Holomorphic Spaces
MSRI Publications
Volume **33**, 1998

Tight Uniform Algebras

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Abstract. We discuss the relationships between a certain class of uniform algebras, called tight uniform algebras, and various concepts from Banach space theory, such as the Dunford–Pettis property, the Pełczyński property, and weak sequential completeness. We also mention some connections with the $\bar{\partial}$-problem, interpolation, pointwise bounded approximation, and inner functions on strictly pseudoconvex domains.

1. Introduction

B. Cole and T. W. Gamelin [1982] introduced a generalized notion of analyticity, which they called tightness. If K is a compact space and $X \subset C(K)$ is a closed subspace (in the uniform norm) we say X is a *tight subspace* if the Hankel-type operator $S_g : X \to C(K)/X$ defined by $f \mapsto fg + X$ is weakly compact for every $g \in C(K)$. Recall that a *uniform algebra* A on K is a closed, separating subalgebra of $C(K)$ which contains the constants. We say a uniform algebra A on K is a *tight uniform algebra* if it is a tight subspace of $C(K)$. The following prototypical example from [Cole and Gamelin 1982] illustrates how tightness could be thought of as an abstract version of the solvability of a $\bar{\partial}$-problem with a mild gain in smoothness.

Let D be a strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary and let $A = A(D)$ be the uniform algebra on $\bar{D}$ of functions analytic in D. Let $K^\infty_{(0,1)}$ be the space of smooth $\bar{\partial}$-closed (0,1)-forms on D. Then there exists a compact linear operator $R : K^\infty_{(0,1)} \to C(\bar{D})$ which solves the $\bar{\partial}$-problem in D; that is, $\bar{\partial} \circ R = I$. The compactness follows from the fact that there exist Hölder estimates on the solutions, hence the mild gain in smoothness. If $g \in C^\infty(\bar{D})$ then we claim S_g can be factored through R and is therefore compact. If we set $T_g(f) = f\bar{\partial}g$ and let $q : C(\bar{D}) \to C(\bar{D})/A$ be the natural quotient map, the diagram

$$\begin{tikzcd} A \arrow[r, "S_g"] \arrow[d, "T_g"'] & C(\bar{D})/A \\ K^\infty_{(0,1)} \arrow[r, "R"] & C(\bar{D}) \arrow[u, "q"'] \end{tikzcd}$$

clearly commutes. It now follows from a density argument that S_g is compact and therefore weakly compact for every g in $C(\bar{D})$. Thus $A(D)$ is a tight uniform algebra.

Another example, also from [Cole and Gamelin 1982], is this. Let K be a compact planar set and let $A = R(K)$ be the uniform algebra of continuous functions on K which are uniform limits of rational functions with poles off K. A similar argument shows that the operators S_g on $R(K)$ are compact for every $g \in C(K)$. If we consider the Vitushkin localization operator $T_g : C(K) \to C(K)$ defined by

$$(T_g f)(\zeta) = f(\zeta)g(\zeta) + \frac{1}{\pi} \iint\limits_K \frac{1}{z-\zeta}\, \bar{\partial} g(z) f(z)\, dx\, dy(z)$$

where $g \in C_c^1(\mathbb{C})$, then T_g is a continuous linear operator under which $R(K)$ is invariant [Gamelin 1969]. If we define $V_g : A \to C(K)$ by

$$(V_g f)(\zeta) = \frac{1}{\pi} \iint\limits_K \frac{1}{z-\zeta}\, \bar{\partial} g(z) f(z)\, dx\, dy(z),$$

it can be seen that V_g is compact. Since the diagram

$$\begin{tikzcd} & A \arrow[dl, "-V_g"'] \arrow[dr, "S_g"] & \\ C(K) \arrow[rr, "q"'] & & C(K)/A \end{tikzcd}$$

commutes (where q is the natural quotient map) it follows that S_g is compact for every $g \in C(K)$. In particular, $R(K)$ is tight for every compact K. If we note that the Cauchy transform solves the $\bar{\partial}$-problem in the plane (see [Gamelin 1969] or [Conway 1991]), then we see the method applied to $R(K)$ is exactly the same as the method applied to $A(D)$.

In all our examples thus far we have found that the operators S_g are compact. We say $X \subseteq C(K)$ is a *strongly tight subspace* if S_g is compact for every g, and similarly we define *strongly tight uniform algebras*.

Clearly any uniform algebra on a compact planar set K which is invariant under T_g for all smooth g is strongly tight. We say A is a *T-invariant uniform algebra* if A is a uniform algebra on a compact planar set K which contains $R(K)$ and is invariant under T_g. For example the algebra $A(K)$ of functions in $C(K)$ which are analytic in the interior of K is T-invariant [Gamelin 1969] and is therefore strongly tight.

One of the main results in [Cole and Gamelin 1982] is that if $C = C(K)$ then A is a tight uniform algebra if and only if $A^{**} + C$ is a (closed) subalgebra of C^{**} ($A^{**} + C$ is always a closed subspace of C^{**}). This result allowed the authors to extend Sarason's theorem about the Hardy space H^∞ on the unit circle to other domains. For example, let K be any compact planar set and let Q be the set of non-peak points for $R(K)$ (z is a *peak point* for A if there exists an $f \in A$ with $f(z) = 1$ and $|f| < 1$ elsewhere). Let λ_Q be Lebesgue measure restricted to Q and define $H^\infty(\lambda_Q)$ to be the weak-star closure of $R(K)$ in $L^\infty(\lambda_Q)$. Cole and Gamelin proved that $H^\infty(\lambda_Q) + C$ is a closed subalgebra of $L^\infty(\lambda_Q)$.

Tight uniform algebras have some Banach space properties that are typical of $C(K)$ spaces. We will discuss these properties and their relation to tightness, and also mention some applications of tightness to pointwise bounded approximation theory and inner functions on domains in $\mathbb{C}^n$.

2. The Dunford–Pettis Property

The following result is from [Dunford and Pettis 1940]. Recall that a continuous linear operator $T : X \to Y$ is *completely continuous* if T takes weakly null sequences in X to norm null sequences in Y.

THEOREM 2.1. *Let (Ω, Σ, μ) be any measure space. Then*:

(a) *If Y is a Banach space and $T : L^1(\mu) \to Y$ is a weakly compact operator then T is completely continuous.*
(b) *If Y is a separable dual space then any bounded linear operator $T : L^1(\mu) \to Y$ is completely continuous.*

Later, Grothendieck [1953] studied Banach spaces X that exhibited property (a) of the above theorem. Following Grothendieck, we say a Banach space X has the *Dunford–Pettis property* if whenever Y is a Banach space and $T : X \to Y$ is a weakly compact linear operator then T is completely continuous.

Part of Grothendieck's work was to provide various characterizations of the Dunford–Pettis property, some of which do not involve operators. It is not difficult to deduce from these characterizations, which we shall present shortly, that part (b) of Theorem 2.1 can be deduced from part (a). Furthermore, by using the result on the factorization of weakly compact operators in [Davis et al. 1974], it can be shown that (a) can also be deduced from (b). These ideas can also be found in [Diestel 1980].

Evidently $L^1(\mu)$ has the Dunford–Pettis property for every μ. From this it can be deduced that $C(K)$ has the Dunford–Pettis property for every compact space K. It has been shown that many uniform algebras, such as the disk algebra, have the Dunford–Pettis property as well. It is easy to see that any infinite dimensional reflexive space fails to have the Dunford–Pettis property. Also, the Hardy space H^1 on the unit circle fails to have the Dunford–Pettis property. To see this, consider the Paley operator $P : H^1 \to l^2$ defined by $f \mapsto (\hat{f}(2^k))_{k=1}^\infty$. It is well-known that P is a bounded linear operator [Pełczyński 1977] and is therefore weakly compact since it maps into a Hilbert space. However, if $f_n(\zeta) = \zeta^{2^n}$ then $\|Pf_n\| = 1$ while $f_n \xrightarrow{w} 0$ in H^1 by the Riemann–Lebesgue Lemma. Therefore P is not completely continuous.

We have mentioned that many uniform algebras have the Dunford–Pettis property. However, this is not true for all uniform algebras. It is easy to see that if X has the Dunford–Pettis property and Y is a complemented subspace of X then Y has the Dunford–Pettis property as well. It is a theorem of Milne [1972] (also, see [Wojtaszczyk 1991]) that every Banach space X is isomorphic to a complemented subspace of a uniform algebra A. The space A can be taken to be the uniform algebra on B_{X^*} (the unit ball in X^* with the weak-star topology) generated by X. If we let $X = l^2$ then A fails to have the Dunford–Pettis property. The author is not aware of any uniform algebra on a compact subset of $\mathbb{R}^n$ which fails the to have the Dunford–Pettis property.

Grothendieck proved the following popular characterizations of the Dunford–Pettis property. (See also [Diestel 1980].) Recall that a sequence $\{x\}$ in a Banach space X is a *weak-Cauchy sequence* if $\lim x^*(x_n)$ exists for every $x^* \in X^*$.

PROPOSITION 2.2. *The following statements are equivalent for any Banach space X.*

(a) *X has the Dunford–Pettis property.*
(b) *If $T : X \to c_0$ is a weakly compact linear operator then T is completely continuous.*
(c) *If $x_n \xrightarrow{w} 0$ in X and $x_n^* \xrightarrow{w} 0$ in X^* then $x_n^*(x_n) \longrightarrow 0$.*
(d) *If $x_n \xrightarrow{w} 0$ in X and $\{x_n^*\}$ is a weak Cauchy sequence in X^* then $x_n^*(x_n) \longrightarrow 0$.*
(e) *If $x_n \xrightarrow{w} 0$ in X and $E \subset X^*$ is relatively weakly compact then*

$$\lim_{n\to\infty} \sup_{x^* \in E} \left| x^*(x_n) \right| = 0.$$

It is well-known that, if K is a compact space, the dual of $C(K)$ is isomorphic to some L^1-space. The following corollary is now an immediate consequence of the proposition and Theorem 2.1.

COROLLARY 2.3. (a) *If X is a Banach space and X^* has the Dunford–Pettis property then X has the Dunford–Pettis property.*
(b) *If K is a compact space then $C(K)$ has the Dunford–Pettis property.*

We mentioned above that the two conclusions of the theorem of Dunford and Pettis are actually equivalent. This is not difficult to prove with the aid of the following important result from [Davis et al. 1974].

THEOREM 2.4. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a weakly compact linear operator. Then there exist bounded linear operators S_1 and S_2 and a reflexive Banach space Z such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "S_1"'] & & Y \\ & Z \arrow[ur, "S_2"'] & \end{tikzcd}$$

commutes.

This theorem can be used to form yet another characterization of the Dunford–Pettis property. We require the following result of Rosenthal and Dor [Diestel 1984]. We say a sequence $\{x_n\}$ in a Banach space X is a c_0-*sequence* if $\{x_n\}$ is equivalent to the unit vector basis of c_0; that is, there exists an isomorphic embedding $T : c_0 \to X$ such that $T(e_n) = x_n$. We define l^1-*sequences* in the same manner. The result is that if X is any Banach space and $\{x_n\}$ is a sequence in B_X which fails to have a weak-Cauchy subsequence then $\{x_n\}$ has an l^1-subsequence.

THEOREM 2.5. *The following statements are equivalent for a Banach space X.*

(a) *X has the Dunford–Pettis property.*
(b) *If $T : X \to Z^*$ is a bounded linear operator and Z contains no copy of l^1 then T is completely continuous.*
(c) *Every bounded linear operator $T : X \to Y$ from X to a separable dual space Y is completely continuous.*
(d) *Every bounded linear operator $T : X \to Y$ from X to a reflexive Banach space is completely continuous.*

PROOF. (a) $\Longrightarrow$(b) Assume X has the Dunford–Pettis property and let $T : X \to Z^*$ be a bounded linear operator where Z contains no copy of l^1. Let $\{x_n\}$ be a weakly null sequence in X and assume $\{Tx_n\}$ fails to tend to zero in norm, so after passing to a subsequence we may assume $\|Tx_n\| > \varepsilon$ for some $\varepsilon > 0$ and for all n. Define $z_n^* = Tx_n$ and let $z_n \in B_Z$ be such that $z_n^*(z_n) > \varepsilon$ for every n. Since Z contains no copy of l^1, by the Rosenthal–Dor Theorem we may assume $\{z_n\}$ is a weak-Cauchy sequence. Let $x_n^* = T^*(z_n)$. Then by Proposition 2.2 we have $x_n^*(x_n) \longrightarrow 0$. However $x_n^*(x_n) = z_n^*(z_n)$, a contradiction.

Now (b) $\Longrightarrow$(c) $\Longrightarrow$(d) follows easily and the rest follows from Theorem 2.4. □

Property (c) provides us with another way of showing the space H^1 fails to have the Dunford–Pettis property. This is because H^1 is separable and is easily seen to be a dual space by the F. and M. Riesz Theorem. If H^1 had the Dunford–Pettis property then by (c) the identity operator on H^1 would be completely continuous. This would say that weakly null sequences in H^1 are norm null. However the characters $f_n(\zeta) = \zeta^n$ are weakly null by the Riemann-Lebesgue Lemma. Thus, H^1 cannot have the Dunford–Pettis property.

We now present a well-known way of detecting when a bounded subset of the dual of a Banach space fails to be weakly compact. We say that a sequence $\{x_n\}$ in a Banach space X is a *weakly unconditionally Cauchy series* (w.u.C. series for short) if $\sum |x^*(x_n)| < \infty$ for every $x^* \in X$. For example, if X is a closed subspace of $C(K)$ then $\{f_n\}$ is a w.u.C. series if and only if $\sum |f_n(z)| \leq C$ for every $z \in K$ for some constant C. We say a continuous linear operator $T : X \to Y$ is an *unconditionally converging operator* if T takes every weakly unconditionally Cauchy series to a series which converges unconditionally in norm. It is a theorem of Bessaga and Pełczyński [1958] that T is an unconditionally converging operator if and only if there does not exist a copy of c_0 in X on which T is an isomorphism.

PROPOSITION 2.6. *Let X be a Banach space and suppose $E \subset X^*$ is a bounded subset. Then the following statements are equivalent*:

(a) *There exists a weakly unconditionally Cauchy series $\sum x_n$ in X such that*

$$\overline{\lim_{n\to\infty}} \sup_{x^*\in E} \left|x^*(x_n)\right| > 0. \tag{2–1}$$

(b) *There exists a c_0-sequence $\{x_n\}$ in X such that* (2–1) *holds.*

If (a) *and* (b) *hold*, *E contains an l^1-sequence. In particular*, *E fails to be relatively weakly compact.*

PROOF. (a) $\Longrightarrow$(b) Assume $\{x_n\}$ is a w.u.C. series in X such that (2–1) holds. Then there exists some $\varepsilon > 0$ and a sequence $\{x_n^*\}$ in E such that, after passing to a subsequence of $\{x_n\}$ if necessary, we have $\left|x_n^*(x_n)\right| > \varepsilon$ for all n. Define $T : X \to l^\infty$ by $T(x) = (x_n^*(x))_{n=1}^\infty$, so $\|Tx_m\| > \varepsilon$ for $m \geq 1$. In particular, the series $\sum Tx_n$ does not converge and so by definition T fails to be an unconditionally converging operator. By the result of Bessaga and Pełczyński mentioned above there exists a subspace $X_0 \subseteq X$ such that X_0 is isomorphic to c_0 and $T|_{X_0}$ is an isomorphic embedding. The unit vector basis in X_0 is the desired c_0-sequence.

Part (a) follows trivially from (b) since the unit vector basis in c_0 is a weakly unconditionally Cauchy series.

To prove the final claim, assume E fails to contain an l^1-sequence. By the Rosenthal–Dor Theorem every sequence in E has a weak-Cauchy subsequence. Let $\{x_n\}$ be a w.u.C. series in X. We now claim that $\lim_{n\to\infty} \sup_{x^*\in E}\left|x^*(x_n)\right| = 0$. To see this, define an operator $T : X^* \to l^1$ by $T(x^*) = (x^*(x_n))_{n=1}^\infty$. It follows from the Closed Graph Theorem that T is a bounded linear operator. Let $K = T(E)$. Recall that l^1 is a Schur space, i.e., the weak and norm convergence of sequences coincide. It follows from this that K has compact closure. In particular, K is totally bounded and it now follows easily that given an $\varepsilon > 0$ there exists an integer N such that $\sum_{k=N}^\infty \left|x^*(x_k)\right| \leq \varepsilon$ for every $x^* \in E$. This proves the claim and finishes the proposition. □

An l^1-sequence $\{x_n^*\}$ in a dual space X^* cannot always be paired with a c_0-sequence in the way described above. For example, let $Y = C[0,1]$ and let $X = Y^*$. The Banach–Mazur Theorem states that every separable Banach space is isometrically isomorphic to a closed subspace of $C[0,1]$ (we say that $C[0,1]$ is a *universal space*; see [Wojtaszczyk 1991]). In particular X^* contains a copy of l^1, but X contains no copy of c_0. To see this, we recall the well-known fact that X is isomorphic to $L^1(\mu)$ for some abstract measure μ. We say a Banach space X is *weakly sequentially complete* if every weak-Cauchy sequence converges weakly in X. Every L^1-space is weakly sequentially complete [Dunford and Schwartz 1958]. However c_0 is not, which implies X contains no copy of c_0.

We will now mention some of the work of J. Bourgain, who showed that certain spaces of continuous functions, such as the ball-algebras and the polydisk algebras, have the Dunford–Pettis property. The following two theorems are easily deduced from the results in [Bourgain 1984a], as was observed in [Cima and Timoney 1987]. Recall the definition of the operators S_g from Section 1.

THEOREM 2.7 [Bourgain 1984a]. *Let X be a closed subspace of $C(K)$ and assume S_g is completely continuous for every $g \in C(K)$. Then*:

(a) *If $f_n \xrightarrow{w} 0$ in X and $E \subset X^*$ is a bounded subset with*

$$\varlimsup_{n\to\infty} \sup_{x^*\in E} \bigl|x^*(f_n)\bigr| > 0,$$

there exists a c_0-sequence $\{x_n\}$ in X failing to tend to zero uniformly on E.

(b) *X has the Dunford–Pettis property.*

The conclusion of (a) implies that E fails to be relatively weakly compact, by our previous proposition. Therefore any weakly null sequence must tend to zero uniformly on relatively weakly compact subsets of X^*. Hence, (a) implies (b) by Proposition 2.2.

THEOREM 2.8 [Bourgain 1984a]. *Let X be a closed subspace of $C(K)$ and assume $(S_g)^{**}$ is completely continuous for every $g \in C(K)$. Then X^* has the Dunford–Pettis property.*

We therefore have the following immediate consequence of Bourgain's work.

THEOREM 2.9. *If X is a strongly tight subspace of $C(K)$ then X and X^* have the Dunford–Pettis property.*

In fact, it follows from the technique mentioned in Theorem 2.7 that X has a property somewhat stronger than the Dunford–Pettis property. We will discuss this more in the next section.

It now follows immediately from [Cole and Gamelin 1982] that if A is any T-invariant uniform algebra (for example $R(K)$ or $A(K)$ for compact planar K) then A and A^* have the Dunford–Pettis property. Cima and Timoney, using different methods, also proved these results by showing S_g^{**} is completely continuous for every $g \in C(K)$ for a T-invariant uniform algebra A. It also follows from the work in [Cole and Gamelin 1982] that when $A = A(D)$ for D strictly pseudoconvex with C^2 boundary, then A and A^* have the Dunford–Pettis property (also, see [Li and Russo 1994]).

Incidentally, if X is a subspace of $C(K)$ we define X_b to be those $g \in C(K)$ such that S_g is completely continuous. Cima and Timoney [1987] showed that X_b is always an algebra, called the *Bourgain algebra of X*. The set of g for which S_g is weakly compact is also an algebra, and likewise the set of g for which S_g is compact. The weakly compact case is done in [Cole and Gamelin 1982] (for the case when $X = A$ itself is an algebra, but the result holds for subspaces) and the compact case is in [Saccone 1995a].

3. The Pełczyński Property

The Pełczyński property involves certain types of series in Banach spaces and how the convergence of these series is affected by linear operators. We begin with a result that follows from the work of Orlicz [1929]: if X and Y are Banach spaces and $T : X \to Y$ is a weakly compact linear operator then T takes weakly unconditionally Cauchy series to series that converge unconditionally in norm. In other words, weakly compact operators are necessarily unconditionally converging operators (defined above). For example, this implies any w.u.C. series in $L^2[0,1]$ must be a norm convergent series.

Now we turn to the paper [Pełczyński 1962], entitled "Banach spaces on which every unconditionally converging operator is weakly compact," where the converse of Orlicz's result is studied. As the title suggests, we say a Banach space X has the *Pełczyński property* if whenever Y is a Banach space and $T : X \to Y$ is an unconditionally converging operator then T is weakly compact. Trivially, every reflexive space has the Pełczyński property. It follows from the result of Bessaga and Pełczyński mentioned in the previous section that X has the Pełczyński property if and only if every non-weakly compact linear operator from X fixes a copy of c_0 (i.e., is an isomorphism on a copy of c_0 in X).

Evidently, when studying the Pełczyński property it is important to know which Banach spaces contain copies of c_0. If (Ω, Σ, μ) is any measure space then $L^1(\mu)$ is weakly sequentially complete and therefore does not contain a copy of c_0. Assume $L^1(\mu)$ has the Pełczyński property. Then the identity operator cannot fix a copy of c_0. Therefore, the identity operator is weakly compact and so $L^1(\mu)$ is reflexive. Furthermore, $L^1(\mu)$ has the Dunford–Pettis property so the identity operator is also completely continuous which implies weakly convergent sequences in $L^1(\mu)$ are norm convergent. It now follows that every bounded sequence in $L^1(\mu)$ has a weakly convergent subsequence (by the reflexivity) and therefore a norm convergent subsequence. Therefore the unit ball in $L^1(\mu)$ is compact and $L^1(\mu)$ is finite-dimensional. Hence, $L^1(\mu)$ has the Pełczyński property if and only if it is reflexive which occurs if and only if it is finite-dimensional.

The Pełczyński property does not share the duality property of the Dunford–Pettis property. It was shown in Pełczyński's original paper [Pełczyński 1962] that if K is any compact space then $C(K)$ has the Pełczyński property. Therefore, any infinite-dimensional L^1-space fails to have the Pełczyński property in spite of the fact that its dual, which is isomorphic to $C(K)$ for some K, has the Pełczyński property. To complete this picture, suppose X is any Banach space such that X and X^* have the Pełczyński property. Then X^* is weakly sequentially complete and therefore contains no copy of c_0. This implies the identity operator on X^* is an unconditionally converging operator. Since X^* has the Pełczyński property, the identity operator must be weakly compact and so X^* is reflexive. It now follows that X and X^* have the Pełczyński property if and only if X is reflexive.

The following result presents some more or less well-known characterizations of the Pełczyński property.

THEOREM 3.1. *If X is a Banach space, the following conditions are equivalent.*

(a) *X has the Pełczyński property.*
(b) *If $T : X \to Y$ is a continuous linear operator which fails to be weakly compact then T is an isomorphism on some copy of c_0 in X.*
(c) *If $E \subseteq X^*$ is a bounded subset and the weak closure of E fails to be weakly compact then there exists a weakly unconditionally Cauchy series $\{x_n\}$ in X which fails to tend to zero uniformly on E.*
(d) (i) *X^* is weakly sequentially complete, and* (ii) *if $\{x_n^*\}$ is an l^1-sequence in X^* then there exists a c_0-sequence $\{x_k\}$ in X such that $\left|x_{n_k}^*(x_k)\right| > \delta > 0$ for all k for some sequence $\{n_k\}$.*

The equivalence of (a) and (b) follows from the remarks above. That of (a) and (c) is due to Pełczyński. The equivalence of (a) and (d) is less well-known, but can be deduced from (c) and the Rosenthal–Dor Theorem.

Note that (i) and (ii) of part (d) are distinct properties. Bourgain and Delbaen [1980] have constructed a Banach space X such that X^* is isomorphic to l^1, while X contains no copy of c_0; thus (i) holds while (ii) fails. R. C. James [1950] constructed a separable Banach space X such that X is nonreflexive and whose natural embedding into X^{**} has a codimension 1 image. In particular X^{**} is separable and therefore X^* contains no copy of l^1 and every sequence in X^* has a weak-Cauchy subsequence. Therefore X satisfies (ii) but fails (i).

As an illustration, consider the following theorem.

THEOREM 3.2 [Mooney 1972]. *Let m be Lebesgue measure on the unit circle Γ and let $H^\infty \subset L^\infty(m)$ be the Hardy space of boundary values of bounded analytic functions in the unit disk. Suppose $\{f_n\}$ is a bounded sequence in $L^1(m)$ such that $\lim_{n\to\infty} \int f_n h\, dm$ exists for every $h \in H^\infty$. Then there exists an element $f \in L^1(m)$ such that*

$$\lim_{n\to\infty} \int f_n h\, dm = \int f h\, dm$$

for every $h \in H^\infty$.

A proof of this can be found in [Garnett 1981]. It uses facts about peak sets in the maximal ideal space of H^∞. Mooney's theorem can easily be related to weak sequential completeness. Since $H^\infty = (L^1/H_0^1)^*$ where H_0^1 is the subspace of the Hardy space H^1 consisting of functions vanishing at the origin, Mooney's theorem is equivalent to the weak sequential completeness of L^1/H_0^1. Let A

be the disk algebra on the unit circle. By the F. and M. Riesz Theorem we have $A^{\perp} = \{f\,dm : f \in H_0^1\}$. It follows that if L is the space of measures singular to Lebesgue measure then A^* is isometrically isomorphic to $L^1/H_0^1 \oplus_{l^1} L$. Furthermore, L is isomorphic to $L^1(\mu)$ for some abstract measure μ. Since $L^1(\mu)$ is weakly sequentially complete, Mooney's theorem is equivalent to the weak sequential completeness of A^*. The disk algebra is an example of a tight uniform algebra and as we note below, all tight uniform algebras have weakly sequentially complete duals.

The Pełczyński property can be related to some ideas in interpolation. It is not hard to see that a bounded sequence $\{x_n\}$ in a Banach space is an l^1-sequence if and only if it interpolates X^*; i.e., for every bounded sequence of scalers $\{\beta_n\}$ there exists an x^* in X^* with $x^*(x_n) = \beta_n$. Consider the following result of P. Beurling (which can be found in [Garnett 1981]). Let D be the open unit disk. If $\{z_n\}$ is a sequence of points in D which interpolates H^∞ then there exists a sequence $\{h_n\}$ in H^∞ such that $\sum |h_n(z)| \le C$ for all $z \in D$ and some constant C and $h_n(z_k) = \delta_{nk}$ where δ is the Kronecker delta function. Note that if A is the disk algebra then the point evaluations in A^* corresponding to the sequence $\{z_n\}$ form an l^1-sequence, and it is not hard to see that the sequence $\{h_n\}$ is a w.u.C. series in H^∞. The Pełczyński property for the disk algebra A offers a similar, but different, result. Given an arbitrary l^1-sequence $\{x_n^*\}$ in the dual of A (not necessarily point evaluations), there exists a c_0-sequence $\{x_n\}$ in A such that $|x_{n_k}^*(x_k)| > \delta > 0$ for some subsequence. This result applies to more general sequences in the dual, but the conclusion is weaker than that of Beurling.

Delbaen [1977] and Kisliakov [1975] independently showed that the disk algebra has the Pełczyński property. Delbaen [1979] extended these results to $R(K)$ for special classes of planar sets K, as did Wojtaszczyk [1979] (although the results of Delbaen were more extensive). It was shown in [Saccone 1995a] that $R(K)$ has the Pełczyński property for every compact planar set K, and also that every T-invariant uniform algebra on a compact planar set has the Pełczyński property. The T-invariant class includes $R(K)$ as well as $A(K)$ for all compact planar sets K. Bourgain [1983] showed that the ball-algebras and the polydisk-algebras have the Pełczyński property. This result was extended in [Saccone 1995a] to $A(D)$ for strictly pseudoconvex domains D in $\mathbb{C}^n$.

It follows from Milne's theorem, mentioned in the previous section, that there exist uniform algebras which fail to have the Pełczyński property. As in the case of the Dunford–Pettis property, the author is not aware of any uniform algebras on compact subsets of $\mathbb{R}^n$ which fail to have the Pełczyński property.

We will now elaborate on work from [Bourgain 1983]. If $X \subseteq C(K)$ we say that $m \in M(K)$ is a *weakly rich measure* for X if, whenever $\{f_n\}$ is a bounded sequence in X such that $\int |f_n|\, d|m| \longrightarrow 0$, the sequence $f_n g + X$ converges weakly to 0 for every $g \in C(K)$. If $f_n g + X$ converges to 0 in norm, we say that m is a *strongly rich measure*. This latter concept was introduced by Bourgain [1984a], who showed that X has the Pełczyński property if there exists a strongly rich measure for X. For example, it is shown that the surface-area measure on the unit sphere in $\mathbb{C}^n$ is a strongly rich measure for the ball-algebras. Note that weakly rich measures on strongly tight spaces (where the operators S_g are compact) are strongly rich.

As in the case of determining whether S_g is weakly compact, compact, or completely continuous, to show m is a weakly or strongly rich measure it suffices to check only a collection of those g which generate $C(K)$ as a uniform algebra. Given a measure $m \in M(K)$ and a closed subspace X of $C(K)$, define $(X,m)_{\mathrm{wr}}$ and $(X,m)_{\mathrm{sr}}$ to be the sets of those $g \in C(K)$ such that $f_n g + X \xrightarrow{w} 0$ and $\|f_n g + X\| \longrightarrow 0$, respectively, whenever $\{f_n\}$ is a bounded sequence in X such that $\int |f_n|\, d|m| \longrightarrow 0$. Then $(X,m)_{\mathrm{wr}}$ and $(X,m)_{\mathrm{sr}}$ are closed subalgebras of $C(K)$ where $(X,m)_{\mathrm{sr}}$ clearly contains X_b [Saccone 1995a]. Furthermore, it follows from Bourgain's result that if X possesses a strongly rich measure then X is a tight subspace.

It was shown in [Saccone 1995a] that every strongly tight uniform algebra on a compact metric space possesses a strongly rich measure and therefore has the Pełczyński property. (The proof actually works for strongly tight subspaces.) As noted in the same paper, it now follows from results in [Cole and Gamelin 1982] that $R(K)$ has the Pełczyński property for every compact planar set, and that $A(D)$ has the Pełczyński property for every strictly pseudoconvex domain D in $\mathbb{C}^n$ with C^2 boundary. It was also noted in [Saccone 1995a] that by examining Bourgain's proof it can be seen that indeed every strongly tight uniform algebra (or subspace) on an arbitrary compact space K has the Pełczyński property.

The following more general result is proved in [Saccone 1997].

THEOREM 3.3. *Let K be a compact space and let X be a tight subspace of $C(K)$. Then X has the Pełczyński property and X^* is weakly sequentially complete.*

If we only assume the operators S_g to be weakly compact instead of compact (that is, if we assume X is tight instead of strongly tight) then Bourgain's results no longer appear to be of use. The basic gliding hump construction used to prove the theorem remains essentially the same, however some calculations in Bourgain's original proof which involved Hilbert space geometry had to be replaced by more general arguments dealing with weak compactness in arbitrary Banach spaces.

COROLLARY 3.4 [Saccone 1995a]. (a) *If K is any compact planar set and A is a T-invariant uniform algebra on K then A has the Pełczyński property. In particular $R(K)$ and $A(K)$ have the Pełczyński property.*
(b) *If D is any strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary then $A(D)$ has the Pełczyński property.*

Although it is now known that a large of class of planar uniform algebras, including $R(K)$ and $A(K)$, have such Banach space properties as the Dunford–Pettis property and the Pełczyński property, it is not known if any of these spaces, when they are not all of $C(K)$, fail to be isomorphic (as Banach spaces) to the disk algebra. For example, it is a theorem of Milutin that if K is any uncountable compact metric space then $C(K)$ is isomorphic to $C[0,1]$; see [Wojtaszczyk 1991]. On the other hand, the polydisk algebras were shown in [Henkin 1968] not to be isomorphic to the ball-algebras in higher dimensions.

It is not currently known whether $H^\infty(\mathbb{B}_n)$ has the Dunford–Pettis property or the Pełczyński property when $n > 1$. Astoundingly enough, Bourgain [1984b] has shown that if A is the disk algebra then all the duals of A have the Dunford–Pettis property and all the even duals of A have the Pełczyński property. The proof involves the theory of ultraproducts of Banach spaces. It follows from this that H^∞ on the unit circle has the Pełczyński property.

We say a Banach space X is a *Grothendieck space* if weak-star null sequences in X^* are weakly null. It is known that any dual space with the Pełczyński property is a Grothendieck space. It follows from Bourgain's work that all the even duals of the disk algebra are Grothendieck spaces, and in particular H^∞ is a Grothendieck space. It is not hard to see that this implies every continuous linear operator from H^∞ to a separable Banach space is weakly compact. In particular, if A is the disk algebra, every continuous linear operator $T : H^\infty \to A$ is weakly compact, and furthermore T^2 must be compact since A has the Dunford–Pettis property.

4. Band Theory

The theory of bands is useful for studying abstract properties of uniform algebras. Good sources for band theory are [Cole and Gamelin 1982] and [Conway 1991].

Let K be a compact Hausdorff space. If $\mathcal{B} \subseteq M(K)$ we say $\mathcal{B}$ is a *band of measures* if $\mathcal{B}$ is a closed subspace of $M(K)$ and has the property that when $\mu \in \mathcal{B}$, $\nu \in M(K)$, and $\nu \ll \mu$, then $\nu \in \mathcal{B}$. There is a Lebesgue decomposition theorem for bands which says that if $\mu \in M(K)$ then μ can be uniquely written as $\mu = \mu_a + \mu_s$ where $\mu_a \in \mathcal{B}$ and μ_s is singular to every element of $\mathcal{B}$. If $\mathcal{B}$ is a band the *complementary band* $\mathcal{B}'$ of $\mathcal{B}$ is the collection of measures singular to every measure in $\mathcal{B}$. It follows from the Lebesgue decomposition that $M(K) = \mathcal{B} \oplus_{l^1} \mathcal{B}'$. It is well known that if $\mathcal{B}$ is a band of measures then there exists some measure space (Ω, Σ, μ) such that $\mathcal{B}$ is isomorphic to $L^1(\mu)$.

If $\mathcal{B}$ is a band we define $L^\infty(\mathcal{B})$ to be the space of uniformly bounded families of functions $F = \{F_\nu\}_{\nu \in \mathcal{B}}$ where $F_\nu \in L^\infty(\nu)$ and $F_\nu = F_\mu$ a.e. with respect to $d\nu$ whenever we have $\nu \ll \mu$. The norm in $L^\infty(\mathcal{B})$ is given by $\|F\| = \sup_{\nu \in \mathcal{B}} \|F_\nu\|_{L^\infty(\nu)}$. The pairing $\langle \nu, F \rangle = \int F_\nu \, d\nu$ for $\nu \in \mathcal{B}$ and $F \in L^\infty(\mathcal{B})$ defines an isometric isomorphism between $L^\infty(\mathcal{B})$ and $\mathcal{B}^*$. If A is a uniform algebra on K we define $H^\infty(\mathcal{B})$ and $H^\infty(\mu)$ to be the weak-star closure of A in

$L^\infty(\mathcal{B})$ and $L^\infty(\mu)$ respectively. If $\mu \in \mathcal{B}$ there is a natural "projection"

$$H^\infty(\mathcal{B}) \xrightarrow{\tau} H^\infty(\mu)$$

defined by $F \mapsto F_\mu$. Since $H^\infty(m)$ is not identified with a subspace of $H^\infty(\mathcal{B})$ the map τ is not a projection in the usual sense. However, A is a closed subspace of $H^\infty(\mathcal{B})$ and we have $\tau(f) = f$ for every $f \in A$. The map τ is the adjoint of the natural injection

$$\frac{L^1(\mu)}{L^1(\mu) \cap A^\perp} \xrightarrow{\sigma} \frac{\mathcal{B}}{\mathcal{B} \cap A^\perp} \tag{4–1}$$

defined by

$$f\,d\mu + L^1(\mu)/L^1(\mu) \cap A^\perp \mapsto f\,d\mu + A^\perp.$$

It is easy to see that the intersection of an arbitrary collection of bands is a band. If $\mathcal{C}$ is an arbitrary subset of $M(K)$ we define the *band generated by* $\mathcal{C}$ to be the smallest band containing $\mathcal{C}$. If $\mu \in M(K)$ then we identify the space $L^1(\mu)$ with the band of measures absolutely continuous with respect to μ. If K is a metric space and $\mathcal{C}$ is a separable subset of $M(K)$ then the band $\mathcal{B}$ generated by $\mathcal{C}$ is separable and a band $\mathcal{B}$ will be separable if and only if there exists some measure $\mu \in M(K)$ such that $\mathcal{B} = L^1(\mu)$.

If A is a uniform algebra on K we define $\mathcal{B}_{A^\perp}$ to be the band generated by the measures in $A^\perp$ and $\mathcal{S}$ to be the band complement of $\mathcal{B}_{A^\perp}$. It follows from the Lebesgue decomposition that

$$A^* \cong \frac{\mathcal{B}_{A^\perp}}{A^\perp} \oplus_{l^1} \mathcal{S}$$

and

$$A^{**} \cong H^\infty(\mathcal{B}_{A^\perp}) \oplus_{l^\infty} L^\infty(\mathcal{S}),$$

where the above isomorphisms are isometries.

We say a band $\mathcal{B}$ is a *reducing band* for A if for any measure $\nu \in A^\perp$ the projection ν_a of ν into $\mathcal{B}$ by the Lebesgue decomposition is also in $A^\perp$. We say $\mathcal{B}$ is a *minimal reducing band* if $\mathcal{B} \neq \{0\}$ while $\{0\}$ is the only reducing band properly contained in $\mathcal{B}$. It is easy to see that the intersection of two reducing bands is a reducing band. Therefore, any two minimal reducing bands are either identical or singular.

If A is a uniform algebra we denote the maximal ideal space of A by $\mathcal{M}_A$. The following version of the abstract F. and M. Riesz Theorem can be found in [Cole and Gamelin 1982].

THEOREM 4.1. *Let A be a uniform algebra and let $\varphi \in \mathcal{M}_A$. Then the band generated by the representing measures for φ is a minimal reducing band.*

We say a point $z \in K$ is a *peak point* for A if there exists an element $f \in A$ such that $f(z) = 1$ and $|f(w)| < 1$ for $w \neq z$. We say z is a *generalized peak point* if the only complex representing measure for z is the point mass at z. The *Choquet boundary* of A is the collection of all generalized peak points.

Now suppose $\mathcal{B}$ is a minimal reducing band such that $\mathcal{B} \subseteq \mathcal{S}$, the singular band to $\mathcal{B}_{A^\perp}$. Then every subband of $\mathcal{B}$ is reducing. By the minimality of $\mathcal{B}$ it can be seen that this implies $\mathcal{B}$ is all multiples of a point mass δ_z at some point $z \in K$. Theorem 4.1 now implies that z is a generalized peak point. Conversely, if z is a generalized peak point then it can be seen that the point mass δ_z at z lies in $\mathcal{S}$ and therefore all multiples of δ_z form a minimal reducing band contained in $\mathcal{S}$. We call these reducing bands *trivial minimal reducing bands* and the others *nontrivial minimal reducing bands.* Note that a minimal reducing band $\mathcal{B}$ is trivial if and only if $\mathcal{B} \cap A^\perp = 0$. Furthermore, since the intersection of two reducing bands is a reducing band, $\mathcal{B}$ is non-trivial if and only if $\mathcal{B} \subseteq \mathcal{B}_{A^\perp}$.

Let $\varphi \in \mathcal{M}_A$ and let $\mathcal{B}_\varphi$ be the band generated by the representing measures for φ. The *Gleason part* of φ is the collection of elements $\psi \in \mathcal{M}_A$ such that ψ has a representing measure in $\mathcal{B}_\varphi$. This is equivalent to saying that $\|\psi - \varphi\|_{A^*} < 2$. If φ and ψ lie in the same Gleason part then $\mathcal{B}_\varphi = \mathcal{B}_\psi$, otherwise $\mathcal{B}_\varphi$ and $\mathcal{B}_\psi$ are singular. We say a Gleason part is a *trivial Gleason part* if it corresponds to a point on the Choquet boundary and a *non-trivial Gleason part* otherwise. Note that a trivial Gleason part is a one-point part, but that there may be some one-point parts which are non-trivial. (This is not standard; usually a Gleason part is called trivial if it simply consists of one point. We therefore have more non-trivial Gleason parts than usual.)

Let z be a point in K and let $\varphi_z \in A^*$ be the point-evaluation at z. Let $\mathcal{B}_z$ be the minimal reducing band generated by the representing measures for z. If z lies off the Choquet boundary then $\mathcal{B}_z$ is non-trivial and so $\mathcal{B}_z \subseteq \mathcal{B}_{A^\perp}$. Therefore, every representing measure for z lies in $\mathcal{B}_{A^\perp}$ and we have $\varphi_z \in \mathcal{B}_{A^\perp}/A^\perp$. Similarly, if z lies on the Choquet boundary then $\varphi_z \in \mathcal{S}$.

If we let $\{\mathcal{B}_\alpha\}$ be the collection of all the non-trivial minimal reducing bands then $\bigoplus_{l^1} \mathcal{B}_\alpha$ is a reducing band contained in $\mathcal{B}_{A^\perp}$. However, this may not be all of $\mathcal{B}_{A^\perp}$. For more information, see [Cole and Gamelin 1982]. The sum $\bigoplus_{l^1} \mathcal{B}_\alpha / \mathcal{B}_\alpha \cap A^\perp$ is now isometric to a closed subspace of A^* which is contained in $\mathcal{B}_{A^\perp}/A^\perp$.

5. Pointwise Bounded Approximation and the Space $\mathcal{B}_{A^\perp}/A^\perp$

Given a uniform algebra A, the space $\mathcal{B}_{A^\perp}/A^\perp$ can be a useful object to study. It controls the uniform algebra in certain ways. For example, representing measures for points off the Choquet boundary lie in $\mathcal{B}_{A^\perp}$, and therefore their corresponding point evaluations lie in $\mathcal{B}_{A^\perp}/A^\perp$. Furthermore, since the dual of $\mathcal{B}_{A^\perp}/A^\perp$ is isometrically isomorphic to $H^\infty(\mathcal{B}_{A^\perp})$, and A is identified isometrically with a subspace of $H^\infty(\mathcal{B}_{A^\perp})$, $\mathcal{B}_{A^\perp}/A^\perp$ is a norming set for A.

It is proved in [Saccone 1997] that when A is a tight uniform algebra on a compact metric space K then $\mathcal{B}_{A^\perp}/A^\perp$ is separable. This separability property gives us even further control over the uniform algebra A. For example, it follows immediately that A has at most countably many non-trivial Gleason parts since if φ and ψ are elements of two different non-trivial Gleason parts, then they both lie in $\mathcal{B}_{A^\perp}/A^\perp$ and $\|\varphi - \psi\|_{A^*} = 2$. The countability of the non-trivial Gleason parts can also be deduced from the fact that every such part corresponds to a distinct nontrivial minimal reducing band. If $\mathcal{B}_{A^\perp}/A^\perp$ is separable then from the comments at the end of the last section we see that there can be at most countably many nontrivial minimal reducing bands.

This separability property can be used to construct special measures for the uniform algebra. We illustrate with the following result of A.M. Davie. Let K be a compact planar set and let $A = R(K)$. Let Q be the set of non-peak points for $R(K)$ and let λ_Q be Lebesgue measure restricted to Q. Davie's result [1972] (see also [Conway 1991]) is that if $f \in H^\infty(\lambda_Q)$ then there exists a sequence $\{f_n\}$ in A with $\|f_n\| \le \|f\|$ that converges to f pointwise a.e. with respect to λ_Q. Such a phenomenon is sometimes referred to as pointwise bounded approximation with a reduction in norm. It is easy to see that there exists such a sequence converging pointwise to f, the hard part is to find a sequence that is bounded. It can be deduced from Davie's result, without much difficulty, that every point in Q has a representing measure absolutely continuous with respect to λ_Q.

We will now study Davie's result by considering the space $H^\infty(\mathcal{B}_{A^\perp})$. The following proposition is not difficult to prove and can be found in [Saccone 1997]. Recall that a linear operator $T : X \to Y$ is a quotient map if the induced injection $S : X/Z \to Y$, where $Z = \ker T$, is an isometry.

Proposition 5.1. *Let A be a uniform algebra on a compact space K and let $m \in \mathcal{B}_{A^\perp}$. The following statements are equivalent*:

(a) *For every $f \in H^\infty(m)$ there exists a sequence $\{f_n\}$ in A with $\|f_n\| \le \|f\|$ such that $f_n \longrightarrow f$ pointwise a.e. with respect to m.*
(b) *The natural projection $H^\infty(\mathcal{B}_{A^\perp}) \xrightarrow{\tau} H^\infty(m)$ is a quotient map.*

If A is a uniform algebra on a compact space K and $m \in \mathcal{B}_{A^\perp}$ we say m is an *ordinary Davie measure* if τ is a quotient map and m is a *strong Davie measure* if τ is an isometry. In general, a linear operator between Banach spaces is an isometric embedding if and only if its dual is a quotient map. Therefore m is an ordinary Davie measure if and only if the map σ in (4–1) (where $\mathcal{B}$ should be taken to be $\mathcal{B}_{A^\perp}$) is an isometric embedding and is a strong Davie measure if and only if σ is a surjective isometry. Since τ is an algebra homomorphism between uniform algebras, τ will be an isometry as soon as it is an isomorphism. Since σ is always injective it follows that m is a strong Davie measure if and only if σ is onto. Since the evaluations for the points off the Choquet boundary lie in $\mathcal{B}_{A^\perp}/A^\perp$, it follows easily that when m is a strong Davie measure then every point off the Choquet boundary has a representing measure absolutely continuous with respect to m.

Interestingly enough, the injectivity of τ is closely related to Bourgain's rich measures, as the next proposition shows. If $m \in M(K)$ let $m = m_a + m_s$ be the Lebesgue decomposition of m with respect to $\mathcal{B}_{A^\perp}$.

Proposition 5.2 [Saccone 1995a]. *Let A be a uniform algebra on a compact space K and let m be an element of $M(K)$. Then the following statements are equivalent*:

(a) *The natural projection $H^\infty(\mathcal{B}_{A^\perp}) \xrightarrow{\tau} H^\infty(m_a)$ is one-to-one.*
(b) *If $\{f_n\}$ is a bounded sequence in A such that $\int |f_n|\, d|m| \longrightarrow 0$ then $f_n \xrightarrow{w^*} 0$ in $L^\infty(\mu)$ for every $\mu \in A^\perp$.*
(c) *m is a weakly rich measure for A.*

If these conditions hold, and K is metrizable, then $\mathcal{B}_{A^\perp}/A^\perp$ is separable.

We now have the following result from [Saccone 1997]. The fact that (b) holds for tight uniform algebras appeared in [Cole and Gamelin 1982], although the proof in [Saccone 1997] is more elementary.

Theorem 5.3. *Let A be a tight uniform algebra on a compact metric space K. Then the following hold.*

(a) *$\mathcal{B}_{A^\perp}/A^\perp$ is separable.*
(b) *A has at most countably many non-trivial Gleason parts and at most countably many nontrivial minimal reducing bands.*
(c) *A has a strong Davie measure m. In particular, every non-peak point for A has a representing measure absolutely continuous with respect to m.*

Part (b) follows from a more general result which is proved in [Saccone 1997]. We say a Banach space X is a *separable distortion of an L^1-space* if $X = M \oplus_{l^1} L^1(\mu)$ where M is separable and μ is some measure. Since every band is isomorphic to $L^1(\mu)$ for some μ, A^* will be isomorphic to a separable distortion of an L^1-space whenever $\mathcal{B}_{A^\perp}/A^\perp$ is separable. The next result from [Saccone 1997] now generalizes the observation implicit in part (b).

Theorem 5.4. *Let A be a uniform algebra and suppose A^* is isomorphic to a closed subspace of a separable distortion of an L^1-space. Then A has at most countably many non-trivial minimal reducing bands and therefore at most countably many non-trivial Gleason parts.*

6. Tightness Versus Strong Tightness

The following problem is open: does there exist a tight uniform algebra which fails to be strongly tight? We need not ask the question of tight subspaces, for if we let $X = l^2$ then X is tight (in any $C(K)$-space). However, from Theorem 2.7 we know that all strongly tight subspaces have the Dunford–Pettis property, a property which l^2 clearly fails to have. Thus l^2 is not strongly tight in any $C(K)$-space.

This problem has been studied in [Carne et al. 1989; Jaramillo and Prieto 1993]. The first of these papers dealt with the following uniform algebra. Let X be a Banach space and let A be the uniform algebra on B_{X^*} (with the weak-star topology) generated by X. Then A consists of analytic functions of possibly infinitely many variables. Recall the result of Milne which states that X is complemented in A (by a norm-one projection in fact). It now follows from Theorem 3.3 that X must have the Pełczyński property if A is to be tight. However, a much stronger result is proved in [Carne et al. 1989], namely that if A is tight then X is reflexive.

We now claim that A is strongly tight if and only if X is finite-dimensional (this result was also proved in [Carne et al. 1989] by a more direct means). Since the ball-algebras are known to be strongly tight (proved in [Cole and Gamelin 1982], for example), we need only prove necessity. If A is strongly tight then A has the Dunford–Pettis property by Corollary 2.3 and therefore X has the Dunford–Pettis property since it is complemented in A. However, by the result in [Carne et al. 1989] , X must also be reflexive, from which it follows that X must be finite-dimensional.

It was shown in [Carne et al. 1989] that if $X = l^2$ then A fails to be tight. However, it is still unknown if there exists an infinite-dimensional (reflexive) space X such that A is tight. In [Jaramillo and Prieto 1993], some strong versions of reflexivity were studied, but no example was produced.

Another example that is not well understood is the space $A(D)$ when D is a bounded domain in $\mathbb{C}^n$. It is known that $A(D)$ is strongly tight whenever the $\bar{\partial}$-problem can be solved in D with Hölder estimates on the solutions; however no satisfactory necessary conditions are known, although the following is proved in [Saccone 1995b].

PROPOSITION 6.1. *Suppose D is a bounded domain in $\mathbb{C}^n$ and let $A = A(D)$. Then the following statements are equivalent*:

(a) *A is strongly tight on $\overline{D}$.*

(b) *A has the property that when $\{f_n\}$ is a bounded sequence in D that tends to zero pointwise in D then we have $\|f_n g + A\| \longrightarrow 0$ for every $g \in C(\overline{D})$.*

7. Inner Functions

Recall the Chang–Marshall Theorem, which states that if m is Lebesgue measure on the unit circle then every closed subalgebra of $L^\infty(m)$ which contains H^∞ is generated by H^∞ and a collection of conjugates of inner functions. The following result shows how this phenomenon breaks down in higher dimensions.

THEOREM 7.1. *Let D be a strictly pseudoconvex domain with C^2 boundary in $\mathbb{C}^n$. Suppose f is an inner function in $H^\infty(\partial D)$. If $f(z_n) \longrightarrow 0$ for some sequence $\{z_n\}$ tending to ∂D then $\bar{f} \notin H^\infty + C$. In particular, if $n > 1$ then $\bar{f} \in H^\infty + C$ if and only if f is constant. If D is the unit disk then $\bar{f} \in H^\infty + C$ if and only if f is a finite Blaschke product.*

The proof is indirect and uses such tools as the Pełczyński property and tight uniform algebras. Given a function $g \in L^\infty(m)$, we define

$$S_{g,H^\infty} : H^\infty(m) \longrightarrow L^\infty(m)/H^\infty(m)$$

by $f \mapsto fg + H^\infty(m)$. It is shown that when $g \in H^\infty + C$ then S_{g,H^∞} is compact and if f is an inner function that tends to zero towards the boundary then $S_{\bar{f},H^\infty}$ is an isomorphism on a copy of c_0 in H^∞.

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