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Elements of Spectral Theory in Terms of the Free Function Model Part I: Basic Constructions

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Abstract. This is a survey of the function model approach to spectral theory, including invariant subspaces, generalized spectral decompositions, similarity to normal operators, stability problems for the continuous spectrum of unitary and selfadjoint operators, and scattering theory.

Part I contains a revised version of the coordinate-free function model of a Hilbert space contraction, based on analysis of functional embeddings related to the minimal unitary dilation of the operator. Using functional embeddings, we introduce and study all other objects of model theory, including the characteristic function, various concrete forms (transcriptions) of the model, one-sided resolvents, and so on. For the case of an inner scalar characteristic function we develop the classical H^∞-calculus up to a local function calculus on the level curves of the characteristic function. The spectrum of operators commuting with the model operator, and in particular functions of the latter, are described in terms of their liftings. A simplified proof of the invariant subspace theorem is given, using the functional embeddings and regular factorizations of the characteristic function. As examples, we consider some compact convolution-type integral operators, and dissipative Schrödinger (Sturm–Liouville) operators on the half-line.

Part II, which will appear elsewhere, will contain applications of the model approach to such topics as angles between invariant subspaces and operator corona equations, generalized spectral decompositions and free interpolation problems, resolvent criteria for similarity to a normal operator, and weak generators of the commutant and the reflexivity property. Classical topics of stability of the continuous spectrum and scattering theory will also be brought into the fold of the coordinate-free model approach.

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Foreword

This text is a detailed and enlarged version of an introductory mini-course on model theory that we gave during the Fall of 1995 at the Mathematical Sciences Research Institute in Berkeley. We are indebted to the Institute for granting us a nice opportunity to work side by side with our colleagues, operator analysts, from throughout the world.

We were rewarded by the highly professional audience that attended our lectures, and are very grateful to our colleagues for making stimulating comments and for encouraging us to give a course to students whose knowledge of the subject was sometimes better than ours. Having no possibility to list all of you, we thankfully mention the initiators of the course, our friends Joseph Ball and Cora Sadosky.

We are extremely indebted to the editors of this volume for inexhaustible patience, to Maria Gamal and Vladimir Kapustin for helping to compile the list of references, and to Alexander Plotkin, David Sherman, and Donald Sarason for their careful reading the manuscript.

Introduction: A Brief Account

About the reader. We hope that most of the material in this paper will be accessible to nonexperts having general knowledge of operator theory and of basic analysis, including the spectral theorem for normal Hilbert space operators. Some more special facts and terminology are listed below in Section 0.7 of this Introduction.

0.1. A bit of philosophy. A *model* of an operator $T : H \to H$ is another operator, say $M : K \to K$, that is in some sense equivalent to T. There exist models up to *unitary equivalence*, $T = U^{-1}MU$ for a unitary operator $U : H \to K$ acting between Hilbert spaces; up to *similarity equivalence*, $T = V^{-1}MV$ for a linear isomorphism $V : H \to K$ acting between Banach spaces; up to quasi-similarity; pseudo-similarity; and other equivalences. So, in fact, these transformations U, V, etc., change the notation, reducing the operator to a form convenient for computations, especially for a functional calculus admitted by the operator. The calculus contains both algebraic and analytic features (in particular, norm estimates of expressions in the operator), and the requirements to simplify them lay the foundation of modern model theories based on *dilations* of the operator under question. As soon as such a dilation is established, any object related to the analysis of our operator can be the subject of a *lifting* up to the level of the dilation (the calculus, the commutant, spectral decompositions, etc.), the level where they can be treated more easily.

The function model of Livsic, de Branges, Sz.-Nagy, and Foiaş follows precisely this course. Starting with a Hilbert space contraction T, or a dissipative operator in the initial Livsic form ($\operatorname{Im}(Tx, x) \geq 0$, for $x \in \operatorname{Dom} T$), it makes use of a unitary (selfadjoint) dilation $\mathcal{U}$ realized by use of the von Neumann spectral theorem as a multiplication operator $\mathcal{U} : f \to e^{it} f$. It then uses advanced trigonometric harmonic analysis on the circle (on the line, in the Livsic case), including Hardy classes, multiplicative Nevanlinna theory and other developed techniques. It is exactly these circumstances that gave B. Sz.-Nagy and C. Foiaş the chance for such an elegant expression as *harmonic analysis of operators* for the branch of operator theory based on the function model, and not the fact that "all is harmonious in the developed theory" as was suggested by M. Krein in his significant preface to the Russian translation of [Sz.-Nagy and Foiaş 1967]. We can even say that such a function model is a kind of noncommutative discrete Fourier transform of an operator, and mention with astonishment that a continuous version of this transform exists only nominally—which fact evidently retards applications to semigroup theory and scattering theory. We will have an opportunity to observe the absence of an "automatic translator" from the circle to the line when dealing with the Cayley transform and Schrödinger operators (Chapter 2).

0.2. What is the free function model? In constructing the free—that is, coordinate-free—function model of a Hilbert space contraction, we follow the Sz.-Nagy–Foiaş turnpike strategy by starting with a *unitary dilation*

$$\mathcal{U} : \mathcal{H} \longrightarrow \mathcal{H}$$

of a given contraction

$$T : H \longrightarrow H,$$

so that we get the dilation (calculus) property

$$p(T) = P_H p(\mathcal{U})|H$$

for all polynomials p. The usual way to continue is to realize the action of $\mathcal{U}$ by use of the spectral theorem, and so to represent T as a compression

$$f \longmapsto P_{\mathcal{K}} z f, \quad \text{for } f \in \mathcal{K},$$

of the multiplication operator $f \mapsto zf$ onto the orthogonal difference $\mathcal{K}$ of two z-invariant subspaces of a weighted space $L^2(\mathbb{T}, \mathcal{E}, W)$, with respect to an operator-valued weight $W(\zeta) : \mathcal{E} \to \mathcal{E}$, in a way similar to the way that the spectral theorem itself represents the unitary operator $\mathcal{U}$. (Here ζ is a complex number on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.)

The idea of the free function model is to stop this construction halfway from the unitary dilation to the final formulae of the Sz.-Nagy–Foiaş model. In other words, we fix neither a concrete spectral representation of a unitary dilation nor the dilation itself, but work directly with an (abstract) dilation equipped with two "functional embeddings" that carry the information on the fine function structure of the operator and the dilation. So, we can say that the free function model of a contraction T having defect spaces E and E_* is a class of in some sense equivalent (see Chapter 1) isometric functional embeddings

$$\pi : L^2(E) \longrightarrow \mathcal{H} \quad \text{and} \quad \pi_* : L^2(E_*) \longrightarrow \mathcal{H}$$

of E- and E_*-valued L^2-spaces ($L^2(E) \stackrel{\text{def}}{=} L^2(\mathbb{T}, E)$) into a Hilbert space $\mathcal{H}$ satisfying a minimality property, in such a way that the product $\pi_*^* \pi$ is a multiplication operator by a function unitarily equivalent to the *characteristic function* Θ_T of T,

$$\Theta_T(z)h = (-T + zD_{T^*}(I - zT^*)^{-1} D_T)h \quad \text{for } h \in E_*,\ z \in \mathbb{D}$$

(where $\mathbb{D} \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C} : |\zeta| < 1\}$). The minimal unitary dilation $\mathcal{U}$ can be defined by the splitting property $\Pi z = \mathcal{U}\Pi$, where

$$\Pi = \pi \oplus \pi_* : L^2(E) \oplus L^2(E_*) \to \mathcal{H}$$

and z stands for the shift operator $z(f \oplus f_*) = zf \oplus zf_*$ on this orthogonal sum of L^2-spaces.

At this stage, we have a great deal of freedom: we can write down or not write down the operator $\mathcal{U}$ using the spectral theorem, where the freedom is in a choice of spectral coordinates; and we can choose in very diverse ways the functional embeddings, keeping $\pi_*^*\pi$ unitarily equivalent to Θ_T (in fact, equivalent with respect to a special class of unitary equivalences). These transcription problems are considering in Chapter 3.

So, we can say that the free model is based on the intrinsic functional structures of T and the unitary dilation $\mathcal{U}$, and it is a kind of virtual function representation of a given operator: we can still stay in the same Hilbert space H where the initial contraction T is defined, but the pair $\{H, T\}$ is now equipped with two isometric embeddings π and π_* satisfying the above-mentioned conditions. Working with an operator on this free function model $\{H, T, \pi_*, \pi\}$ we can at any time transfer (by means of π, π_*) an arbitrary operator computation to functions in the spaces $L^2(E)$ and $L^2(E_*)$, whose elements have all of the advantages of concrete objects with concrete local structures. This is the idea, and in its realization we follow [Vasyunin 1977; Makarov and Vasyunin 1981; Nikolski and Vasyunin 1989; Nikolski and Khrushchev 1987; Nikolski 1994].

As concrete examples of model computations we consider in Chapter 2 two operators: the operator of indefinite integration on the space $L^2((0,1); \mu)$ with respect to an arbitrary finite measure μ, and Schrödinger operators with real potentials and dissipative boundary conditions.

0.3. What is spectral theory? This is a really good question. We can say, having no intention to give a truism and rather looking for an explanation of the title, that for us it is the study of *intrinsic structures* of an operator. So, for instance, even speaking of such well-studied subjects as selfadjoint and more generally of normal operators, the attempts to identify the theory with the spectral theorem are unjustified: this is the main but not sole gist of the theory. In fact, the structures of invariant subspaces and of the restrictions of the operator under consideration to them (the parts of the operator) are far from being straightforward consequences of the spectral theorem, and these subjects should be considered as independent ingredients of spectral theory. All the more, the same is true for the interplay between functional calculus and geometric properties of the operator (recall for example, the problem of density of polynomials in different spaces related to the spectral measure, or the problem of the asymptotic behavior of evolutions defined by the operator).

Beyond normality, there is again the study of invariant subspaces as a path to the parts of an operator, and also the analysis of decompositions into invariant subspaces (series, integrals), and, of course, the functional calculus admitted by an operator, and related studies of its “space action”, that is, properties of the evolution defined by the operator.

(By the way, this is why the famous polar decomposition of a Hilbert space operator, $T = VR$ with $R \geq 0$ and V a partial isometry, is not a crucial reduction of the problem to the selfadjoint and unitary cases: it does not respect any of basic properties and operations mentioned above. Another sentence we have the courage to put in these parentheses is that the previous paragraph explains why, in our opinion, spectral theory is something different from an algebraic approach (say, C^*-algebraic) of replacing an operator by a letter (a symbol) and then studying it: operator structures are lost in the kernel of such a morphism.)

So, in this paper, as a spectral theory, we consider the attempts

- to describe and to study invariant subspaces;
- to use them for decompositions of an operator into sums of its restrictions to such subspaces of a particular type respecting all other operator structures (so-called spectral subspaces);
- to distinguish operators enjoying the best possible spectral decompositions, that is, operators similar to a normal one;
- to study stability properties of fine spectral structures of an operator with respect to small perturbations and to study evolutions defined by it (scattering problems).

Of course, to construct such a spectral theory and to work with it using the function model technique, we need to develop some routine prerequisites, in particular to compute the spectrum and the resolvent of a contraction in terms of our free function model; this is done in Chapter 5 below.

Now we describe briefly how we intend to realize this program.

0.4. Conceptual value of invariant subspaces. The invariant subspaces of an operator, their classification and description, have played, explicitly or implicitly, the central role in operator theory for the last 50 years. The reader may ask, why?

The first and rather formal reason is that invariant subspaces are a direct analogue of the eigenvectors of linear algebra. Indeed, diagonal or Jordan form representations of a matrix are nothing but decompositions of the space an operator acts on into a direct sum of the operator's invariant subspaces of particular types. In the twenties and thirties, the theories of selfadjoint and unitary operators, and later that of normal operators, culminating in various forms of the spectral theorem and its applications, showed that this approach is fruitful. In the fifties and sixties, making use of distribution theory and advanced functional calculi, several generalizations followed this approach: the kernel theorem of Gelfand and L. Schwartz, the spectral operators of Dunford and J. Schwartz, Foiaş's decomposable operators, and some more specialized theories. See [Gel'fand and Vilenkin 1961; Dunford and Schwartz 1971; Dowson 1978; Colojoara and Foiaş 1968].

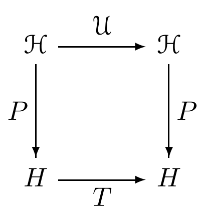
Another, and even double, reason to put invariant subspaces in the foundation of general spectral theory was provided by a real breakthrough that happened fifty years ago when M. Livsic [1946] discovered the notion of the characteristic function and A. Beurling [1949] established a one-to-one correspondence between the invariant subspaces of a Hilbert space isometry and the Nevanlinna inner functions. During the fifties and sixties, mainly due to efforts by M. Livsic and M. Brodski, and V. Potapov (for dissipative operators) and by B. Sz.-Nagy and C. Foiaş, and L. de Branges (for contractions), these observations together became the cornerstone of the theories linking invariant subspaces and factorizations (inner or not) of the characteristic function of an operator. These techniques make it possible to translate spectral decomposition problems for large classes of operators into the function language, and then to use methods of hard analysis (of complex analysis and harmonic analysis) to solve them.

The reader will find in Chapter 6 descriptions of the invariant subspaces of a contraction in terms of its functional embeddings and factorizations of the characteristic function, as well as some concrete examples of such descriptions.

In the Afterword (page 288), which functions as a preview of the Part II of this paper, we will briefly outline how this machinery works for several sample problems: spectral model operators and generalized free interpolation, rational tests for the similarity to a normal operator, stability of the continuous spectrum for trace class perturbations of unitary operators, and scattering theory. The Afterword also contains a brief discussion of factorizations of the characteristic function related to invariant subspaces of different types, and of operator Bezout ("corona") equations.

But first, in the next section, we sketch two of the main technical tools of the model theory: the commutant lifting theorem (CLT) and local functional calculi.

0.5. The commutant lifting theorem (CLT) and local calculi. As we have already mentioned, the main trick of model theory is to consider a unitary dilation $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ of a contractive operator $T : H \to H$, where $H \subset \mathcal{H}$:



to try to "lift" all things related to T up to the level of $\mathcal{U}$, and then to work with these lifted functional objects. One of the main objects related to an operator T is its *commutant* $\{T\}'$, the set of $X : H \to H$ such that $XT = TX$. All functions of an operator, that is, operators included in a functional calculus, belong to $\{T\}'$; so do all projections on an invariant subspace parallel to a complementary invariant subspace (that is, maps P of the form $P(x_1 + x_2) = x_1$ for $x_1 \in H_1$ and $x_2 \in H_2$, where $H = H_1 + H_2$, with $TH_1 \subset H_1$, $TH_2 \subset H_2$); and hence so does the spectral measure of an operator if it exists.

The commutant lifting theorem (CLT) says that the commutant of T can be lifted up to the commutant of $\mathcal{U}$: for $X \in \{T\}'$ there exists an $Y \in \{\mathcal{U}\}'$, a “symbol of X”, such that

$$X = P_H Y|H \quad \text{and} \quad \|X\| = \|Y\|.$$

This fundamental result was proved by B. Sz.-Nagy and C. Foiaş [1968], preceded by the important partial case of an inner scalar characteristic function Θ_T, discovered by D. Sarason [1967]. For more details we refer to Chapter 4 below, and for the history and more references refer to [Sz.-Nagy and Foiaş 1967].

The functional structure of Y, related to the functional embeddings of the model, is important, as are the links between this lifting process and the fundamental Nehari theorem on “Laurent extension” (lifting) of a Hankel type form. Without entering into these details now, we refer to [Nikolski 1986; Pták and Vrbová 1988] for the interplay between the CLT and Nehari type theorems and to [Cotlar and Sadosky 1984/85; 1986a; 1986b; 1988; 1992] for an extensive theory unifying these two techniques. To finish with our appreciation of the CLT, we mention its basic role for control theory and interpolation by analytic, rational, and other functions with constraints (of the Carathéodory or Nevanlinna–Pick type). A vast array of techniques have been developed for these applications: the Adamyan–Arov–Krein step-by-step extension process, choice sequences by Apostol, Foiaş et al., Schur parameters techniques, and more; for details we refer to [Foiaş and Frazho 1990; Ball et al. 1990; Bakonyi and Constantinescu 1992; Nikolski 1986; Alpay $\geq$ 1998].

However, we will avoid these deep theories and instead give, in Chapter 4, the simplest direct proof of the CLT based on S. Parrott’s approach [1970] (see also [Davidson 1988]). This approach makes use of a version of the CLT: the Ando theorem on commuting unitary dilations of two commuting contractions [Ando 1963], which says that, if $T_1 : H \to H$ and $T_2 : H \to H$ satisfy $\|T_1\| \leq 1$, $\|T_2\| \leq 1$, and

$$T_1 T_2 = T_2 T_1,$$

there exist commuting unitary dilations $\mathcal{U}_1$ and $\mathcal{U}_2$:

$$\mathcal{U}_1 \mathcal{U}_2 = \mathcal{U}_2 \mathcal{U}_1.$$

We give a short new proof of Ando’s theorem and derive the CLT and describe the admissible symbols Y in functional terms characterizing the functional (model) parameters of Y related to the functional embeddings π and π_*. Also, we give a formula expressing the norm of X as the distance

$$\|X\| = \inf\{\|Y + \pi \Gamma \pi_*^*\| : \Gamma \in H^\infty(E_* \to E)\},$$

where $H^\infty(E_* \to E)$ stands for the space of bounded holomorphic functions in the unit disc $\mathbb{D}$ taking as values operators $f(z) : E_* \to E$ from E_* to E (see Chapter 4).

For *two-sided inner* characteristic functions Θ, that is, for H^∞-functions whose boundary values $\Theta(\zeta)$ (where $\zeta \in \mathbb{T}$) are unitary almost everywhere on

$\mathbb{T}$, the CLT is a partial case of the vector-valued version of the Nehari theorem: if $X \in \{T\}'$ and $\mathbf{H} = \pi^* X P_H \pi_*$, then $\mathbf{H}$ is a Hankel operator (that is, $\mathbf{H} : H^2(E_*) \to H^2_-(E)$ and $\mathbf{H}z = P_- z\mathbf{H}$), and the function $\pi^* Y \pi_*$ is a symbol of $\mathbf{H}$ (that is, $\mathbf{H}f = P_-(\pi^* Y \pi_*)f$ for $f \in H^2(E_*)$). The formula for the norm of X is the usual Nehari formula

$$\|X\| = \operatorname{dist}_{L^\infty(E_* \to E)}(\pi^* Y \pi_*, H^\infty(E_* \to E));$$

see [Nikolski 1986].

The H^∞-calculus exists for every completely nonunitary contraction already by the initial unitary dilation theorem. It takes the following simplified form for two-sided inner characteristic functions:

$$f \longmapsto f(T) \stackrel{\text{def}}{=} P_H \pi_* f \pi_*^* | H,$$
$$\|f(T)\| = \operatorname{dist}_{L^\infty(E_* \to E)}(\Theta^* f, H^\infty(E_* \to E)).$$

Dealing with the calculus, and even with the whole commutant, it is important to localize, if possible, the expressions for $f(T)$ and for the norm $\|f(T)\|$, so as to make visible their dependence on the local behavior of f near the spectrum of T (whereas the definition and aforementioned formulas depend on the global behavior of f in the disc $\mathbb{D}$ and on the torus $\mathbb{T}$). Following [Nikolski and Khrushchev 1987] we give such expressions for operators T with scalar inner characteristic functions Θ, showing, for instance, an estimate on level sets of Θ:

$$\|f(T)\| \le C_\varepsilon \sup\{|f(z)| : z \in \mathbb{D},\ |\Theta(z)| < \varepsilon\}, \quad \text{for } f \in H^\infty.$$

Later on we use these estimates for our treatment of free interpolation problems (Chapter 8).

That is all we include in Part I of the article.

0.6. Prehistory. When writing an expository paper, one automatically accepts an extra assignment to comment on the history of the subject. Frankly speaking, we would like to avoid this obligation and to restrict ourselves to sporadic remarks. In fact, model theory is a quite recent subject, and we hope that careful references will be enough. Nonetheless, we will recall a kind of prehistory of the theory. This consists of four fundamental results by V. I. Smirnov (1928), A. I. Plessner (1939), M. A. Naimark (1943), and G. Julia (1944), which anticipated the subsequent developments of the field and ought to been have its clear landmarks, but were not identified properly and are still vaguely cited.

Smirnov's results. The following result, usually attributed to A. Beurling, is Theorem 2 of [Smirnov 1928a]: If $f \in H^2$ is an outer function (see Section 0.7 below), then

$$\operatorname*{span}_{H^2}(z^n f : n \ge 0) = H^2.$$

In [Smirnov 1932, Section 10] there is another proof of this fact—essentially the same that Beurling published in 1949. Another fundamental result, the inner-outer factorization $f = f_{\text{inn}} \cdot f_{\text{out}}$ of H^p functions (see Section 0.7 for definitions), appears in [1928b, Section 5]. Together, these two results immediately imply the well-known formula for the invariant subspace generated by a function $f \in H^2$:

$$\operatorname*{span}_{H^2}(z^n f : n \geq 0) = f_{\text{inn}} H^2.$$

This is the main ingredient of the famous Beurling paper [1949] on the shift operator, which stimulated the study of invariant subspaces. Unfortunately, in Smirnov's time the very notion of a bounded operator was not completely formulated—Banach's book dates from 1932—and, what is more, during the next ten years nobody thought about the shift operator, until H. Wold and A. I. Plessner.

Plessner's results. In 1939, A. I. Plessner constructed the H^∞-calculus for isometric operators allowing a unitary extension with an absolutely continuous spectrum. His paper [1939a] contains a construction of L^∞-functions of a maximal symmetric operator $A : H \to H$ having an absolutely continuous selfadjoint extension $\mathcal{A} : \mathcal{H} \to \mathcal{H}$:

$$f(A) = P_H f(\mathcal{A})|H;$$

the multiplicativity is proved for H^∞ functions.

Plessner [1939b] also proved that any Hilbert space isometry $V : H \to H$ is unitarily equivalent to an orthogonal sum of a unitary operator U and a number of copies of the (now standard!) H^2-shift operator $S : H^2 \to H^2$, given by $Sf = zf$:

$$V \simeq U \oplus \left(\sum_i \in I \oplus S \right), \quad \text{with } \operatorname{card}(I) = \dim(H \ominus VH).$$

To apply to isometries the calculus from the first paper, Plessner proved that an isometry V having an absolutely continuous unitary part U is an inner function of a maximal symmetric operator of the previous type: $V = f(A)$.

Clearly, the missing link to complete the construction of the H^∞-calculus for contractions would be the existence of isometric dilations (and their absolute continuity). The first step to such a dilation was accomplished by G. Julia in 1944, but his approach remained overlooked until the mid-fifties.

Julia's result. Julia [1944a; 1944b; 1944c] discovered a "one-step isometric dilation" of a Hilbert space contraction $T : H \to H$. Precisely, he observed that the operator defined on $H \oplus H$ by the block matrix

$$V = \begin{pmatrix} T & D_{T^*} \\ -D_T & T^* \end{pmatrix}$$

is an isometry, and that $T = P_H V|H$.

Of course, V is not necessarily a "degree-two dilation" $T^2 = P_H V^2|H$, nor a full polynomial dilation $T^n = P_H V^n|H$, for $n \geq 0$. We know now that to get such a dilation one has to extend the isometry V by the shift operator of multiplicity H, or simply take

$$\overline{V}(x, x_1, x_2, \dots) = (Tx, D_T x, x_1, x_2, \dots)$$

for $(x, x_1, \dots) \in H \oplus H \oplus \cdots$. But Julia did not make by this observation.

In fact, isometric and even unitary dilations were discovered ten years later by B. Sz.-Nagy [1953], ignoring Julia's step but using Naimark's dilation theorem.

Naimark's result. Historically, this was the last latent cornerstone of model theory. M. A. Naimark [1943] proved that every operator-valued positive measure $\mathcal{E}(\sigma) : H \to H$ taking contractive values, $0 \leq \mathcal{E}(\sigma) \leq I$, has a dilation that is a spectral measure: there exists an orthoprojection-valued measure $E(\sigma) : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} \supset H$, such that $\mathcal{E}(\sigma) = P_H E(\sigma)|H$ for all σ.

Later, Sz.-Nagy [1953] derived from this result the existence of the minimal unitary dilation for an arbitrary contraction, and the history of model theory started. In fact, as already mentioned, it started seven years earlier, with Livsic [1946], motivated rather by the theory of selfadjoint extensions of symmetric operators (developed by Krein, Naimark, and many others) than by the dilation philosophy.

0.7. Prerequisites. For the reader's convenience, we collect here some more specialized facts and terminology used throughout the paper.

General terminology. All vector spaces are considered over the complex numbers $\mathbb{C}$; a subspace of a normed space means a closed vector subspace. For a subset A of a normed space X, we denote by

$$\operatorname{span}(A) \stackrel{\text{def}}{=} \operatorname{clos}(\operatorname{Lin}(A))$$

the closed linear hull of A, and by

$$\operatorname{dist}_X(x, A) \stackrel{\text{def}}{=} \inf\{\|x - y\| : y \in A\}$$

the distance of a vector $x \in X$ from A. For a subspace $E \subset H$, the orthogonal complement is denoted by $E^\perp$ or $H \ominus E$, and the orthogonal projection on E by P_E. The orthogonal sum of a family $\{E_n\}$ of subspaces of a Hilbert space is denoted by

$$\sum_n \oplus E_n \stackrel{\text{def}}{=} \bigl\{ \textstyle\sum_n x_n : x_n \in E_n \text{ and } \sum_n \|x_n\|^2 < \infty \bigr\}.$$

Operators on a Hilbert space. "Operator" means a bounded linear operator unless otherwise indicated; $L(E \to F)$ stands for the space of all operators from E to F. A subspace $E \subset H$ is called *invariant* with respect to an operator $T : H \to H$ if $TE \subset E$; the restriction of T to E is denoted by $T|E$ and T is called an *extension* of $T|E$. The set of all T-invariant subspaces is denoted by

$\operatorname{Lat} T$ (or $\operatorname{Lat}(T)$). *Reducing* subspaces are subspaces from $\operatorname{Lat} T \cap \operatorname{Lat} T^*$; T^* stands for the (hermitian) adjoint of T. We denote by $\operatorname{Range}(T)$ the range of T, that is, the linear set $\{Th : h \in H\}$, and by $\operatorname{Ker} T \stackrel{\text{def}}{=} \{x : Tx = 0\}$ for the kernel of T.

A *partial isometry* $V : H \to K$ is an operator between Hilbert spaces such that $V|(\operatorname{Ker} V)^\perp$ is an isometry; the subspaces $(\operatorname{Ker} V)^\perp$ and $\operatorname{Range} V$ are called, respectively, the *initial* and *final* subspaces of V. The final subspace of a partial isometry V is the initial subspace of the adjoint V^*, and vice versa; moreover $V^*V = P_{(\operatorname{Ker} V)^\perp}$.

The *polar decomposition* of an operator $T : H \to K$ is the representation $T = VR$, where $R = |T| \stackrel{\text{def}}{=} (T^*T)^{1/2} \geq 0$ is the positive square root of T^*T (the modulus of T) and $V : H \to K$ a partial isometry with the initial space $\operatorname{Range} R = (\operatorname{Ker} T)^\perp$ and the final space $\operatorname{Range} T$. The equation $A^*A = B^*B$ is equivalent to saying that $B = VA$, where V stands for a partial isometry with the initial space $\operatorname{Range} A$ and the final space $\operatorname{Range} B$.

The *spectral theorem* (in the von Neumann form) says that a *normal operator* $N : H \to H$ (that is, one such that $N^*N = NN^*$) is unitarily equivalent to the multiplication operator

$$f(z) \longmapsto zf(z)$$

on the space

$$\{f \in L^2(H, \mu_N) : f(z) \in E_N(z)H \text{ a.e. with respect to } \mu_N\},$$

where $E_N(\,\cdot\,)$ stands for a mesurable family of orthoprojections on H, and μ_N stands for a so-called scalar spectral measure of N carried by the spectrum $\sigma(N)$; the class of measures equivalent to μ_N and the dimension function $z \mapsto \dim H(z)$ defined μ_N-a.e. are complete unitary invariants of N.

Isometries and co-isometries. The Wold–Kolmogorov theorem says that an isometry $V : H \to H$ gives rise to an orthogonal decomposition

$$H = H_\infty \oplus \left(\sum_n \geq 0 \oplus V^n E \right),$$

where $H_\infty = \bigcap_{n \geq 0} V^n H$ and $E = H \ominus VH = \operatorname{Ker} V^*$. The subspace H_∞ reduces V and the restriction $V|H_\infty$ is unitary; any other subspace with these properties is contained in H_∞. An isometry V with $H_\infty = \{0\}$ is called *pure* (or an *abstract shift operator*), and a subspace $L \subset H$ satisfying $V^n L \perp V^m L$ for $n \neq m \geq 0$ is called a *wandering subspace* of V; in particular, $E = H \ominus VH$ is a wandering subspace. A pure isometry V is unitarily equivalent to the *shift operator* of multiplicity $\dim E = \dim \operatorname{Ker} V^*$, defined by

$$f \longmapsto zf \quad \text{for } f \in H^2(E),$$

where

$$H^2(E) = \big\{ f = \textstyle\sum_{n\geq 0} \hat{f}(n) z^n : \hat{f}(n) \in E \text{ and } \sum_{n\geq 0} \|\hat{f}(n)\|^2 < \infty \big\}$$

stands for E-valued *Hardy space* (see also below).

A *co-isometry* $V : H \to K$ is an operator (a partial isometry) whose adjoint V^* is an isometry; it is called *pure* if V^* is pure.

Contractions. A *contraction* $T : H \to K$ is an operator with $\|T\| \leq 1$. The *defect operators* of T are $D_T = (I - T^*T)^{1/2}$ and $D_{T^*} = (I - TT^*)^{1/2}$; the closures of their ranges $\mathcal{D}_T \stackrel{\text{def}}{=} \operatorname{clos} \operatorname{Range} D_T$ and $\mathcal{D}_{T^*}$ are called the *defect subspaces*, and $\dim \mathcal{D}_T = \operatorname{rank}(I - T^*T)$ and $\dim \mathcal{D}_{T^*}$ are the *defect numbers*. The defect operators are intertwined by the contraction, $TD_T = D_{T^*}T$, and hence the restriction $T|\mathcal{D}_T^\perp$ is an isometry from $\mathcal{D}_T^\perp$ to $\mathcal{D}_{T^*}^\perp$. A contraction T is called *completely nonunitary* if there exists no reducing subspace (or invariant subspace) where T acts as a unitary operator.

Function spaces and their operators. Lebesgue spaces of vector-valued functions are defined in the standard way: the notation $L^p(\Omega, E, \mu)$ means the L^p-space of E-valued functions on a mesure space (Ω, μ). We can abbreviate $L^p(\Omega, E, \mu)$ as $L^p(E, \mu)$, $L^p(E)$, $L^p(\Omega, E)$, or $L^p(\Omega)$ (this last when $E = \mathbb{C}$).

Our standard case is $\Omega = \mathbb{T}$, endowed with normalized Lebesgue measure $\mu = m$. The *Hardy subspace* of $L^p(E) = L^p(\mathbb{T}, E)$ is defined in the usual way,

$$H^p(E) = \{ f \in L^p(E) : \hat{f}(n) = 0 \text{ for } n < 0 \},$$

where the $\hat{f}(n)$, for $n \in \mathbb{Z}$, are the Fourier coefficients. Obviously, $L^2(E \oplus F) = L^2(E) \oplus L^2(F)$. Also as usual, $H^p(E)$ is identified with the space of boundary functions of the corresponding space of functions holomorphic in the unit disc $\mathbb{D}$ (for $p = 2$, see above). The *reproducing kernel* of the space $H^2(E)$ is defined by the equality

$$\Big(f, \frac{e}{1 - \bar{\lambda} z} \Big)_{H^2(E)} = (f(\lambda), e)_E \quad \text{for } f \in H^2(E) \text{ and } e \in E.$$

The *Riesz projections* $P_{H^2} = P_+$ and $P_- = I - P_+$ are defined on $L^2(E)$ by

$$P_+ \Big(\sum_{n \in \mathbb{Z}} \hat{f}(n) z^n \Big) = \sum_{n \geq 0} \hat{f}(n) z^n, \quad \text{for } |z| = 1.$$

Clearly, $L^2(E) = H^2(E) \oplus H^2_-(E)$, where $H^2_-(E) = P_- L^2(E)$.

The L^p and H^p spaces with values in the space $L(E \to F)$ of all operators from E to F are denoted by $L^p(E \to F)$ and $H^p(E \to F)$ respectively. Any operator A acting from $L^2(E_1)$ to $L^2(E_2)$ and intertwining multiplication by z on these spaces (which means that $Az = zA$) is the multiplication operator induced by a function $\Theta \in L^\infty(E_1 \to E_2)$, namely $(Af)(z) = \Theta(z) f(z)$; moreover, $\|A\| = \|A|H^2(E_1)\| = \|\Theta\|_{L^\infty(E_1 \to E_2)}$. We often identify A and Θ; we have $\Theta \in H^\infty(E_1 \to E_2)$ if and only if $\Theta \in L^\infty(E_1 \to E_2)$ and $\Theta H^2(E_1) \subset H^2(E_2)$. A contractive-valued function $\Theta \in H^\infty(E_1 \to E_2)$ is called *pure* if it does not reduce to a constant isometry on any subspace of E_1, or, equivalently, if $\|\Theta(0)e\|_{E_2} < \|e\|_{E_1}$ for all nonzero $e \in E_1$.

The last point is to recall the *Nevanlinna–Riesz–Smirnov canonical factorization* of scalar H^1-functions:

$$\Theta = BS \cdot F = \Theta_{\mathrm{inn}} \cdot \Theta_{\mathrm{out}},$$

where B is a Blaschke product, S a singular inner function, and F an outer function. We have

$$B(z) = \prod_{\lambda \in \mathbb{D}} b_\lambda^{k(\lambda)}(z),$$

where

$$b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda} z}, \quad \text{with } |\lambda| < 1,$$

is a Blaschke factor and $k(\lambda)$ is the divisor of zero multiplicities, satisfying the Blaschke condition $\sum_{\lambda \in \mathbb{D}} k(\lambda)(1 - |\lambda|) < \infty$. We also have

$$S(z) = \exp\Bigl(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu_s(\zeta)\Bigr),$$

μ_s being a positive measure singular with respect to Lebesgue measure m, and

$$F(z) = \exp\Bigl(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \frac{1}{|\Theta(\zeta)|}\, dm(\zeta)\Bigr).$$

Let δ_λ be the measure of mass 1 concentrated at the point λ (Dirac measure). We associate with the function Θ the measure on the closed unit disc defined by

$$d\mu_\Theta = \log \frac{1}{|\Theta|}\, dm + d\mu_s + \sum_{\lambda \in \mathbb{D}} k(\lambda)(1 - |\lambda|^2)\delta_\lambda.$$

This is often called the *representing measure* of Θ.

Chapter 1. Construction of the Function Model

1.1. Unitary dilation. We start constructing the function model for a contraction on a Hilbert space with an explicit description of its minimal unitary dilation. An operator $\mathcal{U}$ acting on a Hilbert space $\mathcal{H}$ is called a *dilation* of an operator T acting on H, where $H \subset \mathcal{H}$, if

$$T^n = P_H \mathcal{U}^n | H \quad \text{for all } n \geq 1.$$

The dilation is called *minimal* if $\operatorname{span}\{\mathcal{U}^n H : n \in \mathbb{Z}\} = \mathcal{H}$. It is called *unitary* if $\mathcal{U}$ is a unitary operator.

The structure of the space $\mathcal{H}$ where a dilation of an operator acts is given by the following lemma of Sarason [1965].

1.2. Lemma. *$\mathcal{U} : \mathcal{H} \to \mathcal{H}$ is a dilation of $T : H \to H$ if and only if*

$$\mathcal{H} = G_* \oplus H \oplus G, \tag{1.2.1}$$

with

$$\mathcal{U}G \subset G, \quad \mathcal{U}^* G_* \subset G_*, \quad \textit{and} \quad P_H \mathcal{U}|H = T.$$

G and G_* are called the *outgoing* and the *incoming* subspace, respectively.

Proof. To prove the "only if" part, put

$$G = \operatorname{span}\{\mathcal{U}^n H : n \geq 0\} \ominus H,$$
$$G_* = \mathcal{H} \ominus \operatorname{span}\{\mathcal{U}^n H : n \geq 0\}.$$

If $g = \lim p_n(\mathcal{U}) h_n$ is orthogonal to H, $h_n \in H$, and $f \in H$, then

$$\begin{aligned}(\mathcal{U}g, f) &= \lim(\mathcal{U}p_n(\mathcal{U})h_n, f) = \lim(Tp_n(T)h_n, f) = \lim(p_n(T)h_n, T^* f) \\ &= \lim(p_n(\mathcal{U})h_n, T^* f) = (g, T^* f) = 0,\end{aligned}$$

that is, $\mathcal{U}G \subset G$. The inclusion $\mathcal{U}^* G_* \subset G_*$ is obvious.

For the converse, we use the block matrix representation of $\mathcal{U}$ with respect to the decomposition (1.2.1). We get

$$\mathcal{U} = \begin{pmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{pmatrix} \Longrightarrow \mathcal{U}^n = \begin{pmatrix} * & 0 & 0 \\ * & T^n & 0 \\ * & * & * \end{pmatrix} \Longrightarrow P_H \mathcal{U}^n | H = T^n,$$

for all $n \geq 1$. □

1.3. The matrix of a unitary dilation. We need a unitary operator $\mathcal{U} : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} = G_* \oplus H \oplus G$, of the form:

$$\mathcal{U} = \begin{pmatrix} \mathcal{E}_* & 0 & 0 \\ A & T & 0 \\ C & B & \mathcal{E} \end{pmatrix}.$$

In matrix form, the conditions $\mathcal{U}^*\mathcal{U} = I$ and $\mathcal{U}\mathcal{U}^* = I$ become

$$\begin{pmatrix} \mathcal{E}_*^* \mathcal{E}_* + A^*A + C^*C & A^*T + C^*B & C^*\mathcal{E} \\ T^*A + B^*C & T^*T + B^*B & B^*\mathcal{E} \\ \mathcal{E}^*C & \mathcal{E}^*B & \mathcal{E}^*\mathcal{E} \end{pmatrix} = I,$$

$$\begin{pmatrix} \mathcal{E}_*\mathcal{E}_*^* & \mathcal{E}_*A^* & \mathcal{E}_*C^* \\ A\mathcal{E}_*^* & AA^* + TT^* & AC^* + TB^* \\ C\mathcal{E}_*^* & CA^* + BT^* & CC^* + BB^* + \mathcal{E}\mathcal{E}^* \end{pmatrix} = I.$$

We put $D_T \overset{\text{def}}{=} (I - T^*T)^{1/2}$ and $\mathcal{D}_T \overset{\text{def}}{=} \operatorname{clos} D_T H$. Twelve different entries give twelve different equations. Here are ten of them:

$$\begin{cases} \mathcal{E}^*\mathcal{E} = I & \Longrightarrow \ \mathcal{E} \text{ is an isometry;} \\ \mathcal{E}_*\mathcal{E}_*^* = I & \Longrightarrow \ \mathcal{E}_* \text{ is a co-isometry;} \\ AA^* + TT^* = I & \Longrightarrow \ A^* = V_* D_{T^*} \text{ (polar decomposition);} \\ \mathcal{E}_* A^* = 0 & \Longrightarrow \ V_* : \mathcal{D}_{T^*} \to \operatorname{Ker} \mathcal{E}_*; \\ T^*T + B^*B = I & \Longrightarrow \ B = V D_T \text{ (polar decomposition);} \\ \mathcal{E}^* B = 0 & \Longrightarrow \ V : \mathcal{D}_T \to \operatorname{Ker} \mathcal{E}^*; \\ CA^* + BT^* = 0 & \Longrightarrow \ CV_* + VT^* = 0 \ \Longrightarrow \ C = -VT^*V_*^* + C_0, \\ & \qquad\qquad\qquad\qquad\qquad C_0 | \operatorname{Range} V_* = 0; \\ A^*T + C^*B = 0 & \Longrightarrow \ C_0^* V D_T = 0 \ \Longrightarrow \ C_0^* | \operatorname{Range} V = 0; \\ \mathcal{E}_*^*\mathcal{E}_* + A^*A + C^*C = I & \Longrightarrow \ V_*V_*^* + C_0^*C_0 = P_{\operatorname{Ker} \mathcal{E}_*}; \\ CC^* + BB^* + \mathcal{E}\mathcal{E}^* = I & \Longrightarrow \ VV^* + C_0C_0^* = P_{\operatorname{Ker} \mathcal{E}^*}. \end{cases}$$

The last two of these identities mean that C_0 is a partial isometry with initial space $\operatorname{Ker} \mathcal{E}_* \ominus \operatorname{Range} V_*$ and final space $\operatorname{Ker} \mathcal{E}^* \ominus \operatorname{Range} V$; therefore, the two other identities $\mathcal{E}^* C = 0$ and $C\mathcal{E}_*^* = 0$ are fulfilled automatically.

Now, if we introduce

$$G^{(1)} = \sum_n \geq 0 \oplus \mathcal{E}^n V \mathcal{D}_T, \qquad G_*^{(1)} = \sum_n \geq 0 \oplus \mathcal{E}_*^{*n} V_* \mathcal{D}_{T^*},$$
$$G^{(2)} = G \ominus G^{(1)}, \qquad G_*^{(2)} = G_* \ominus G_*^{(1)},$$

then $G_*^{(2)} \oplus G^{(2)}$ is a reducing subspace of $\mathcal{U}$:

$$\mathcal{U} | G_*^{(2)} \oplus G^{(2)} = \begin{pmatrix} \mathcal{E}_* | G_*^{(2)} & 0 \\ C_0 & \mathcal{E} | G^{(2)} \end{pmatrix}.$$

Therefore, the operator

$$\mathcal{U} | G_*^{(1)} \oplus H \oplus G^{(1)} = \begin{pmatrix} \mathcal{E}_* | G_*^{(1)} & 0 & 0 \\ D_{T^*} V_*^* & T & 0 \\ -VT^*V_*^* & V D_T & \mathcal{E} | G^{(1)} \end{pmatrix}$$

is also a unitary dilation of T. Moreover, the dilation is minimal. Indeed, since

$$\mathcal{U} H = \begin{pmatrix} 0 \\ T \\ V D_T \end{pmatrix} H,$$

we see that $\operatorname{span}\{\mathcal{U}H, H\} = H \oplus V\mathcal{D}_T$, whence $\operatorname{span}\{\mathcal{U}^n H : n \geq 0\} = H \oplus G^{(1)}$. In a similar way we obtain $\operatorname{span}\{\mathcal{U}^n H : n \leq 0\} = G_*^{(1)} \oplus H$. So the dilation is minimal if and only if $\mathcal{E}$ and $\mathcal{E}_*^*$ are pure isometries and $\operatorname{Range} V = \operatorname{Ker} \mathcal{E}^*$, $\operatorname{Range} V_* = \operatorname{Ker} \mathcal{E}_*$.

Thus, we arrive at the following theorem.

1.4. Theorem. *An operator* $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ *is a minimal unitary dilation of* $T : H \to H$ *if and only if there exist subspaces* G *and* G_* *of* $\mathcal{H}$ *such that*

$$\mathcal{H} = G_* \oplus H \oplus G,$$

and, with respect to this decomposition, $\mathcal{U}$ *has matrix*

$$\mathcal{U} = \begin{pmatrix} \mathcal{E}_* & 0 & 0 \\ D_{T^*}V_*^* & T & 0 \\ -VT^*V_*^* & VD_T & \mathcal{E} \end{pmatrix},$$

where $\mathcal{E} = \mathcal{U}|G$ *and* $\mathcal{E}_*^* = \mathcal{U}^*|G_*$ *are pure isometries,* V *is a partial isometry with initial space* $\mathcal{D}_T$ *and final space* $\operatorname{Ker}\mathcal{E}^*$, *and* V_* *is a partial isometry with initial space* $\mathcal{D}_{T^*}$ *and final space* $\operatorname{Ker}\mathcal{E}_*$. □

1.5. First glimpse into the function model. In principle, the construction of the minimal unitary dilation could be split into two steps: first we can construct a minimal isometric dilation (or a minimal co-isometric extension), and then apply the same procedure to the adjoint of the operator obtained. As a result, we get a minimal unitary dilation. It is worth mentioning that in our terms the restriction of $\mathcal{U}$ to $H \oplus G$ is a minimal isometric dilation of T and the compression of $\mathcal{U}$ onto $G_* \oplus H$ is a minimal co-isometric extension of T.

A function model of a contraction T arises whenever we realize $\mathcal{E}$ and $\mathcal{E}_*^*$, which are abstract shift operators, as the operators of multiplication by z on appropriate Hardy spaces. Such a realization provides us with two functional embeddings of the corresponding Lebesgue spaces in the space of the minimal unitary dilation. All other objects needed for handling such a model operator (the characteristic function, formulas for the projection onto H, for the dilation, and for the operator itself, etc.) are computed in terms of these embeddings. We present the necessary constructions in the rest of this chapter. In Chapter 3 we give some explicit formulas for the model as examples of the above-mentioned functional realizations of the coordinate-free model in terms of multiplication by z on certain L^2-spaces.

1.6. Functional embeddings. As already mentioned, after fixing a minimal unitary dilation of a given contraction T, we begin with a spectral representation of the pure isometry $\mathcal{E}$ and the pure co-isometry $\mathcal{E}_*$ using the Wold decomposition. Let E and E_* be two auxiliary Hilbert spaces such that

$$\dim E = \dim \mathcal{D}_T = \dim \operatorname{Ker}\mathcal{E}^*,$$
$$\dim E_* = \dim \mathcal{D}_{T^*} = \dim \operatorname{Ker}\mathcal{E}_*.$$

Since $\operatorname{Ker}\mathcal{E}^* = G \ominus \mathcal{E}G = G \ominus \mathcal{U}G$ and $\operatorname{Ker}\mathcal{E}_* = G_* \ominus \mathcal{E}_*^*G = G_* \ominus \mathcal{U}^*G_*$, there exist unitary mappings

$$v : E \to G \ominus \mathcal{U}G \quad \text{and} \quad v_* : E_* \to G_* \ominus \mathcal{U}^*G_* \tag{1.6.1}$$

identifying these spaces. Now we can define a mapping

$$\Pi = (\pi_*, \pi) : L^2(E_*) \oplus L^2(E) \to \mathcal{H} \tag{1.6.2}$$

by the formulas

$$\pi\Big(\sum_{k\in\mathbb{Z}} z^k e_k\Big) = \sum_{k\in\mathbb{Z}} \mathcal{U}^k v e_k \qquad \text{for } e_k \in E,$$

$$\pi_*\Big(\sum_{k\in\mathbb{Z}} z^k e_{*k}\Big) = \sum_{k\in\mathbb{Z}} \mathcal{U}^{k+1} v_* e_{*k} \quad \text{for } e_{*k} \in E_*,$$

$$\Pi(f_* \oplus f) = \pi_* f_* + \pi f.$$

The operators Π, π, and π_* are called *functional mappings* or *functional embeddings*. Immediately from the above definition we deduce the following properties of the embeddings π and π_*:

(i) $\pi^*\pi = I_{L^2(E)}$ and $\pi_*^*\pi_* = I_{L^2(E_*)}$.
(ii) $\pi z = \mathcal{U}\pi$ and $\pi_* z = \mathcal{U}\pi_*$.
(iii) $\pi H^2(E) = G$ and $\pi_* H^2_-(E_*) = G_*$.

Property (i) is a consequence of the fact that $G \ominus \mathcal{U}G$ and $G_* \ominus \mathcal{U}^*G_*$ are wandering subspaces for $\mathcal{U}$; that is,

$$\mathcal{U}^n(G \ominus \mathcal{U}G) \perp \mathcal{U}^m(G \ominus \mathcal{U}G) \quad \text{and} \quad \mathcal{U}^n(G_* \ominus \mathcal{U}^*G_*) \perp \mathcal{U}^m(G_* \ominus \mathcal{U}^*G_*)$$

for distinct $n, m \in \mathbb{Z}$. Property (ii) is obvious, as are the relations $\pi z^n = \mathcal{U}^n\pi$ and $\pi_* z^n = \mathcal{U}^n\pi_*$ for all $n \in \mathbb{Z}$. To check (iii) we write

$$\begin{aligned}
\pi H^2(E) &= \big\{\pi \textstyle\sum_{k\ge 0} z^k e_k : e_k \in E,\ \sum_{n\ge 0} \|e_k\|^2 < \infty\big\} \\
&= \big\{ \textstyle\sum_{k\ge 0} \mathcal{U}^k v e_k : e_k \in E,\ \sum_{n\ge 0} \|e_k\|^2 < \infty\big\} \\
&= \textstyle\sum_k \ge 0 \oplus \mathcal{U}^k(G \ominus \mathcal{U}G) = G, \\
\pi_* H^2_-(E_*) &= \big\{\pi_* \textstyle\sum_{k<0} z^k e_{*k} : e_{*k} \in E_*,\ \sum_{n\ge 0} \|e_{*k}\|^2 < \infty\big\} \\
&= \big\{ \textstyle\sum_{k<0} \mathcal{U}^{k+1} v_* e_{*k} : e_{*k} \in E_*,\ \sum_{n\ge 0} \|e_{*k}\|^2 < \infty\big\} \\
&= \textstyle\sum_k \ge 0 \oplus \mathcal{U}^{*k}(G_* \ominus \mathcal{U}^*G_*) = G_*.
\end{aligned}$$

In general, the space $\operatorname{clos}\operatorname{Range}\Pi$ may be different from the entire space $\mathcal{H}$ of the minimal unitary dilation of T. Now we describe this range as follows.

1.7. Lemma. *The orthogonal complement*

$$H_u \overset{\text{def}}{=} (\operatorname{Range}\Pi)^\perp = \big(\pi L^2(E) + \pi_* L^2(E_*)\big)^\perp$$

is contained in H and is a reducing subspace of T. Moreover, H_u is maximal among all T-invariant subspaces $H' \subset H$ for which the restriction $T|H'$ is unitary.

PROOF. Since $G = \pi H^2(E) \subset \operatorname{Range}\Pi$ and $G_* = \pi_* H^2_-(E_*) \subset \operatorname{Range}\Pi$, we have $H_u \subset H$. The intertwining property (ii) implies that H_u reduces $\mathcal{U}$. Therefore, the operator $T|H_u = \mathcal{U}|H_u$ is unitary.

It remains to prove the maximality of H_u. Let $TH' \subset H' \subset H$, and let $T|H'$ be unitary. Then for $h \in H'$ and $n \geq 0$ we have

$$\|h\| = \|T^n h\| = \|P_H \mathcal{U}^n h\| \leq \|\mathcal{U}^n h\| = \|h\|.$$

Therefore, $\mathcal{U}^n h = P_H \mathcal{U}^n h \in H$, whence $\mathcal{U}^n H' \subset H$, $n \geq 0$. Thus, $\mathcal{U}^n H' \perp \pi H^2(E)$ and $H' \perp \mathcal{U}^{*n} \pi H^2(E)$, which yields $H' \perp \pi L^2(E)$ because

$$\operatorname{span}\{\mathcal{U}^{*n}\pi H^2(E) : n \geq 0\} = \operatorname{span}\{\pi \bar{z}^n H^2(E) : n \geq 0\} = \pi L^2(E).$$

Similarly, the relations

$$\|h\| = \|T^{*n} h\| = \|P_H \mathcal{U}^{*n} h\| \leq \|\mathcal{U}^{*n} h\| = \|h\|$$

imply that $H' \perp \pi_* L^2(E_*)$, whence $H' \subset H_u$. □

The lemma yields the well-known decomposition of a contraction into a unitary and a completely nonunitary part. We recall that a contraction is called *completely nonunitary* if $H_u = \{0\}$, that is, if the restriction of it to any nontrivial invariant subspace is not a unitary operator.

1.8. Corollary. *A contraction T on H can be uniquely represented in the form $T = T_u \oplus T_0$, where $T_u = T|H_u$ is unitary and $T_0 \stackrel{\text{def}}{=} T|H_0$, for $H_0 \stackrel{\text{def}}{=} H \ominus H_u$, is completely nonunitary. Therefore, a contraction is completely nonunitary if and only if*

(iv) $\operatorname{clos}\operatorname{Range}\Pi = \mathcal{H}$.

In what follows we shall deal with completely nonunitary contractions only.

1.9. Characteristic function. We continue the construction of the function model of a given completely nonunitary contraction T. We use the functional embeddings to introduce the central object of model theory, the *characteristic function* of an operator.

We define

$$\Theta \stackrel{\text{def}}{=} \pi_*^* \pi : L^2(E) \to L^2(E_*). \qquad (1.9.1)$$

By property (ii) on page 228, the mapping Θ intertwines the operators of multiplication by z in the two L^2-spaces above:

$$\Theta z = z\Theta.$$

We saw in Section 0.7 that such a Θ is the operator of multiplication by a bounded operator-valued function, which we denote by the same symbol:

$$(\Theta f)(\zeta) = \Theta(\zeta) f(\zeta) \quad \text{for } \zeta \in \mathbb{T}.$$

We shall write $\Theta \in L^\infty(E \to E_*)$, which means that Θ is a measurable function defined and bounded almost everywhere on the unit circle whose values are operators acting from E to E_*. Moreover, this function is contractive-valued ($\|\Theta\| \le \|\pi_*^*\| \cdot \|\pi\| = 1$) and analytic. Indeed, we have

$$\pi H^2(E) = G \perp G_* = \pi_* H^2_-(E_*),$$

whence

$$\Theta H^2(E) = \pi_*^* \pi H^2(E) \subset H^2(E_*);$$

that is, $\Theta \in H^\infty(E \to E_*)$.

Definition. The function Θ defined by (1.9.1) is called the *characteristic function* of the given completely nonunitary contraction T.

Thus, formally, the "characteristic function" of a contraction T is a family of functions (depending on $\mathcal{U}$ and Π). Our next goal is to describe the family of characteristic functions corresponding to all operators unitarily equivalent to a given operator. Later, in Remark 1.12, we shall make somewhat more precise the notion of a characteristic function.

1.10. Equivalence relations. The definition of the functional embeddings, and therefore that of the characteristic function, involve two arbitrary Hilbert spaces E and E_*, only their dimensions being essential. So, it is natural to regard all the objects obtained as equivalent if a pair of spaces E, E_* is replaced by another one, say E', E'_*, of the same dimension. *By definition*, two functions $\Theta \in L^\infty(E \to E_*)$ and $\Theta' \in L^\infty(E' \to E'_*)$ are said to be *equivalent* if there exist unitary mappings

$$u : E \to E' \quad \text{and} \quad u_* : E_* \to E'_* \tag{1.10.1}$$

such that

$$\Theta' u = u_* \Theta. \tag{1.10.2}$$

In what follows, being in the framework of Hilbert space theory, we shall often view our initial object, a Hilbert space contraction, up to unitary equivalence. The minimal unitary dilation of such a contraction is also defined up to the corresponding unitary equivalence $W : \mathcal{H} \to \mathcal{H}'$ intertwining $\mathcal{U}$ and $\mathcal{U}'$ and preserving the structure (1.2.1), that is, satisfying

$$WG = G', \quad WG_* = G'_*, \quad WH = H'. \tag{1.10.3}$$

Therefore, the following definition is natural. Two functional embeddings Π and Π' are said to be *equivalent* if there exist unitary mappings (1.10.1) and a unitary operator (1.10.3) such that

$$\Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} = W\Pi.$$

1.11. Theorem. *Let T, T' be two completely nonunitary contractions; we denote by Π, Π' their functional embeddings and by Θ, Θ' their characteristic functions. The following assertions are equivalent.*

(i) *T and T' are unitarily equivalent.*
(ii) *Π and Π' are equivalent.*
(iii) *Θ and Θ' are equivalent.*

Proof. (1) $\implies$ (2). Let W_0 be a unitary operator, $W_0T = T'W_0$. We define W on finite sums $\sum \mathcal{U}^n h_n$, where $h_n \in H$, by the formula

$$W\big(\sum \mathcal{U}^n h_n\big) = \sum \mathcal{U}'^n W_0 h_n.$$

We check that W is norm-preserving; this will imply that W is well-defined and can be extended to an isometry acting on the whole of $\mathcal{H}$. Since the finite sums $\sum \mathcal{U}'^n W_0 h_n$ are dense in $\mathcal{H}'$, such an extension will be surjective and, therefore, unitary. We have

$$\begin{aligned}\left\| W\left(\sum_n \mathcal{U}'^n h_n\right)\right\|^2 &= \sum_n \|\mathcal{U}'^n W_0 h_n\|^2 + 2\operatorname{Re}\sum_{k>l}({\mathcal{U}'}^{k-l} W_0 h_k,\, W_0 h_l)\\ &= \sum_n \|h_n\|^2 + 2\operatorname{Re}\sum_{k>l}({T'}^{k-l} W_0 h_k, W_0 h_l)\\ &= \sum_n \|\mathcal{U}^n h_n\|^2 + 2\operatorname{Re}\sum_{k>l}(\mathcal{U}^{k-l} h_k, h_l) = \left\|\sum_n \mathcal{U}^n h_n\right\|^2.\end{aligned}$$

From the definition we see that $W\mathcal{U} = \mathcal{U}'W$ and $WH = W_0H = H'$. Moreover,

$$\begin{aligned}W(G\oplus H) &= W\operatorname{span}\{\mathcal{U}^n H : n\ge 0\} = \operatorname{span}\{{\mathcal{U}'}^n W_0 H : n \ge 0\}\\ &= \operatorname{span}\{{\mathcal{U}'}^n H' : n\ge 0\} = G'\oplus H';\end{aligned}$$

that is, $WG = G'$, whence $W(G\ominus \mathcal{U}G) = G'\ominus \mathcal{U}'G'$. Similarly, $WG_* = G'_*$ and $W(G_*\ominus \mathcal{U}^*G_*) = G'_*\ominus \mathcal{U}'^*G'_*$. Using the unitary mappings v and v_* of (1.6.1), we can define

$$u = {v'}^* W v : E\to E', \quad u_* = {v'}_*^* W v_* : E_*\to E'_*.$$

Now we show that these operators provide an equivalence between Π and Π':

$$\begin{aligned}W\pi\left(\sum z^k e_k\right) &= W\sum \mathcal{U}^k v e_k = \sum {\mathcal{U}'}^k W v e_k\\ &= \sum {\mathcal{U}'}^k v' u e_k = \pi'\left(\sum z^k u e_k\right) = \pi' u\left(\sum z^k e_k\right).\end{aligned}$$

Thus $W\pi = \pi' u$. That $W\pi_* = \pi'_* u_*$ can be checked similarly.

(2) $\implies$ (3). We have

$$\Theta' u = {\pi'}_*^*\pi' u = {\pi'}_*^* W\pi = (W^*\pi'_*)^*\pi = (\pi_* u_*^*)^*\pi = u_*\pi_*^*\pi = u_*\Theta.$$

(3) $\Longrightarrow$ (2). Defining an operator W on the dense set $\Pi L^2(E_* \oplus E)$ by the identity

$$W\Pi = \Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix}$$

we check that W is norm-preserving. As in the first part of the proof, this will imply that W is well-defined and extends to a unitary operator.

First, we note that property (i) of page 228, combined with the definition of Θ, can be written as

$$\Pi^*\Pi = \begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix}.$$

Using this relation and the identity $\Theta' u = u_* \Theta$, we obtain

$$\begin{aligned} \|W\Pi x\|^2 &= \left\| \Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} x \right\|^2 = \left(\begin{pmatrix} u_*^* & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} I & \Theta' \\ \Theta'^* & I \end{pmatrix} \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} x,\ x \right) \\ &= \left(\begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix} x,\ x \right) = \|\Pi x\|^2, \end{aligned}$$

which completes the proof that Π and Π' are equivalent.

(2) $\Longrightarrow$ (1). We put

$$\Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} = W\Pi.$$

Then $WG = W\pi H^2(E) = \pi' u H^2(E) = \pi' H^2(E') = G'$ and, similarly, $WG_* = G'_*$, whence $WH = H'$. Thus, $W_0 \overset{\text{def}}{=} W|H$ defines a unitary operator from H to H'. Since

$$W\mathcal{U}\Pi = W\Pi z = \Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} z = \mathcal{U}'\Pi' \begin{pmatrix} u_* & 0 \\ 0 & u \end{pmatrix} = \mathcal{U}' W\Pi$$

and $\operatorname{Range}\Pi$ is dence in $\mathcal{H}$, we have $\mathcal{U}'W = W\mathcal{U}$. This yields

$$W_0 T = W_0 P_H \mathcal{U}|H = P_{H'} W\mathcal{U}|H = P_{H'}\mathcal{U}'W|H = T'W_0;$$

that is, T and T' are unitarily equivalent. □

1.12. The dilations and the characteristic function of a given contraction. The same arguments show that a minimal unitary dilation of a given contraction T is unique up to a unitary equivalence $W\mathcal{U} = \mathcal{U}'W$ such that $WG = G'$, $WG_* = G'_*$, and $W|H = I|H$.

Returning to the main definition 1.9 and taking Theorem 1.11 into account, we see that it is natural that *the characteristic function* of a contraction should mean any function Θ defined by (1.9.1) for any operator T unitarily equivalent to the given one.

Now we find an expression for the characteristic function of a completely nonunitary contraction in terms of the contraction itself.

1.13. Theorem. *The characteristic function Θ of a completely nonunitary contraction T is equivalent to the function in $H^\infty(\mathcal{D}_T \to \mathcal{D}_{T^*})$ defined on the unit disc by the formula*

$$\Theta_T(\lambda) = \big(-T + \lambda D_{T^*}(I-\lambda T^*)^{-1}D_T\big)|\mathcal{D}_T \quad \text{for } \lambda \in \mathbb{D}. \tag{1.13.1}$$

Proof. Let Θ be the function (1.9.1) defined by the embeddings (1.6.2) related to the unitary dilation described in Theorem 1.4. We shall prove that

$$\Theta(\lambda) = \Omega_*^*\big(-T + \lambda D_{T^*}(I-\lambda T^*)^{-1}D_T\big)\Omega$$

where

$$\Omega = V^* v : E \to H \quad \text{and} \quad \Omega_* = V_*^* v_* : E_* \to H,$$

V, V_* being defined in Theorem 1.4 and v, v_* in (1.6.1). The above Ω and Ω_* map E and E_* isometrically onto $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, respectively. In particular, choosing $E = \mathcal{D}_T$, $E_* = \mathcal{D}_{T^*}$ and $v = V|\mathcal{D}_T$, $v_* = V_*|\mathcal{D}_{T^*}$, we get the function Θ_T as one of the possible choices among equivalent representations of the characteristic function.

Let $e \in E$ and $e_* \in E_*$. Then, for $|\lambda| < 1$,

$$\begin{aligned}(\Theta(\lambda)e, e_*)_{E_*} &= \left(\Theta e,\, \frac{e_*}{1-\bar\lambda z}\right)_{L^2(E_*)} = \left(\pi e,\, \pi_*\frac{e_*}{1-\bar\lambda z}\right)_{\mathcal{H}} \\ &= \left(ve,\, \pi_*\sum_{k\ge 0}\bar\lambda^k z^k e_*\right)_{\mathcal{H}} = \left(ve,\, \sum_{k\ge 0}\bar\lambda^k \mathcal{U}^{k+1} v_* e_*\right)_{\mathcal{H}} \\ &= \left(v_*^*\sum_{k\ge 0}\lambda^k \mathcal{U}^{*(k+1)} ve,\, e_*\right)_{E_*}.\end{aligned}$$

Therefore,

$$\Theta(\lambda) = v_*^*\sum_{k\ge 0}\lambda^k\mathcal{U}^{*(k+1)}v = v_*^*\,\frac{(I-\lambda\mathcal{U}^*)^{-1} - I}{\lambda}\,v = \frac{1}{\lambda}\,v_*^*(I-\lambda\mathcal{U}^*)^{-1}v,$$

because $\operatorname{Range} v \perp \operatorname{Range} v_*$ (the latter fact follows from the relations $G \perp G_*$ and $\operatorname{Range} v \subset G$, $\operatorname{Range} v_* \subset G_*$).

The proof of the following formula for the inverse of a block matrix operator is left to the reader:

$$\begin{pmatrix} X & A & B \\ 0 & Y & C \\ 0 & 0 & Z\end{pmatrix}^{-1} = \begin{pmatrix} X^{-1} & -X^{-1}AY^{-1} & X^{-1}(-B + AY^{-1}C)Z^{-1} \\ 0 & Y^{-1} & -Y^{-1}CZ^{-1} \\ 0 & 0 & Z^{-1}\end{pmatrix}.$$

We apply this formula to the operator

$$I - \lambda\mathcal{U}^* = \begin{pmatrix} I - \lambda\mathcal{E}_*^* & -\lambda V_* D_{T^*} & \lambda V_* T V^* \\ 0 & I - \lambda T^* & -\lambda D_T V^* \\ 0 & 0 & I - \lambda\mathcal{E}^*\end{pmatrix}$$

(compare the expression for $\mathcal{U}$ in Theorem 1.4). This results in the relation

$$\begin{aligned}\Theta(\lambda) &= \frac{1}{\lambda}\,(v_*^*, 0, 0)(I-\lambda\mathcal{U}^*)^{-1}\begin{pmatrix}0\\0\\v\end{pmatrix}\\ &= \frac{1}{\lambda}\,v_*^*(I-\lambda\mathcal{E}_*^*)^{-1}\big(-\lambda V_*TV^* + (-\lambda V_* D_{T^*})(I-\lambda T^*)^{-1}(-\lambda D_T V^*)\big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times (I-\lambda\mathcal{E}^*)^{-1}v\\ &= \Omega_*^*\big(-T + \lambda D_{T^*}(I-\lambda T^*)^{-1}D_T\big)\Omega,\end{aligned}$$

because $v_*^*\mathcal{E}_*^* = 0$ and $\mathcal{E}^* v = 0$. □

1.14. Two more expressions for Θ_T. Multiplying (1.13.1) from the right by D_T we get the following useful formula

$$\Theta_T(\lambda)D_T = D_{T^*}(I-\lambda T^*)^{-1}(\lambda I - T); \tag{1.14.1}$$

and multiplying from the left by D_{T_*} we get

$$D_{T^*}\Theta_T(\lambda) = (\lambda I - T)(I-\lambda T^*)^{-1}D_T|\mathcal{D}_T. \tag{1.14.2}$$

1.15. Corollary. *The characteristic function is a pure contractive-valued function; that is,*

$$\|\Theta(0)e\|_{E_*} < \|e\|_E \quad \textit{for any nonzero } e \in E.$$

Indeed, if $\|\Theta(0)e\|_{E_*} = \|e\|_E$, then

$$\|\Omega e\| = \|e\| = \|\Theta(0)e\| = \|\Omega_*^* T\Omega e\| = \|T\Omega e\|,$$

that is, $\Omega e \in \operatorname{Ker} D_T$. However, since $\Omega e \in \mathcal{D}_T$, we have $\Omega e = 0$, whence $e = 0$.

1.16. Coordinate-free function model. Now we are ready to construct the coordinate-free function model for a contraction on a Hilbert space. This will help us to solve the following *inverse problem*: given a purely contractive-valued function Θ analytic in the unit disc, find a completely nonunitary contraction whose characteristic function is equivalent to Θ.

To this end, the following steps can be taken.

(i) We take a function $\Theta \in H^\infty(E \to E_*)$ such that $\|\Theta\|_\infty \le 1$ and $\|\Theta(0)e\|_{E_*} < \|e\|_E$ for all nonzero $e \in E$.

(ii) We take any functional embedding $\Pi = \pi_* \oplus \pi$ acting from $L^2(E_* \oplus E)$ to an arbitrary Hilbert space with the prescribed modulus

$$\Pi^*\Pi = \begin{pmatrix} I & \Theta \\ \Theta^* & I\end{pmatrix}. \tag{1.16.1}$$

(iii) We put $\mathcal{H} = \operatorname{clos}\operatorname{Range}\Pi$.

(iv) We introduce a unitary operator $\mathcal{U}$ on $\mathcal{H}$ by the relation

$$\mathcal{U}\Pi = \Pi z.$$

(v) We introduce the *model subspace*

$$\mathcal{K}_\Theta \stackrel{\text{def}}{=} \mathcal{H} \ominus \big(\pi H^2(E) \oplus \pi_* H^2_-(E_*)\big),$$

where, as before, $\pi = \Pi|L^2(E)$ and $\pi_* = \Pi|L^2(E_*)$.

(vi) Finally we define a *model operator* by the formula

$$\mathcal{M}_\Theta \stackrel{\text{def}}{=} P_\Theta \mathcal{U}|\mathcal{K}_\Theta,$$

where P_Θ stands for the orthogonal projection from $\mathcal{H}$ onto $\mathcal{K}_\Theta$.

The next theorem shows that these six steps really solve the inverse problem in question.

1.17. Theorem. $\mathcal{M}_\Theta$ *is a completely nonunitary contraction with the characteristic function* Θ. *The operator* $\mathcal{U}$ *is the minimal unitary dilation of* $\mathcal{M}_\Theta$.

PROOF. First of all, it is worth mentioning that the operator $\mathcal{U}$ described in step (4) above is well-defined and unitary. Clearly, the norm is preserved, since

$$\|\mathcal{U}\Pi x\|^2 = \|\Pi z x\|^2 = \left(\begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix} zx,\, zx\right) = \left(\begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix} x,\, x\right) = \|\Pi x\|^2;$$

therefore, $\mathcal{U}$ is well-defined and isometric. By step (3), $\mathcal{U}$ is densely defined and possesses a dense range; hence, it admits a unitary extension to $\mathcal{H}$.

Now, we prove that $\mathcal{U}$ is a minimal unitary dilation of $\mathcal{M} = \mathcal{M}_\Theta$. From the definition it is clear that $\mathcal{U}$ is a dilation. More precisely, this follows from the fact that the subspaces $G \stackrel{\text{def}}{=} \pi H^2(E)$ and $G_* \stackrel{\text{def}}{=} \pi_* H^2_-(E_*)$ are invariant under $\mathcal{U}$ and $\mathcal{U}^*$, respectively, which shows that $\mathcal{H} = G_* \oplus \mathcal{K}_\Theta \oplus G$ is the decomposition from Lemma 1.2. Thus, all we need to verify is minimality.

We check the identity

$$\mathcal{U} P_\Theta \pi \bar{z} e + P_\Theta \pi_* \Theta_0 e = \pi(I - \Theta_0^* \Theta_0) e \quad \text{for } e \in E, \tag{1.17.1}$$

where $\Theta_0 \stackrel{\text{def}}{=} \Theta(0)$. Using step (v), we get for the orthogonal projection onto $\mathcal{K}_\Theta$ the formula

$$P_\Theta = I - \pi P_+ \pi^* - \pi_* P_- \pi_*^*,$$

where P_+ and P_- stand for the Riesz projections (see Section 0.7). Thus, we obtain

$$\mathcal{U} P_\Theta \pi \bar{z} e = \mathcal{U}(\pi P_- - \pi_* P_- \Theta)\bar{z} e = \mathcal{U}(\pi \bar{z} e - \pi_* \Theta_0 \bar{z} e) = \pi e - \pi_* \Theta_0 e,$$
$$P_\Theta \pi_* \Theta_0 e = (\pi_* P_+ - \pi P_+ \Theta^*)\Theta_0 e = \pi_* \Theta_0 e - \pi \Theta_0^* \Theta_0 e,$$

which implies (1.17.1).

Since Θ is pure, the operator $I - \Theta_0^* \Theta_0$ has a dense range, whence

$$\pi E \subset \operatorname{span}\{\mathcal{K}_\Theta, \mathcal{U}\mathcal{K}_\Theta\},$$

and

$$G = \pi H^2(E) = \operatorname{span}\{\mathcal{U}^n \pi E : n \ge 0\} \subset \operatorname{span}\{\mathcal{U}^n \mathcal{K}_\Theta : n \ge 0\}.$$

In a similar way, using the identity

$$P_\Theta \pi_* e_* + \mathcal{U} P_\Theta \pi \bar{z} \Theta_0^* e_* = \pi_*(I - \Theta_0 \Theta_0^*) e_*,$$

we get

$$G_* = \pi_* H^2_-(E_*) = \operatorname{span}\{\mathcal{U}^n \pi_* E_* : n < 0\} \subset \operatorname{span}\{\mathcal{U}^n \mathcal{K}_\Theta : n \le 0\}.$$

Hence, $\mathcal{H} = G_* \oplus \mathcal{K}_\Theta \oplus G \subset \operatorname{span}\{\mathcal{U}^n \mathcal{K}_\Theta : n \in \mathbb{Z}\}$, so $\mathcal{U}$ is a minimal dilation.

By construction, π and π_* are functional embeddings with $v = \pi|E$ and $v_* = \pi_* \bar{z}|E_*$. Therefore, Θ is the characteristic function of $\mathcal{M}_\Theta$. □

Now comes the main result of the Chapter, to complete the construction of the function model.

1.18. Theorem. *If $\Theta = \Theta_T$ is the characteristic function of a completely nonunitary contraction T, then T is unitarily equivalent to the model operator $\mathcal{M}_\Theta$ constructed with the help of the six steps in* 1.16.

PROOF. Obvious from Theorems 1.11 and 1.17. □

Let T be a completely nonunitary contraction and let $\Theta = \Theta_T$ be the characteristic function of T. We say that $\mathcal{M}_\Theta$ is a *coordinate-free model* of T acting on the model space $\mathcal{K}_\Theta$. Some explicit formulas for the function model will be obtained as transcriptions of this free model by specifying a representation of the Hilbert space $\mathcal{H}$ and a solution Π of equation (1.16.1). Several examples of such transcriptions are given in Chapter 3.

1.19. Residual subspaces. In this section we introduce and briefly discuss two more functional mappings related to the function model of a contraction; these are quite useful for the study of the commutant lifting (see Chapter 4) and invariant subspaces (see Chapter 6). These mappings, denoted below by τ and τ_*, arise necessarily when one studies the absolutely continuous spectrum of a contraction, because they give spectral representations of the restrictions of the unitary dilation $\mathcal{U}$ to the so called *residual* and *$*$-residual* parts of $\mathcal{H}$, that is, to

$$\mathcal{R} \stackrel{\text{def}}{=} \mathcal{H} \ominus \pi_* L^2(E_*) \quad \text{and} \quad \mathcal{R}_* \stackrel{\text{def}}{=} \mathcal{H} \ominus \pi L^2(E).$$

Clearly,

$$\mathcal{R} = (I - \pi_* \pi_*^*)\mathcal{H} = \operatorname{clos}(I - \pi_* \pi_*^*) \pi L^2(E) = \operatorname{clos}(\pi - \pi_* \Theta) L^2(E).$$

Since

$$(\pi - \pi_*\Theta)^*(\pi - \pi_*\Theta) = (\pi^* - \Theta^* \pi_*^*)(\pi - \pi_*\Theta) = I - \Theta^*\Theta = \Delta^2,$$

the polar decomposition

$$\pi - \pi_* \Theta = \tau \Delta \tag{1.19.1}$$

provides us with a partial isometry τ acting from $L^2(E)$ to $\mathcal{H}$, which is isometric on $L^2(\Delta E) \stackrel{\text{def}}{=} \operatorname{clos} \Delta L^2(E)$ and whose range is $\mathcal{R}$. It turns out to be more convenient to view τ as defined only on $L^2(\Delta E)$. Then τ is an isometry that intertwines z on $L^2(\Delta E)$ with $\mathcal{U}|\mathcal{R}$; that is, it provides a unitary equivalence of these operators. Algebraically, these properties can be written as follows:

$$\begin{aligned}\tau^*\tau &= I,\\ \tau\tau^* &= I - \pi_*\pi_*^*,\\ \tau z &= \mathcal{U}\tau.\end{aligned}$$

In a similar way,

$$\mathcal{R}_* = \tau_* L^2(\Delta_* E_*),$$

where τ_* is the partial isometry occurring in the polar decomposition

$$\pi_* - \pi\Theta^* = \tau_*\Delta_*. \tag{1.19.2}$$

Then

$$\begin{aligned}\tau_*^*\tau_* &= I,\\ \tau_*\tau_*^* &= I - \pi\pi^*,\\ \tau_* z &= \mathcal{U}\tau_*.\end{aligned}$$

Below we list some more relations for the embeddings τ, τ_* and π, π_*, which are simple consequences of the definitions:

$$\begin{aligned}\tau^*\pi &= \Delta, & \tau_*^*\pi_* &= \Delta_*,\\ \tau^*\pi_* &= 0, & \tau_*^*\pi &= 0,\\ \tau^*\tau_* &= -\Theta^*, & \tau_*^*\tau &= -\Theta.\end{aligned}$$

Chapter 2. Examples

In this chapter, we give two examples where the characteristic function is computed. Both deal with dissipative operators, rather than contractions. However, as is well known, the theories of these two classes are related by the Cayley transform, which allows us to transfer information obtained for dissipative operators to contractions and vice versa. This transfer is not completely automatic, and we start with some prerequisites.

2.1. Definition. A densely defined operator A is said to be *dissipative* if

$$\operatorname{Im}(Ax, x) \geq 0 \quad \text{for all } x \in \operatorname{Dom} A.$$

A dissipative operator is *maximal* if it has no proper dissipative extension. A dissipative operator is *completely nonselfadjoint* if it has no selfadjoint restriction on a nonzero invariant subspace.

The next lemma collects some properties of dissipative operators and their relations to contractions. The proof of the lemma is classical; see [Sz.-Nagy and Foiaş 1967], for example.

2.2. Lemma. 1. *If A is a dissipative operator, the operator*

$$T = \mathcal{C}(A) \stackrel{\text{def}}{=} (A - iI)(A + iI)^{-1}$$

is a contraction acting from $(A + iI)\operatorname{Dom} A$ to $(A - iI)\operatorname{Dom} A$ and such that $1 \notin \sigma_p(T)$. Conversely, if T is a contraction such that 1 is not an eigenvalue of T, the operator

$$A = \mathcal{C}^{-1}(T) = i(I + T)(I - T)^{-1}$$

is well-defined on $\operatorname{Dom} A = (I - T)\operatorname{Dom} T$ and is dissipative. The operator $\mathcal{C}(A)$ is called the Cayley transform of A, and $\mathcal{C}^{-1}(T)$ is called the Cayley transform of T.

2. *Every dissipative operator has a maximal dissipative extension. A maximal dissipative operator is closed. A dissipative operator A is maximal if and only if $\operatorname{Dom} \mathcal{C}(A) = H$.*
3. *The operator A is selfadjoint if and only if $\mathcal{C}(A)$ is unitary.*
4. *Two dissipative operators A_1 and A_2 are unitarily equivalent if and only if are $\mathcal{C}(A_1)$ and $\mathcal{C}(A_2)$ so.*

A bounded operator A is maximal dissipative if and only if it is defined on the whole space ($\operatorname{Dom} A = H$) and its imaginary part is nonnegative ($\operatorname{Im} A = \frac{1}{2i}(A - A^*) \geq 0$). In this simplest case, we compute the defect operators and the characteristic function of the Cayley transform.

2.3. Lemma. *Let A be a bounded dissipative operator and let $T = \mathcal{C}(A)$ be the Cayley transform of A. Then*

$$D_T^2 = 2i(A^* - iI)^{-1}(A^* - A)(A + iI)^{-1},$$
$$D_{T^*}^2 = 2i(A + iI)^{-1}(A^* - A)(A^* - iI)^{-1}.$$

Moreover, there exist partial isometries V and V_* with initial space

$$\operatorname{clos} \operatorname{Range}(\operatorname{Im} A)$$

and final spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$, respectively, such that

$$D_T = V 2(\operatorname{Im} A)^{1/2}(A + iI)^{-1} = (A^* - iI)^{-1} 2(\operatorname{Im} A)^{1/2} V^*, \tag{2.3.1}$$
$$D_{T^*} = V_* 2(\operatorname{Im} A)^{1/2}(A^* - iI)^{-1} = (A + iI)^{-1} 2(\operatorname{Im} A)^{1/2} V_*^*. \tag{2.3.2}$$

In particular, $\operatorname{rank} D_T = \operatorname{rank} D_{T^*} = \operatorname{rank}(\operatorname{Im} A)$.

Proof. The first two formulas are straightforward consequences of the definitions of D_T, D_{T^*}, and $\mathcal{C}(A)$. The second two relations follow from the polar decomposition; see Section 0.7. □

2.4. Lemma. *The characteristic function $\Theta_T(z)$ is equivalent to the function*

$$\mathcal{S}_A\Big(i\,\frac{1+z}{1-z}\Big),$$

where $z \in \mathbb{D}$ *and*

$$\mathcal{S}_A(\zeta) \overset{\text{def}}{=} \bigl(I + i(2\operatorname{Im} A)^{1/2}(A^* - \zeta I)^{-1}(2\operatorname{Im} A)^{1/2}\bigr)\big|\operatorname{Range}(\operatorname{Im} A), \quad \textit{for } \operatorname{Im}\zeta > 0. \tag{2.4.1}$$

Proof. Using the the expressions for $T = \mathcal{C}(A)$ and formulas (2.3.1)–(2.3.2) we can rewrite formula (1.14.1) for Θ_T as follows:

$$\begin{aligned}
&V_*^*\Theta_T(z)D_T\\
&\quad = V_*^* D_{T^*}(I - zT^*)^{-1}(zI - T)\\
&\quad = 2(\operatorname{Im} A)^{1/2}(A^* - iI)^{-1}\bigl(I - z(A^* + iI)(A^* - iI)^{-1}\bigr)^{-1}\bigl(zI - (A - iI)(A + iI)^{-1}\bigr)\\
&\quad = 2(\operatorname{Im} A)^{1/2}\bigl((A^* - iI) - z(A^* + iI)\bigr)^{-1}\bigl(z(A + iI) - (A - iI)\bigr)(A + iI)^{-1}\\
&\quad = -2(\operatorname{Im} A)^{1/2}\Bigl(A^* - i\,\frac{1+z}{1-z}\,I\Bigr)^{-1}\Bigl(A - i\,\frac{1+z}{1-z}\,I\Bigr)(A + iI)^{-1}\\
&\quad = -2(\operatorname{Im} A)^{1/2}(A^* - \zeta I)^{-1}(A - \zeta I)(A + iI)^{-1}\\
&\quad = -2(\operatorname{Im} A)^{1/2}(A^* - \zeta I)^{-1}(2i\operatorname{Im} A + A^* - \zeta I)(A + iI)^{-1}\\
&\quad = -\bigl(I + 2i(\operatorname{Im} A)^{1/2}(A^* - \zeta I)^{-1}(\operatorname{Im} A)^{1/2}\bigr)2(\operatorname{Im} A)^{1/2}(A + iI)^{-1}.
\end{aligned}$$

Taking into account (2.3.1), we get for Θ_T an expression equivalent to (2.4.1). $\square$

The function (2.4.1) is called *the characteristic function of the dissipative operator A*.

2.5. Example: The dissipative integration operator. Let μ be a positive finite Borel measure on the interval [0,1]. We consider the integration operator A acting on the space $L^2(\mu)$ and defined by

$$(Af)(x) = i\int_{[0,x\}} f(t)\,d\mu(t) \overset{\text{def}}{=} i\int_{[0,x)} f(t)\,d\mu(t) + \frac{i}{2}\,\mu(\{x\})f(x) \quad \text{for } x \in [0,1].$$

Obviously, A is well-defined and bounded (even compact) on $L^2(\mu)$, and its adjoint operator is given by

$$(A^*f)(x) = -i\int_{\{x,1]} f(t)\,d\mu(t).$$

Therefore, A is a dissipative operator with rank-one imaginary part:

$$(2\operatorname{Im} A)f = \int_{[0,1]} f(t)\,d\mu(t) = (f,\mathbf{1})\mathbf{1},$$

$$\bigl((2\operatorname{Im} A)f, f\bigr) = |(f,\mathbf{1})|^2 \ge 0 \quad \text{for } f \in L^2(\mu).$$

Our goal is to construct the model for the Cayley transform $\mathcal{C}(A)$ of the operator A. In accordance with the general theory of Chapter 1, the only thing we need is to find the completely nonunitary part of $\mathcal{C}(A)$ and to compute its characteristic function. However, it turns out that A is completely nonselfadjoint, so $\mathcal{C}(A)$ is completely nonunitary (see part 3 of Lemma 2.2). Taking into account Lemma 2.4, we will compute not the characteristic function of $\mathcal{C}(A)$, but the characteristic function of A itself.

2.6. Theorem. *The operator A is a completely nonselfadjoint dissipative operator with characteristic function*

$$\mathcal{S}_A(\zeta) = \Bigg(\prod_{0\le t\le 1} \frac{\zeta - \frac{i}{2}\mu(\{t\})}{\zeta + \frac{i}{2}\mu(\{t\})}\Bigg) \exp\Big(-i\,\frac{\mu_c([0,1])}{\zeta}\Big). \tag{2.6.1}$$

To check that A is a completely nonselfadjoint operator and to compute its characteristic function $\mathcal{S}_A$, we need to compute the resolvent $(A^* - \zeta I)^{-1}$, that is, to find a solution f of the equation

$$(A^* - \zeta I)f = h, \tag{2.6.2}$$

or

$$-i \int_{\{x,1]} f(t)\,d\mu(t) - \zeta f(x) = h(x).$$

Putting $M = \mu([0,1])$, we introduce the map $\varphi : [0,1] \to [0,M]$ given by

$$\varphi(x) = \begin{cases} \frac{1}{2}\mu(\{0\}) & \text{if } x = 0, \\ \mu([0,x)) + \frac{1}{2}\mu(\{x\}) & \text{if } x > 0, \end{cases}$$

and the map $\psi : [0,M] \to [0,1]$ given by

$$\psi(t) = \begin{cases} \inf\{x : \mu([0,x)) > t\} & \text{if } t < \mu([0,1)), \\ 1 & \text{if } t \ge \mu([0,1)). \end{cases}$$

Then the solution of (2.6.2) can be described as follows.

2.7. Lemma. *If* $\operatorname{Re}\zeta \ne 0$ *and* $h \in L^2(\mu)$, *equation* (2.6.2) *has a unique solution* $f \in L^2(\mu)$, *which can be recovered from the solution g of the differential equation*

$$g'(\tau) = \frac{g(\tau) - h(\psi(\tau))}{\tau - \varphi(\psi(\tau)) - i\zeta} \tag{2.7.1}$$

satisfying the initial condition $g(M) = 0$. *Namely, the function*

$$f \stackrel{\text{def}}{=} \frac{1}{\zeta}(g\circ\varphi - h)$$

solves (2.6.2).

Proof. For ζ away from the imaginary line, the function in the denominator of (2.7.1) is bounded away from zero; hence equation (2.7.1) has an absolutely continuous solution g. Thus we need only to check that the function $f(x) = \frac{1}{\zeta}\big(g(\varphi(x)) - h(x)\big)$ solves equation (2.6.2).

Further, we note that φ is continuous at every point x of the interval $[0,1]$ for which $\mu(\{x\})=0$, and that each jump of φ, that is, each point mass of the measure μ, corresponds to an interval $\big(\varphi(x_{-0}),\,\varphi(x_{+0})\big)=\big(\mu([0,x)),\,\mu([0,x])\big)$ where the function ψ is constant. In turn, the function ψ is continuous everywhere off the set of points t such that $\varphi(x_1)=\varphi(x_2)=t$ for at least two different points $x_1\neq x_2$. If x is not a mass point for μ and $x_1=\inf\{x:\varphi(x)=t\}$, $x_2=\sup\{x:\varphi(x)=t\}$, then ψ has a jump at t with $\psi(t_{-0})=x_1$ and $\psi(t_{+0})=x_2$.

We introduce functions f_1, h_1, and g on $[0,M]$ by the formulas

$$f_1(t)=f(\psi(t)),\quad h_1(t)=h(\psi(t)),\quad g(t)=-i\int_t^M f_1(s)\,ds.$$

From the definition we see that g is an absolutely continuous function, piecewise linear on the intervals where ψ is constant. We prove that g coincides with $h_1+\zeta f_1$ on the image of φ.

Changing the variable, we get the relations

$$\int_0^{\varphi(x)} f_1(s)\,ds=\int_{[0,x\}} f(t)\,d\mu(t)\quad\text{and}\quad\int_{\varphi(x)}^{M} f_1(s)\,ds=\int_{\{x,1]} f(t)\,d\mu(t).$$

In particular,

$$g(\varphi(x))=-i\int_{\{x,1]} f(t)\,d\mu(t)=h(x)+\zeta f(x). \tag{2.7.2}$$

If τ is an arbitrary point in $[0,M]$ and $x=\psi(\tau)$, then $f_1(s)=f_1(\tau)=f(x)$ on the interval $s\in(\varphi(x_{-0}),\varphi(x_{+0}))$, and

$$\begin{aligned}g(\tau)&=-i\int_\tau^M f_1(s)\,ds=-i\int_{\varphi(x)}^M f_1(s)\,ds-i\int_\tau^{\varphi(x)} f_1(s)\,ds\\&=-i\int_{\{x,1]} f(t)\,d\mu(t)-i(\varphi(x)-\tau)f_1(\tau)=h(x)+\zeta f(x)-i(\varphi(x)-\tau)f_1(\tau)\\&=h_1(\tau)+i(\tau-\varphi(\psi(\tau))-i\zeta)f_1(\tau),\end{aligned}$$

which yields

$$if_1(\tau)=\frac{g(\tau)-h_1(\tau)}{\tau-\varphi(\psi(\tau))-i\zeta}.$$

Since the definition of g is equivalent to the equation $g'=if_1$ with the initial condition $g(M)=0$, the latter relation implies that g is a solution of (2.7.1). Therefore, the conclusion of the lemma follows from (2.7.2). □

2.8. Corollary. *The operator A is completely nonselfadjoint.*

Proof. Let H_0 be the maximal invariant subspace such that the restriction $A|H_0$ is selfadjoint. Then H_0 is a reducing subspace for A (see Lemma 2.2 and Corollary 1.8), hence $\sigma(A|H_0) \subset \sigma(A^*)$. By Lemma 2.7, equation (2.6.2) has an L^2-solution for all h and for all ζ, $\operatorname{Re}\zeta \neq 0$; that is, A^* has no real spectrum, except, maybe, the point $\zeta = 0$. Thus, to check that A is completely nonselfadjoint it suffices to check that the kernel of A^* is trivial.

Let $A^* f = 0$, that is,

$$\frac{1}{2}\mu(\{x\})f(x) + \int\limits_{(x,1]} f(t)\,d\mu(t) = 0 \quad \forall x \in [0,1].$$

Putting $x + \varepsilon$ in place of x and letting ε tend to zero, we obtain

$$\int\limits_{(x,1]} f(t)\,d\mu(t) = 0 \quad \text{for all } x \in [0,1],$$

or

$$\int\limits_{(x,y]} f(t)\,d\mu(t) = 0 \quad \text{for all } x, y \in [0,1],$$

which implies that f vanishes μ-a.e. □

We mention that T. Kriete [1972] found a criterion for complete nonselfadjointness for a class of dissipative operators with rank-one imaginary part. Our operator A corresponding to an absolutely continuous measure μ is contained in this class.

2.9. Proof of Theorem 2.6. That A is completely nonselfadjoint has already been proved. Now we compute the characteristic function $\mathcal{S}_A$:

$$\begin{aligned}\mathcal{S}_A(\zeta) &= \big(\big(I + i\sqrt{2\operatorname{Im} A}(A^* - \zeta I)^{-1}\sqrt{2\operatorname{Im} A}\big)\mathbf{1},\, \mathbf{1}\big)\,\|\mathbf{1}\|^{-2}\\ &= 1 + i\|\mathbf{1}\|^{-2}\big((A^* - \zeta I)^{-1}\sqrt{2\operatorname{Im} A}\,\mathbf{1},\, \sqrt{2\operatorname{Im} A}\,\mathbf{1}\big)\\ &= 1 + i\big((A^* - \zeta I)^{-1}\mathbf{1},\, \mathbf{1}\big),\end{aligned}$$

because $\sqrt{2\operatorname{Im} A}\,f = \|\mathbf{1}\|^{-1}(f,\mathbf{1})\mathbf{1}$.

So, for computing $\mathcal{S}_A(\zeta)$ we need the solution f of (2.6.2) with $h = \mathbf{1}$. Actually, we need not f itself, but

$$\int\limits_{[0,1]} f(t)\,d\mu(t) = \int\limits_0^M f_1(s)\,ds = ig(0),$$

where g is the solution of

$$g'(\tau) = \frac{g(\tau) - 1}{\tau - \varphi(\psi(\tau)) - i\zeta} \quad \text{with } g(M) = 0.$$

Putting $g_1 = 1 - g$ and $\omega(\tau) = (\tau - \varphi(\psi(\tau)) - i\zeta)^{-1}$, we see that $\mathcal{S}_A(\zeta) = 1 - g(0) = g_1(0)$ and that

$$g_1'(\tau) = \omega(\tau) g_1(\tau), \quad g_1(M) = 1.$$

Thus

$$g_1(\tau) = \exp\Big(-\int_\tau^M \omega(t)\,dt\Big),$$

and

$$\mathcal{S}_A(\zeta) = \exp\Big(-\int_0^M \omega(t)\,dt\Big). \tag{2.9.1}$$

As mentioned in Section 2.7, ψ is constant on the intervals

$$\Omega_x \overset{\text{def}}{=} \big(\varphi(x_{-0}),\, \varphi(x_{+0})\big) = \big(\mu([0,x)),\, \mu([0,x])\big),$$

where its value is x; that is, $\omega(t) = (t - \varphi(x) - i\zeta)^{-1}$ for $t \in \Omega_x$. Set $\Omega \overset{\text{def}}{=} \bigcap\{\Omega_x : \mu(\{x\}) > 0\}$. For almost all $t \notin \Omega$ (in fact for all t except for the ends of Ω_x) we have $\varphi(\psi(t)) = t$, that is, $\omega(t) = \text{const} = -1/(i\zeta)$ for $t \in [0,M] \setminus \Omega$, whence

$$\int_0^M \omega(t)\,dt = -\frac{M}{i\zeta} + \int_\Omega (\omega(t) + \frac{1}{i\zeta})dt = -\frac{\mu([0,1])}{i\zeta} + \sum_{x:\mu(\{x\})>0} \int_{\Omega_x} (\omega(t) + \frac{1}{i\zeta})\,dt.$$

The integral over Ω can naturally be split into the sum of the integrals over the intervals $\Omega_x = \big(\varphi(x_{-0}),\, \varphi(x_{+0})\big)$ on which $\omega(t) = (t - \varphi(x) - i\zeta)^{-1}$, and

$$\begin{aligned}\int_{\varphi(x_{-0})}^{\varphi(x_{+0})} (\omega(t) + \frac{1}{i\zeta})\,dt &= \int_{\varphi(x_{-0})}^{\varphi(x_{+0})} \frac{ds}{s - \varphi(x) - i\zeta} + \frac{1}{i\zeta}\big(\varphi(x_{+0}) - \varphi(x_{-0})\big)\\ &= \log\frac{\varphi(x_{+0}) - \varphi(x) - i\zeta}{\varphi(x_{-0}) - \varphi(x) - i\zeta} + \frac{1}{i\zeta}\big(\varphi(x_{+0}) - \varphi(x_{-0})\big)\\ &= \log\frac{1 - i\mu(\{x\})/(2\zeta)}{1 + i\mu(\{x\})/(2\zeta)} + \frac{\mu(\{x\})}{i\zeta},\end{aligned}$$

because $\varphi(x_{+0}) - \varphi(x) = \varphi(x) - \varphi(x_{-0}) = \frac{1}{2}\mu(\{x\})$. Finally, substituting this in (2.9.1), we obtain

$$\mathcal{S}_A(\zeta) = \Bigg(\prod_{0\le t\le 1} \frac{\zeta - \frac{i}{2}\mu(\{t\})}{\zeta + \frac{i}{2}\mu(\{t\})}\Bigg) \exp\Big(-i\,\frac{\mu_c([0,1])}{\zeta}\Big),$$

where μ_c is the continuous part of μ:

$$\mu_c([0,1]) = \mu([0,1]) - \sum \mu(\{t\}). \qquad \square$$

2.10. Unitary classification. Thus, the characteristic function of A is a scalar inner function whose singular part has only one singularity, at $\zeta = 0$ (with mass $\mu_c([0,1])$) and whose zeros on the imaginary axis are at $\zeta = \frac{i}{2}\mu(\{t\})$ and have multiplicity equal to the number of points on $[0,1]$ with the same mass $\mu(\{t\})$. We see that S_A is independent of the distribution of μ on the interval $[0,1]$, depending only on the total continuous mass $\mu_c([0,1])$ and on the values of the point masses $\mu(\{t\})$ (regardless of their location). Therefore, two operators A determined by measures μ_1 and μ_2 are unitarily equivalent if and only if

$$\mu_{1c}([0,1]) = \mu_{2c}([0,1])$$

and

$$\operatorname{card}\{t : \mu_1(\{t\}) = \lambda\} = \operatorname{card}\{t : \mu_2(\{t\}) = \lambda\} \quad \text{for all } \lambda > 0.$$

We know the characteristic function of the operator, so we can describe its spectrum (see Chapter 5). In our case the spectrum of the operator A consists of the eigenvalues of multiplicity one at the points $i\lambda$ such that

$$k(\lambda) \stackrel{\text{def}}{=} \operatorname{card}\{t : \mu(\{t\}) = 2\lambda\} > 0.$$

Moreover, $k(\lambda)$ is the size of the corresponding Jordan block. If A is not a finite rank operator—that is, if the support of μ is not a finite set—the point $\lambda = 0$ is the sole point of the essential spectrum of A.

2.11. The matrix case. As an illustration to the previous computations, we rewrite a partial case of the operator A in a matrix form.

Set $\mu = \sum_{k\geq 1} \mu_k \delta_{t_k}$, where $\mu_k > 0$, $t_k > 0$ and $\sum_{k\geq 1}\mu_k < \infty$. Then our operator A is unitarily equivalent to the operator $\mathcal{A} : \ell^2(\mu_k) \to \ell^2(\mu_k)$ defined by the formula

$$\mathcal{A}f = \bigl\{ i\bigl(\textstyle\sum_{t_j<t_k} f_j\mu_j + \frac{1}{2} f_k\mu_k\bigr) : k \geq 1 \bigr\}$$

on the sequence space

$$\ell^2(\mu_k) = \bigl\{ f = (f_k)_{k\geq 1} : \textstyle\sum_{k\geq 1} |f_k|^2 \mu_k < \infty \bigr\}.$$

Taking the unitary transformation $V : \ell^2(\mu_k) \to \ell^2$ given by

$$Vf = (a_k f_k)_{k\geq 1},$$

where the a_k are complex numbers satisfying $|a_k|^2 = \mu_k$ for $k \geq 1$, we get a unitarily equivalent operator $\mathfrak{J}_a : \ell^2 \to \ell^2$ given by

$$\mathfrak{J}_a x = \bigl\{ i\bigl(\textstyle\sum_{t_j<t_k} a_k \bar{a}_j x_j + \frac{1}{2}|a_k|^2 x_k\bigr) : k \geq 1 \bigr\} \quad \text{for } x \in \ell^2.$$

From the preceding discussion we know that two operators $\mathfrak{J}_a$ and $\mathfrak{J}_b$, with $a, b \in \ell^2$, are unitarily equivalent if and only if the decreasing rearrangements of $|a|$ and $|b|$ coincide, and that the spectrum of $\mathfrak{J}_a$ is $\sigma(\mathfrak{J}_a) = \frac{i}{2}\operatorname{Range}|a|^2 = \frac{i}{2}\{0,\, |a_1|^2,\, |a_2|^2, \ldots\}$. Every invariant subspace of $\mathfrak{J}_a$ is generated by the eigenvectors and root vectors it contains; see Section 6.20.

For two special distributions of t_k the operators $\mathcal{J}_a$ are related to the triangular truncation of matrices on the space ℓ^2. If we take an increasing sequence, $t_k < t_{k+1}$ for $k \geq 1$ the corresponding operator $\frac{1}{i}\mathcal{J}_a$ is a kind of lower truncation of a selfadjoint matrix $\mathbf{a} = (a_k \bar{a}_j)_{k,j\geq 1}$:

$$\mathcal{J}_a^{(l)} = i \begin{pmatrix} \frac{1}{2}|a_1|^2 & 0 & 0 & 0 & \cdots \\ a_2\bar{a}_1 & \frac{1}{2}|a_2|^2 & 0 & 0 & \cdots \\ a_3\bar{a}_1 & a_3\bar{a}_2 & \frac{1}{2}|a_3|^2 & 0 & \cdots \\ a_4\bar{a}_1 & a_4\bar{a}_2 & a_4\bar{a}_3 & \frac{1}{2}|a_4|^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Likewise, if we take a decreasing sequence, $\frac{1}{i}\mathcal{J}_a$ acts as upper truncation:

$$\mathcal{J}_a^{(u)} = i \begin{pmatrix} \frac{1}{2}|a_1|^2 & a_1\bar{a}_2 & a_1\bar{a}_3 & a_1\bar{a}_4 & \cdots \\ 0 & \frac{1}{2}|a_2|^2 & a_2\bar{a}_3 & a_2\bar{a}_4 & \cdots \\ 0 & 0 & \frac{1}{2}|a_3|^2 & a_3\bar{a}_4 & \cdots \\ 0 & 0 & 0 & \frac{1}{2}|a_4|^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is clear that the matrix $\mathbf{a}$ represents the rank-one operator $(\cdot, a)a$, and that the operators $\mathcal{J}_a^{(u)}$ and $\mathcal{J}_a^{(l)}$ are unitarily equivalent.

2.12. Example 2: The dissipative Sturm–Liouville operator. We consider the differential operator ℓ_h arising from the differential expression

$$\ell y = -y'' + qy,$$

where q is a real function, and by the boundary condition $y'(0) = hy(0)$, where h is a complex number. The operator ℓ_h has domain $\operatorname{Dom}(\ell_h) = \{y \in W^2_{2,\mathrm{loc}}(\mathbb{R}_+) : \ell y \in L^2(\mathbb{R}_+),\ y'(0) = hy(0)\}$, and is defined by

$$\ell_h(y) = \ell y \in L^2(\mathbb{R}_+). \tag{2.12.1}$$

The general facts about the operators ℓ_h can be found in [Reed and Simon 1975] or [Atkinson 1964]. In particular,

$$\ell_h^* = \ell_{\bar{h}}.$$

Moreover, ℓ_h is a rank-one perturbation of the operator ℓ_∞ defined by the differential expression ℓ and by the boundary condition $y(0) = 0$. The operator ℓ_∞ is selfadjoint. The operator ℓ_h is dissipative if and only if $\operatorname{Im} h > 0$.

Our goal is to compute the characteristic function $\mathcal{S}_{\ell_h}$ of ℓ_h and, thus, to include the study of ℓ_h in the model theory.

To this end, we consider two solutions φ_ζ and ψ_ζ of the equation $\ell y = \zeta y$, satisfying the boundary conditions

$$\begin{cases} \varphi_\zeta(0) = 0, \\ \varphi'_\zeta(0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} \psi_\zeta(0) = -1, \\ \psi'_\zeta(0) = 0. \end{cases}$$

It is well known that for every ζ with $\operatorname{Re}\zeta \neq 0$ there exists a unique L^2-solution y_ζ of our equation $\ell y = \zeta y$ representable as a linear combination of φ_ζ and ψ_ζ:

$$y_\zeta = \psi_\zeta + m(\zeta)\varphi_\zeta.$$

The function $m(\zeta)$ determined by this condition for all $\zeta \neq \bar{\zeta}$ is called the *Weyl function*.

Following B. Pavlov [1976], we can compute the characteristic function of ℓ_h in terms of the Weyl function m.

An important remark is that now we cannot use the formulas from Lemmas 2.3 and 2.4 for the defect operators and for the characteristic function, because our operator $A = \ell_h$ is unbounded, its domain is different from the domain of the adjoint, and the imaginary part is not well-defined. This difficulty can be overcome; we refer the reader to [Solomyak 1989], for example.

If $T = \mathcal{C}(A) = (A - iI)(A + iI)^{-1}$, then $T^* = (A^* + iI)(A^* - iI)^{-1}$. Taking an arbitrary vector $f \in \operatorname{Dom} A$ and putting $x = (A + iI)f$, we have $Tx = (A - iI)f$, whence

$$D_T^2 x = (A + iI)f - T^*(A - iI)f = (A + iI)f - (A^* + iI)(A^* - iI)^{-1}(A - iI)f.$$

Set $g \stackrel{\text{def}}{=} (A^* - iI)^{-1}(A - iI)f$. Then

$$(A - iI)f = (A^* - iI)g,$$

or

$$\ell(f - g) = i(f - g),$$

which implies that $f - g = cy_i$, and

$$D_T^2 x = (A + iI)f - (A^* + iI)g = (\ell + iI)(f - g) = 2icy_i.$$

Thus, we have checked that the defect subspace $\mathcal{D}_T$ of the Cayley transform $T = \mathcal{C}(\ell_h)$ is the one-dimensional subspace

$$\mathcal{D}_T = \operatorname{span}\{y_i\}$$

generated by the solutions y_ζ for $\zeta = i$. In a similar way, the defect subspace of the adjoint operator T^* is also one-dimensional:

$$\mathcal{D}_{T^*} = \operatorname{span}\{y_{-i}\}.$$

Next, we note that $Ty_i = 0$ and $T^* y_{-i} = 0$, whence $D_T y_i = y_i$ and $D_{T^*} y_{-i} = y_{-i}$. Now, instead of (2.4.1), we use expression (1.14.2), namely,

$$D_{T^*}\Theta_T(z) = (zI - T)(I - zT^*)^{-1} D_T,$$

which can be rewritten in terms of $A = \ell_h$ as follows:

$$D_{T^*}\Theta_T(z) = -(A - \zeta I)(A + iI)^{-1}(A^* - iI)(A^* - \zeta I)^{-1} D_T,$$

where, as before, $\zeta = i(1+z)/(1-z)$. We apply this operator to the vector y_i and employ the formula

$$(\ell_h - \lambda I)^{-1} y_\mu = \frac{y_\mu + c y_\lambda}{\mu - \lambda},$$

(it will be used twice); here c is chosen so as to ensure that the vector above belongs to $\operatorname{Dom}(\ell_h)$, namely,

$$c = c(\lambda, \mu, h) = -\frac{m(\mu) + h}{m(\lambda) + h}.$$

As a result, we obtain

$$\begin{aligned} D_{T^*}\Theta_T(z) y_i &= -(\ell_h - \zeta I)(\ell_h + iI)^{-1}(\ell_{\bar h} - iI)(\ell_{\bar h} - \zeta I)^{-1} y_i \\ &= -(\ell_h - \zeta I)(\ell_h + iI)^{-1}(\ell_{\bar h} - iI)\,\frac{y_i + c(\zeta, i, \bar h) y_\zeta}{i - \zeta} \\ &= (\ell_h - \zeta I)(\ell_h + iI)^{-1} c(\zeta, i, \bar h) y_\zeta \\ &= (\ell_h - \zeta I) c(\zeta, i, \bar h)\,\frac{y_\zeta + c(-i, \zeta, h) y_{-i}}{\zeta + i} \\ &= -c(\zeta, i, \bar h) c(-i, \zeta, h) y_{-i} = c_0\,\frac{m(\zeta) + h}{m(\zeta) + \bar h}, \end{aligned}$$

where $c_0 = -\big(m(i) + \bar h\big)/\big(m(-i) + h\big)$ is a unimodular constant. Since $y_{-i} = \bar y_i$ and, therefore, $\|y_i\| = \|y_{-i}\|$, we see that the characteristic function is equivalent to the simple expression

$$\mathcal{S}_{\ell_h}(\zeta) = \frac{m(\zeta) + h}{m(\zeta) + \bar h}.$$

Thus, we have proved the following theorem.

2.13. Theorem. *The characteristic function $\Theta_{\mathcal{C}(\ell_h)}$ of the Cayley transform of a dissipative Sturm–Liouville operator* (2.12.1) *is*

$$\Theta_{\mathcal{C}(\ell_h)}(z) = \mathcal{S}_{\ell_h}(\zeta) = \frac{m(\zeta) + h}{m(\zeta) + \bar h},$$

where $\zeta = i\,\dfrac{1+z}{1-z}$.

Chapter 3. Transcriptions of the Model

A coordinate transcription of the function model arises whenever we choose a spectral representation of the minimal unitary dilation $\mathcal{U}$ and a solution Π of the equation

$$\Pi^*\Pi = W_\Theta \overset{\text{def}}{=} \begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix}. \tag{3.0}$$

3.1. Multiplicity of the minimal dilation. To begin with, we mention that for a given completely nonunitary contraction T the spectral measure $E_{\mathcal{U}}$ of the minimal unitary dilation $\mathcal{U}$ and Lebesgue measure on the unit circle are mutually absolutely continuous. Moreover, the local spectral multiplicity of $E_{\mathcal{U}}$ is at least $\max(\partial, \partial_*)$ and at most $\partial + \partial_*$, where $\partial = \dim \mathcal{D}_T$ and $\partial_* = \dim \mathcal{D}_{T^*}$. This is an immediate consequence of the existence of the embeddings π and π_* intertwining $\mathcal{U}$ and z on $L^2(E)$ and $L^2(E_*)$, respectively, and the completeness property $\mathcal{H} = \operatorname{clos} \operatorname{Range} \Pi$.

3.2. Choosing a space of the minimal dilation. Thus, the minimal unitary dilation $\mathcal{U}$ is unitarily equivalent to the operator of multiplication by z on any weighted space

$$L^2(E_* \oplus E, W) \stackrel{\text{def}}{=} \bigl\{ f : \textstyle\int_{\mathbb{T}} \bigl(W(\zeta) f(\zeta), f(\zeta)\bigr)\, dm(\zeta) < \infty \bigr\}$$

whenever the operator-valued weight $W(\zeta) : E_* \oplus E \to E_* \oplus E$ satisfies the spectral multiplicity condition $\operatorname{rank} W(\zeta) = \operatorname{rank} E_{\mathcal{U}}(\zeta)$ for a.e. $\zeta \in \mathbb{T}$.

We could choose another coefficient space instead of $E_* \oplus E$; however, the latter space is natural and minimal among those ensuring that $\dim(E_* \oplus E) = \partial_* + \partial \geq \operatorname{rank} E_{\mathcal{U}}$.

3.3. Intermediate space. In order to separate the role of the weight and to facilitate computation we assume that the embeddings π and π_* are continuous as mappings into the nonweighted spaces $L^2(E) \to L^2(E_* \oplus E)$ and $L^2(E_*) \to L^2(E_* \oplus E)$, respectively. The corresponding adjoint mappings will be denoted by the symbols π^+ and π_*^+. To start with, we also assume that the weight W is bounded. The operator adjoint to the natural embedding $L^2(E_* \oplus E) \to L^2(E_* \oplus E, W)$ is the operator of multiplication by W. Thus, we have $\pi^* = \pi^+ W$ and $\pi_*^* = \pi_*^+ W$, that is, $\Pi^* = \Pi^+ W$.

3.4. Choosing functional embeddings. So, we must choose an operator Π satisfying (3.0). To solve this equation we rewrite it in terms of Π^+:

$$\Pi^* \Pi = W_\Theta \quad \Longleftrightarrow \quad \begin{cases} \pi^+ W \pi = I, \\ \pi_*^+ W \pi_* = I, \\ \pi_*^+ W \pi = \Theta. \end{cases} \tag{3.4.1}$$

Here we shall not dwell on the description of all solutions of this system, nor do we discuss in detail the possible preferences in choosing a specific transcription; this can be found in [Nikolski and Vasyunin 1989]. We only present some ideas concerning what could be required from a transcription and describe three most popular transcriptions that are used in numerous papers of many authors.

3.5. The Szőkefalvi-Nagy–Foiaş transcription. First of all, we can prefer to work in a nonweighted L^2-space. Though it is not always possible to put $W = I$, in any case as $\mathcal{H}$ we can choose a subspace of $L^2(E_* \oplus E)$. This means the chosen W is a projection: $W(\zeta) = P_{\mathrm{Range}\,\Pi(\zeta)}$, where $\zeta \in \mathbb{T}$. Moreover, we can try to take π_* (or π) to be the natural embedding.

Solving (3.4.1) under the assumptions

$$\pi_* = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} X \\ Y \end{pmatrix}, \quad W = W^2 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

we obtain

$$\begin{aligned} \pi_*^+ W \pi_* = I &\implies A = I, \\ W^2 = W &\implies B = 0 \text{ and } C^2 = C, \\ \pi_*^+ W \pi = \Theta &\implies X = \Theta, \\ \pi^+ W \pi = I &\implies Y^* C Y = \Delta^2 \stackrel{\text{def}}{=} I - \Theta^*\Theta. \end{aligned}$$

The usual choice is

$$Y = \Delta, \quad C(\zeta) = P_{\mathrm{Range}\,\Delta(\zeta)}.$$

In this way we arrive at the *Szőkefalvi-Nagy–Foiaş incoming transcription* of the model:

$$\pi = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix}, \quad \pi_* = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad W = \begin{pmatrix} I & 0 \\ 0 & P_{\mathrm{Range}\,\Delta} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L^2(E_*) \\ L^2(\Delta E) \end{pmatrix}, \tag{3.5.1}$$

where $L^2(\Delta E) \stackrel{\text{def}}{=} \operatorname{clos} \Delta L^2(E)$;

$$G = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(E), \quad G_* = \begin{pmatrix} H^2_-(E_*) \\ 0 \end{pmatrix}, \quad \mathcal{K}_\Theta = \begin{pmatrix} H^2(E_*) \\ L^2(\Delta E) \end{pmatrix} \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(E),$$

$$\mathcal{M}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} zf - \Theta[z(\Theta^* f + \Delta g)]\widehat{\ }(0) \\ zg - \Delta[z(\Theta^* f + \Delta g)]\widehat{\ }(0) \end{pmatrix}, \quad \mathcal{M}_\Theta^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \dfrac{f - f(0)}{z} \\ \bar{z} g \end{pmatrix}.$$

Choosing π (rather than π_*) to be the natural embedding, we obtain the *Szőkefalvi-Nagy–Foiaş outgoing transcription* of the model:

$$\pi = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \pi_* = \begin{pmatrix} \Delta_* \\ \Theta^* \end{pmatrix}, \quad W = \begin{pmatrix} P_{\mathrm{Range}\,\Delta_*} & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L^2(\Delta_* E_*) \\ L^2(E) \end{pmatrix},$$

$$G = \begin{pmatrix} 0 \\ H^2(E) \end{pmatrix}, \quad G_* = \begin{pmatrix} \Delta_* \\ \Theta^* \end{pmatrix} H^2_-(E_*), \quad \mathcal{K}_\Theta = \begin{pmatrix} L^2(\Delta_* E_*) \\ H^2_-(E) \end{pmatrix} \ominus \begin{pmatrix} \Delta_* \\ \Theta^* \end{pmatrix} H^2_-(E_*),$$

$$\mathcal{M}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} zf \\ zg - [zg]\widehat{\ }(0) \end{pmatrix}, \quad \mathcal{M}_\Theta^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{z} f - \bar{z}\Delta_*[\Delta_* f + \Theta g]\widehat{\ }(0) \\ \bar{z} g - \bar{z}\Theta^*[\Delta_* f + \Theta g]\widehat{\ }(0) \end{pmatrix}.$$

In the case of an inner characteristic function Θ (that is, $\Delta = 0$) the first transcription becomes especially simple:

$$\mathcal{K}_\Theta = H^2(E_*) \ominus \Theta H^2(E),$$

and the operator adjoint to the model operator is the restriction on $\mathcal{K}_\Theta$ of the backward shift

$$\mathcal{M}_\Theta^* f = \frac{f - f(0)}{z}.$$

If the characteristic function is $*$-inner (that is, $\Delta_* = 0$), then for the second transcription we have

$$\mathcal{K}_\Theta = H^2_-(E) \ominus \Theta^* H^2_-(E_*), \quad \mathcal{M}_\Theta g = zg - [zg]\widehat{\ }(0);$$

that is, now the model operator $\mathcal{M}_\Theta$ itself is a restriction of the backward shift. However, if we wish to work in the more usual space consisting of analytic functions rather than of anti-analytic functions, we can apply the transformation J: $(Jh)(z) = \bar{z}h(\bar{z})$, obtaining

$$\mathcal{K}_\Theta = H^2(E) \ominus \widetilde{\Theta} H^2(E_*), \qquad \text{where} \quad \widetilde{\Theta} = J\Theta J, \quad \text{that is,} \quad \widetilde{\Theta}(z) = \Theta^*(\bar{z}),$$

and

$$\mathcal{M}_\Theta f = \frac{f - f(0)}{z}.$$

In this representation, the minimal unitary dilation of $\mathcal{M}_\Theta$ is the operator of multiplication by $\bar{z}$ on $\mathcal{H} = L^2(E)$.

3.6. The Pavlov transcription. It seems that the most natural way of choosing our embeddings π and π_* is to decide that both of them are the identity embeddings $\pi : L^2(E) \to L^2(E_* \oplus E)$ and $\pi_* : L^2(E_*) \to L^2(E_* \oplus E)$. However, in this case we cannot avoid some complications related to the weight W. Indeed, if we put $\Pi = \mathrm{id}$, then $\Pi^* = \Pi^+ W = W$, and (3.0) implies that $W = W_\Theta$,

$$\mathcal{H} = L^2\left(E_* \oplus E, \begin{pmatrix} I & \Theta \\ \Theta^* & I \end{pmatrix}\right), \quad G = \begin{pmatrix} 0 \\ H^2(E) \end{pmatrix} \quad , G_* = \begin{pmatrix} H^2_-(E_*) \\ 0 \end{pmatrix},$$

$$\mathcal{M}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} zf \\ zg - [z(\Theta^* f + g)]\widehat{\ }(0) \end{pmatrix}, \quad \mathcal{M}^*_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{z}f - \bar{z}[f + \Theta g]\widehat{\ }(0) \\ \bar{z}g \end{pmatrix}.$$

This version of the model was proposed by B. Pavlov [1975] for the investigation of the problems of scattering theory, where the incoming and outgoing subspaces play an essential role. In this representation, these subspaces have the simplest possible form. However, this must be paid for with the complexity of computations in $\mathcal{K}_\Theta$. The vectors in $\mathcal{H}$ are no longer pairs of L^2-functions (this is due to possible degeneracy of the weight and the necessity of completion); moreover, it may happen that the model space contains no vector representable as a pair of L^2-functions, except, of course, the zero vector.

3.7. The de Branges–Rovnyak transcription. If we prefer the model subspace to consist of analytic functions only, we can choose $\Pi^* = \mathrm{id}$. Then (3.0) implies $\Pi = W_\Theta$, which yields $W = W_\Theta^{[-1]}$ (for a selfadjoint operator A, we denote by $A^{[-1]}$ the operator equal to zero on $\operatorname{Ker} A$ and to the left inverse of A on $\operatorname{Range} A$). Thus,

$$\mathcal{H} = L^2\big(E_* \oplus E, W_\Theta^{[-1]}\big), \quad G = \begin{pmatrix} \Theta \\ I \end{pmatrix} H^2(E), \quad G_* = \begin{pmatrix} I \\ \Theta^* \end{pmatrix} H^2_-(E_*).$$

Now the model space consists of pairs of analytic and anti-analytic functions:

$$\mathcal{K}_\Theta = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in H^2(E),\ g \in H^2_-(E_*),\ g - \Theta^* f \in \Delta L^2(E) \right\}.$$

The action of the model operator is not more involved than in other transcriptions; we have

$$\mathcal{M}_\Theta \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} zf - \Theta[zg]\widehat{\ }(0) \\ zg - [zg]\widehat{\ }(0) \end{pmatrix}, \quad \mathcal{M}_\Theta^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \frac{f-f(0)}{z} \\ \bar{z}g - \Theta^* \bar{z} f(0) \end{pmatrix}.$$

However, the verification that a given pair of functions belongs to $\mathcal{K}_\Theta$ and the computation of the norm become rather difficult in this representation.

To identify this transcription with the original de Branges–Rovnyak form of the model we need the following description of the model space given in [de Branges and Rovnyak 1966]:

$$\mathcal{H}(\Theta) \stackrel{\text{def}}{=} (I - \Theta P_+ \Theta^*)^{1/2} H^2(E_*)$$

(this space endowed with the *range norm*),

$$\mathcal{D}(\Theta) \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in \mathcal{H}(\Theta),\ g \in H^2(E),\ \text{and } z^n f - \Theta P_+ z^n J g \in \mathcal{H}(\Theta) \text{ for } n \geq 0 \right\},$$

where $(Jh)(\bar{z}) \stackrel{\text{def}}{=} \bar{z} h(\bar{z})$, as in Section 3.5. The norm on $\mathcal{D}(\Theta)$ is defined by

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 \stackrel{\text{def}}{=} \lim \bigl(\| z^n f - \Theta P_+ z^n J g \|^2_{\mathcal{H}(\Theta)} + \| P_+ z^n J g \|^2_{H^2(E)} \bigr).$$

The original de Branges–Rovnyak model operator is

$$\mathrm{BR} \begin{pmatrix} f \\ g \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{f-f(0)}{z} \\ zg - \Theta(\bar{z})^* f(0) \end{pmatrix}, \quad \text{with } \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{D}(\Theta).$$

3.8. Proposition.

$$\mathcal{K}_\Theta = \mathcal{J}\mathcal{D}(\Theta) \quad \textit{and} \quad \mathcal{J}\mathcal{M}_\Theta^* \mathcal{J} = \mathrm{BR},$$

where

$$\mathcal{J} = \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix} : \mathcal{H}(\Theta) \oplus L^2(E) \to \mathcal{H}(\Theta) \oplus L^2(E).$$

The proof can be found in [Nikolski and Vasyunin 1989].

Chapter 4. The Commutant Lifting Theorem and Calculi

Our first goal in this chapter is to give a simple proof of the Sz.-Nagy–Foiaş commutant lifting theorem (CLT) and to describe the parametrizations of the lifting operators in terms of the coordinate-free functional model. The second theme is the classical H^∞-function calculus for a completely nonunitary contraction, along with refinements pertaining to locally defined versions of the calculus.

We derive the CLT from the following theorem of T. Ando.

4.1. Theorem. *Any two commuting contractions have commuting unitary dilations.*

The idea for proving the commutant lifting theorem via the Ando theorem goes back to S. Parrott [1970]. Here we present it with all details, including a new simple proof of the Ando theorem. This simplicity makes the approach quite attractive.

We start by introducing the necessary terminology and proving an "abstract" version of the lifting theorem.

4.2. Definition. Let T be a contraction on H; suppose that $X : H \to H$ belongs to the commutant of T:

$$X \in \{T\}' \stackrel{\text{def}}{=} \{A : AT = TA\}.$$

Next, let $\mathcal{H} = G_* \oplus H \oplus G$ be the space of the minimal unitary dilation $\mathcal{U}$ of T. An operator Y acting on $\mathcal{H}$ is called a *lifting* of X if Y commutes with $\mathcal{U}$, $YG \subset G$, $Y^*G_* \subset G_*$, and $X = P_H Y|H$.

In other words, that Y is a lifting of X means that $Y \in \{\mathcal{U}\}'$ and Y is a dilation of X, that is, that the operator Y has the following matrix representation with respect to the decomposition $\mathcal{H} = G_* \oplus H \oplus G$:

$$Y = \begin{pmatrix} * & 0 & 0 \\ * & X & 0 \\ * & * & * \end{pmatrix} \begin{pmatrix} G_* \\ H \\ G \end{pmatrix} \longrightarrow \begin{pmatrix} G_* \\ H \\ G \end{pmatrix}. \tag{4.2.1}$$

4.3. Commutant Lifting Theorem. *Let T be a contraction on H. Then an operator X on H is a contraction commuting with T if and only if there exists a contractive lifting Y of X.*

4.4. Parametrization Theorem. *Let T be a contraction on a Hilbert space H. Then Y is a lifting of an operator commuting with T if and only if there exist two bounded analytic functions $A \in H^\infty(E \to E)$, $A_* \in H^\infty(E_* \to E_*)$ that are intertwined by the characteristic function of T,*

$$\Theta A = A_* \Theta, \tag{4.4.1}$$

and a bounded function $B \in L^\infty(\Delta_* E_* \to \Delta E)$ such that

$$Y = \pi_* A_* \pi_*^* + \tau \Delta A \pi^* + \tau B \tau_*^*. \tag{4.4.2}$$

Here π, π_* *and* τ, τ_* *are the functional embeddings defined in* 1.6 *and* 1.19, *respectively, and* $\Delta = (I - \Theta^*\Theta)^{1/2}$. *Furthermore, if* $X = P_H Y|H$, *then*

$$\{Y + \pi\Gamma\pi_*^* : \Gamma \in H^\infty(E_* \to E)\}$$

is the set of all liftings of X, *and*

$$\begin{aligned}\|X\|_H &= \inf\{\|Y + \pi\Gamma\pi_*^*\|_{\mathcal{H}} : \Gamma \in H^\infty(E_* \to E)\}\\ &= \operatorname{dist}_{\mathcal{H}}\big(Y, \pi H^\infty(E_* \to E)\pi_*^*\big).\end{aligned} \tag{4.4.3}$$

The infimum is attained at an H^∞*-function* Γ.

This parametrization differs from the original one given by B. Sz.-Nagy and C. Foiaş [1973] in one point: here we have one free parameter B instead of two matrix entries subject to a certain relation in the Sz.-Nagy–Foiaş parametrization.

We would like to underline here that (4.4.2) describes *liftings* of operators commuting with T, rather than all operators commuting with $\mathcal{U}$ whose compressions to H commute with T. It is very essential that liftings are not arbitrary operators commuting with $\mathcal{U}$, but those respecting the triangular structure of $\mathcal{U}$ given by (1.2.1), that is, the operators described in (4.4.2) leave invariant the subspaces G and $H \oplus G$.

Now we begin to realize the program outlined above by proving the Ando theorem. As already mentioned, the theorem on the existence of a unitary dilation can be proved in two steps: first, we construct a co-isometric extension of a given contraction, and second, we apply the same step to the adjoint of this extension (we recall that an operator $A : \mathcal{H} \to \mathcal{H}$ is called an *extension* of $B : H \to H$ if $H \subset \mathcal{H}$ is invariant subspace of A and $B = A|H$).

We use this approach to prove the Ando theorem (Section 4.6) and the commutant lifting theorem (Section 4.8). The first step is as follows.

4.5. Lemma. *Any two commuting contractions have commuting co-isometric extensions.*

Proof. Let T_1, T_2 be two commuting contractions on a Hilbert space H. It is always possible to find co-isometric extensions V_1, V_2 of T_1, T_2, respectively, acting on one and the same space $\mathcal{H}$. Indeed, if V_1 acts on a space $H \oplus H_1$ and V_2 on $H \oplus H_2$, we can take $\mathcal{H} = H \oplus H_1 \oplus H_2$, defining $V_1|H_2$ and $V_2|H_1$ as the identity operators.

Furthermore, we may assume that the operators V_1V_2 and V_2V_1 are unitarily equivalent. Indeed, V_1V_2 and V_2V_1 are two co-isometric extensions of the operator $T_1T_2 = T_2T_1$. Let V_0 be a minimal co-isometric extension of T_1T_2 (minimality means that the space where V_0 acts is the smallest subspace reducing V_0 and containing H). Then the operators V_1V_2 and V_2V_1 are unitarily equivalent to certain orthogonal sums:

$$V_1V_2 \backsimeq V_0 \oplus V_{12}, \quad V_2V_1 \backsimeq V_0 \oplus V_{21},$$

where V_{12}, V_{21} are co-isometries acting on the corresponding spaces H_{12} and H_{21}. Now, we extend the operators V_1 and V_2 from $\mathcal{H}$ to $\mathcal{H}\oplus\sum_1^\infty\oplus(H_{12}\oplus H_{21})$, one of them by the identity operator on $\sum_1^\infty\oplus(H_{12}\oplus H_{21})$, and another by the infinite orthogonal sum of the operators $V_{12}\oplus V_{21}$. The operators obtained are also co-isometric extensions of T_i, but their products are unitarily equivalent, namely, they are equivalent to V_0 plus the infinite orthogonal sum of the operators $V_{12}\oplus V_{21}$.

Thus, we assume that V_1V_2 and V_2V_1 are unitarily equivalent, that is, there exists a unitary operator U such that

$$V_1V_2U = UV_2V_1.$$

Put $W_1 = V_1U^*$, $W_2 = UV_2$. Then W_1 and W_2 commute. Observing that U intertwines the minimal parts of V_1V_2 and V_2V_1 and that a unitary operator intertwining any two minimal co-isometric extensions of a contraction on H can be chosen as the identity on H, we conclude that the restriction $U|H$ is the identity operator, whence $W_i|H = V_i|H = T_i$, that is, the W_i are the required co-isometric extensions of T_i, for $i = 1, 2$. □

Now the Ando theorem follows easily.

4.6. Proof of Theorem 4.1. If the operators T_i of Lemma 4.5 are isometries, then the extensions V_i can be taken unitary. (It is easily seen that the minimal co-isometric extension of an isometry is unitary.) Then the final commuting isometries W_i will also be unitary. So the theorem is proved for co-isometries.

As for arbitrary contractions, we can apply the result obtained above to the commuting isometries W_1^*, W_2^*. Their commuting unitary extensions are the desired dilations of T_1^*, T_2^*. □

We start proving Theorem 4.3 with the following lemma.

4.7. Lemma. *Let T and X be commuting contractions on H, and let V be the minimal co-isometric extension of T acting on $\mathcal{H}_+$. Then there exists a contraction Y_+ on $\mathcal{H}_+$ commuting with V and extending the operator X.*

Proof. Let V_T and V_X be arbitrary commuting co-isometric extensions of T and X, respectively. (Such extensions exist by Lemma 4.5). Putting $\mathcal{H}_+ = \operatorname{span}\{V_T^{*n}H : n \ge 0\}$, we see that $V_T\mathcal{H}_+ \subset \mathcal{H}_+$ and $V = V_T|\mathcal{H}_+$ is a minimal co-isometric extension of T. Let $Y_+ = P_{\mathcal{H}_+}V_X|\mathcal{H}_+$. Since $\mathcal{H}_+$ is a reducing subspace for V_T, we have

$$VY_+ = V_TP_{\mathcal{H}_+}V_X|\mathcal{H}_+ = P_{\mathcal{H}_+}V_TV_X|\mathcal{H}_+ = P_{\mathcal{H}_+}V_XV_T|\mathcal{H}_+ = Y_+V$$

and, moreover,

$$Y_+|H = P_{\mathcal{H}_+}V_X|H = P_{\mathcal{H}_+}X = X. \qquad \square$$

4.8. Proof of Theorem 4.3. By Lemma 4.7, given $X \in \{T\}'$, we can find a contractive extension Y_+ of X^*, commuting with the minimal co-isometric extension V of T^*. We recall that $V = P_{H\oplus G}\mathcal{U}^*|H \oplus G$ (see Remark 1.5). Applying Lemma 4.7 once again, now to Y_+^* and V^*, we get the required dilation Y. Indeed, since the minimal unitary dilation $\mathcal{U}$ of T is the minimal co-isometric extension of V^*, we have $Y\mathcal{U} = \mathcal{U}Y$. Furthermore, being an extension of X^*, the operator Y_+ has the matrix structure

$$Y_+ = \begin{pmatrix} * & 0 \\ * & X^* \end{pmatrix} : \begin{pmatrix} G \\ H \end{pmatrix} \longrightarrow \begin{pmatrix} G \\ H \end{pmatrix},$$

or

$$Y_+^* = \begin{pmatrix} X & 0 \\ * & * \end{pmatrix} : \begin{pmatrix} H \\ G \end{pmatrix} \longrightarrow \begin{pmatrix} H \\ G \end{pmatrix}.$$

The operator Y, being an extension of Y_+^*, is of the form

$$Y = \begin{pmatrix} * & 0 & 0 \\ * & X & 0 \\ * & * & * \end{pmatrix} : \begin{pmatrix} G_* \\ H \\ G \end{pmatrix} \longrightarrow \begin{pmatrix} G_* \\ H \\ G \end{pmatrix},$$

which means, in accordance with Definition 4.2, that Y is a lifting of X.

The converse is obvious: if Y is a contractive lifting of $X = P_H Y|H$, then X is a contraction and

$$\begin{aligned} XT &= P_H YT = P_H Y P_H \mathcal{U}|H \\ &= P_H Y(I - P_G - P_{G_*})\mathcal{U}|H \quad (\text{since } \mathcal{U}H \subset H \oplus G \text{ and } YG \subset G) \\ &= P_H Y\mathcal{U}|H = P_H \mathcal{U}Y|H \\ &= P_H \mathcal{U}(P_H + P_G + P_{G_*})Y|H \quad (\text{since } YH \subset H \oplus G \text{ and } \mathcal{U}G \subset G) \\ &= P_H \mathcal{U} P_H Y|H = TX. \end{aligned}$$ □

Now we employ the function model to give a functional parametrization of the commutant.

4.9. Proof of Theorem 4.4. Since the functional embeddings π, π_*, τ, and τ_* all intertwine the dilation $\mathcal{U}$ and the operator of multiplication by z, expression (4.4.2) provides an operator commuting with $\mathcal{U}$ for every triple of operator-valued functions (A, A_*, B). Furthermore, (4.4.2) implies that $\pi_*^* Y = A_* \pi_*^*$; that is, $Y^* G_* = Y^* \pi_* H^2_-(E_*) = \pi_* A_*^* H^2_-(E_*) \subset \pi_* H^2_-(E_*) = G_*$, and $Y\pi = \pi_* A_* \Theta + \tau\Delta A$. By (4.4.1), the latter relation can be rewritten as $Y\pi = (\pi_*\Theta + \tau\Delta)A = \pi A$. Then, obviously, $YG = Y\pi H^2(E) = \pi A H^2(E) \subset \pi H^2(E) = G$.

So, we have proved that any operator Y of the form (4.4.2) is a lifting of an operator belonging to the commutant of T. Now we prove the converse.

Let Y be a lifting of an operator belonging to the commutant of T; this means that $Y\mathcal{U} = \mathcal{U}Y$, $YG \subset G$, and $Y^* G_* \subset G_*$. First, we use the inclusion $YG \subset G$, which we rewrite in the form $Y\pi H^2(E) \subset \pi H^2(E)$. The relations $\pi^*\pi = I$, $\pi\pi_*^* + \tau_*\tau_*^* = I$, and $\tau_*^*\pi = 0$ show that the latter inclusion is equivalent to the inclusion $\pi^*Y\pi H^2(E) \subset H^2(E)$ together with the identity $\tau_*^*Y\pi|H^2(E) = 0$. Since Y commutes with $\mathcal{U}$, the operators $\pi^*Y\pi$ and $\tau_*^*Y\pi$ commute with multiplication by z; that is, they are operators of multiplication by certain operator-valued functions, which we denote by the same symbols. Moreover, the inclusion $\pi^*Y\pi H^2(E) \subset H^2(E)$ means that $A \stackrel{\text{def}}{=} \pi^*Y\pi \in H^\infty(E \to E)$, and the identity $\tau_*^*Y\pi|H^2(E) = 0$ shows that $\tau_*^*Y\pi = 0$ everywhere. This yields $Y\pi = (\pi\pi^* + \tau_*\tau_*^*)Y\pi = \pi\pi^*Y\pi = \pi A$; that is, we have the intertwining relation

$$\pi A = Y\pi. \tag{4.9.1}$$

Similarly, introducing $A_* \stackrel{\text{def}}{=} \pi_*^*Y\pi_*$, we deduce from the inclusion $Y^*G_* \subset G_*$ that $A_* \in H^\infty(E_* \to E_*)$ and that

$$\pi_*^*Y = A_*\pi_*^*. \tag{4.9.2}$$

Multiplying (4.9.1) by π_*^* from the left and (4.9.2) by π from the right, we arrive at (4.4.1). Furthermore,

$$\begin{aligned} Y &= (\pi_*\pi_*^* + \tau\tau^*)Y = \pi_*A_*\pi_*^* + \tau\tau^*Y(\pi\pi^* + \tau_*\tau_*^*) \\ &= \pi_*A_*\pi_*^* + \tau\tau^*\pi A\pi^* + \tau(\tau^*Y\tau_*)\tau_*^* = \pi_*A_*\pi_*^* + \tau\Delta A\pi^* + \tau B\tau_*^*, \end{aligned} \tag{4.9.3}$$

where we have put $B \stackrel{\text{def}}{=} \tau^*Y\tau_*$. Since B intertwines the operators of multiplication by z on the spaces $L^2(\Delta_*E_*)$ and $L^2(\Delta E)$, B is the operator of multiplication by a function belonging to $L^\infty(\Delta_*E_* \to \Delta E)$. For convenience, B may be regarded as a function in $L^\infty(E_* \to E)$ equal to zero on $\operatorname{Ker}\Delta_*$.

To complete the proof we must describe all liftings of a given operator and check the formula for the norm. First, we note that if Y is a lifting of X, then, clearly, $\|X\| \le \|Y\|$. By Theorem 4.3, there exists a contractive lifting of the operator $X\|X\|^{-1}$. Multiplying it by $\|X\|$, we get a lifting Y of X with norm at most $\|X\|$. Therefore,

$$\|X\| = \inf\{\|Y\| : Y \text{ is a lifting of } X\}$$

and to prove formula (4.4.3) it suffices to check that the set of all liftings of X is of the form

$$\{Y + \pi\Gamma\pi_*^* : \Gamma \in H^\infty(E_* \to E)\},$$

where Y is an arbitrary lifting. In other words, we need to check that the set

$$\{\pi\Gamma\pi_*^* : \Gamma \in H^\infty(E_* \to E)\}$$

is merely the set of liftings of the zero operator. The latter assertion is proved in Lemma 4.10 below.

4.10. Lemma. *The following assertions are equivalent.*

(1) Y *is a lifting of the zero operator.*
(2) $Y = \pi\Gamma\pi_*^*$ *for some* Γ, $\Gamma \in H^\infty(E_* \to E)$.

(3) *For some function* $\Gamma \in H^\infty(E_* \to E)$, *the operator* Y *is representable as in* (4.4.2), *with*

$$A = \Gamma\Theta, \quad A_* = \Theta\Gamma, \quad B = \Delta\Gamma\Delta_*. \tag{4.10.1}$$

Here π *and* π_* *are the functional embeddings defined in Section* 1.6, $\Delta = (I - \Theta^*\Theta)^{1/2}$, *and* $\Delta_* = (I - \Theta\Theta^*)^{1/2}$.

Proof. (1) $\Longrightarrow$ (2). If $P_H Y|H = 0$, then $Y(G \oplus H) \subset G$. Since $G = \pi H^2(E)$ and $G \oplus H = \mathcal{H} \ominus G_* = \pi_* H^2(E_*) \oplus \tau L^2(\Delta E)$, we can rewrite the latter inclusion in the form

$$Y(\pi_* H^2(E_*) \oplus \tau L^2(\Delta E)) \subset \pi H^2(E). \tag{4.10.2}$$

In particular,

$$Y\pi_* H^2(E_*) \subset \pi H^2(E),$$

which implies that $\Gamma \overset{\text{def}}{=} \pi^* Y \pi_* \in H^\infty(E_* \to E)$. Furthermore, applying the operators $\mathcal{U}^{*n}$ to the both sides of (4.10.2) and then letting n tend to infinity, we get

$$Y\mathcal{H} \subset \pi L^2(E);$$

that is, $\tau_*^* Y = 0$. Similarly, applying the operators $\mathcal{U}^n$ to the same inclusion and then letting n tend to infinity, we get

$$Y\tau L^2(\Delta E) = \{0\};$$

that is, $Y\tau = 0$. Therefore,

$$Y = (\pi\pi^* + \tau_*\tau_*^*)Y(\pi_*\pi_*^* + \tau\tau^*) = \pi(\pi^* Y \pi_*)\pi_*^* = \pi\Gamma\pi_*^*.$$

(2) $\Longleftrightarrow$ (3). Since $\pi = \pi_*\Theta + \tau\Delta$ and $\pi_* = \pi\Theta^* + \tau_*\Delta_*$, we have

$$\begin{aligned}\pi\Gamma\pi_*^* &= (\pi_*\Theta + \tau\Delta)\Gamma\pi_*^* = \pi_*\Theta\Gamma\pi_* + \tau\Delta\Gamma(\Theta\pi^* + \Delta_*\tau_*^*) \\ &= \pi_*(\Theta\Gamma)\pi_*^* + \tau\Delta(\Gamma\Theta)\pi^* + \tau(\Delta\Gamma\Delta_*)\tau_*^*;\end{aligned}$$

that is, the identity $Y = \pi\Gamma\pi_*^*$ is equivalent to (4.4.2) with $A = \Gamma\Theta$, $A_* = \Theta\Gamma$, $B = \Delta\Gamma\Delta_*$.

(2) + (3) $\Longrightarrow$ (1). By the already proved part of Theorem 4.4 the operator Y defined by formula (4.4.2) is a lifting of the operator $X = P_H Y|H$. To compute X, take a vector $h \in H$. Then $\pi_*^* h \in H^2(E_*)$, whence

$$Xh = P_H Y h = P_H \pi\Gamma\pi_*^* h \in P_H \pi H^2(E) = \{0\},$$

so that $X = 0$. ☐

4.11. Another expression for a lifting. The representation (4.4.2) can be rewritten in the form

$$Y = \pi A \pi^* + \pi_* A_* \Delta_* \tau_*^* + \tau B \tau_*^*. \tag{4.11.1}$$

The existence of the two representations (4.4.2) and (4.11.1) of the same operator is founded on the duality between the operators and their adjoints. We can prove this formula in the same way as (4.9.3):

$$\begin{aligned} Y &= Y(\pi\pi^* + \tau_*\tau_*^*) = \pi A \pi^* + (\pi_*\pi_*^* + \tau\tau^*) Y \tau_* \tau_*^* \\ &= \pi A \pi^* + \pi_* A_* \pi_*^* \tau_* \tau_*^* + \tau(\tau^* Y \tau_*)\tau_*^* = \pi A \pi^* + \pi_* A_* \Delta_* \tau_*^* + \tau B \tau_*^*. \end{aligned}$$

Moreover, we note that formulas (4.4.2) and (4.11.1) represent one and the same operator if and only if relation (4.4.1) is fulfilled. Indeed, the difference of (4.4.2) and (4.11.1) is equal to

$$\begin{aligned} (\pi_* A_* \pi_*^* + \tau\Delta A \pi^*) - (\pi A \pi^* + \pi_* A_* \Delta_* \tau_*^*) &= \pi_* A_* (\pi_*^* - \Delta_* \tau_*^*) - (\pi - \tau\Delta) A \pi^* \\ &= \pi_* (A_* \Theta - \Theta A) \pi^*. \end{aligned}$$

4.12. More function parameters. We introduce two more functions related to a lifting operator:

$$\begin{aligned} C &\overset{\text{def}}{=} \tau^* Y \tau \in L^\infty(\Delta E \to \Delta E) \\ C_* &\overset{\text{def}}{=} \tau_*^* Y \tau_* \in L^\infty(\Delta_* E \to \Delta_* E). \end{aligned}$$

These functions satisfy the relations

$$C = \Delta A \Delta - B\Theta, \tag{4.12.1}$$

$$C_* = \Delta_* A_* \Delta_* - \Theta B, \tag{4.12.2}$$

$$Y\tau = \tau C, \tag{4.12.3}$$

$$\tau_*^* Y = C_* \tau_*^*. \tag{4.12.4}$$

Indeed, multiplying (4.4.2) by τ from the right, we obtain

$$Y\tau = (\pi_* A_* \pi_*^* + \tau \Delta A \pi^* + \tau B \tau_*^*)\tau = \tau(\Delta A \Delta - B\Theta),$$

which yields (4.12.1) and (4.12.3). Similarly, multiplying (4.11.1) by τ_*^* from the left, we get (4.12.2) and (4.12.4). □

It should be noted that the parameters C and C_* of a lifting Y of an operator X are *uniquely determined by* X. Indeed, if $X = 0$, then $Y = \pi \Gamma \pi_*^*$, whence

$$C = \tau^* Y \tau = 0, \quad C_* = \tau_*^* Y \tau_* = 0.$$

4.13. Multiplication Theorem. *Let X_1 and X_2 be operators commuting with T. For $i = 1, 2$, let Y_i be the lifting of X_i with parameters $A_i, A_{*i}, B_i, C_i, C_{*i}$.*

Then the operator $Y \stackrel{\text{def}}{=} Y_2 Y_1$ *is a lifting of the product* $X_2 X_1$, *and the parameters of* Y *are*

$$A = A_2 A_1, \qquad A_* = A_{*2} A_{*1}, \tag{4.13.1}$$

$$B = \Delta A_2 \Theta^* A_{*1} \Delta_* + B_2 \Delta_* A_{*1} \Delta_* + \Delta A_2 \Delta B_1 - B_2 \Theta B_1, \tag{4.13.2}$$

$$C = C_2 C_1, \qquad C_* = C_{*2} C_{*1}. \tag{4.13.3}$$

Proof. Relation (4.9.1) yields

$$Y\pi = Y_2 Y_1 \pi = Y_2 \pi A_1 = \pi A_2 A_1,$$

so that $A = A_2 A_1$. Similarly, $A_* = A_{*2} A_{*1}$ is a consequence of (4.9.2). In the same way, (4.12.3) and (4.12.4) imply (4.13.3). To check (4.13.2) we combine (4.9.1) and (4.9.2) with (4.12.1) and (4.12.3):

$$\begin{aligned} B &= \tau^* Y \tau_* = \tau^* Y_2 (\pi_* \pi_*^* + \tau \tau^*) Y_1 \tau_* = \tau^* Y_2 (\pi_* A_{*1} \pi_*^* \tau_* + \tau B_1) \\ &= \tau^* Y_2 (\pi \Theta^* + \tau_* \Delta_*) A_{*1} \Delta_* + \tau^* \tau C_2 B_1 \\ &= (\Delta A_2 \Theta^* + B_2 \Delta_*) A_{*1} \Delta_* + (\Delta A_2 \Delta - B_2 \Theta) B_1. \qquad \square \end{aligned}$$

4.14. Intertwining two contractions. From the very beginning we could have considered the liftings of the operators intertwining two arbitrary contractions $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ instead of those of the operators intertwining a contraction with itself. More precisely, we mean the operators $X_{21} : H_1 \to H_2$ satisfying

$$T_2 X_{21} = X_{21} T_1. \tag{4.14.1}$$

An operator $Y_{21} : \mathcal{H}_1 \to \mathcal{H}_2$ is said to be a lifting of X_{21} if $X_{21} = P_{H_2} Y_{21} | H_1$, $Y_{21} G_1 \subset G_2$, and $Y_{21}^* G_{*2} \subset G_{*1}$. For such a lifting problem all results would be the same. The only difference is that in this more general situation we have different spaces and different function models from the right and from the left. For example, instead of (4.4.2), for a lifting Y_{21} of the intertwining operator X_{21} we have the following formula

$$Y_{21} = \pi_{2*} A_{21*} \pi_{1*}^* + \tau_2 \Delta_2 A_{21} \pi_1^* + \tau_2 B_{21} \tau_{1*}^* \tag{4.14.2}$$

acting between the corresponding spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ of the minimal unitary dilations of T_1 and T_2.

Moreover, such a generalization is an immediate consequence of the lifting theorem for the commutant. Indeed, having the intertwining relation (4.14.1), we can introduce on the space $H = H_1 \oplus H_2$ the commuting operators

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 0 \\ X_{21} & 0 \end{pmatrix} : \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \to \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}.$$

Let Y be a lifting of X. Then the operator $Y_{21} \stackrel{\text{def}}{=} P_{\mathcal{H}_2} Y|\mathcal{H}_1$ is a lifting of X_{21}. Indeed,

$$Y_{21}G_1 = P_{\mathcal{H}_2} Y G_1 \subset P_{\mathcal{H}_2} Y(G_1 \oplus G_2) \subset P_{\mathcal{H}_2}(G_1 \oplus G_2) = G_2,$$

and, similarly,

$$Y_{21}^* G_{*2} = P_{\mathcal{H}_1} Y^* G_{*2} \subset P_{\mathcal{H}_1} Y^*(G_{*1} \oplus G_{*2}) \subset P_{\mathcal{H}_1}(G_{*1} \oplus G_{*2}) = G_{*1}.$$

The relations $X_{21} = P_{H_2} X|H_1 = P_{H_2} Y|H_1 = P_{H_2} Y_{21}|H_1$ are clear.

4.15. The special case of an inner characteristic function. All formulas become simpler under the assumption that $\Theta = \Theta_T$ is an inner or $*$-inner function. We recall that this is equivalent to saying that the imbedding π_* (or, respectively, π) is onto. Indeed, Θ is inner means $\Delta = 0$, which is equivalent to saying that $\tau = 0$, and the latter is possible (see Section 1.19) if and only if π_* is a co-isometry, that is, unitary. Similarly,

$$\Theta \text{ is } *\text{-inner} \iff \Delta_* = 0 \iff \tau_* = 0 \iff \pi\pi^* = I \iff \pi \text{ is unitary.}$$

In this section we assume that π_* is unitary, that is, that Θ is an inner function. In this case, formula (4.4.2) for a lifting becomes

$$Y = \pi_* A_* \pi_*^*,$$

where the sole free parameter A_* runs over all functions in $H^\infty(E_* \to E_*)$ satisfying $A \stackrel{\text{def}}{=} \Theta^* A_* \Theta \in H^\infty(E \to E)$.

The distance formula (4.4.3) becomes

$$\begin{aligned} \|X\| &= \inf\{\|Y + \pi\Gamma\pi_*^*\| : \Gamma \in H^\infty(E_* \to E)\} \\ &= \inf\{\|A_* + \Theta\Gamma\| : \Gamma \in H^\infty(E_* \to E)\} \\ &= \operatorname{dist}\big(A_*, \Theta H^\infty(E_* \to E)\big). \end{aligned}$$

4.16. Lifting for the Sz.-Nagy–Foiaş model. Let

$$H = \mathcal{K}_\Theta = \begin{pmatrix} H^2(E_*) \\ L^2(\Delta E) \end{pmatrix} \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(E)$$

be the Sz.-Nagy–Foiaş transcription of the model, with

$$T = \mathcal{M}_\Theta = P_\Theta z|\mathcal{K}_\Theta.$$

In this model, the liftings take the form of the operator of multiplication by the matrix function

$$Y = \begin{pmatrix} A_* & 0 \\ \Delta A \Theta^* + B\Delta_* & C \end{pmatrix},$$

where C is defined by (4.12.1). Indeed, in the Sz.-Nagy–Foiaş transcription the functional embeddings are

$$\pi_* = \begin{pmatrix} I \\ 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} \Theta \\ \Delta \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad \tau_* = \begin{pmatrix} \Delta_* \\ -\Theta^* \end{pmatrix}.$$

Therefore, (4.4.2) turns into

$$Y = \begin{pmatrix} I \\ 0 \end{pmatrix} A_*(I,0) + \begin{pmatrix} 0 \\ I \end{pmatrix} \Delta A(\Theta^*, \Delta) + \begin{pmatrix} 0 \\ I \end{pmatrix} B(\Delta_*, -\Theta)$$
$$= \begin{pmatrix} A_* & 0 \\ \Delta A \Theta^* + B\Delta_* & \Delta A \Delta - B\Theta \end{pmatrix},$$

as claimed (see (4.12.1)).

In the case of an inner characteristic function we have $\mathcal{H} = L^2(E_*)$, $\pi_* = I$, and

$$H = \mathcal{K}_\Theta = H^2(E_*) \ominus \Theta H^2(E).$$

Then, the lifting Y is simply the operator of multiplication by A_*, and we get

$$X = P_\Theta Y | \mathcal{K}_\Theta = (I - \Theta P_+ \Theta^*) A_* | \mathcal{K}_\Theta.$$

4.17. The special case of a scalar characteristic function. Now we consider a scalar characteristic function Θ; that is, we assume that $\dim E = \dim E_* = 1$; we identify $E = E_* = \mathbb{C}$. We put $a = A = A_*$. In this case, as a second parameter it is convenient to take the right bottom entry $c = \Delta^2 a - \Theta B$ satisfying the condition $a - c \in \Theta L^\infty(\Delta)$, where under the symbol $L^\infty(\Delta)$ we mean the space of all essentially bounded functions with respect to the measure $\Delta\, dm$. Then

$$Y = \begin{pmatrix} a & 0 \\ \Delta(a-c)\Theta^{-1} & c \end{pmatrix}, \tag{4.17.1}$$

where $a \in H^\infty$ and $c \in a + \Theta L^\infty(\Delta)$. Now the formula for the norm can be rewritten as

$$\|X\| = \|P_\Theta Y | \mathcal{K}_\Theta\| = \inf\left\{ \left\| \begin{pmatrix} a + \Theta\Gamma & 0 \\ \Delta(a-c)\Theta^{-1} + \Delta\Gamma & c \end{pmatrix} \right\|_\infty : \quad \Gamma \in H^\infty \right\}.$$

This expression can be estimated as follows:

$$\max\left\{ \operatorname{dist}\left(\frac{a - \Delta^2 c}{\Theta}, H^\infty\right),\, \|c\|_{L^\infty(\Delta)} \right\} \le \|X\| \le \operatorname{dist}\left(\frac{a-\Delta^2 c}{\Theta}, H^\infty\right) + 2\|c\|_{L^\infty(\Delta)}.$$

For functions of the model operator, that is, in the case where $c = a$, the estimate above takes the form

$$\max\{\operatorname{dist}(a\overline{\Theta}, H^\infty), \|a\|_{L^\infty(\Delta)}\} \le \|a(\mathcal{M}_\Theta)\| \le \operatorname{dist}(a\overline{\Theta}, H^\infty) + 2\|a\|_{L^\infty(\Delta)}.$$

For details, refer to [Nikolski and Khrushchev 1987].

4.18. The special case of a scalar inner characteristic function. For a scalar inner function Θ_T we have $|\Theta_T| = 1$ and $\Delta = 0$ a.e. on $\mathbb{T}$, and the Sz.-Nagy–Foiaş model reduces to

$$\mathcal{K}_\Theta = H^2 \ominus \Theta H^2,$$
$$\mathcal{M}_\Theta f = P_\Theta z f \quad \text{for } f \in \mathcal{K}_\Theta.$$

The commutant lifting theorem for this case was discovered by D. Sarason [1967] and served as a model for proving the general result [Sz.-Nagy and Foiaş 1968] presented in this chapter. The Sarason theorem says that whatever is $X \in \{\mathcal{M}_\Theta\}'$ there exists a function $Y = a \in H^\infty$ such that

$$X = P_\Theta a | \mathcal{K}_\Theta = a(\mathcal{M}_\Theta), \quad \|X\| = \|a\|_\infty.$$

In general, for $\varphi \in H^\infty$ we have

$$\|\varphi(\mathcal{M}_\Theta)\| = \operatorname{dist}_{L^\infty}(\varphi\bar{\Theta}, H^\infty) \le \|\varphi\|_\infty.$$

One can observe that this scalar version of the CLT says nothing but that the commutant $\{\mathcal{M}_\Theta\}'$ is reduced to functions of the model operator defined by the H^∞-calculus.

Now, we enter in some more details of the H^∞-calculus.

4.19. H^∞-calculus for completely nonunitary contractions. Roughly speaking, a calculus for an operator $T : H \to H$ is an algebra homomorphism extending the standard polynomial calculus $p \mapsto p(T)$. More precisely, let $\mathcal{A}$ be a topological function algebra containing the complex polynomials; an *$\mathcal{A}$-calculus* for a Hilbert space operator $T : H \to H$ is a continuous algebra homomorphism $f \mapsto f(T) \in L(H \to H)$, where $f \in \mathcal{A}$, such that $z^n(T) = T^n$ for $n \ge 0$.

The existence of unitary dilations for completely nonunitary contractions gives an easy possibility to define a rich functional calculus. Indeed, let $\mathcal{U}$ be the minimal unitary dilation of a completely nonunitary contraction $T : H \to H$. For any $f \in L^\infty$, we put

$$f(T) = P_H f(\mathcal{U}) | H, \tag{4.19.1}$$

where $f(\mathcal{U})$ is well-defined because of absolute continuity of $\mathcal{U}$. Observe that $z^n(T) = T^n$ for $n \ge 0$, and that

$$\|f(T)\| \le \|f\|_\infty \tag{4.19.2}$$

for every f. We note that the mapping $f \mapsto f(\mathcal{U})$ is a calculus, but $f \mapsto f(T)$ is not. However, the restriction of this mapping to H^∞ is a calculus, called *the H^∞- or Sz.-Nagy–Foiaş calculus* for the completely nonunitary contraction T.

One of the easiest ways to check the multiplicativity in (4.19.1) for H^∞-functions is to observe that, due to the von Neumann spectral theorem for $\mathcal{U}$ (see Section 0.7), we have

$$(f(T)x, y) = (f(\mathcal{U})x, y) = \int_{\mathbb{T}} f(\zeta)\,\big(x(\zeta), y(\zeta)\big)\, dm(\zeta)$$

for all $x, y \in H$ and $f \in L^\infty$, where $(x(\,\cdot\,), y(\,\cdot\,)) \in L^1$. Therefore, the mapping $f \mapsto f(T)$ is continuous with respect to the w^*-topology of L^∞ and the weak operator topology of $L(H \to H)$. Clearly, having $(pq)(T) = p(T)q(T)$ for all complex polynomials p, q, we get first $(pf)(T) = p(T)f(T)$ for $f \in H^\infty$ (for example, using the Fejér sums approximations for f), whence $(gf)(T) = g(T)f(T)$ for all pairs $f, g \in H^\infty$.

It can be proved that for a function f different from the zero function the equality $f(\mathcal{M}_\Theta) = 0$ occurs if and only if Θ is two-sided inner and $f \cdot I \in \Theta H^\infty(E)$.

Further, from the asserted property of w^*-continuity, it is clear that the H^∞ calculus is compatible with any other calculus, continuous in a stronger sense (for instance, with the classical Riesz–Dunford holomorphic calculus). However, for many purposes (e. g., for applications to free interpolation, or for similarity problems; see Part II) we need calculi for functions defined locally, on a kind of neighborhood of the spectrum (ideally, on the spectrum itself, as for normal operators) and not on the entire unit disc $\mathbb{D}$. Clearly, such a local calculus requires a stronger upper estimate for the norms $\|f(T)\|$ than is given by (4.19.2). We give an example of such an estimate in the next section.

4.20. Level curves estimate. In this section, following [Nikolski and Khrushchev 1987], we obtain an estimate of $\|f(T)\|$ for the case of a scalar inner characteristic function $\Theta = \Theta_T$. This estimate depends on the values of f on the level sets $L(\Theta, \varepsilon)$ of Θ. The latter are defined as follows:

$$L(\Theta, \varepsilon) \stackrel{\text{def}}{=} \{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\} \quad \text{for } 0 < \varepsilon < 1.$$

It is well known (and will be proved in Chapter 5) that the spectrum of the operator under consideration, $T \simeq \mathcal{M}_\Theta$, coincides with "zeros of Θ" in the sense that

$$\sigma(\mathcal{M}_\Theta) = \{\zeta \in \bar{\mathbb{D}} : \varliminf_{\substack{z \to \zeta \\ z \in \mathbb{D}}} |\Theta(z)| = 0\}.$$

Therefore, it is natural to consider the level sets $L(\Theta, \varepsilon)$ as "fine neighborhoods" of $\sigma(\mathcal{M}_\Theta)$ (it is clear that whatever is a neighborhood V of the spectrum, one has $L(\Theta, \varepsilon) \subset V$ for ε small enough). The following theorem [Nikolski and Khrushchev 1987] gives us an estimate of $\operatorname{dist}_{L^\infty}(f\bar{\Theta}, H^\infty)$ in terms of the smallness of f on the level curves $\partial L(\Theta, \varepsilon)$.

4.21. Theorem. *Let Θ be a scalar inner function and $0 < \varepsilon < 1$. There exists a constant $A = A(\varepsilon)$ such that*

$$\|f(\mathcal{M}_\Theta)\| \le A \cdot \sup\{|f(z)| : z \in L(\Theta, \varepsilon)\}, \quad f \in H^\infty.$$

Proof. The proof is based on the existence of so-called Carleson contours [Carleson 1962; Garnett 1981, Chapter 8, section 5]. There exists a constant $p \ge 1$ such that for every $\varepsilon \in (0, 1)$ and for every H^∞-function Θ, one can find a contour γ_ε splitting the disc $\mathbb{D}$ into two parts: one, call it Ω, is contained in $\mathbb{D} \setminus L(\Theta, \varepsilon^p)$, and the other (the complement of $\operatorname{clos}\Omega$) is contained in $L(\Theta, \varepsilon)$, and such that the arclength on γ_ε is a Carleson measure with embedding norm C, $H^1|\gamma_\varepsilon \subset L^1(\gamma_\varepsilon, |dz|)$, depending only on ε. (Recall, that a measure μ on $\mathbb{D}$ is called a Carleson measure if the restriction on $\operatorname{supp}\mu$ is a continuous embedding of H^1 in $L^2(\mu)$.)

Let $\Omega_r = \Omega \cap \{|z| < \varepsilon\}$; the boundary $\partial\Omega_r$ consists of a part $\gamma_{\varepsilon,r}$ of γ_ε and a part $\Gamma_r = \Omega_r \cap \mathbb{T}_r$ of the circle $\mathbb{T}_r = r\mathbb{T}$.

Let g be a polynomial. Then

$$\int_{\gamma_{\varepsilon,r}} \frac{fg}{\Theta}\, dz = -\int_{\Gamma_r} \frac{fg}{\Theta}\, dz,$$

and hence

$$\left|\frac{1}{2\pi i}\int_{\Gamma_r} \frac{fg}{\Theta}\, dz\right| \le \varepsilon^{-p} \sup_{\gamma_{\varepsilon,r}} |f| \cdot \frac{1}{2\pi}\int_{\gamma_{\varepsilon,r}} |g|\,|dz| \le \varepsilon^{-p} C \|g\|_1 \cdot \sup_{L(\Theta,\varepsilon)} |f|.$$

Let E_r be the radial projection of the set Γ_r on the circle $\mathbb{T}$. Then

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{fg}{\Theta}\, dz = \frac{r}{2\pi i}\int_{\mathbb{T}} \frac{f(r\zeta)g(r\zeta)}{\Theta}(r\zeta)\chi_{E_r}(\zeta)\, d\zeta.$$

It is clear that this integral tends to

$$\int_{\mathbb{T}} \frac{f}{\Theta} gz\, dm$$

as $r \to 1$. (Indeed, $|\Theta(r\zeta)| \ge \varepsilon^p$ for $\zeta \in E_r$ and $\chi_{E_r} \to 1$ a.e. on $\mathbb{T}$, because $\lim_{r\to 1} |\Theta(r\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$.) But, it is clear from duality arguments that

$$\operatorname{dist}_{L^\infty}(f\bar{\Theta}, H^\infty) = \sup\left\{\left|\int_{\mathbb{T}} f\bar{\Theta} gz\, dm\right| : \|g\|_1 \le 1\right\},$$

and we get the required inequality with $A = \varepsilon^{-p}C$. □

4.22. Local functional calculi. The estimates we have obtained actually allow us to extend considerably the functional calculus (4.19.1) from the algebra H^∞ to the algebra $H^\infty\big(L(\Theta,\varepsilon)\big)$ of bounded holomorphic functions on a level set $L(\Theta,\varepsilon)$, $\varepsilon > 0$. This is done in the following theorem [Nikolski and Khrushchev 1987].

4.23. Theorem. *Let Θ be a scalar inner function and $0 < \varepsilon < 1$. For every $f \in H^\infty\big(L(\Theta,\varepsilon)\big)$ there exists a function $\varphi \in H^\infty$ such that $f - \varphi \in \Theta H^\infty\big(L(\Theta,\varepsilon)\big)$ and $\|\varphi\|_\infty \le A\|f\|_{H^\infty(L(\Theta,\varepsilon))}$ (with the same constant A as in Theorem 4.21). The mappping*

$$f \mapsto \varphi(\mathcal{M}_\Theta) \overset{\text{def}}{=} [f](\mathcal{M}_\Theta) \quad \text{for } f \in H^\infty\big(L(\Theta,\varepsilon)\big),$$

is a well-defined calculus. Its kernel consists (precisely) of the functions of the form Θh, $h \in H^\infty\big(L(\Theta,\varepsilon)\big)$. For the functions in H^∞ this calculus coincides with the Sz.-Nagy–Foiaş one, $[f](\mathcal{M}_\Theta) = f(\mathcal{M}_\Theta)$, and, considered on $\bigcup_{\varepsilon>0} H^\infty\big(L(\Theta,\varepsilon)\big)$, it contains also the Riesz–Dunford calculus.

For the proof, we need the following remarkable lemma by L. Carleson [1962].

4.24. Lemma. *Let B be a finite Blaschke product with simple zeros $\lambda_1, \dots, \lambda_n$, and let f be a function from $H^\infty\big(L(B,\varepsilon)\big)$, where $\varepsilon > 0$. Then there exists a function $\varphi \in H^\infty$ such that $\varphi(\lambda_i) = f(\lambda_i)$ for $1 \le i \le n$ and such that $\|\varphi\|_\infty \le C\varepsilon^{-p}\|f\|_{H^\infty(L(B,\varepsilon))}$, where C is an absolute constant and p is the exponent in the definition of Carleson contours.*

Sketch of proof. (See [Carleson 1962] for the full version.) The general form of H^∞-functions φ interpolating $f(\lambda_i)$ is given by $\varphi = \varphi_0 + Bh$, with $h \in H^\infty$, so that

$$\begin{aligned}\inf\{\|\varphi\|_\infty : \varphi(\lambda_i) = f(\lambda_i) \text{ for } 1 \le i \le n\}& \\ &= \sup\Big\{\Big|\frac{1}{2\pi i}\int_{\mathbb{T}} \bar{B}\varphi_0 g\,dz\Big| : g \in H^1 \text{ with } \|g\|_1 \le 1\Big\} \\ &= \sup\Big\{\Big|\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{\varphi_0 g}{B}dz\Big| : g \in H^1 \text{ with } \|g\|_1 \le 1\Big\} \\ &= \sup\Big\{\Big|\frac{1}{2\pi i}\int_{\gamma_\varepsilon} \frac{fg}{B}dz\Big| : g \in H^1 \text{ with } \|g\|_1 \le 1\Big\} \\ &\le C\varepsilon^{-p}\|f\|_{H^\infty(L(B,\varepsilon))}. \qquad \Box\end{aligned}$$

4.25. Proof of Theorem 4.23. The existence of the requsted function φ can be derived from Lemma 4.24 in a standard way by using the fact that the constants C and p are independent of the function B. First, Θ is assumed to be a Blaschke product B with simple zeros. We approximate B by its partial products B_n (so that, $|B| \le |B_n|$ in $\mathbb{D}$). Then we pass to the limit in the equalities $f - \varphi_n = B_n h_n$ (which follow from Lemma 4.24) using the compactness principle.

Next, an arbitrary inner function Θ can be uniformly approximated by Blaschke products with simple zeros—for instance, by its "Frostman shifts" $B = (\Theta - \lambda_n)(1 - \bar{\lambda}_n\Theta)^{-1}$, where the λ_n converge to zero and $\{z : \Theta(z) = \lambda_n,\ \Theta'(z) = 0\} = \varnothing$ for each n; see, for example, [Nikolski 1986, Chapter 2, Section 5]. Repeated application of the compactness principle proves the existence of the requested φ.

Moreover, if $\psi \in H^\infty$ is another function corresponding to the same f, then $(\varphi - \psi)/\Theta$ is analytic and bounded on $L(\Theta, \varepsilon)$ by the definition of φ and ψ, and on $\mathbb{D} \setminus L(\Theta,\varepsilon)$ by the definition of $L(\Theta,\varepsilon)$. Hence, $(\varphi - \psi)/\Theta \in H^\infty$ and $\varphi(\mathcal{M}_\Theta) = \psi(\mathcal{M}_\Theta)$. Thus, the mapping $f \mapsto \varphi(\mathcal{M}_\Theta) = [f](\mathcal{M}_\Theta)$ is well-defined and bounded.

The multiplicativity of $f \mapsto [f](\mathcal{M}_\Theta)$ and other properties are obvious.

4.26. Explicit formula. In fact, one can prove [Nikolski and Khrushchev 1987] the following explicit formula for the local calculus of Theorem 4.23:

$$[f](\mathcal{M}_\Theta)x = \frac{\Theta}{2\pi i}\int_{\gamma_\varepsilon} \frac{f(\zeta)x(\zeta)}{\Theta(\zeta)(\zeta - z)}\,d\zeta \quad \text{for } x \in \mathcal{K}_\Theta,$$

where γ_ε is the same contour as in the proof of Theorem 4.21.

The spectral mapping theorem for the H^∞-calculus and for our local calculi will be proved in Chapter 5.

Chapter 5. Spectrum and Resolvent

Now we use the description of the commutant to obtain a formula for the resolvent of a given contraction in its model representation. The regular points of a contraction T will be characterized by the existence and some analytic continuation properties of the inverse $\Theta(z)^{-1}$ of the characteristic function $\Theta = \Theta_T$.

The notation is the same as in the previous chapters, namely, $T : H \to H$ is a completely nonunitary contraction, $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ its minimal unitary dilation, π and π_* the corresponding functional embeddings, $\Theta = \Theta_T$ the characteristic function of T, etc.

5.1. Theorem. *A point λ with $|\lambda| < 1$ belongs to the spectrum of T if and only if $\Theta(\lambda)$ is not invertible. Moreover, if $\Theta(\lambda)$ is invertible, the resolvent $R_\lambda \overset{\text{def}}{=} (T - \lambda I)^{-1}$ of T at the point λ commutes with T and has a lifting Y with the following parameters*:

$$A = \frac{I - \Theta(\lambda)^{-1}\Theta}{z - \lambda}, \tag{5.1.1}$$

$$A_* = \frac{I - \Theta\Theta(\lambda)^{-1}}{z - \lambda}, \tag{5.1.2}$$

$$B = -\frac{\Theta^* + \Delta\Theta(\lambda)^{-1}\Delta_*}{z - \lambda}. \tag{5.1.3}$$

The resolvent can be written in the form

$$R_\lambda = P_H(\mathcal{U} - \lambda I)^{-1}(I - \pi\Theta(\lambda)^{-1}\pi_*^*)|H. \tag{5.1.4}$$

Proof. Assume that λ is a regular point of the contraction T, so there exists an operator R_λ such that

$$R_\lambda(T - \lambda I) = (T - \lambda I)R_\lambda = I.$$

Since $R_\lambda \in \{T\}'$, Theorems 4.3 and 4.4 imply that $R_\lambda = P_H Y|H$, where Y is a lifting of R_λ determined by certain parameters A, A_*, B (see formula (4.4.2)). The operator $T - \lambda I$ also belongs to the commutant of T and has a lifting with the parameters $A = (z-\lambda)I_E$, $A_* = (z-\lambda)I_{E_*}$, and $B = -(z-\lambda)\Theta^*$. Therefore, by the multiplication theorem (Theorem 4.13), the formulas

$$\begin{aligned}
&A_0 = I - A(z - \lambda), \qquad A_{*0} = I - A_*(z - \lambda),\\
&B_0 = -\Theta^* - (\Delta A\Theta^*(z-\lambda)\Delta_* + B\Delta_*(z-\lambda)\Delta_* - \Delta A\Delta\Theta^*(z-\lambda) + B\Theta\Theta^*(z-\lambda))\\
&\quad = -\Theta^* - B(z - \lambda)
\end{aligned}$$

provide us with the parameters of a lifting of the zero operator $I - R_\lambda(T-\lambda)$. Hence, by Lemma 4.11, there exists a function $\Gamma \in H^\infty(E_* \to E)$ such that

$$I - A(z-\lambda) = \Gamma\Theta, \tag{5.1.5}$$

$$I - A_*(z-\lambda) = \Theta\Gamma, \tag{5.1.6}$$

$$-\Theta^* - B(z-\lambda) = \Delta\Gamma\Delta_*. \tag{5.1.7}$$

Evaluating (5.1.5) and (5.1.6) at the point λ, we conclude that the operator $\Theta(\lambda)$ is invertible and $\Gamma(\lambda) = \Theta(\lambda)^{-1}$.

Conversely, assuming that $\Theta(\lambda)$ is invertible, we prove the existence of the resolvent at the point λ. To this end we need to choose an analytic function Γ such that equations (5.1.5)–(5.1.7) are solvable with respect to A, A_*, B. The simplest choice is the constant function Γ: $\Gamma(z) = \Theta(\lambda)^{-1}$, $|z| < 1$. In fact, clearly, any function $\Gamma \in H^\infty(E_* \to E)$ satisfying $\Gamma(\lambda) = \Theta(\lambda)^{-1}$ gives a desired solution

$$A = \frac{I - \Gamma\Theta}{z-\lambda} \in H^\infty(E \to E),$$
$$A_* = \frac{I - \Theta\Gamma}{z-\lambda} \in H^\infty(E_* \to E_*),$$
$$B = -\frac{\Theta^* + \Delta\Gamma\Delta_*}{z-\lambda} \in L^\infty(\Delta_* E_* \to \Delta E).$$

To obtain formula (5.1.4) we plug the parameters (5.1.1)–(5.1.3) into formula (4.4.2), that is, we consider the following lifting of R_λ:

$$\begin{aligned} Y &= \pi_* \frac{I - \Theta\Theta(\lambda)^{-1}}{z-\lambda}\pi_*^* + \tau\Delta\frac{I - \Theta(\lambda)^{-1}\Theta}{z-\lambda}\pi^* - \tau\,\frac{\Theta^* + \Delta\Theta(\lambda)^{-1}\Delta_*}{z-\lambda}\tau_*^* \\ &= (\mathcal{U} - \lambda I)^{-1}(\pi_*\pi_*^* + \tau(\Delta\pi^* - \Theta^*\tau_*^*) - \pi_*\Theta\Theta(\lambda)^{-1}\pi_*^* - \tau\Delta\Theta(\lambda)^{-1}(\Theta\pi^* + \Delta_*\tau_*^*)) \\ &= (\mathcal{U} - \lambda I)^{-1}(\pi_*\pi_*^* + \tau\tau^* - (\pi_*\Theta + \tau\Delta)\Theta(\lambda)^{-1}\pi_*^*) \\ &= (\mathcal{U} - \lambda I)^{-1}(I - \pi\Theta(\lambda)^{-1}\pi_*^*). \end{aligned}$$

This yields

$$R_\lambda = P_H(\mathcal{U} - \lambda I)^{-1}(I - \pi\Theta(\lambda)^{-1}\pi_*^*)|H,$$

as claimed. □

5.2. One-sided spectrum. Actually, formula (5.1.4) contains more information than is asserted in Theorem 5.1. Indeed, if $\Theta(\lambda)^{-1}$ means only a left or a right inverse to $\Theta(\lambda)$, denoted in what follows by $[\Theta(\lambda)]_l^{-1}$ and $[\Theta(\lambda)]_r^{-1}$, respectively, then (5.1.4) yields a left or a right inverse to $T - \lambda I$, respectively. More precisely:

5.3. Theorem. *For* $|\lambda| < 1$, *the operator* $T - \lambda I$ *has a left inverse if and only if* $\Theta(\lambda)$ *is left invertible, and* $T - \lambda I$ *has a right inverse if and only if* $\Theta(\lambda)$ *is right* *invertible. In both cases the corresponding one-sided inverses can be expressed by the same formulas*:

$$[\Theta(\lambda)]_{l,r}^{-1} = \big(-T^* - D_T[T-\lambda I]_{l,r}^{-1} D_{T^*}\big)|\mathcal{D}_{T^*}, \tag{5.3.1}$$

$$[T-\lambda I]_{l,r}^{-1} = P_H(\mathcal{U}-\lambda I)^{-1}(I-\pi[\Theta(\lambda)]_{l,r}^{-1}\pi_*^*)|H. \tag{5.3.2}$$

Proof. For example, assume that $\Theta(\lambda)$ is left invertible. Then, denoting by $\mathcal{R}_{l,r}$ the right-hand side of (5.3.2), we have

$$\begin{aligned}\mathcal{R}_l(T-\lambda I) &= P_H(\mathcal{U}-\lambda I)^{-1}(I-\pi[\Theta(\lambda)]_l^{-1}\pi_*^*)P_H(\mathcal{U}-\lambda I)|H\\ &= P_H(I-\pi[\Theta(\lambda)]_l^{-1}\pi_*^*)|H\\ &\qquad - P_H(\mathcal{U}-\lambda I)^{-1}(I-\pi[\Theta(\lambda)]_l^{-1}\pi_*^*)P_G(\mathcal{U}-\lambda I)|H\\ &\qquad - P_H(\mathcal{U}-\lambda I)^{-1}(I-\pi[\Theta(\lambda)]_l^{-1}\pi_*^*)P_{G_*}(\mathcal{U}-\lambda I)|H.\end{aligned}$$

We compute the three terms of the above sum separately. The first one is the identity operator. Indeed,

$$\pi[\Theta(\lambda)]_l^{-1}\pi_*^* H \subset \pi[\Theta(\lambda)]_l^{-1} H^2(E_*) \subset \pi H^2(E) = G \perp H. \tag{5.3.3}$$

The two other summands are zero operators, because

$$\begin{aligned}(\mathcal{U}-\lambda I)^{-1}(I-\pi[\Theta(\lambda)]_l^{-1}\pi_*^*)G &= (\mathcal{U}-\lambda I)^{-1}\pi(I-[\Theta(\lambda)]_l^{-1}\Theta)H^2(E)\\ &= \pi\frac{I-[\Theta(\lambda)]_l^{-1}\Theta}{z-\lambda}H^2(E) \subset \pi H^2(E) = G\perp H\end{aligned}$$

and

$$(\mathcal{U}-\lambda I)H \subset G\oplus H \perp G_*.$$

So $\mathcal{R}_l(T-\lambda I) = I$, as desired.

Conversely, let $T-\lambda I$ be left invertible, and let $[T-\lambda I]_l^{-1}$ be a left inverse to $T-\lambda I$ (this inverse is not necessarily equal to (5.3.2)). We check that the operator

$$\Lambda = \big(-T^* - D_T[T-\lambda I]_l^{-1}D_{T^*}\big)|\mathcal{D}_{T^*}, \tag{5.3.4}$$

is a left inverse to $\Theta(\lambda)$.

Since the operator (5.3.4) acts from $\mathcal{D}_{T^*}$ to $\mathcal{D}_T$, we can use formula (1.13.1) for the characteristic function Θ_T. This yields

$$\begin{aligned}\Lambda\Theta_T(\lambda) &= \big(-T^* - D_T[T-\lambda I]_l^{-1}D_{T^*}\big)\big(-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T\big)\\ &= I - D_T^2 - \lambda T^* D_{T^*}(I-\lambda T^*)^{-1}D_T + D_T[T-\lambda I]_l^{-1}D_{T^*}T\\ &\qquad - \lambda D_T[T-\lambda I]_l^{-1}D_{T^*}^2(I-\lambda T^*)^{-1}D_T\\ &= I - D_T[T-\lambda I]_l^{-1}\big\{(T-\lambda I)(I-\lambda T^*) + (T-\lambda I)\lambda T^*\\ &\qquad - T(I-\lambda T^*) + \lambda(I-T^*T)\big\}(I-\lambda T^*)^{-1}D_T\\ &= I,\end{aligned}$$

whence Λ is a left inverse to $\Theta_T(\lambda)$. ☐

5.4. Resolvent and characteristic function off the unit disc. First, we observe that formula (1.13.1) allows us to define $\Theta(z)$ when $|z|>1$ and $1/z\notin\sigma(T^*)$, by the same expression:

$$\Theta_T(z)\overset{\text{def}}{=}\big(-T+zD_{T^*}(I-zT^*)^{-1}D_T\big)|\mathcal{D}_T\quad\text{for } |z|>1. \tag{5.4.1}$$

Also, it is clear that

$$\Theta_T(z)^*=\Theta_{T^*}(\bar z), \tag{5.4.2}$$

for $|z|<1$ as well as for $|z|>1$. Comparing these formulas with (5.3.1) for $|\lambda|<1$, $\lambda\notin\sigma(T)$, we get

$$\Theta_T(\lambda)^{-1}=\Theta_T(\lambda^*)^*=\Theta_{T^*}(1/\lambda), \tag{5.4.3}$$

where $\lambda^*=1/\bar\lambda$.

As for the resolvent, if $|\lambda|>1$, then, clearly,

$$(T-\lambda I)^{-1}=-\frac{1}{\lambda}\sum_{n\ge 0}\Big(\frac{T}{\lambda}\Big)^n=-\frac{1}{\lambda}P_H\sum_{n\ge 0}\Big(\frac{\mathcal{U}}{\lambda}\Big)^n|H=P_H(\mathcal{U}-\lambda I)^{-1}|H, \tag{5.4.4}$$

which formally coincides with (5.1.4). Indeed, taking (5.4.3) into account, as in (5.3.3) we have

$$\pi\Theta(\lambda^*)^*\pi_*^*H\subset\pi\Theta(\lambda^*)^*H^2(E_*)\subset\pi H^2(E)=G,$$

and $(\mathcal{U}-\lambda I)^{-1}G\subset G$, whence $P_H(\mathcal{U}-\lambda I)^{-1}\pi\Theta(\lambda^*)^*\pi_*^*)|H=0$ and

$$P_H(\mathcal{U}-\lambda I)^{-1}(I-\pi\Theta(\lambda^*)^*\pi_*^*)|H=P_H(\mathcal{U}-\lambda I)^{-1}|H.$$

It is worth mentioning that (5.4.1) and (1.13.1), formally coinciding, can give two different analytic functions. For example, let $\Theta=\frac{1}{2}(1-z)$. According to (5.4.2) this is the characteristic function of a contraction unitarily equivalent to its adjoint; that is, the adjoint operator has the same characteristic function. However, if we substitute this expression in (5.4.3), we obtain

$$\Theta(\lambda)^{-1}=\frac{2}{1-\lambda}\ne\frac{\lambda-1}{2\lambda}=\Theta\Big(\frac{1}{\lambda}\Big).$$

But for $|z|>1$ formula (5.4.1) gives us another expression, namely, $\Theta_T(z)=2z/(z-1)$. Inversion of this expression as the left-hand term in (5.4.3) gives a correct equality.

In what follows we shall see that (1.13.1) and (5.4.1) define the same analytic function if there exists a regular point of T on the unit circle.

5.5. Boundary spectrum. Now we consider the spectrum on the unit circle $\mathbb{T}$. Formulas (5.3.2) and (5.4.4) cannot be directly extended to $|\lambda| = 1$, because the entire unit circle is the spectrum of $\mathcal{U}$. However, rewriting these formulas in an appropriate form, we can describe the boundary spectrum $\sigma(T) \cap \mathbb{T}$ in terms of analytic continuation of Θ_T across the boundary.

A new local form of the resolvent is as follows.

5.6. Lemma. *Let $x \in H$. Then*

$$(T - \lambda I)^{-1}x = \begin{cases} (\mathcal{U} - \lambda I)^{-1}\big(x - \pi\Theta(\lambda)^{-1}(\pi_*^* x)(\lambda)\big) & \text{if } |\lambda| < 1,\ \lambda \notin \sigma(T), \\ (\mathcal{U} - \lambda I)^{-1}\big(x - \pi(\pi^* x)(\lambda)\big) & \text{if } |\lambda| > 1. \end{cases} \tag{5.6.1}$$

Proof. For $|\lambda| < 1$, the inclusion

$$(\mathcal{U} - \lambda I)^{-1}(I - \pi\Theta(\lambda)^{-1}\pi_*^*)H \subset H \oplus G$$

can be verified by straightforward computation; also, it follows from the definition of lifting (Definition 4.2), because the operator on the left in the above inclusion is a lifting of $(T - \lambda I)^{-1}$. Therefore, for $|\lambda| < 1$ we have

$$\begin{aligned} R_\lambda x &= (I - P_G)(\mathcal{U} - \lambda I)^{-1}\big(x - \pi\Theta(\lambda)^{-1}\pi_*^* x\big) \\ &= (\mathcal{U} - \lambda I)^{-1}\big(x - \pi\Theta(\lambda)^{-1}\pi_*^* x\big) - \pi P_+ \frac{\pi^* x - \Theta(\lambda)^{-1}\pi_*^* x}{z - \lambda} \\ &= (\mathcal{U} - \lambda I)^{-1}\big(x - \pi\Theta(\lambda)^{-1}\pi_*^* x\big) + \pi\Theta(\lambda)^{-1}\frac{\pi_*^* x - (\pi_*^* x)(\lambda)}{z - \lambda} \\ &= (\mathcal{U} - \lambda I)^{-1}\big(x - \pi\Theta(\lambda)^{-1}(\pi_*^* x)(\lambda)\big). \end{aligned}$$

Similarly, for $|\lambda| > 1$,

$$\begin{aligned} R_\lambda x &= (I - P_G)(\mathcal{U} - \lambda I)^{-1}x = (\mathcal{U} - \lambda I)^{-1}x - \pi P_+ \frac{\pi^* x}{z - \lambda} \\ &= (\mathcal{U} - \lambda I)^{-1}x + \pi\,\frac{(\pi^* x)(\lambda)}{z - \lambda} = (\mathcal{U} - \lambda I)^{-1}\big(x - \pi(\pi^* x)(\lambda)\big). \qquad \square \end{aligned}$$

To show that formulas (5.6.1) can be extended to regular points of the operator T on the unit circle we need the following property.

5.7. Lemma. *Assume that there exists a neighbourhood $\mathcal{O}_\lambda$ of a point $\lambda \in \Theta$ such that Θ admits analytic continuation to $\mathcal{O}_\lambda$ and that there exists a left* (*right*) *inverse $[\Theta]_l^{-1}$* (*respectively*, *$[\Theta]_r^{-1}$*), *coinciding with Θ^* on $\mathcal{O}_\lambda \cap \mathbb{T}$. Then for every vector $x \in H$ the function $\pi^* x$* (*respectively*, *$\pi_*^* x$*) *admits analytic continuation to $\mathcal{O}_\lambda$.*

Proof. The hypotheses of the lemma imply $\Delta(\zeta) = 0$ on $\mathcal{O}_\lambda \cap \mathbb{T}$. Therefore, since $\pi = \pi_*\Theta + \tau\Delta$, on $\mathcal{O}_\lambda$ we have

$$\begin{aligned} (\pi^* x)(\zeta) &= \Theta(\zeta)^*(\pi_*^* x)(\zeta) + \Delta(\zeta)(\tau^* x)(\zeta) = \Theta(\zeta)^*(\pi_*^* x)(\zeta) \\ &= [\Theta(\zeta)]_l^{-1}(\pi_*^* x)(\zeta), \end{aligned} \tag{5.7.1}$$

and the right-hand side in (5.7.1) gives an analytic continuation of $\pi^* x$ to $\mathcal{O}_\lambda$. If $[\Theta(\lambda)]_r^{-1}$ is a right inverse to $\Theta(\lambda)$, then $\Delta_*^2(\zeta) = I - \Theta(\zeta)\Theta(\zeta)^* = I - \Theta(\zeta)[\Theta(\zeta)]_r^{-1} = 0$ on $\mathcal{O}_\lambda \cap \mathbb{T}$, that is,

$$(\pi_*^* x)(\zeta) = \Theta(\zeta)(\pi^* x)(\zeta) + \Delta_*(\zeta)(\tau_*^* x)(\zeta) = \Theta(\zeta)(\pi^* x)(\zeta), \tag{5.7.2}$$

and again the right-hand side gives an analytic continuation of $\pi_*^* x$ to $\mathcal{O}_\lambda$. □

5.8. Theorem. *A point λ, $|\lambda| = 1$, belongs to the resolvent set of the operator T if and only if the characteristic function $\Theta = \Theta_T$ admits analytic continuation to a neighbourhood $\mathcal{O}_\lambda$ of the point λ, and $\Theta(\zeta)^{-1} = \Theta(\zeta)^*$ for $\zeta \in \mathcal{O}_\lambda \cap \mathbb{T}$.*

Proof. If λ with $|\lambda| = 1$ is a regular point for T, then $\bar{\lambda}$ is a regular point for T^*, and formula (1.13.1) determines an analytic continuation to a neighbourhood of λ; moreover, identity (5.4.3) turns into $\Theta(\zeta)^{-1} = \Theta(\zeta)^*$ for $\zeta \in \mathcal{O}_\lambda \cap \mathbb{T}$.

Conversely, suppose that Θ is analytic in a neighbourhood of a point λ and that $\Theta(\zeta)^{-1} = \Theta(\zeta)^*$ for $\zeta \in \mathcal{O}_\lambda \cap \mathbb{T}$. Then, as before, each expression on the right in (5.6.1) is the resolvent at the point λ. Indeed, for every $x \in H$ evaluation at the point λ is well defined by Lemma 5.7, and by (5.7.1)–(5.7.2) we have $\Theta(\lambda)(\pi^* x)(\lambda) = (\pi_*^* x)(\lambda)$, that is, the two expressions on the right in (5.6.1) coincide. We shall check that $y \stackrel{\text{def}}{=} x - \pi(\pi^* x)(\lambda) \in (\mathcal{U} - \lambda I)H$; this will imply that (5.6.1) is well-defined at such a point λ and, after analytic continuation, gives us a formula for the resolvent at that point.

We have

$$\pi^* y = \pi^* x - (\pi^* x)(\lambda) \in (z - \lambda)H_-^2(E).$$

Since $\Delta_*(\zeta) = 0$ for $\zeta \in \mathcal{O}_\lambda \cap \mathbb{T}$, the operator of multiplication by $1/(z - \lambda)$ is bounded on $L^2(\Delta_* E_*)$, whence

$$y = (\mathcal{U} - \lambda I)y', \quad \text{where } y' = \pi \frac{\pi^* x - (\pi^* x)(\lambda)}{z - \lambda} + \tau_* \frac{\tau_*^* y}{z - \lambda},$$

and

$$\pi^* y' = \frac{\pi^* x - (\pi^* x)(\lambda)}{z - \lambda} \in H_-^2(E),$$

$$\pi_*^* y' = \Theta \frac{\pi^* y}{z - \lambda} + \Delta_* \frac{\tau_*^* y}{z - \lambda} = \frac{\pi_*^* y}{z - \lambda} = \frac{\pi_*^* x - \Theta\Theta(\lambda)^{-1}(\pi_*^* x)(\lambda)}{z - \lambda} \in H^2(E_*).$$

Consequently, $y' \in H$ and $y \in (\mathcal{U} - \lambda I)H$. □

Now we consider the discrete spectrum of T and the corresponding eigen- and root-subspaces.

5.9. Theorem. *A point λ is an eigenvalue of T if and only if $|\lambda| < 1$ and* $\operatorname{Ker}\Theta(\lambda) \neq \{0\}$. *Moreover,*

$$\operatorname{Ker}(T - \lambda I) = \pi \frac{\operatorname{Ker}\Theta(\lambda)}{z - \lambda}.$$

Proof. Let $x \in H$. Then $x \in \operatorname{Ker}(T - \lambda I)$ if and only if $(\mathcal{U} - \lambda I)x \in G$, that is, there exists a function $e \in H^2(E)$ such that $(\mathcal{U} - \lambda I)x = \pi e$, whence

$$(z-\lambda)\pi^* x = e,$$
$$(z-\lambda)\pi_*^* x = \Theta e.$$

Since $\pi^* x \in H^2_-(E)$, the function e is a constant vector in E. And since $\pi_*^* x \in H^2(E)$, putting $z = \lambda$ in the second of the above relations, we get $e \in \operatorname{Ker}\Theta(\lambda)$. All arguments can be reversed; that is, if $e \in \operatorname{Ker}\Theta(\lambda)$, then the vector $x = \pi e/(z-\lambda)$ is in H and, moreover, $x \in \operatorname{Ker}(T - \lambda I)$. □

As for the root subspaces of T, they will be described in the next chapter, after we introduce the notion of regular factorization.

5.10. The spectral mapping theorem. Now, we prove a spectral mapping theorem for the H^∞ functional calculus in the case of a scalar characteristic function. Moreover, we compute the spectrum of an arbitrary operator in the commutant (in terms of the corresponding functional parameters of its lifting, see Section 4.17 above). To this end, given a function $\Theta \in H^\infty$, let us define the Θ-*range of a function* $f \in H^\infty$ by the formula

$$\operatorname{Range}_\Theta f = \bigl\{\lambda \in \mathbb{C} : \inf_{z\in\mathbb{D}} (|f(z) - \lambda| + |\Theta(z)|) = 0\bigr\}.$$

Also, we recall the notion of the *essential range* of a measurable function g on a measure space:

$$\operatorname{Range}(g) = \{\lambda \in \mathbb{C} : \operatorname{essinf} |g - \lambda| = 0\}.$$

5.11. Theorem. *Let $\Theta \not\equiv 0$ be a scalar characteristic function. Take $X \in \{\mathcal{M}_\Theta\}'$ and let Y be the lifting of X written in the form* (4.17.1):

$$Y = \begin{pmatrix} a & 0 \\ \Delta(a-c)\Theta^{-1} & c \end{pmatrix}, \tag{5.11.1}$$

where $a \in H^\infty$ and the entries on the bottom row lie in $L^\infty(\Delta)$, with $\Delta^2 = 1 - |\Theta|^2$ and $L^\infty(\Delta)$ the space of all essentially bounded functions with respect to the measure $\Delta\, dm$ (m being Lebesgue measure on $\mathbb{T}$). Then

$$\sigma(X) = (\operatorname{Range}_\Theta a) \cup \bigl(\operatorname{Range}(c)\bigr),$$

where $\operatorname{Range}(c)$ *is taken with respect to the measure $\Delta\, dm$.*

Proof. Essentially, we repeat the scheme of the proof of Theorem 5.1. If X is invertible, then $X^{-1} \in \{\mathcal{M}_\Theta\}'$, and, by the commutant lifting theorem, $X^{-1} = P_\Theta Y'$, where $P_\Theta = P_{\mathcal{K}_\Theta}$ and the matrix Y' is of the form (5.11.1) with $a' \in H^\infty$ and $c' \in L^\infty(\Delta)$. By direct computation (or by using the Multiplication Theorem 4.13) we see that the matrix $I - YY'$ is of the form (5.11.1) with the parameters aa' and cc' instead of a and c, respectively. Since $I - YY'$ is a lifting of the zero operator, by Theorem 4.10 we have

$$I - YY' = \begin{pmatrix} \Theta\Gamma & 0 \\ \Delta\Gamma & 0 \end{pmatrix}, \tag{5.11.2}$$

for some $\Gamma \in H^\infty$. Thus, we have $1 - aa' = \Theta\Gamma$ and $cc' = 1$ a.e. with respect to $\Delta\, dm$. Hence, $0 \notin \mathrm{Range}_\Theta \cup \mathrm{Range}(c)$.

Conversely, if the equations $aa' + \Theta\Gamma = 1$ and $cc' = 1$ are solvable with a', $\Gamma \in H^\infty$, and $c' \in L^\infty(\Delta)$, the matrix

$$Y' = \begin{pmatrix} a' & 0 \\ \Delta(a' - c')/\Theta & c' \end{pmatrix} \tag{5.11.3}$$

determines a lifting of the operator inverse to X. First we have to check that Y' is a lifting of an operator in $\{\mathcal{M}_\Theta\}'$. To this end, we need to verify that $a' - c' \in \Theta L^\infty(\Delta)$ (see Section 4.17). Since

$$ac(a' - c') = (1 - \Theta\Gamma)c - a = (c - a) - \Theta\Gamma \in \Theta L^\infty(\Delta),$$

we need to check that the functions a and c are bounded away from zero on a subset of Θ where $\mathbb{T}$ is small, say, $|\Theta(\zeta)| < \frac{1}{2}\|\Gamma\|_\infty$. For such a ζ we have

$$|a| \geq \frac{|aa'|}{\|a'\|_\infty} = \frac{|1 - \Theta\Gamma|}{\|a'\|_\infty} \geq \frac{1 - |\Theta| \cdot \|\Gamma\|}{\|a'\|_\infty} \geq \frac{1}{2\|a'\|_\infty},$$
$$|c| \geq \frac{1}{\|c'\|_{L^\infty(\Delta)}},$$

whence $a' - c' \in \Theta L^\infty(\Delta)$ and (5.11.3) defines a lifting of an operator. To check that this operator is inverse of X, we multiply the matrices (5.11.1) and (5.11.3), obtaining (5.11.2).

To conclude we recall that, by the Carleson corona theorem (see [Garnett 1981] or [Nikolski 1986], for example), the existence of the needed solutions a' and Γ is equivalent to $\inf_{\mathbb{D}}(|a| + |\Theta|) > 0$. ☐

5.12. The spectrum of $\mathcal{M}_\Theta$. Clearly, Theorem 5.11 contains also the well-known formula for the spectrum of the model operator itself:

$$\sigma(\mathcal{M}_\Theta) = \{\lambda \in \overline{\mathbb{D}} : \varliminf_{\substack{z \in \mathbb{D} \\ z \to \lambda}} |\Theta(z)| = 0\} \cup \mathrm{supp}(\Delta) = \mathrm{supp}(\mu_\Theta),$$

μ_Θ being the representing measure of Θ in the Nevanlinna–Riesz–Smirnov sense (for the definition see Section 0.7).

This should be compared with the general case described in Theorems 5.1 and 5.8.

5.13. Corollary. *For $X = P_\Theta Y|\mathcal{K}_\Theta \in \{\mathcal{M}_\Theta\}'$, we have*

$$\|X\| \geq \sup\bigl\{|\lambda| : \lambda \in \sigma(X)\bigr\} = \max\bigl(\|c\|_{L^\infty(\Delta)}, \max\{|\lambda| : \lambda \in \mathrm{Range}_\Theta\, a\}\bigr).$$

An upper estimate of the same type as Corollary 5.13 (that is, in terms of the values of the functions a and c on the spectrum $\sigma(\mathcal{M}_\Theta)$) does not seem possible. Recall, that in Section 4.21, in the case $a = c = f$, we already presented an upper estimate depending on the values $a(\zeta)$ in some narrow "neighborhood" $L(\Theta, \varepsilon)$ of the spectrum $\sigma(\mathcal{M}_\Theta)$.

Chapter 6. Invariant Subspaces

In this chapter we describe the lattice of invariant subspaces of a completely nonunitary contraction in terms of certain functional embeddings and the characteristic function. In fact, the description that we are going to discuss is a direct consequence of the classical Wold–Kolmogorov formula, as is the case for the invariant subspaces of unitary operators. We recall the following well-known description of the lattice $\operatorname{Lat}\mathcal{U}$ of a unitary operator.

6.1. Lemma. *Let $\mathcal{U}$ be a unitary operator on a Hilbert space $\mathcal{H}$, and let $L \in \operatorname{Lat}\mathcal{U}$. Then*

$$L = R \oplus \eta H^2(F),$$

where $F = L \ominus \mathcal{U}L$, $R = \bigcap_{n\geq 0} \mathcal{U}^n L$, and $\eta : L^2(F) \to \mathcal{H}$ is a "functional embedding" intertwining z and $\mathcal{U}$; that is, $\eta z = \mathcal{U}\eta$. Moreover, R is a reducing subspace of $\mathcal{U}$.

Proof. The Wold–Kolmogorov lemma implies that $L = R \oplus \left(\sum_n \geq 0 \oplus \mathcal{U}^n F\right)$; of course, this is equivalent to the assertion claimed, with the natural definition of η, namely,

$$\eta\left(\sum_{k\in\mathbb{Z}} a_k e^{ikt}\right) = \sum_{k\in\mathbb{Z}} \mathcal{U}^k a_k, \quad \text{with } a_k \in F. \qquad \square$$

Also, it is worth recalling that reducing subspaces are well-behaved with respect to the spectral decomposition of $\mathcal{U}$. In particular, the description of such subspaces becomes explicit when one uses the von Neumann spectral theorem: L is a reducing subspace of a unitary operator $\mathcal{U}$ if and only if $L = \{f \in L^2(\mathcal{H}, \mu_{\mathcal{U}}) : f(\zeta) \in P_L(\zeta)\mathcal{H}\ \mu_{\mathcal{U}}\text{-a.e.}\}$, where P_L stands for a projection-valued function on $\mathbb{T}$ subordinate to $E_{\mathcal{U}}$, that is, $P_L(\zeta) \leq E_{\mathcal{U}}(\zeta)\ \mu_{\mathcal{U}}$-a.e.

Now, turning to the invariant subspaces of completely nonunitary contractions, we can benefit from the coordinate-free function model approach, because using the language of this approach we need only follow precisely the same lines as in Lemma 6.1. In order to make this point even more transparent, we start with an outline of the description of $\operatorname{Lat} T$, paralleling the steps of this description with the proof of the leading partial case, Beurling's theorem [1949].

6.2 The description of the invariant subspaces: the first draft. The steps of the proof, which will be formalized later, will be listed in a table. On the left-hand side, we deal with a completely nonunitary contraction T and its function model $\{H, T, \mathcal{H}, \mathcal{U}, \pi, \pi_*, \ldots\}$, and on the right-hand side we have the standard shift operator S on the (scalar) Hardy space H^2, given by $Sf = zf$ for $f \in H^2$.

1. $T: H \to H$ a completely nonunitary contraction; $\mathcal{U}: \mathcal{H} \to \mathcal{H}$ the minimal unitary dilation; π, π_* the functional embeddings; $\Theta_T = \pi_*^* \pi$	1. $S: H^2 \to H^2$, $Sf = fz$ for $f \in H^2$; $\mathcal{H} = L^2(\mathbb{T})$, $\mathcal{U}f = zf$ for $f \in L^2$; $E = \{0\}$, $\pi = 0$; $E_* = \mathbb{C}$, $G_* = L^2$, $\pi_* = I$; $\Theta_T = 0_{1\times 0}: \mathbb{C} \to \{0\}$
2. Let $L \in \operatorname{Lat} T$. Then $L \oplus G \in \operatorname{Lat} \mathcal{U}$ (Lemma 6.3). Take F: $\dim F = \dim\big((L \oplus G) \ominus \mathcal{U}(L \oplus G)\big)$; $\eta: L^2(F) \to \mathcal{H}$, $\eta F = \big((L \oplus G) \ominus \mathcal{U}(L \oplus G)\big)$, $\eta z = \mathcal{U} \eta$	2. Let $L \in \operatorname{Lat} S$. Let $F = L \ominus SL$, $\dim F = 1$; $\eta: L^2 \to L^2$ as in Lemma 6.1, $\eta z = \mathcal{U} \eta$
3. Apply the Wold–Kolmogorov lemma: $L \oplus G = \mathcal{R} \oplus \eta H^2(F)$	3. $L = \mathcal{R} \oplus \eta H^2$, $\mathcal{R} \subset H^2 \implies \mathcal{R} = \{0\}$
4. The mapping $\eta^* \pi: L^2(E) \to L^2(F)$ commutes with z and is analytic, because $\eta^* \pi H^2(E) \subset H^2(F)$ (Section 6.5)	4. $\pi = 0$, hence $\eta^* \pi = 0$
5. The same for $\pi_*^* \eta: L^2(F) \to L^2(E_*)$	5. $\pi_*^* \eta = \eta: L^2 \to L^2$
6. Put $\Theta_1 = \eta^* \pi \in H^\infty(E \to F)$, $\Theta_2 = \pi_*^* \eta \in H^\infty(F \to E_*)$, $\Theta_T = \pi_*^* \pi = \pi_*^* (\eta \eta^*) \pi = \Theta_2 \Theta_1$ (cf. (6.5.3)), $L = \operatorname{span}\big(\pi L^2(E), \eta L^2(F)\big) \ominus$ $\big(\pi H^2(E) \oplus \eta H^2_-(F)\big)$ (cf. (6.7.1))	6. $\Theta_1 = 0_{1\times 0}: \mathbb{C} \to \{0\}$, $\Theta_2 = \eta \in H^\infty$, $\Theta_T = \Theta_2 \Theta_1 = 0_{1\times 0}: \mathbb{C} \to \{0\}$, $L = \Theta_2 H^2$

Now we check the proposed program step by step and describe the factorizations $\Theta_T = \Theta_2 \Theta_1$ of the characteristic function appearing in this way.

6.3. Lemma. $L \in \operatorname{Lat} T$ *if and only if* $L \oplus G \in \operatorname{Lat} \mathcal{U}$.

Proof. Let $L \in \operatorname{Lat} T$. Since $G \in \operatorname{Lat} \mathcal{U}$, it suffices to check that $\mathcal{U}L \subset L \oplus G$. The inclusions $\mathcal{U}L \subset \mathcal{U}(H \oplus G) \subset H \oplus G$ imply that

$$\mathcal{U}L \subset P_H \mathcal{U}L \oplus P_G \mathcal{U}L \subset TL \oplus G \subset L \oplus G.$$

The converse is also obvious: $TL = P_H \mathcal{U}L \subset P_H(L \oplus G) = L$. □

6.4. An additional functional embedding arises if the completely nonunitary (pure) part of the isometry $\mathcal{U}|L \oplus G$ is realized as the multiplication by z in a suitable H^2-space. More precisely, we can take an auxiliary Hilbert space F such that

$$\dim F = \dim\bigl((L \oplus G) \ominus \mathcal{U}(L \oplus G)\bigr)$$

and an isometry $\eta_L : L^2(F) \to \mathcal{H}$ with the following properties:

η_L intertwines $\mathcal{U}$ and z, that is, $\eta_L z = \mathcal{U}\eta_L$;
$\eta_L H^2(F)$ is the subspace of $L \oplus G$ where the pure part of $\mathcal{U}|(L \oplus G)$ acts.

Therefore, the reducing part of $\mathcal{U}|(L \oplus G)$ acts on the subspace

$$\mathcal{R}_L \stackrel{\text{def}}{=} (L \oplus G) \ominus \eta_L H^2(F); \tag{6.4.1}$$

in other words, the formula

$$L \oplus G = \mathcal{R}_L \oplus \eta_L H^2(F) \tag{6.4.2}$$

provides the Wold–Kolmogorov decomposition for the isometry $\mathcal{U}|(L \oplus G)$. Since $\mathcal{R}_L$ is reducing, it is orthogonal not only to $\eta_L H^2(F)$, but to the whole of $\eta_L L^2(F)$, that is, $\eta_L^* \mathcal{R}_L = \{0\}$.

In what follows we omit the index L and write simply η if we have no need to emphasize the fact that this embedding is generated by an invariant subspace L.

6.5. Properties of the additional embedding. It is clear that the operator

$$\Theta_1 \stackrel{\text{def}}{=} \eta^* \pi \tag{6.5.1}$$

intertwines the multiplications by z in $L^2(E)$ and $L^2(F)$, that is, $\Theta_1 \in L^\infty(E \to F)$ (as usual, we identify an operator of multiplication with the corresponding function). Moreover, the function Θ_1 is analytic, that is, $\Theta_1 \in H^\infty(E \to F)$, because

$$\Theta_1 H^2(E) = \eta^* \pi H^2(E) = \eta^* G \subset \eta^*(\eta H^2(F) \oplus \mathcal{R}_L) = H^2(F).$$

Similarly, the operator

$$\Theta_2 \stackrel{\text{def}}{=} \pi_*^* \eta \tag{6.5.2}$$

is also (an operator of multiplication by) a bounded analytic function:

$$\Theta_2^* H^2_-(E_*) = \eta^* \pi_* H^2_-(E^*) = \eta^* G_* \subset H^2_-(F),$$

because $G_* \perp H \oplus G \supset L \oplus G \supset \eta H^2(F)$.

As the third property of η we mention the identity

$$\pi_*^*(I - \eta\eta^*)\pi = 0. \tag{6.5.3}$$

For the proof, note that $\pi L^2(E) \subset \eta L^2(F) \oplus \mathcal{R}_L$, that is, $(I - \eta\eta^*)\pi L^2(E) \subset \mathcal{R}_L$. It suffices to show that $\pi_*^* \mathcal{R}_L = \{0\}$. Since $\mathcal{R}_L$ reduces $\mathcal{U}$, the relation $\mathcal{R}_L \perp G_*$ implies that $\mathcal{R}_L \perp \pi_* L^2(E_*)$, that is, $\pi_*^* \mathcal{R}_L = \{0\}$.

6.6. Definition. An isometry η from $L^2(F)$ to $\mathcal{H} = \operatorname{span}\{\pi L^2(E), \pi_* L^2(E_*)\}$ is said to be *compatible with* π *and* π_* if the following four conditions are fulfilled:

(a) $\eta z = \mathcal{U}\eta$ (that is, η is a $\mathcal{U}$-functional embedding);
(b) $\pi H^2(E) \perp \eta H^2_-(F)$;
(c) $\eta H^2(F) \perp \pi_* H^2_-(E_*)$;
(d) $\pi_*^*(I - \eta\eta^*)\pi = 0$.

In these terms, in Section 6.5 we proved that any T-invariant subspace L generates an embedding $\eta = \eta_L$ compatible with π and π_*.

6.7. Theorem. *Let T be a completely nonunitary contraction, π and π_* its functional embeddings. The mapping $L \mapsto \eta_L$ is a bijection of* $\operatorname{Lat} T$ *onto the set of isometries compatible with π and π_*. Moreover, the inverse mapping $\eta \mapsto L_\eta$ can be defined by the formula*

$$L_\eta = \operatorname{span}\{\pi L^2(E), \eta L^2(F)\} \ominus \big(\pi H^2(E) \oplus \eta H^2_-(F)\big). \tag{6.7.1}$$

Proof. First, we check that $L_\eta \in \operatorname{Lat} T$ for any isometry η compatible with π and π_*. To this end, we rewrite (6.7.1) in the form

$$L_\eta = \big(\eta H^2(F) \oplus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E)\big) \ominus \pi H^2(E). \tag{6.7.2}$$

Clearly, this representation implies that L_η is T-invariant. Indeed, since

$$\mathcal{U}L_\eta \subset \eta H^2(F) \oplus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E),$$

after projecting onto H we find ourselves in the orthogonal complement of $G = \pi H^2(E)$, that is, in L_η.

Now we check that $L = L_{\eta_L}$ for every invariant subspace L. Comparing (6.7.2) with the definition of the subspace $\mathcal{R}_L$ in (6.4.1), we see that we must verify the formula

$$\mathcal{R}_L = \operatorname{clos}(I - \eta\eta^*)\pi L^2(E),$$

where $\eta = \eta_L$. The inclusion $\operatorname{clos}(I - \eta\eta^*)\pi L^2(E) \subset \mathcal{R}_L$ was already proved in Section 6.5. Now we consider the $\mathcal{U}$-reducing subspace $\mathcal{R}_L \ominus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E)$ and verify that it is contained in L. If this is done, the complete nonunitarity of T will imply that the above subspace is, in fact, the zero subspace.

Let $x \in \mathcal{R}_L \ominus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E)$. Since $x \in \mathcal{R}_L$, we have $\eta^* x = 0$. The property $x \perp (I - \eta\eta^*)\pi L^2(E)$ means $\pi^*(I - \eta\eta^*)x = 0$, so $\pi^* x = \pi^*\eta\eta^* x = 0$. Therefore, $x \perp G$; since $x \in L \oplus G$, this is equivalent to $x \in L$.

If we start with a compatible embedding η and construct the corresponding invariant subspace L_η as in (6.7.2), then, as is easily seen, the additional embedding constructed by L_η coincides with (is equivalent to) the initial η; that is, $\eta_{L_\eta} = \eta$. Indeed, by (6.7.2),

$$L_\eta \oplus G = \eta H^2(F) \oplus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E).$$

This representation is the Wold–Kolmogorov decomposition (6.4.2) with $\eta_L = \eta$. □

6.8. Regular factorizations. Using definitions (6.5.1) and (6.5.2) we can rewrite identity (6.5.3) in the form

$$\Theta = \Theta_2 \Theta_1. \tag{6.8.1}$$

This means that any compatible embedding η induces a factorization of the characteristic function into two contractive H^∞-factors. Among all possible contractive H^∞-factorizations of Θ, the factorizations induced by compatible embeddings are distinguished by the particular property of being *regular factorizations*. We can define regular factorization as a factorization (6.8.1) generated by a compatible embedding η by formulas (6.5.1) and (6.5.2). However, it is important that there exists another description of such factorizations that does not use the notion of functional embedding and, moreover, can be applied to a factorization of an arbitrary contraction.

We start by restating the invariant subspace theorem (Theorem 6.7) in terms dating back to Sz.-Nagy and Foiaş [1967]; then we present the above-mentioned characterization of regular factorizations (Section 6.10). To distinguish the definition of regularity via an additional embedding and the definition given in Section 6.10 we shall call the latter one sometimes a *locally regular factorization*. After some preparation (Lemmas 6.11–6.14) we shall be able to prove (Theorem 6.15) that these two definitions of regularity coincide.

6.9. Theorem. *Let T be a completely nonunitary contraction, and let π, π_*, $\Theta = \Theta_T$, etc., have the usual meanings. There exists a bijection between the lattice* $\operatorname{Lat} T$ *and the set of all regular factorizations*

$$\Theta = \Theta_2 \Theta_1$$

of the characteristic function Θ. *This bijection is given by the formula*

$$L_{\Theta_2,\Theta_1} = \big(\eta H^2(F) \oplus \operatorname{clos}(I - \eta\eta^*)\pi L^2(E)\big) \ominus \pi H^2(E), \tag{6.9.1}$$

where η is the compatible embedding producing the functions Θ_i via (6.5.1) *and* (6.5.2). *In the original Sz.-Nagy–Foiaş transcription the representation given by* (6.9.1) *takes the form*

$$L_{\Theta_2,\Theta_1} = \begin{pmatrix} \Theta_2 \\ Z_2^* \Delta_2 \end{pmatrix} H^2(F) \oplus \begin{pmatrix} 0 \\ Z_1^* \end{pmatrix} L^2(\Delta_1 E) \ominus \begin{pmatrix} \Theta \\ \Delta \end{pmatrix} H^2(E),$$

where the Z_i and Δ_i are certain operator-valued functions to be defined in the next section.

Proof. The bijection is the same as in Theorem 6.7, namely, $\eta \leftrightarrow L_\eta$. The formula in the Sz.-Nagy–Foiaş transcription follows from expression (6.13.1) for η combined with the definition of τ (1.19.1) and formulas (3.5.1) for the embeddings π and π_* in the Sz.-Nagy–Foiaş representation. □

Now we introduce the language for the local description of the regular factorizations.

6.10. Definition. Let A and B be two Hilbert space contractions such that A acts into the space where B is defined, that is, the product BA is well defined. Let $\mathcal{D}_A$, $\mathcal{D}_B$, and $\mathcal{D}_{BA}$ be the defect spaces of these operators. We define an operator

$$Z : \mathcal{D}_{BA} \to \begin{pmatrix} \mathcal{D}_A \\ \mathcal{D}_B \end{pmatrix}$$

by the identity

$$ZD_{BA} \stackrel{\text{def}}{=} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} D_{BA} = \begin{pmatrix} D_A \\ D_B A \end{pmatrix}. \tag{6.10.1}$$

The factorization BA is called *regular* if Z is unitary.

For a factorization $\Theta = \Theta_2\Theta_1$, the definition of Z becomes

$$Z\Delta = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2\Theta_1 \end{pmatrix} \tag{6.10.2}$$

where $\Delta_i = (I - \Theta_i^*\Theta_i)^{1/2}$, and all functions are regarded as multiplication operators on the corresponding spaces. It should be mentioned at once that a function $Z \in L^\infty(E_1 \to E_2)$ is unitary as an operator from $L^2(E_1)$ to $L^2(E_2)$ if and only if the operators $Z(\zeta) : E_1 \to E_2$ are unitary for almost all $\zeta \in \mathbb{T}$. This implies that *a factorization $\Theta = \Theta_2\Theta_1$ is regular if and only if the factorizations $\Theta(\zeta) = \Theta_2(\zeta)\Theta_1(\zeta)$ are regular for almost all $\zeta \in \mathbb{T}$.*

The above observation motivates the use of the adverb *locally* when talking about regular function factorizations $\Theta = \Theta_2\Theta_1$. In what follows, we shall often use the term *locally regular factorization* instead of "regular factorization", emphasizing the fact that we deal with functions; also, this will allow us to distinguish (temporarily) between this new wording and the previous one defined in terms of compatible embeddings.

Our next observation is that Z in (6.10.1) is *always* an isometry.

6.11. Lemma. *The operator Z defined by* (6.10.1) (*and*, *therefore*, *by* (6.10.2)) *is an isometry.*

Proof. It suffices to check that Z is norm-preserving on a dense set of vectors. By the definition of $\mathcal{D}_{BA}$, the set of vectors $\{D_{BA}x : x \in H\}$ is dense there. On this dense set we have

$$\begin{aligned} \|ZD_{BA}x\|^2 &= \|D_A x\|^2 + \|D_B Ax\|^2 \\ &= \|x\|^2 - \|Ax\|^2 + \|Ax\|^2 - \|BAx\|^2 = \|D_{BA}x\|^2. \qquad \square \end{aligned}$$

Thus, to prove that Z is unitary it suffices to check that this operator is a co-isometry, that is, that Z_1 and Z_2 are co-isometries. Moreover, the following assertion is true.

6.12. Lemma. *Z is a co-isometry if and only if Z_2 is.*

Proof. If Z_2 is a co-isometry, then

$$\begin{aligned} 0 < &\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 - \left\| Z^* \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 \\ &= \|x_1\|^2 + \|x_2\|^2 - \|Z_1^* x_1\|^2 - 2\operatorname{Re}(Z_1^* x_1, Z_2^* x_2) - \|Z_2^* x_2\|^2 \\ &= \|x_1\|^2 - \|Z_1^* x_1\|^2 - 2\operatorname{Re}(Z_1^* x_1, Z_2^* x_2), \end{aligned}$$

which is possible if and only if $Z_1 Z_2^* = 0$. Hence,

$$\begin{pmatrix} 0 \\ \mathcal{D}_B \end{pmatrix} = \begin{pmatrix} 0 \\ \operatorname{Range} Z_2 \end{pmatrix} = \operatorname{Range} Z Z_2^* Z_2 \subset \operatorname{Range} Z,$$

whence

$$\operatorname{Range} Z = \begin{pmatrix} \operatorname{clos} \operatorname{Range} Z_1 \\ \mathcal{D}_B \end{pmatrix} = \begin{pmatrix} \mathcal{D}_A \\ \mathcal{D}_B \end{pmatrix}.$$

Thus, Z is unitary. □

6.13. Lemma. *Let η be an embedding compatible with π and π_*, and let $\Theta = \Theta_2\Theta_1$ be the corresponding factorization of the characteristic function (that is, Θ_1 is defined by* (6.5.1) *and Θ_2 by* (6.5.2)*). Then*

$$\eta = \pi_* \Theta_2 + \tau Z_2^* \Delta_2, \tag{6.13.1}$$

where Z_2 is defined by (6.10.2) *and $\Delta_2 \stackrel{\text{def}}{=} (I - \Theta_2^* \Theta_2)^{1/2}$, or symmetrically,*

$$\eta = \pi \Theta_1^* + \tau_* Z_{*1}^* \Delta_{*1}, \tag{6.13.2}$$

where $\Delta_{*1} \stackrel{\text{def}}{=} (I - \Theta_1 \Theta_1^*)^{1/2}$, and Z_{*1} is the component of the operator Z_* corresponding to the factorization $\Theta^* = \Theta_1^* \Theta_2^*$ in accordance with (6.10.1):

$$Z_* \Delta_* \stackrel{\text{def}}{=} \begin{pmatrix} Z_{*2} \\ Z_{*1} \end{pmatrix} \Delta_* = \begin{pmatrix} \Delta_{*2} \\ \Delta_{*1} \Theta_2^* \end{pmatrix}. \tag{6.13.3}$$

Proof. Since

$$\eta = (\pi_* \pi_*^* + \tau \tau^*) \eta = \pi_* \Theta_2 + \tau(\tau^* \eta),$$

relation (6.13.1) will be proved if we check that $\tau^* \eta = Z_2^* \Delta_2$, or $\eta^* \tau = \Delta_2 Z_2$. We have

$$\eta^* \tau \Delta = \eta^* (\pi - \pi_* \Theta) = \Theta_1 - \Theta_2^* \Theta = \Delta_2^2 \Theta_1 = \Delta_2 Z_2 \Delta.$$

Applying $I = \pi\pi^* + \tau_* \tau_*^*$ to η, we obtain the symmetric formula (6.13.2). □

Now we prove that, conversely, a contractive H^∞-factorization $\Theta = \Theta_2\Theta_1$ (not necessarily locally regular) gives rise to a functional embedding η (which is not necessarily isometric).

6.14. Lemma. *Let $\Theta = \Theta_2\Theta_1$ be a contractive H^∞-factorization. Then the operators defined by* (6.13.1) *and* (6.13.2) *coincide and possess properties* a)–d) *of Definition* 6.6.

Proof. First, let us check that formulas (6.13.1) and (6.13.2) are equivalent. Assuming, for example, that (6.13.2) defines an operator η, we check that (6.13.1) is also true, with the same η. We have

$$\begin{aligned}\pi_*^*\eta &= \pi_*^*(\pi\Theta_1^* + \tau_* Z_{*1}^*\Delta_{*1}) = \Theta\Theta_1^* + \Delta_* Z_{*1}^*\Delta_{*1}\\ &= \Theta\Theta_1^* + \Theta_2\Delta_{*1}\Delta_{*1} = \Theta_2(\Theta_1\Theta_1^* + \Delta_{*1}^2) = \Theta_2.\end{aligned}$$

Here we have used definition (6.13.3). Now, using the formula (to be proved below)

$$Z_{*1}\Theta = \Theta_1 Z_1 - \Delta_{*1}\Delta_2 Z_2, \tag{6.14.1}$$

we get

$$\begin{aligned}\tau^*\eta &= \tau^*(\pi\Theta_1^* + \tau_* Z_{*1}^*\Delta_{*1}) = \Delta\Theta_1^* - \Theta^* Z_{*1}^*\Delta_{*1}\\ &= \Delta\Theta_1^* - (Z_1^*\Theta_1^* - Z_2^*\Delta_2\Delta_{*1})\Delta_{*1} = (I - Z_1^* Z_1)\Delta\Theta_1^* + Z_2^*\Delta_2\Delta_{*1}^2\\ &= Z_2^* Z_2\Delta\Theta_1^* + Z_2^*\Delta_2\Delta_{*1}^2 = Z_2^*\Delta_2(\Theta_1\Theta_1^* + \Delta_{*1}^2) = Z_2^*\Delta_2.\end{aligned}$$

Therefore,

$$\eta = (\pi_*\pi_*^* + \tau\tau^*)\eta = \pi_*\Theta_2 + \tau Z_2^*\Delta_2.$$

We complete the proof that (6.13.1) and (6.13.2) are equivalent by checking formula (6.14.1):

$$\begin{aligned}(\Theta_1 Z_1 - \Delta_{*1}\Delta_2 Z_2)\Delta &= \Theta_1\Delta_1 - \Delta_{*1}\Delta_2^2\Theta_1 = \Delta_{*1}(I - \Delta_2^2)\Theta_1\\ &= \Delta_{*1}\Theta_2^*\Theta_2\Theta_1 = Z_{*1}\Delta_*\Theta = Z_{*1}\Theta\Delta.\end{aligned}$$

Now, if a contractive factorization $\Theta = \Theta_2\Theta_1$ is given and if η is defined by (6.13.1)–(6.13.2), then, clearly, $\pi^*\eta = \Theta_1^*$ and $\pi_*^*\eta = \Theta_2$; that is, formulas (6.13.1)–(6.13.2) determine the bijection inverse to that given by (6.5.1)–(6.5.2). Property (a) is obvious; properties (b) and (c) of Definition 6.6 are equivalent to the analyticity of Θ_1 and Θ_2, respectively. Property d) means merely that $\Theta = \Theta_2\Theta_1$. □

So, in talking about a contractive H^∞-factorization of Θ, we deal with an embedding η satisfying conditions a)–d) of Definition 6.6. Thus, to get a compatible embedding we need only to check that η is an isometry. This property of η is equivalent to the property that the factorization $\Theta = \Theta_2\Theta_1$ is locally regular.

6.15. Theorem. *If $\Theta = \Theta_2\Theta_1$ is a locally regular factorization, formula* (6.13.1) (*or* (6.13.2)) *defines an embedding η compatible with π and π_*. Conversely, if η is a compatible embedding, then the factorization $\Theta = \Theta_1\Theta_2$ with the factors defined by* (6.5.1), (6.5.2) *is locally regular. Moreover, formulas* (6.5.1), (6.5.2), *and* (6.13.1) (*or* (6.13.2)) *determine a one-to-one correspondence between the set of all compatible embeddings and the set of all regular factorizations.*

PROOF. As already mentioned, after establishing Lemmas 6.13 and 6.14, it only remains to prove that η is an isometry if and only if the corresponding factorization is locally regular.

We have

$$\begin{aligned} I-\eta^*\eta &= I-(\Theta_2^*\pi_*^*+\Delta_2 Z_2\tau^*)(\pi_*\Theta_2+\tau Z_2^*\Delta_2) \\ &= I-\Theta_2^*\Theta_2-\Delta_2 Z_2 Z_2^*\Delta_2=\Delta_2(I-Z_2Z_2^*)\Delta_2. \end{aligned}$$

Hence, η is an isometry if and only if Z_2 is a co-isometry; by Lemma 6.12, the latter is equivalent to the fact that Z is a co-isometry. We conclude that η is an isometry if and only if $\Theta=\Theta_2\Theta_1$ is a locally regular factorization. □

The factors in a regular factorization of a characteristic function Θ_T are very deeply related with the parts of the operator T induced by the corresponding invariant subspace $L=L_{\Theta_2,\Theta_1}$, namely, with the restriction $T_1=T|L$ and the compression to the orthogonal complement $T_2=P_{L^\perp}T|L^\perp$.

6.16. Theorem. *Let T be a contraction on a Hilbert space, $\Theta=\Theta_T$ the characteristic function of T. If L is a T-invariant subspace and $\Theta=\Theta_2\Theta_1$ is the corresponding regular factorization of Θ, then the pure part of Θ_1 is the characteristic function of the operator $T_1=T|L$ and the pure part of Θ_2 is the characteristic function of the operator $T_2=P_{L^\perp}T|L^\perp$.*

OUTLINE OF THE PROOF. Let η be the additional functional embedding corresponding to the invariant subspace L (see Section 6.4). Putting

$$\begin{aligned} E_1 &\stackrel{\text{def}}{=} E\ominus\pi^*(\pi E\cap\eta F), \\ E_{*1} &\stackrel{\text{def}}{=} F\ominus\eta^*(\pi E\cap\eta F), \end{aligned}$$

we introduce two functional embeddings $\pi_1\stackrel{\text{def}}{=}\pi|L^2(E_1)$ and $\pi_{*1}\stackrel{\text{def}}{=}\eta|L^2(E_{*1})$. It is not very difficult to verify that π_1 and π_{*1} form the pair of "canonical" functional embeddings related with the operator T_1 and, therefore, the characteristic function Θ_{T_1} of T_1 is equal to $\pi_{*1}^*\pi_1$, however the latter operator is the pure part of Θ_1.

In a similar way, for T_2 we put

$$\begin{aligned} E_2 &\stackrel{\text{def}}{=} F\ominus\eta^*(\pi_*E_*\cap\eta F), \\ E_{*2} &\stackrel{\text{def}}{=} E_*\ominus\pi_*^*(\pi_*E_*\cap\eta F), \end{aligned}$$

and the corresponding embeddings are $\pi_2\stackrel{\text{def}}{=}\eta|L^2(E_2)$ and $\pi_{*2}\stackrel{\text{def}}{=}\pi_*|L^2(E_{*2})$. □

As an application of the above description of the invariant subspaces we find the factorizations corresponding to the root subspaces, in particular, to the eigenspaces, described in Theorem 5.10.

6.17. Theorem. *For every* $\lambda \in \mathbb{D}$ *we have*

$$\operatorname{Ker}(T-\lambda I)^n = \pi \sum_{k}^{n-1} = 0 \oplus \frac{\vartheta_k^* \operatorname{Ker}\Theta_k(\lambda)}{z-\lambda}, \tag{6.17.1}$$

where $\Theta_T = \Theta_k \vartheta_k$ *is the* (*locally*) *regular factorization corresponding to the invariant subspace* $\operatorname{Ker}(T-\lambda I)^k$, $0 \le k \le n-1$ (*for* $k=0$ *we take* $\Theta_0 = \Theta_T$, $\vartheta_0 = I$).

Outline of the proof. We work by induction on n. The case $n=1$ is contained in Theorem 5.10 with $\Theta_0 \stackrel{\text{def}}{=} \Theta$ and $\vartheta_0 \stackrel{\text{def}}{=} I$. If $H_1 \stackrel{\text{def}}{=} \operatorname{Ker}(T-\lambda I)$ and $P_1 \stackrel{\text{def}}{=} P_{\operatorname{Ker}\Theta(\lambda)}$, then

$$H_1 \oplus G = \pi \frac{\operatorname{Ker}\Theta(\lambda)}{z-\lambda} \oplus \pi H^2(E) = \pi\big(\bar{b}_\lambda H^2(P_1 E) \oplus H^2((I-P_1)E)\big).$$

Hence, operating as in steps 2 and 3 of Section 6.2, we can take $F=E$ and $\eta = \pi\big(\bar{b}_\lambda P_1 + (I-P_1)\big)$. In accordance with (6.5.1) the factor ϑ_1 occurring in the corresponding regular factorization $\Theta = \Theta_1\vartheta_1$ is as follows:

$$\vartheta_1 = \eta^*\pi = b_\lambda P_1 + (I-P_1).$$

In order to describe $\operatorname{Ker}(T-\lambda I)^2$ we consider the triangulation (that is, the block matrix form) of T with respect to the orthogonal decomposition $H = H_1^\perp \oplus H_1$. Putting $T_1 \stackrel{\text{def}}{=} (I-P_1)T|H_1^\perp$, we get $\operatorname{Ker}(T-\lambda I)^2 = \operatorname{Ker}(T_1-\lambda I) \oplus H_1$; now we can apply the same Theorem 5.10 to the operator T_1:

$$\operatorname{Ker}(T_1-\lambda I) = \pi_1 \frac{\operatorname{Ker}\Theta_1(\lambda)}{z-\lambda}.$$

The next observation is that, up to passage to the pure parts, the characteristic function of T_1 is the factor Θ_1, and the role of the canonical embedding π_1 is played by the embedding $\eta = \pi\vartheta_1^*$. Therefore, we have

$$\operatorname{Ker}(T_1-\lambda I) = \pi \frac{\vartheta_1^* \operatorname{Ker}\Theta_1(\lambda)}{z-\lambda}.$$

By induction, if the subspace $H_k = \operatorname{Ker}(T-\lambda I)^k$ and the corresponding factorization $\Theta = \Theta_k\vartheta_k$ has been described, then we can write

$$H_{k+1} \stackrel{\text{def}}{=} \operatorname{Ker}(T-\lambda I)^{k+1} = H_k \oplus \pi \frac{\vartheta_k^* \operatorname{Ker}\Theta_k(\lambda)}{z-\lambda},$$

and compute the next regular factor $\vartheta_k = (b_\lambda P_k + (I-P_k))$, where $P_k \stackrel{\text{def}}{=} P_{\operatorname{Ker}\Theta_{k-1}(\lambda)}$. This gives us a recursive procedure for computing the regular factors ϑ_k for $0 \le k \le n-1$, and, finally, to prove formula (6.17.1).

We omit some details of the proof, because they require more information on regular factorizations than we have prepared in this chapter. ☐

6.18. Example: Scalar characteristic function. We recall that this term refers to the case where $\dim E = \operatorname{rank}(I - T^*T) = 1$ and $\dim E_* = \operatorname{rank}(I - TT^*) = 1$; thus, we can make the identifications $E = E_* = \mathbb{C}$ and $H^\infty(\mathbb{C} \to \mathbb{C}) = H^\infty$.

Let $\Theta_1 \in H^\infty(\mathbb{C}^1 \to F)$ and $\Theta_2 \in H^\infty(F \to \mathbb{C}^1)$ be contractive functions. The definition of regular factorizations (Section 6.10) and Lemma 6.11 imply that in the finite-dimensional case a factorization $\Theta = \Theta_2\Theta_1$ is regular if and only if

$$\operatorname{rank}\Delta(\zeta) = \operatorname{rank}\Delta_1(\zeta) + \operatorname{rank}\Delta_2(\zeta) \quad \text{for a.e. } \zeta \in \mathbb{T}. \tag{6.18.1}$$

In this and only in this case the final space of the isometry $Z(\zeta)$ has the same dimension as its initial space, so that Z is unitary. Whence the condition $0 \le \dim F \le 2$ is necessary for the factorization $\Theta = \Theta_2\Theta_1$ to be regular. We consider these three possibilities separately.

$\underline{\dim F = 0}$.

This case can occur for the function $\Theta = 0 : \mathbb{C}^1 \to \mathbb{C}^1$ only. Indeed, if $\dim F = 0$, then $\Theta_1 = 0 : \mathbb{C}^1 \to \{0\}$, $\Theta_2 = 0 : \{0\} \to \mathbb{C}^1$ and $\Theta = \Theta_2\Theta_1 = 0$. The factorization

$$0_{1\times 1} = 0_{1\times 0}0_{0\times 1}.$$

is clearly regular, because here we have

$$\operatorname{rank}\Delta(\zeta) = \operatorname{rank}\Delta_1(\zeta) = 1, \quad \operatorname{rank}\Delta_2(\zeta) = 0.$$

The operator corresponding to this characteristic function is

$$\mathcal{M}_\Theta = S \oplus S^* \overset{\text{def}}{=} z|H^2 \oplus P_- z|H^2_-.$$

In this case, the underlying geometry is very transparent:

$$\mathcal{H} = \begin{pmatrix} L^2 \\ L^2 \end{pmatrix}, \ \mathcal{K}_\Theta = \begin{pmatrix} H^2 \\ H^2_- \end{pmatrix}, \ \pi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ G = \begin{pmatrix} 0 \\ H^2 \end{pmatrix}, \ \pi_* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ G_* = \begin{pmatrix} H^2_- \\ 0 \end{pmatrix}.$$

Since $F = \{0\}$, from (6.7.1) we have

$$L = \pi L^2 \ominus \pi H^2 = \pi H^2_- = \begin{pmatrix} 0 \\ H^2_- \end{pmatrix},$$

whence we see that $L = \{0\} \oplus H^2_-$ is the reducing subspace where the operator $S^* = P_- z|H^2_-$ acts.

$\underline{\dim F = 1}$.

Now Θ_1 and Θ_2 are ordinary scalar functions belonging to the unit ball of H^∞. The factorization $\Theta = \Theta_2\Theta_1$ is regular if and only if

$$\max\{|\Theta_1(\zeta)|, |\Theta_2(\zeta)|\} = 1 \quad \text{for a.e. } \zeta \in \mathbb{T}. \tag{6.18.2}$$

Indeed, since $\operatorname{rank}\Delta(\zeta) \le 1$, (6.18.1) implies that $\min\{\Delta_1(\zeta), \Delta_2(\zeta)\} = 0$, which is equivalent to (6.18.2). Conversely, if, for example, $|\Theta_1(\zeta)| = 1$, then

$|\Theta(\zeta)| = |\Theta_2(\zeta)|$, $\Delta(\zeta) = \Delta_2(\zeta)$, $\Delta_1(\zeta) = 0$, and (6.18.1) follows. Similarly for $|\Theta_2(\zeta)| = 1$.

Condition (6.18.2) means that the inner part of Θ can be arbitrarily factored in two inner factors, and the factorizations of the outer part are determined by the Borel subsets $\sigma \subset \mathbb{T}$ in such a way that

$$|\Theta_1(\zeta)| = \begin{cases} |\Theta(\zeta)| & \text{if } \zeta \in \sigma, \\ 1 & \text{if } \zeta \in \mathbb{T} \setminus \sigma, \end{cases} \qquad |\Theta_2(\zeta)| = \begin{cases} |\Theta(\zeta)| & \text{if } \zeta \in \mathbb{T} \setminus \sigma, \\ 1 & \text{if } \zeta \in \sigma. \end{cases}$$

Particular case: $\Theta = 0$. If $\Theta = 0$, then the only possibilities are $0 = 0 \cdot \vartheta$ or $0 = \vartheta \cdot 0$, with an inner ϑ. The corresponding invariant subspaces are as follows

$$0 = 0 \cdot \vartheta \Longleftrightarrow L = \begin{pmatrix} 0 \\ H^2_- \ominus \vartheta^* H^2_- \end{pmatrix}, \quad 0 = \vartheta \cdot 0 \Longleftrightarrow L = \begin{pmatrix} \vartheta H^2 \\ H^2_- \end{pmatrix}.$$

We recall that in this case

$$\mathcal{K}_\Theta = \begin{pmatrix} H^2 \\ H^2_- \end{pmatrix}, \quad \mathcal{M}_\Theta = S \oplus S^* = \begin{pmatrix} z \\ P_- z \end{pmatrix}.$$

$\dim F = 2$.

In this case the factorizations under study are of the form

$$\Theta = \Theta_2\Theta_1 = (f_1,\ f_2) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = f_1 g_1 + f_2 g_2.$$

Such a factorization is regular if and only if $|\Theta(\zeta)| < 1$ for a.e. $\zeta \in \mathbb{T}$, Θ_2 is co-inner, and Θ_1 is inner, that is, $|f_1(\zeta)|^2 + |f_2(\zeta)|^2 = 1$ and $|g_1(\zeta)|^2 + |g_2(\zeta)|^2 = 1$ for a.e. $\zeta \in \mathbb{T}$. Indeed, since $\operatorname{rank}\Theta_2(\zeta) \le \dim E_* = 1$, we have $\operatorname{rank}\Delta_2(\zeta) \ge \dim F - \operatorname{rank}\Theta_2(\zeta) \ge 1$. Hence, (6.18.1) implies that the case $\dim F = 2$ is possible only if $\operatorname{rank}\Delta(\zeta) \ge 1$, that is, we neccessarily have $|\Theta(\zeta)| < 1$ and $\operatorname{rank}\Delta(\zeta) = 1$ for a.e. ζ. Whence (6.18.1) is equivalent to the conditions $\operatorname{rank}\Delta_2(\zeta) = 1$ and $\operatorname{rank}\Delta_1(\zeta) = 0$, which means that Θ_2 is co-inner and Θ_1 is inner, correspondingly.

It should be noted that even the existence of factorizations of this sort are not completely obvious. However, they do exist for any Θ satisfying $|\Theta(\zeta)| < 1$ a.e.; moreover, there exist infinitely many such factorizations. To see that, we take any function φ on $\mathbb{T}$ whose values are ± 1 (that is, this is a function satisfying the condition $\varphi^2(\zeta) = 1$ a.e.). Then, assuming that $\Theta \not\equiv 0$, we take two outer functions f_1, f_2 such that

$$|f_1|^2 = \frac{1 - \varphi\Delta}{2}, \quad |f_2|^2 = \frac{1 + \varphi\Delta}{2}.$$

This is indeed possible, because

$$\log \frac{1 \pm \varphi\Delta}{2} \ge \log \frac{1 - \Delta}{2} \ge \log \frac{1 - \Delta^2}{4} = \log \frac{|\Theta|^2}{4} \in L^1.$$

Since

$$|f_1|^2 |f_2|^2 = \frac{1 - \varphi^2\Delta^2}{4} = \frac{1 - \Delta^2}{4} = \frac{|\Theta|^2}{4},$$

for the outer part of Θ we have $\Theta_{\text{out}} = 2f_1f_2$. Hence, taking $g_1 = f_2\Theta_{\text{inn}}$, $g_2 = f_1\Theta_{\text{inn}}$, we get a solution; here Θ_{inn} stands for the inner part of Θ.

The case $\Theta = 0$ is treated later on.

6.19. Problem. How can one describe all factorizations of the latter type, that is, the (1×2)–(2×1) factorizations of a scalar function Θ satisfying $|\Theta| < 1$ a.e.? We make some comments on this problem.

(a) Let a function Θ with $|\Theta(\zeta)| < 1$ a.e. be given. The problem consists in describing all regular factorizations of Θ with $\dim F = 2$ (that is, in completing the description of the invariant subspaces of a given operator with the characteristic function $\Theta_T = \Theta$). In detail, the problem is as follows:

Find all pairs of H^∞-functions f_1, f_2 and g_1, g_2 such that $f_1g_1 + f_2g_2 = \Theta$ and

$$|f_1(\zeta)|^2 + |f_2(\zeta)|^2 = |g_1(\zeta)|^2 + |g_2(\zeta)|^2 = 1 \quad \textit{for a.e. } \zeta \in \mathbb{T}.$$

(b) Let a (2×1)-inner function Θ_1 be given. The problem consists in the description of all Θ such that $\Theta_1 = \binom{g_1}{g_2}$ is a right regular factor of Θ (that is, in obtaining a description of the operators having a scalar characteristic function and an invariant subspace restriction that is unitarily equivalent to the given operator $\mathcal{M}_{\Theta_1}$). In detail:

Find all H^∞-functions Θ, $|\Theta(\zeta)| < 1$ a.e., for which there exists a pair of H^∞-functions f_1, f_2 such that $f_1g_1 + f_2g_2 = \Theta$ and $|f_1(\zeta)|^2 + |f_2(\zeta)|^2 = 1$ a.e.

A particular case: $\Theta = 0$. Again, let $\Theta = 0$. Then it can be shown that all regular factorizations of $\dim F = 2$ type admit the following parametrization: if $\vartheta = \binom{\alpha}{\beta}$ is an arbitrary (2×1)-function which is inner and co-outer (that is, the inner parts of α and β are coprime), and ϑ_1, ϑ_2 are two arbitrary scalar inner functions, then the representation

$$0 = (\vartheta_2\beta, -\vartheta_2\alpha)\begin{pmatrix}\vartheta_1\alpha\\ \vartheta_1\beta\end{pmatrix}$$

is a required factorization. All (1×2)–(2×1) regular factorizations of $\Theta = 0$ are of this type.

Moreover, for $\Theta = 0$ we can answer question (b) as well: every operator with a (2×1)-inner characteristic function can be realized as the restriction of the operator $S \oplus S^*$ to an invariant subspace.

The latter statement is not true for general functions Θ. For example, an operator with a nonzero scalar characteristic function possesses an invariant subspace where it acts as the unilateral shift S if and only if $\log\Delta \in L^1$. Indeed, to get such an invariant subspace we need a regular factorization of the form

$$\Theta = (f_1,\ f_2)\begin{pmatrix}1\\ 0\end{pmatrix},$$

that is, we need $f_1 = \Theta$ and $|f_2| = (1 - |f_1|^2)^{1/2} = \Delta$, $f_2 \in H^\infty$, and this is possible if and only if $\log \Delta \in L^1$.

6.20. Example: Indefinite integration operator. We now return to the operator

$$A : f \mapsto i \int_{[0,x\}} f(t)\, d\mu(t);$$

the corresponding characteristic function $\mathcal{S}_A$, computed in Theorem 2.6 is a scalar inner function. A description of the lattice of the invariant subspaces of the operator A was given by Leggett [1973]. We derive a description of this lattice as an immediate consequence of the description of invariant subspaces obtained in this chapter.

As is clear from Section 6.18, all regular factorizations of a scalar inner function are factorizations in two scalar inner factors. Assume that the continuous part μ_c of μ is nonzero. Then there exists a continuous nest of invariant subspaces corresponding to the factors

$$\mathcal{S}_\alpha = \exp\Big(-\frac{i\alpha}{\zeta}\Big) \quad \text{for } 0 \le \alpha \le \mu_c([0,1]).$$

They are the only invariant subspaces of A if and only if μ is a continuous measure ($\mu = \mu_c$). Clearly, in this case the lattice $\operatorname{Lat} A$ is completely ordered by inclusion, or, in other words, A is a unicellular operator.

If μ has a nontrivial discrete part, then A has isolated eigenvalues on the imaginary axis: $\lambda = \frac{i}{2}\mu(\{t\})$. For every such λ there exists a unique eigenvector, and the corresponding Jordan block has dimension $\operatorname{card}\{t : \frac{i}{2}\mu(\{t\}) = \lambda\}$.

If μ is a purely discrete measure ($\mu_c = 0$), then $\mathcal{S}_A$ is a Blaschke product, and hence every inner factorization of $\mathcal{S}_A$, $\mathcal{S}_A = \mathcal{S}_2\mathcal{S}_1$, is a factorization in two Blaschke products. Therefore, the characteristic function of the restriction $A|L$ to an invariant subspace L is always a Blaschke product. We recall that an operator with a scalar characteristic function Θ has a complete family of eigen- and root-vectors if and only if Θ is a Blaschke product (see [Nikolski 1986]). Thus, for the operator A we have the following spectral synthesis theorem: *if μ is a discrete measure, then every invariant subspace of A is generated by the eigen- and root-vectors it contains.*

If the measure μ has infinitely many point masses together with a nonzero continuous part μ_c, then the angle between any "continuous" invariant subspace and any infinite dimensional "discrete" subspace is equal to zero, because

$$\inf_{\operatorname{Im}\zeta > 0} (|\mathcal{S}_\alpha(\zeta)| + |B_\sigma(\zeta)|) = 0,$$

where

$$B_\sigma \overset{\text{def}}{=} \prod_{t\in\sigma} \frac{\zeta - \frac{i}{2}\mu(\{t\})}{\zeta + \frac{i}{2}\mu(\{t\})}.$$

Why this infimum determines the angle between the invariant subspaces will be explained in Chapter 7, in Part II of this article.

Afterword: Outline of Part II

The second part of this paper, "The Function Model in Action", will be published elsewhere. For the reader who expected to find applications of the function model here, now we describe briefly how the model works in spectral theory.

A.1. Angles between invariant subspaces, and operator Bezout equations. As is well known, angles between invariant subspaces are the key tool for studying all kinds of spectral decompositions. Recall that the *angle* $\alpha = \alpha(K, K')$ between two subspaces $K, K' \subset H$ of a Hilbert space H is, by definition, the number $\alpha \in [0, \pi/2]$ satisfying

$$\cos\alpha = \sup\{|(x,y)| : x \in K,\ y \in K',\ \|x\| = \|y\| = 1\}.$$

For instance, eigenvectors $(x_n)_{n\geq 1}$, or any other vectors of a Hilbert space, form a basis if and only if the angles between the subspaces $K_n = \operatorname{span}(x_k : 1 \leq k \leq n)$ and $K'_n = \operatorname{span}(x_k : k > n)$ are uniformly bounded away from zero. This is equivalent to saying that the *projections* $\mathcal{P}_n$ *on* K_n *parallel to* K'_n, defined by

$$\mathcal{P}_n(x + x') = x \quad \text{for } x \in K_n,\ x' \in K'_n,$$

are uniformly bounded: $\sup_n \|\mathcal{P}_n\| < \infty$. The same equivalence exists for unconditional bases, if we replace K_n by $K_\sigma = \operatorname{span}(x_k : k \in \sigma)$ and K'_n by $K'_\sigma = \operatorname{span}(x_k : k \notin \sigma)$, and finally $\mathcal{P}_n$ by $\mathcal{P}_\sigma$; the basis condition is

$$\sup\{\|\mathcal{P}_\sigma\| : \sigma \subset \mathbb{N}\} < \infty.$$

The same is valid for bases of subspaces, standard or unconditional. Taking spectral subspaces of a given operator (see below), we obtain a kind of spectral decomposition, and considering all families of spectral subspaces we arrive at Dunford spectral operators; see [Dunford and Schwartz 1971; Dowson 1978; Nikolski 1986] for details. This scheme explains why angles are important.

As was mentioned above, invariant subspaces of a contraction T correspond to regular factorizations of its characteristic function Θ_T, see Chapter 6 for details. Hence, to work with spectral decompositions in terms of the function model, we need to know which factorizations $\Theta_T = \Theta_2\Theta_1$ and $\Theta_T = \Theta'_2\Theta'_1$ correspond to invariant subspaces $TK \subset K$ and $TK' \subset K'$ having a positive angle between them. In some partial cases—for inner factorizations, for instance—this problem was solved by P. Fuhrmann [1981] and Teodorescu [1975]. A criterion that holds in full generality was proved in [Vasyunin 1994], using the CLT: the projection $\mathcal{P}$ is bounded if and only if *two* Bezout equations have bounded solutions: the "standard" one

$$W\Theta_1 + W'\Theta'_1 = I,$$

where W, W' are H^∞ operator-valued functions; and another "complementary" Bezout equation to be solved in L^∞ operator-valued functions. In Chapter 7 we will discuss this in detail; here we only mention that the obvious necessary condition, namely, the uniform local left invertibility condition

$$\|\Theta_1(z)x\| + \|\Theta_1'(z)x\| \geq \varepsilon\|x\| \quad \text{for all } z \in \mathbb{D} \text{ and all } x,$$

is, in general, not sufficient [Treil 1989].

A.2. Spectral subspaces and generalized free interpolation. The *free interpolation problem* of complex analysis, stated for a class X of functions holomorphic on a domain Ω, consists of a description of those subsets $\Lambda \subset \Omega$ for which the restriction space $X|\Lambda$ is free of traces of holomorphy, that is, is an ideal space of functions on Λ, in the sense that $a \in X|\Lambda$ and $|b(z)| \leq |a(z)|$, for $z \in \Lambda$, imply $b \in X|\Lambda$.

The interplay between this problem and operator theory goes back to the late sixties, when it was understood that the freedom property of a restriction space is adequate for the unconditional convergence of a spectral decomposition, or to the existence of a spectral measure [Nikolski and Pavlov 1968; 1970]. In fact, both of these properties are equivalent to the boundedness of the corresponding L^∞-calculus; for instance, it is clear that the space $X|\Lambda$ is an ideal if and only if every $\ell^\infty(\Lambda)$-function is a multiplier of $X|\Lambda$: $a \in X|\Lambda$ and $m \in \ell^\infty(\Lambda)$ imply $ma \in X|\Lambda$. The joint studies in interpolation theory and the function model have led to solutions of several concrete problems in both subjects, and to a considerable evolution of the very notion of interpolation, transforming it to what we call now generalized free interpolation. We present these results in Chapter 8, and now, to summarize them briefly, we start with the operator theory part of the problem, namely spectral subspaces.

Originally, spectral subspaces were introduced as a substitute for the spectral measure of a normal operator to serve more general classes of operators appearing in perturbation theory. Given a closed set $\sigma \subset \mathbb{C}$ and an operator T, the *spectral subspace* $K(\sigma)$ over σ can be defined as the set of x such that the local resolvent $\lambda \mapsto (\lambda I - T)^{-1}x$ admits an analytic extension to $\mathbb{C} \setminus \sigma$.

Another way to define such a subspace is to postulate its invariance and maximality properties: a subspace satisfying $TK(\sigma) \subset K(\sigma)$ and the corresponding spectrum inclusion $\sigma(T|K(\sigma)) \subset \sigma$, and such that $TE \subset E$ and $\sigma(T|E) \subset \sigma$ imply $E \subset K(\sigma)$.

Obviously, for normal operators, the subspaces $K(\sigma)$ coincide with the ranges of the spectral measure, $K(\sigma) = \operatorname{Range} E(\sigma)$, and therefore they exist for σ taken from the whole Borel σ-algebra $\mathfrak{B}$. However, for general operators, it is not clear how to define subspaces $K(\sigma)$ for σ from some more or less rich σ-algebra of subsets of the complex plane $\mathbb{C}$ (any complete substitute of the spectral measure would be defined, of course, on the entire algebra $\mathfrak{B}$). This obstruction limits the use and the significance of this notion to general operators, endowed with no additional structure.

At this stage, the function model shows some of its advantages: using the multiplicative structure of H^∞- and H^2-functions, we can explicitly define an analogue of spectral subspaces for every Borel set σ and for every contraction whose characteristic function allows a "scalar multiple". For example, for the simplest case of a scalar characteristic function $\Theta_T \not\equiv 0$, when $I - T^*T$ and $I - TT^*$ both have rank 1, one can show, first, that the spectral subspaces $K(\sigma)$ over *closed* subsets σ coincide with invariant subspaces corresponding to factorizations

$$\Theta = \Theta_{\sigma'}\Theta_\sigma,$$

where Θ_σ and $\Theta_{\sigma'}$ stand for the *parts of* Θ_T *over* σ *and* $\sigma' \stackrel{\text{def}}{=} \mathbb{C} \setminus \sigma$, respectively. To define Θ_σ, we use the canonical *Nevanlinna–Riesz–Smirnov representation*,

$$\Theta_T(z) = \Biggl(\prod_{\lambda\in\mathbb{D}} b_\lambda^{k(\lambda)}(z)\Biggr)\exp\Bigl(-\int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\Bigr),$$

putting

$$\Theta_\sigma(z) = \Biggl(\prod_{\lambda\in\sigma} b_\lambda^{k(\lambda)}(z)\Biggr)\exp\Bigl(-\int_{\sigma\cap\mathbb{T}} \frac{\zeta+z}{\zeta-z}\,d\mu(\zeta)\Bigr), \text{with} \quad |z|<1.$$

And then, we can *define* an analogue of spectral subspaces, called *prespectral*, as invariant subspaces corresponding to factorizations $\Theta = \Theta_{\sigma'}\Theta_\sigma$ with an arbitrary Borel set σ. This is a simple but principal step, because it supplies us with a "space-valued analogue" of spectral measure. Now, to develop a kind of spectral decomposition, nothing remains but to check the angles between $K(\sigma)$ and $K(\sigma')$ for σ from a given σ-algebra.

One can join spectral decompositions and complex interpolation in at least two ways. The first way uses the commutant lifting theorem and H^∞-calculus mentioned above. The key observation is the following: if an operator A, induced on an algebraic sum $K(\sigma) + K(\sigma')$ by two restrictions $f_\sigma(T)|K(\sigma)$ and $f_{\sigma'}(T)|K(\sigma')$, where $f_\sigma,\ f_{\sigma'} \in H^\infty$, is well-defined and bounded, then, due to the CLT, there exists a function $f \in H^\infty$ "interpolating" the operator A in the sense $A = f(T)$. This equality is equivalent to saying that

$$f - f_\sigma \in \Theta_{\sigma'}H^\infty \quad \text{and} \quad f - f_{\sigma'} \in \Theta_\sigma H^\infty.$$

For the case when Θ_T is a Blaschke product with simple zeros, these inclusions can be interpreted in terms of classical interpolation theory. In more general situations they lead to what we call *generalized interpolation*, namely, to interpolation by germs of H^∞- and H^2-functions, for an arbitrary Blaschke product Θ_T; to a kind of asymptotic interpolation, for a point mass singular inner function Θ_σ; and to tangential interpolation on the boundary of the unit disc, for outer Θ_σ.

Continuing in this way and using the Carleson corona theorem [Garnett 1981; Nikolski 1986], we prove in Chapter 8 the following *generalized free interpolation theorem*, which is essentially contained in [Vasyunin 1978; Nikolski 1978b]. Given a bounded sequence $(\Theta_k)_{k\geq 1}$ of H^∞-functions, the following conditions are equivalent:

(1) For every $0-1$ sequence $(\varepsilon_k)_{k\geq 1}$ there exists an H^∞-function f such that $f-\varepsilon_k \in \Theta_k H^\infty$ for $k \geq 1$.

(2) For every sequence of functions $f_k \in H^\infty$, where $k \geq 1$, satisfying

$$\sup_k \operatorname{dist}_{L^\infty}(f_k \overline{\Theta}_k, H^\infty) < \infty,$$

there exists an H^∞-function f such that $f - f_k \in \Theta_k H^\infty$ for $k \geq 1$ and

$$\sup_k \left\| \frac{f-f_k}{\Theta_k} \right\|_\infty < \infty.$$

(3) There exists a uniformly equivalent sequence $(\theta_k)_{k\geq 1}$ (that is, one satisfying

$$0 < \varepsilon < \left|\frac{\Theta_k(z)}{\theta_k(z)}\right| < \varepsilon^{-1}$$

for all $k \geq 1$, $z \in \mathbb{D}$, and a fixed $\varepsilon > 0$), making the product $\theta = \prod_{k\geq 1} \theta_k$ convergent and satisfying the *generalized Carleson condition*

$$|\theta(z)| \geq \varepsilon \cdot \inf_k |\theta_k(z)| \quad \text{for } z \in \mathbb{D}.$$

As an operator-theoretic consequence of this theory we can mention a criterion for the Dunford spectrality (see [Dunford and Schwartz 1971] for definitions) of a contraction with defect numbers $(1,1)$: it is necessary and sufficient that $\inf_{\mathbb{T}} |\Theta_T|$ be positive, that the singular measure μ_s be purely atomic, that $\mu_s = \sum_{n\geq 1} a_n \delta_{\zeta_n}$ for $\zeta_n \in \mathbb{T}$, and that the sequence $(\theta_k)_{k\geq 1}$ consisting of the corresponding point mass singular inner functions $\exp\bigl(-a_n(\zeta_n+z)(\zeta_n-z)\bigr)$ and all Blaschke factors $b_\lambda^{k(\lambda)}$ satisfy the generalized Carleson condition.

Another way to join spectral decompositions and interpolation is to go down from the calculus level to the individual vector level and to use N. Bari's characterization of unconditional bases of Hilbert space. Namely, a sum of subspaces $K(\sigma_1) + K(\sigma_2) + \cdots$ is an unconditionally convergent decomposition of the subspace $\operatorname{span}(K(\sigma_i) : i \geq 1)$ if and only if an approximate Parseval identity holds: there exists a constant $c > 0$ such that

$$c \sum_i \|x_i\|^2 \leq \Bigl\| \sum_i x_i \Bigr\|^2 \leq c^{-1} \sum_i \|x_i\|^2$$

for all sequences $x_i \in K(\sigma_i)$, $i \geq 1$. Now, at least for the case of *inner* characteristic functions Θ_T, one can use the interpolation meaning of the spectral projections f_i, where $f = \sum_i f_i$, corresponding to the *dual* biorthogonal decomposition $K(\sigma_1)' + K(\sigma_2)' + \ldots$, and add the following assertion, equivalent to (1)–(3) listed above:

(4) The functions $f_i \in H^2$ satisfy $\sum_i \|f_i\|^2_{H^2/\Theta_i H^2} < \infty$ if and only if there exists a function $f \in H^2$ interpolating the f_i, in the sense that $f - f_i \in \Theta_i H^2$ for $i \geq 1$.

In particular, we get a *generalized embedding theorem*: the generalized Carleson condition implies the convergence

$$\sum_i \|f\|^2_{H^2/\Theta_i H^2} < \infty$$

for every $f \in H^2$.

These results, in which we follow [Nikolski 1978a; 1987; Nikolski and Khrushchev 1987], were generalized in [Hartmann 1996] for all H^p with $1 < p < \infty$.

The last subject of interpolation theory that we will treat in Chapter 8 is a local estimate of interpolation data and locally defined data. Generally speaking, the idea is that the distances $\operatorname{dist}_{L^\infty}(f_i\overline{\Theta}_i, H^\infty)$ from the theorem above, in fact, coincide with operator norms of the restrictions $\|f_i(T)|K(\sigma_i)\|$ and hence should be expressed in local terms related to the behavior of f_i near the spectrum of $T|K(\sigma_i)$ (that is, $\operatorname{clos}(\sigma(T) \cap \sigma_i)$). For the inner function case, an appropriate technical tool is a local function calculus developed in Chapter 4 and especially an estimate of the mentioned operator norm by the local norm $\sup\{|f_i(z)| : z \in \Omega(\Theta_i, \varepsilon)\}$, where $\Omega(\Theta_i, \varepsilon) = \{z \in \mathbb{D} : |\Theta_i(z)| < \varepsilon\}$ stands for a level set of the function Θ_i; the latter essentially coincides with the characteristic function of the restriction $T|K(\sigma_i)$. This leads directly to the following *local interpolation* assertion which is stated to be equivalent to assertions (1)–(4) above:

(5) For all sequences of functions $f_i \in H^\infty(\Omega(\Theta_i, \varepsilon))$, where $i \geq 1$, such that $\sup_i \|f_i\|_{H^\infty(\Omega(\Theta_i,\varepsilon))} < \infty$, there exists an $H^\infty(\mathbb{D})$-function f *interpolating* the f_i in the sense that

$$f - f_i \in \Theta_i H^\infty(\Omega(\Theta_i, \varepsilon)) \quad \text{for } i \geq 1,$$

and

$$\sup_i \left\| \frac{f - f_i}{\Theta_i} \right\|_{H^\infty(\Omega(\Theta_i,\varepsilon))} < \infty.$$

It is curious that, for the Blaschke product case, where $\Theta_T = \prod b_\lambda^{k(\lambda)}$ and $\Theta_i = b_\lambda^{k(\lambda)}$, $\lambda = \lambda_i$, the corresponding level sets $\Omega(\Theta_i, \varepsilon)$ are noneuclidean discs $\{z : |b_\lambda(z)| < \varepsilon^{1/k(\lambda)}\}$, and that for this case it is well known [Vinogradov and Rukshin 1982] that the considered sets $\Omega(\Theta_i, \varepsilon)$ cannot be replaced by smaller ones. Namely, for discs $\{z : |b_{\lambda_i}(z)| < r_i\}$ with $r_i = o(\varepsilon^{1/k(\lambda_i)})$ for $i \to \infty$, the generalized Carleson condition does not imply the conclusion of assertion 5).

A.3. Similarity to a normal operator. It is clear from the previous discussion that we believe that spectral decompositions are one of the most powerful tools of spectral theory. Therefore, we rank operators admitting the same quality spectral decompositions as normal ones, as the best possible operators on

Hilbert space. As to normals, they can be characterized by any of the following properties, referring not only to decompositions but also to the calculus.

(1) An operator N on a Hilbert space H is normal if and only if there exists a projection-valued *contractive* measure $E(\cdot)$ such that $Nx = \int_\sigma z\,dE(z)x$ (Riemann convergent integral for every $x \in H$).
(2) An operator N on a Hilbert space H is normal if and only if there exists an *isometric* functional calculus $f \mapsto f(N)$ defined on the algebra $C(\sigma)$ of all continuous complex functions on a suitable compact set $\sigma \subset \mathbb{C}$, that is $\|f(T)\| = \|f\|_{C(\sigma)} \overset{\text{def}}{=} \sup_\sigma |f|$ for all $f \in C(\sigma)$.

John Wermer [1954] has discovered the remarkable fact that the passage to *operators similar to a normal one*, $N \mapsto T = V^{-1}NV$, transforms the picture in such a way that, to keep the criteria, we need simply to replace all equalities by norm inequalities. Precisely, the following theorems hold true.

(1′) An operator T on a Hilbert space H is similar to a normal one if and only if there exists a *bounded* projection-valued measure $\mathcal{E}(\cdot)$ such that $Tx = \int_\sigma z\,d\mathcal{E}(z)x$ (the Riemann convergent integral for every $x \in H$).
(2′) An operator T on a Hilbert space H is similar to a normal one if and only if there exists a *bounded* functional calculus $f \mapsto f(T)$ defined on the algebra $C(\sigma)$ of a suitable compact set $\sigma \subset \mathbb{C}$, that is

$$\|f(T)\| \le C\|f\|_{C(\sigma)} \quad \text{for } f \in C(\sigma). \tag{A.3.1}$$

Nevertheless, an inconvenience of (A.3.1) as a sufficient condition for similarity to a normal operator is obvious. Indeed, what we have at hand when dealing with an operator T, are *rational* expressions in T, that is the resolvent

$$R(\lambda, T) = (\lambda I - T)^{-1} \quad \text{for } \lambda \in \mathbb{C} \setminus \sigma(T),$$

and hence $f(T)$ for f in the set $\mathrm{Rat}(\mathbb{C} \setminus \sigma(T))$ of rational functions having poles on $\mathbb{C} \setminus \sigma(T)$. The knowledge of other continous functions of T, required by(A.3.1), is still mostly implicit and the very definition of them is hiding in the existence of the functional calculus. For all that, for operators with a "thin" spectrum, rational functions are sufficient. For instance, if the set $\mathrm{Rat}(\mathbb{C} \setminus \sigma)$ is norm dense in the space $C(\sigma)$, then it suffices to require (A.3.1) for $f \in \mathrm{Rat}(\mathbb{C} \setminus \sigma)$ in order to guarantee similarity to a normal operator and the inclusion $\sigma(T) \subset \sigma$. However, for most concrete situations, for example those coming from perturbation theory, there are too many rational functions in $\mathrm{Rat}(\mathbb{C} \setminus \sigma)$ to consider even this restricted version of (A.3.1) as a practical test for similarity.

The idea of *rational tests* for similarity, presented in Chapter 9 following [Benamara and Nikolski 1997], is to reduce the number of test functions $\mathrm{Rat}(\mathbb{C} \setminus \sigma)$ to a reasonable part of it. A natural choice for such a reduction is to consider estimates (A.3.1) for $f \in \mathrm{Rat}_n(\mathbb{C} \setminus \sigma)$, the set of all rational functions of degree at most $n = 1, 2, \ldots$ and with poles in $\mathbb{C} \setminus \sigma$. The case $n = 1$, also called the *resolvent criterion*, is the most popular and consists of testing for (A.3.1) rational functions f of degree 1 only: thus $f = 1/(\lambda - z)$, with

$$\left\| \frac{1}{\lambda - z} \right\|_{C(\sigma)} = \frac{1}{\operatorname{dist}(\lambda, \sigma)}.$$

In this case, the problem can be stated as follows: for which classes of operators is the *linear growth of the resolvent* (LGR), when approaching to the spectrum,

$$\|R(\lambda, T)\| \le \frac{\text{const}}{\operatorname{dist}(\lambda, \sigma(T))} \quad \text{for } \lambda \in \mathbb{C} \setminus \sigma(T), \tag{A.3.2}$$

sufficient for T to be similar to a normal operator? It is well known (A. Markus) that in general this is not the case even for operators with the real spectrum, $\sigma(T) \subset \mathbb{R}$; in [Benamara and Nikolski 1997] it is shown that no spectral restrictions, excepting only a finite spectrum $\sigma(T)$, together with the LGR, implies similarity to a normal operator. On the other hand, B. Sz.-Nagy and C. Foiaş [1967] proved that for *contractions* T with a unitary spectrum, $\sigma(T) \subset \mathbb{T}$, the LGR implies similarity to a normal, and hence to a unitary operator.

In Chapter 9, following [Benamara and Nikolski 1997], we explain why the resolvent criterion is still true for contractions T with $\sigma(T) \ne \overline{\mathbb{D}}$ and $\operatorname{rank}(I - T^*T) < \infty$, $\operatorname{rank}(I - TT^*) < \infty$, and fails for T with $\sigma(T) \ne \overline{\mathbb{D}}$ and $(I - T^*T) \in \bigcap_{p>1} \mathfrak{S}_p$, $(I - TT^*) \in \bigcap_{p>1} \mathfrak{S}_p$, where $\mathfrak{S}_p$ stands for the Schatten–von Neumann ideals of compact operators. The case of $p = 1$, trace class perturbations of unitary operators, is still open. Our technique is based on free interpolation results (Chapter 8) and on estimates of angles between invariant subspaces (Chapter 7).

Some higher rational tests, that is, estimates (A.3.1) for $f \in \operatorname{Rat}_n(\mathbb{C} \setminus \sigma)$, $n > 1$, are also considered (and also following [Nikolski $\ge$ 1998]). For this case, a kind of Bernstein inequality is proved for operators with unitary or real spectra: it turns out that (A.3.2) implies inequality (A.3.1) for all $n = 2, 3, \ldots$ with constants const $= c_n$ depending on n. So, one can say that the $\operatorname{Rat}_n(\mathbb{C} \setminus \mathbb{T})$ and $\operatorname{Rat}_n(\mathbb{C} \setminus \mathbb{R})$ tests provide us with no new cases of similarity to a normal operator with respect to the simplest resolvent test. Of course, if the constants are uniformly bounded, $\sup_n c_n < \infty$, the operator is similar to a unitary (for $\sigma(T) \subset \mathbb{T}$) or a selfadjoint operator (for $\sigma(T) \subset \mathbb{R}$) by Wermer's theorem.

A.4. Stability of the continuous spectrum. The problem of stability of the continuous spectrum goes back to H. Weyl (1909) and J. von Neumann (1935) for the selfadjoint case, and to A. Weinstein (1937), N. Aronszajn and A. Weinstein (the forties) and S. Kuroda (the fifties and sixties) for more general settings; see [Akhiezer and Glazman 1966; Kato 1967] for references and initial results. The *continuous spectrum* $\sigma_c(T)$ of an operator T is the spectrum with isolated eigenvalues of finite algebraic multiplicity removed, that is, eigenvalues whose Riesz projection is of finite rank. It is well known that if the resolvent set

$\mathbb{C} \setminus \sigma(T)$ is connected, the continuous spectrum is *stable* with respect to compact perturbations:

$$\sigma_c(T+K) = \sigma_c(T)$$

for all $K \in \mathfrak{S}_\infty$. If $\mathbb{C} \setminus \sigma(T)$ contains nontrivial bounded components, it may happen that $\sigma_c(T+K) \neq \sigma_c(T)$ even for finite rank operators K. In this case, at least one bounded component of $\mathbb{C} \setminus \sigma(T)$ is filled in by a layer of eigenvalues of perturbed operator $T+K$ (the so-called *singular case* of Fredholm perturbation theory, see [Kato 1967]). The problem is to distinguish operators T allowing the singular case from those whose continuous spectrum is stable with respect to perturbations from a given class of operators.

One of the classic examples of instability [Kato 1967] is a rank-one perturbation of the standard shift operator $\mathcal{S} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$:

$$\sigma_c(\mathcal{S}+K) = \overline{\mathbb{D}},$$

where $Kf = -(f, \bar{z})1$ for $f \in L^2(\mathbb{T})$ (this contrasts with $\sigma_c(\mathcal{S}) = \mathbb{T}$). Clearly, the same construction works for any unitary operator U (instead of $\mathcal{S}$) whose spectral measure E_U dominates Lebesgue measure m of the unit circle $\mathbb{T}$ ($E_U \succeq m$). The inverse is also true, but is not so obvious. More precisely, it is proved in [Nikolski 1969] that these statements are equivalent for a given unitary operator U:

(1) $\sigma_c(U+K) = \sigma_c(U)$ for all rank-one operators K;
(2) $\sigma_c(U+K) = \sigma_c(U)$ for all $K \in \mathfrak{S}_1$;
(3) E_U does not dominate m, that is there exists a spectral gap $\sigma \subset \mathbb{T}$, $\sigma \in \mathfrak{B}$, of positive Lebesgue measure ($E_U(\sigma) = 0$ but $m\sigma > 0$);
(4) U belongs to the weak closed operator algebra generated by U^{-1};
(5) $\operatorname{Lat} U^{-1} \subset \operatorname{Lat} U$, where $\operatorname{Lat} U$ stands for the lattice of invariant subspaces of U.

The stability problem for bigger perturbations leads to a rougher picture:

$$\sigma_c(U+K) = \sigma_c(U) \quad \text{for all } K \in \mathfrak{S}_p \text{ with } p > 1$$

if and only if $\sigma(U) \neq \mathbb{T}$.

In the framework of perturbation theory, a natural problem was to find a stability criterion for trace class perturbations T of unitary operators, $T = U + C$, $C \in \mathfrak{S}_1$. The solution was found in [Makarov and Vasyunin 1981] by using the function model. The first problem here was to insert a noncontraction $T = U + C$ into model theory. This is done by considering the function model for an auxiliary "nearest to T" contraction T_0 and realizing T on this model. Another problem was to find a proper analogue of condition (3) above to express a spectral gap of the operator under question. The language of the function model helps once more, and the true expression for a completely nonunitary operator turns out to be the following:

(3′) the characteristic function Θ_T is J_T-unitary on a set $\sigma \subset \mathbb{T}$ of positive Lebesgue measure, where

$$\Theta_T(z) \overset{\text{def}}{=} (-TJ_T + zD_{T^*}(I - zT^*)^{-1}D_T)|\,\text{Range}\,D_T,$$

and $D_A \overset{\text{def}}{=} |I - A^*A|^{1/2}$, $J_A \overset{\text{def}}{=} \text{sign}(I - A^*A)$ (the square root $|\cdot|^{1/2}$ and the sign function $\text{sign}(t) = t/|t|$ applied to the selfadjoint operator $I - A^*A$).

With these refinements, it will be proved in Chapter 10 that assertions (1)–(5) are still equivalent for an invertible operator $T = U + C$, $C \in \mathfrak{S}_1$ instead of U, if we add to condition 3) for the unitary part of T condition (3′) for its completely nonunitaty part.

A.5. Scattering and other subjects. The quick development and the very rise of mathematical scattering theory in the late fifties was motivated by influences of both physical scattering theory and several mathematical theories. Among the latter, a leading role is playing by both of the subjects considered above, namely, by stability problems of the continuous spectrum and similarity problems. Of course, as we are speaking here about predecessors of scattering theory, we have in mind the classical framework of selfadjoint and unitary operators, in which domain the foundation of both theories was laid by H. Weyl (1909), J. von Neumann and K. Friedrichs (in the thirties through the fifties). Without entering into technical details, we trace here an approach to scattering problems that is adjustable to the use of the function model.

The main goal of scattering theory is to compare the asymptotic behavior for $t \to \pm\infty$ of two continuous groups on a Hilbert space: the "perturbed" group $S(t)$ and the "nonperturbed" group $S_0(t)$ ("free evolution"). Both continuous, $t \in \mathbb{R}$, and discrete, $t \in \mathbb{Z}$, times are considered. Being motivated by quantum physics scattering phenomena, mathematical scattering theory was started with the so-called *nonstationary approach* which consists of the following. Let S, S_0 be unitary groups on a Hilbert space H whose selfadjoint generators, A and A_0 respectively, differ by a "small", say of finite rank, operator, $A = A_0 + \Delta$ where the spectral measures of A, A_0 are absolutely continuous with respect to Lebesgue measure ($E_A,\ E_{A_0} \preceq m$). It turns out that under these hypotheses the asymptotic behavior of the groups are the same: for every $x \in H$ there exist unique vectors $x_0^\pm \in H$ such that $\lim_{t\to\pm\infty} \|S(t)x - S_0(t)x_0^\pm\| = 0$. The operators

$$x = \lim_{t\to\pm\infty} S(t)^{-1}S_0(t)x_0^\pm \overset{\text{def}}{=} W_\pm(A, A_0)x_0^\pm$$

are called the *wave operators* of the pair A, A_0 and $\mathbf{S} = W_+^*W_-$ is called the *scattering operator*. Under the above hypotheses, all three are unitary operators. The operator $\mathbf{S}$ links the "free asymptotics" for $t \to -\infty$ and $t \to \infty$, that is, $\mathbf{S}x_0^- = x_0^+$, and it commutes with A_0, whereas $W_\pm$ establish the similarity—in fact, the unitary equivalence—of A and A_0,

$$AW_\pm = W_\pm A_0,$$

and the semigroups themselves (the so-called *intertwining properties*).

In general, for instance for trace class selfadjoint perturbations Δ, the wave limits $W_\pm$ exist only on the absolutely continuous subspace of A_0, $H_0^{ac} = E_{A_0}^{ac}(H)$, and map it onto $H^{ac} = E_A^{ac}(H)$, establishing a unitary equivalence between the absolutely continuous parts $A|H^{ac}$, $A_0|H_0^{ac}$ and, *a fortiori*, the stability of the absolutely continuous spectrum with respect to nuclear perturbations. For the case of discrete time, all is similar, except we have no generators and deal directly with the unitary operators $U^n = S(1)^n$ and $U_0^n = S_0(1)^n$, for $n \in \mathbb{Z}$, and their absolutely continuous parts.

There exist several realizations of the second, *stationary approach* to scattering theory, that is, to find the wave operators $W_\pm$ avoiding the wave limits of the initial definition. The idea, coming back to K. Friedrichs, is to define these operators as solutions of some operator equations and then prove the above intertwining properties and the existence of the wave limits. The initial Friedrichs observation is that, for operators with absolutely continuous spectra,

$$W_\pm = I + \Gamma^\pm(\Delta W_\pm)$$

where $\Gamma^\pm(X) = \int_0^{\pm\infty} S_0(-t)XS_0(t)\,dt$. For methods of solving these Friedrichs (Γ)-equations we refer to [Kato 1967; Dunford and Schwartz 1971]. For other stationary approaches, making use of some resolvent equations (in a sense, the Fourier–Laplace transform of the Friedrichs ones) instead of the (Γ)-equations, see [Reed and Simon 1979; Yafaev 1992].

Function models are ideally adapted to stationary methods of scattering theory, because, as before, using the local function structure, one can guess explicit formulas conjecturally solving the needed operator equations, and then prove that they really provide the solutions. Another advantage is that the model approach is also well-adapted for an important passage from scattering for unitary (semi)groups to scattering for contractive semigroups. The pioneering role in applying function models to scattering problems was played by L. de Branges [de Branges 1962; de Branges and Rovnyak 1966; de Branges and Shulman 1968]. In Part II of this paper, we analize the de Branges and other model approaches to scattering theory, deriving from them the main facts of the theory. Here we simply mention some other sources: [Lax and Phillips 1967; Adamyan and Arov 1966; Naboko 1980; 1987].

Among other subjects that we plan to include in Part II of the paper, we can mention some properties of the weak star closed algebra $\operatorname{alg} M_\Theta$ generated by the model operator M_Θ. For instance, we consider the problem of weak generators of this algebra, closely related to properties of the lattice $\operatorname{Lat} M_\Theta$ of invariant subspaces of M_Θ; the problem, solved in [Kapustin 1992], of the reflexivity of M_Θ; and the problem of the "invisible spectrum" for $\operatorname{alg} M_\Theta$.

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