Phi_0$. In this situation we write $\Phi \sim (\Phi_1, \Phi_2, \Phi_0)$. Such a GTK

Φ is completely positive definite if and only if Φ_1 and Φ_2 are completely positive definite and $\Phi_0 \le (\Phi_1, \Phi_2)$ in $X_1 \times X_2$, that is,

$$\left|\sum_{j,k} \langle \Phi_0(x_j, y_k)(A_j, B_k)\xi_j, \eta_k\rangle\right|^2 \\ \le \left(\sum_{j,k} \langle \Phi_1(x_j, x_k)(A_j, A_k)\xi_j, \xi_k\rangle\right)\left(\sum_{j,k} \langle \Phi_2(y_j, y_k)(B_j, B_k)\eta_j, \eta_k\rangle\right) \quad (4\text{–}9)$$

for all finite sets $\{x_j\} \subset X_1$, $\{y_j\} \subset X_2$, $\{A_j\}, \{B_j\} \subset \mathcal{A}$, $\{\xi_j\}, \{\eta_j\} \subset H$.

For G a group and $\alpha : G \to \mathcal{L}(\mathcal{A})$ a representation of G by linear maps $\alpha_\gamma : \mathcal{A} \to \mathcal{A}$, for $\gamma \in G$, we call $\{\mathcal{A}, G, \alpha\}$ a *linear dynamical system*. Here we restrict ourselves to the case when G is an amenable group and $\alpha_\gamma(A) = U_\gamma A$, for $\gamma \mapsto U_\gamma$, a unitary representation of G satisfying $U_\gamma(\mathcal{A}) \subset \mathcal{A}$ for all $\gamma \in G$.

A kernel Φ is called *invariant with respect to such linear dynamical system*, or, simply, α-*invariant* if

$$\Phi(x, y)\big(\alpha_\gamma(A), \alpha_\gamma(B)\big) = \Phi(x, y)(A, B) \quad \text{for all } \gamma \in G,\ A, B \in \mathcal{A},\ x, y \in X. \quad (4\text{–}10)$$

For instance, if Φ is of Holevo type, it is α-invariant for $\alpha_\gamma : A \mapsto U_\gamma A$, for all $U_\gamma \in \mathcal{A}$ unitary. But such invariance does not, in general, imply that Φ must be of Holevo type.

There is a large class of linear dynamical systems $\{\mathcal{A}, G, \alpha\}$, with $\alpha_\gamma(A) = U_\gamma A$, such that α-invariance of a kernel Φ *is* equivalent to Φ being of Holevo type. We call systems of this class *reducing*. Examples of reducing systems and their representations are (i) the Schrödinger representation, when $\mathcal{A}$ is the algebra of Hilbert–Schmidt operators acting in $L^2(\mathbb{R})$, and (ii) a unitary representation of a nilpotent Lie group G, when $\mathcal{A}$ is the Schwartz algebra of operators $\{A_f : f \in \mathcal{S}(G)\}$, for $A_f = \int_G f(\gamma)U_\gamma d\gamma$, and $\mathcal{S}(G)$ the Schwartz class of functions in G (compare [du Cloux 1989]).

Given a completely positive definite kernel Φ, set $Y = X \otimes \mathcal{A} \otimes H$ and define a numerical kernel $K : Y \times Y \to \mathbb{C}$ by

$$K((x, A, \xi), \big(y, B, \eta)) := \langle \xi,\ \Phi(x, y)(A, B)\eta\rangle \quad (4\text{–}11)$$

for all $(x, A, \xi), (y, B, \eta) \in Y$. Let $\mathcal{H}$ be the Hilbert space spanned by the functions $K_{xA\xi} : Y \to \mathbb{C}$ given by

$$K_{xA\xi}(y, B, \eta) = K((x, A, \xi),\ (y, B, \eta)) \quad (4\text{–}12)$$

with the scalar product

$$\langle K_{xA\xi}, K_{yB\eta}\rangle = K_{xA\xi}(y, B, \eta).$$

A bijection $\sigma : X \to X$ gives rise to a unitary operator $\sigma : \mathcal{H} \to \mathcal{H}$, defined by

$$\sigma(K_{xA\xi}) := K_{\sigma(x)A\xi} \quad \text{for all } x \in X, A \in \mathcal{A},\ \xi \in H. \quad (4\text{–}13)$$

If the completely positive definite Φ is a GTK, that is, if $\Phi \sim (\Phi_1, \Phi_2, \Phi_0)$ for two Toeplitz kernels and a Hankel kernel in $X = X_1 \cup X_2$, then it defines a pair of scattering systems $[\mathcal{H}_1; W_1^+, W_1^-; \sigma_1]$ and $[\mathcal{H}_2; W_2^+, W_2^-; \sigma_2]$, where Φ_1, Φ_2 are the reproducing kernels for $\mathcal{H}_1$ and $\mathcal{H}_2$, σ_1 and σ_2 are the corresponding unitary operators, for which the forms $B_i : \mathcal{H}_i \times \mathcal{H}_i \to \mathbb{C}$ defined by

$$B_i(K^i_{xA\xi}, K^i_{yB\eta}) := \langle \xi, \Phi_i(x,y)(A,B)\eta \rangle \tag{4–14}$$

are Toeplitz. It is not difficult to check that for W_i^+, W_i^- corresponding to the subspaces of $\mathcal{H}_i$ spanned by $\{K^i_{xA\xi} \in \mathcal{H}_i : x \in X_i\}$, for $i = 1, 2$, a scattering system is obtained in which the form $B_0 : W_1^+ \times W_2^- \to \mathbb{C}$ defined by K^1 and K^2 in a way similar to (4–14) is Hankel. Then the Lifting Theorem 2.2 yields a lifting theorem for α-invariant completely positive definite GTKs:

THEOREM 4.1 [C-S 1994c]. *Given a linear dynamical system $\{\mathcal{A}, G, \alpha\}$ and an α-invariant completely positive definite GTK $\Phi \sim (\Phi_1, \Phi_2, \Phi_0) : X \times X \to \mathcal{L}(\mathcal{A}\times\mathcal{A}, \mathcal{L}(H))$, there exists an α-invariant Toeplitz $\Phi' : X\times X \to \mathcal{L}(\mathcal{A}\times\mathcal{A}, \mathcal{L}(H))$ such that $\Phi'|(X_1\times X_2) = \Phi_0$ and $\Phi' \le (\Phi_1, \Phi_2)$ in all of $X \times X$. Furthermore, if $\{\mathcal{A}, G, \alpha\}$ is reducing and Φ is of Holevo type, Φ' is also of Holevo type.*

In the special case when $X = \mathbb{Z}$, $X_1 = \mathbb{Z}_1$, $X_2 = \mathbb{Z}_2$, there is a precise integral representation, closely related to the GBT.

THEOREM 4.2 (INTEGRAL REPRESENTATION FOR α-INVARIANT COMPLETELY POSITIVE DEFINITE GTK DEFINED IN $\mathbb{Z}\times\mathbb{Z}$ [C-S 1994c]). *Set $\Omega = \{1, 2\}$. Given a completely positive definite GTK $\Phi \sim (\Phi_1, \Phi_2, \Phi_0) : \mathbb{Z} \times \mathbb{Z} \to \mathcal{L}(\mathcal{A}\times\mathcal{A}, \mathcal{L}(H))$, there exists a 2×2 matrix of measures $\mu = (\mu_{jk})$ defined on $\mathbb{T}$ such that, for every Borel set $D \subset \mathbb{T}$, $\mu(D) : \Omega \times \Omega \to \mathcal{L}(\mathcal{A}\times\mathcal{A}, \mathcal{L}(H))$ is a completely positive definite kernel, satisfying*

$$\Phi_i(m,n)(A,B) = \hat{\mu}_{ii}(m,n)(A,B) = \int_{\mathbb{T}} e^{i(m-n)t}\, d\mu_{ii}(A,B)$$

for all $m, n \in \mathbb{Z}$ and $i = 1, 2$, and

$$\Phi_0(m,n)(A,B) = \hat{\mu}_{12}(m,n)(A,B) \quad \text{for } m \in \mathbb{Z}_1,\ n \in \mathbb{Z}_2.$$

Furthermore, the measures μ_{11} and μ_{22} are α-invariant.

These lifting theorems provide results in dilation theory, including one on intertwining contractions coupled with $*$-representations generalizing the Sz.-Nagy–Foiaş theorem. These results from [C-S 1994c] will not be stated here.

The last result of this section reduces to the Nehari–Page theorem [Page 1970] in the case $\dim \mathcal{A} = 1$.

THEOREM 4.3 (NEHARI THEOREM FOR SEQUENCES DEFINED IN C^*-ALGEBRAS [C-S 1994c]). *Let $\mathcal{A} \subset \mathcal{L}(H)$ be the C^*-algebra of operators defined in the*

Schwartz space $\mathcal{S}(G)$, for G a nilpotent Lie group, through a unitary representation. Given a sequence of linear maps, $s_n : \mathcal{A} \to \mathcal{L}(H)$, for $n = 1, 2, \ldots$, and a $*$-representation $\theta : \mathcal{A} \to \mathcal{L}(H)$ of $\mathcal{A}$, the following conditions are equivalent:

(i) *There exists a function $f : \mathbb{T} \to \mathcal{L}(\mathcal{A}, \mathcal{L}(H))$ completely contractive with respect to θ, that is, satisfying*

$$\Bigl|\sum_{j,k} \langle f(t)(A_k^* A_j)\, \xi_k, \xi_j\rangle\Bigr| \le \sum_j \|\theta(A_j)\xi_j\|^2,$$

and such that $\hat{f}(-n) = s_n$ for $n = 1, 2, \ldots$.

(ii) *The Hankel kernel $\{s(m+n) : m, n > 0\}$ satisfies*

$$\Bigl|\sum_{m,n>0} \langle s(m+n)(A_n^* A_m)\eta_n,\, \xi_m\rangle\Bigr|^2 \le \Bigl(\sum_{m>0} \|\theta(A_m)\xi_m\|^2\Bigr)\Bigl(\sum_{n>0} \|\theta(A_n)\eta_n\|^2\Bigr).$$

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