Tight and Taut Submanifolds
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Taut and Dupin Submanifolds

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Abstract. This is a survey of the closely related fields of taut submanifolds and Dupin submanifolds of Euclidean space. The emphasis is on stating results in their proper context and noting areas for future research; relatively few proofs are given. The important class of isoparametric submanifolds is surveyed in detail, as is the relationship between the two concepts of taut and Dupin. Also included is a brief introduction to submanifold theory in Lie sphere geometry, which is needed to state many known results on Dupin submanifolds accurately. The paper concludes with detailed descriptions of the main known classification results for both Dupin and taut submanifolds.

Dupin [1822] determined which surfaces M embedded in Euclidean three-space $\mathbb{R}^3$ can be obtained as the envelope of the family of spheres tangent to three fixed spheres. These surfaces, known as the *cyclides of Dupin*, can all be constructed by inverting a torus of revolution, a circular cylinder or a circular cone in a sphere. The cyclides of Dupin were studied extensively in the nineteenth century (see, for example, [Cayley 1873; Liouville 1847; Maxwell 1867]). They have several other important characterizations. They are the only surfaces M in $\mathbb{R}^3$ whose focal set consists of two curves, which must, in fact, be a pair of focal conics. This is equivalent to requiring that M have two distinct principal curvatures at every point, each of which is constant along each of its corresponding lines of curvature. It is also equivalent to the condition that all lines of curvature in both families are circles or straight lines.

The cyclides reappeared in modern differential geometry in a paper by Banchoff [1970]. He considered compact surfaces M embedded in $\mathbb{R}^3$ with the property that every metric sphere in $\mathbb{R}^3$ cuts M into at most two pieces; this is called the *spherical two-piece property*, or STPP. For surfaces, the STPP is equivalent to requiring that M be *taut*, i.e., that every nondegenerate Euclidean distance function $L_p(x) = |p - x|^2$, where $p \in \mathbb{R}^3$, has the minimum number of critical points allowed by the Morse inequalities. Banchoff showed that tautness implies

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that M must be a metric sphere or a cyclide of Dupin, and the close link between these notions was established.

A hypersurface M in $\mathbb{R}^n$ is said to be *Dupin* if, along each curvature surface, the corresponding principal curvature is constant. A Dupin hypersurface M is called *proper Dupin* if the number of distinct principal curvatures is constant on M. These concepts can both be generalized in a natural way to submanifolds of codimension greater than one in $\mathbb{R}^n$. A fundamental result in the theory due to Pinkall [1986] is that a taut submanifold must be Dupin. Conversely, the work of Thorbergsson [1983a] and Pinkall [1986] shows that a compact proper Dupin submanifold embedded in $\mathbb{R}^n$ must be taut. A major open question is whether the condition that the number of distinct principal curvatures is constant can be dropped; in other words, does Dupin imply taut? These results are discussed in Section 4.

This paper is a survey of the major results on taut and Dupin submanifolds. We concentrate on stating the results in their proper context and noting areas for future research and give very few proofs. In particular, we do not repeat proofs of many fundamental results in the field that can be found in [Cecil and Ryan 1985], and we will concentrate on work done since that reference appeared. We will not attempt to cover the related field of tight immersions, since that is done in the paper by Banchoff and Kühnel [1997] in this volume.

Important examples of taut submanifolds are the isoparametric submanifolds. These will be reviewed in Section 2, but the reader is referred to the excellent article [Terng 1993] for an in-depth survey of that field. There is also extensive research on real hypersurfaces with constant principal curvatures in complex space forms, which is covered by the paper by Niebergall and Ryan [1997] in this volume, and which will not be discussed here. We now give a brief overview of the contents of this article.

In Section 1 we review the critical point theory and submanifold theory needed to formulate the definition of a taut submanifold, and we list some basic results and methods for constructing taut embeddings. In Section 2 we list the primary known results for isoparametric hypersurfaces in spheres, which play an important role in the theory of Dupin hypersurfaces.

In Section 3 we give the definition of a Dupin submanifold and review Pinkall's standard local constructions of proper Dupin hypersurfaces with an arbitrary number of distinct principal curvatures and respective multiplicities. In Section 4 we discuss the relationship between the taut and Dupin conditions in detail.

Many of the main classifications of proper Dupin submanifolds are done in the context of Lie sphere geometry. In Section 5 we give a brief introduction to this theory in order to be able to explain these classifications accurately.

Section 6 is a survey of the known results on compact proper Dupin hypersurfaces. Thorbergsson [1983a] applied the work of Münzner [1980; 1981] to show that the number g of distinct principal curvatures of such a hypersurface must be 1, 2, 3, 4, or 6, the same as for an isoparametric hypersurface in a sphere. For some time, it was conjectured that every compact proper Dupin hypersurface is equivalent by a Lie sphere transformation to an isoparametric hypersurface. However, this is not the case, as examples constructed by Pinkall and Thorbergsson [1989a] and by Miyaoka and Ozawa [1989] demonstrate. We describe these examples in detail.

In Section 7 we study the local classifications of proper Dupin hypersurfaces that have been obtained using Lie sphere geometry. We describe the known results and mention several areas for further research.

Finally in Section 8 we survey the known classifications of taut embeddings. To some extent, this section can be read independent of the rest of the paper, although some references to the previous sections are necessary.

1. Taut Submanifolds

We begin with a brief review of the critical point theory and submanifold theory needed to formulate the definition of tautness. In this paper, all manifolds are assumed to be connected unless explicitly stated otherwise. Let M be a smooth, connected n-dimensional manifold, and let ϕ be a smooth real-valued function defined on M. A point $x \in M$ is a *critical point* of ϕ if the differential ϕ_* is zero at x. The critical point x is *nondegenerate* if the Hessian H of ϕ is a nondegenerate bilinear form at x, and otherwise it is said to be *degenerate*. The *index* of a nondegenerate critical point x is equal to the index of H as a bilinear form, that is, the dimension of a maximal subspace on which H is negative-definite. The function ϕ is called a *Morse function* or *nondegenerate function* if it has only nondegenerate critical points on M.

Let ϕ be a Morse function on M such that the set

$$M_r(\phi) = \{x \in M : \phi(x) \leq r\}$$

is compact for all $r \in \mathbb{R}$. Of course, this is true for any Morse function on a compact manifold M. Let $\mu_k(\phi, r)$ be the number of critical points of ϕ of index k on $M_r(\phi)$. If M is compact, let $\mu_k(\phi)$ denote the number of critical points of index k on M. For a field F, let

$$\beta_k(\phi, r, F) = \dim_F H_k(M_r(\phi); F)$$

be the k-th F-Betti number of $M_r(\phi)$, and let $\beta_k(M; F)$ be the k-th F-Betti number of a compact M. Then the *Morse inequalities* (see [Morse and Cairns 1969, p. 270], for example) state that

$$\mu_k(\phi, r) \geq \beta_k(\phi, r, F)$$

for all F, k, r, and for a compact M,

$$\mu_k(\phi) \geq \beta_k(M; F)$$

for all F, k. A Morse function ϕ on M is said to be *perfect* if ϕ has the minimum number of critical points possible by the Morse inequalities, that is, if there exists a field F such that

$$\mu_k(\phi, r) = \beta_k(\phi, r, F)$$

for all k, r. For a compact manifold M, this is equivalent to the condition $\mu_k(\phi) = \beta_k(M; F)$ for all k. Equivalently, it can be shown (see [Morse and Cairns 1969, p. 260], for example) that a Morse function ϕ on a compact manifold M is perfect if there exists a field F such that, for all k, r, the map on homology

$$H_k(M_r(\phi); F) \to H_k(M; F) \tag{1.1}$$

induced by the inclusion of $M_r(\phi)$ in M is injective. This formulation has proved to be quite useful in the theory of tight and taut immersions.

Let $f : M \to \mathbb{R}^n$ be a smooth immersion of a manifold M into n-dimensional Euclidean space. Since f is an immersion, it is an embedding on a suitably small neighborhood of any point $x \in M$. Thus, for local calculations, we often identify the tangent space T_xM with its image $f_*(T_xM)$ under the differential f_* of f. Suppose that $X \in T_xM$ and that ξ is a field of unit normal vectors to $f(M)$ defined on a neighborhood of x. Then we have the fundamental equation

$$D_X\xi = -A_\xi X + \nabla^\perp_X \xi,$$

where $-A_\xi X$ is the component of $D_X\xi$ tangent to M, and $\nabla^\perp_X\xi$ is the component normal to M. Here A_ξ is a symmetric tensor of type $(1, 1)$ on M called the *shape operator* determined by ξ, and $\nabla^\perp$ is a covariant derivative operator in the normal bundle of M called the *normal connection*. The eigenvalues of A_ξ are called the *principal curvatures* of A_ξ. When $f(M)$ is a hypersurface, a local field of unit normal vectors ξ is determined up to a sign. In this case, the shape operator A_ξ is often denoted simply by A, and the eigenvalues of A_ξ are determined up to a sign, depending on the choice of ξ. In that case, these eigenvalues are called the principal curvatures of M or of f.

The *normal exponential map* F from the normal bundle $N(M)$ to $\mathbb{R}^n$ is defined by

$$F(x, \eta) = f(x) + \eta,$$

where η is a normal vector to $f(M)$ at $f(x)$. A point $p \in \mathbb{R}^n$ is called a *focal point of multiplicity* m *of* (M, x) if $p = F(x, \xi)$ and the differential F_* has nullity m at (x, ξ). A point $p \in \mathbb{R}^n$ is called a focal point of M if p is a focal point of (M, x) for some $x \in M$. The set of all focal points of M is called the *focal set* of M. Since $N(M)$ and $\mathbb{R}^n$ have the same dimension, Sard's Theorem implies that the focal set of M has measure zero in $\mathbb{R}^n$. A direct computation (see [Milnor 1963, p. 34], for example) shows that if $p = F(x, t\xi)$, where $|\xi| = 1$, then p is a focal point of (M, x) of multiplicity m if and only if $1/t$ is a principal curvature of A_ξ of multiplicity m. In this paper, we often consider an immersion $f : M \to S^n$ into the unit sphere S^n in $\mathbb{R}^{n+1}$. In that case, one can also define the notions of normal exponential map and focal point in a manner analogous to the definitions given here for submanifolds of Euclidean space.

A *Euclidean distance function* is a function $L_p : \mathbb{R}^n \to \mathbb{R}$ given by the formula $L_p(q) = |p - q|^2$, where $p \in \mathbb{R}^n$. The level sets of L_p are spheres centered at the point p. Let f be an immersion of a smooth manifold M into $\mathbb{R}^n$. We consider the restriction of L_p to M defined by $L_p(x) = \left|p - f(x)\right|^2$. It is well-known (see [Milnor 1963, pp. 33–38], for example) that L_p has a critical point at $x \in M$ if and only if p lies along the normal line to $f(M)$ at $f(x)$. The critical point x is degenerate precisely when p is a focal point of (M, x). If L_p has a nondegenerate critical point at x, its index is the number of focal points of (M, x) on the line segment from p to $f(x)$, taking into account multiplicities. Since the set of focal points of f has measure zero in $\mathbb{R}^n$, L_p is a Morse function for almost all $p \in \mathbb{R}^n$. The immersion f is said to be *taut* if every Morse function of the form L_p is perfect, that is, there exists a field F such that

$$\mu_k(L_p, r) = \beta_k(M_r(L_p); F)$$

for every Morse function of the form L_p and for every k, r. This definition makes sense for noncompact manifolds, and there do exist taut immersions of noncompact manifolds—for example, a circular cylinder in $\mathbb{R}^3$—but most results deal with compact manifolds.

As in the theory of tight immersions, there is a formulation of tautness due to Kuiper in terms of Čech homology that has proved to be very useful in establishing certain fundamental results (see [Cecil and Ryan 1985, Section 2.1] for more detail). So far the field $F = \mathbb{Z}_2$ has been sufficient for almost all considerations, so we will use it exclusively here. Recall that a map $f : M \to \mathbb{R}^n$ is *proper* if $f^{-1}K$ is compact for every compact subset K of $\mathbb{R}^n$. Using 1.1 and the Čech theory, one can show that a proper immersion f of a manifold M into $\mathbb{R}^n$ is taut if and only if, for every closed ball B in $\mathbb{R}^n$, the induced homomorphism

$$H_i(f^{-1}B) \to H_i(M) \tag{1.2}$$

in Čech homology with $\mathbb{Z}_2$-coefficients is injective for every i. The use of Čech homology allows one to use all closed balls in $\mathbb{R}^n$ rather than only those determined by level sets of nondegenerate distance functions. Note that this formulation of tautness makes sense even if f is only assumed to be a proper continuous map and M a topological space. In that case, f is called a *taut map*.

A few key facts follow quickly from the definition. First, a taut immersion must be an embedding. In fact, this is true even if f is only assumed to be *0-taut*, i.e., the induced homomorphism 1.2 is injective for $i = 0$. For a compact manifold M, 0-tautness is equivalent to the *spherical two-piece property* (STPP) of Banchoff [1970], which requires that $f^{-1}\Omega$ be connected whenever Ω is a closed ball, the complement of an open ball, or a closed half-space. This is also equivalent to the condition that every Morse function of the form L_p have exactly one local maximum and one local minimum, which is equivalent to tautness for compact manifolds of dimension two by the Morse inequalities. The STPP is quite strong, and for a long time, every known STPP embedding was actually taut, but Curtin [1991] showed that there exist STPP embeddings that are not taut. Specifically, an embedding $f : M \to \mathbb{R}^n$ is said to be *k-taut* if the induced homomorphism 1.2 is injective for all $i \leq k$. Curtin found substantial embeddings of S^n into S^{n+d}, for $d \geq 1$, that are k-taut but not $(k+1)$-taut for every $n \geq 3$ and every $k \geq 0$ provided that $(d+1)(k+2) \leq n+1$. He also produced k-taut embeddings of manifolds other than spheres. Later [Curtin 1994] he also introduced a notion of tautness for manifolds with boundary.

As noted above, tautness can be studied for maps defined on spaces that are not manifolds. In fact, the first paper on tautness [Banchoff 1970] determined all STPP subsets of the plane. Later, Kuiper [1984] determined all taut subsets of $\mathbb{R}^2$ and all compact taut ANR (absolute neighborhood retract) subsets of $\mathbb{R}^3$.

Next, a taut embedding f of a compact manifold M into $\mathbb{R}^n$ must be *tight*, that is, every nondegenerate linear height function $l_p(x) = \langle p,\, f(x)\rangle$, for p a unit vector in $\mathbb{R}^n$, must be perfect. This is easily shown using Čech homology, since a closed half-space can be obtained as the limit of closed balls. A map f of a compact topological space into $\mathbb{R}^n$ is said to have the *two-piece property*, or TPP, if $f^{-1}h$ is connected for every closed half-space h. Of course, the STPP implies the TPP. As it turns out, tautness is a much stronger condition than tightness.

It is sometimes said that taut is equivalent to the combination of tight and spherical. This is true in the following sense. First, suppose that f is an embedding of a compact manifold M into $\mathbb{R}^{n+1}$ that lies in the unit sphere S^n in $\mathbb{R}^{n+1}$, in which case we say that f is *spherical*. Then, if f is a tight immersion into $\mathbb{R}^{n+1}$, it must also be taut, because the intersection of any closed ball B with S^n can be realized as the intersection of a closed half-space with S^n. Note also that the distance in S^n from p to $f(x)$ is given by the *spherical distance function* $d_p(x) = \arccos l_p(x)$, which has the same critical points as l_p. Thus, for simplicity, we usually use linear height functions rather than spherical distance functions in treating taut submanifolds of S^n. Next, if $P_q : S^n - \{q\} \to \mathbb{R}^n$ is stereographic projection with pole q not in $f(M)$, then $P_q \circ f$ is a taut embedding of M into $\mathbb{R}^n$, since P_q maps a metric ball in S^n to either a closed ball, the complement of an open ball, or a closed half-space in $\mathbb{R}^n$. Thus, f is tight and spherical if and only if $P_q \circ f$ is taut. This was first observed in [Banchoff 1970].

Hence, the theory of taut embeddings of compact manifolds is essentially the same whether one maps into $\mathbb{R}^n$ or S^n, and we work with whichever ambient space is more convenient for the problem at hand. By the same type of reasoning, tautness is easily shown to be invariant under *Möbius transformations*, that is, conformal transformations of S^n onto itself, since such transformations transformations map hyperspheres to hyperspheres in S^n.

These considerations lead to a fundamental result on the bound on the codimension of a taut embedding. Recall that an immersion $f : M \to \mathbb{R}^n$ is said to be *substantial* if the image $f(M)$ does not lie in any affine hyperplane in $\mathbb{R}^n$.

In one of the most remarkable results in the theory of tight immersions, Kuiper [1962] showed that if $f : M \to \mathbb{R}^n$ is a substantial TPP immersion of a compact manifold M of dimension k, then $n \leq k(k+3)/2$, and if equality holds, then f must be a Veronese embedding of the real projective space P^k. (The case where equality holds is due to Little and Pohl [1971] for $k > 2$.) Since tautness implies tightness, Banchoff [1970] observed that the following basic theorem holds for compact manifolds; see [Cecil and Ryan 1985, p. 124] for a proof. Carter and West [1972] extended the result to noncompact manifolds.

THEOREM 1.1. *Let $f : M \to \mathbb{R}^n$ be a substantial taut embedding of a k-dimensional manifold M.*

(a) *If M is compact, then $n \leq k(k+3)/2$. If $n = k(k+3)/2$, then f is a spherical Veronese embedding of a real projective space P^k.*
(b) *If M is noncompact, then $n < k(k+3)/2$. If $n = k(k+3)/2 - 1$, then $f(M)$ is the image under stereographic projection of a Veronese manifold, where the pole of the projection is on the Veronese manifold.*

Actually, in the compact case only the STPP is required, and in the noncompact case all that is required is that every nondegenerate distance function L_p have exactly one local minimum and no local maxima on M.

Suppose that f is a taut embedding of a compact $(n-1)$-dimensional manifold M into S^n. Then $f(M)$ is orientable, so let ξ be a field of unit normal vectors to $f(M)$ in S^n. The *parallel hypersurface* to f at signed distance t is given by the map $f_t : M \to S^n$ with equation

$$f_t(x) = \cos t\, f(x) + \sin t\, \xi(x). \tag{1.3}$$

Thus, $f_t(x)$ is obtained by travelling a signed distance t along the normal geodesic to $f(M)$ through $f(x)$. For sufficiently small values of t, f_t is also an embedding of M, and f is taut if and only if f_t is taut, since each linear height function l_p has the same critical points on $f_t(M)$ as on $f(M)$. This type of consideration is also valid for submanifolds of codimension greater than one in S^n. If $\phi : V \to S^n$ is a compact submanifold of codimension greater than one, we consider the tube ϕ_t of radius t around $\phi(V)$ in S^n. For sufficiently small t, the map ϕ_t is an embedding of the unit normal bundle B^{n-1} of $\phi(V)$ into S^n. Furthermore, ϕ is a taut embedding of V if and only if ϕ_t is a taut embedding of B^{n-1}. To see this, one first computes that every nondegenerate height function l_p has twice as many critical points on the tube ϕ_t as it has on ϕ. Since the sum of the $\mathbb{Z}_2$-Betti numbers of B^{n-1} is twice the sum of the $\mathbb{Z}_2$-Betti numbers of V [Pinkall 1986], tautness is preserved.

Thus, in a certain sense, tautness is preserved by the group of Lie sphere transformations, since this group is generated by Möbius transformations and parallel transformations (those that map a hypersurface to a parallel hypersurface). In fact, one can show that any Lie sphere transformation T is of the form $T = \phi P_t \psi$, where ϕ and ψ are Möbius transformations, and P_t is parallel transformation with respect to either the spherical metric on S^n, a Euclidean metric on $S^n - \{p\}$, or a hyperbolic metric on an open hemisphere in S^n [Cecil and Chern 1987; Cecil 1992, p. 63]. As with the spherical metric, if $f(M)$ is taut, a parallel hypersurface in the Euclidean or hyperbolic metric is also taut if it is an immersed hypersurface. Thus, as long as $f(M)$ lies in the appropriate space and the image $Tf(M)$ is an immersed hypersurface, tautness is preserved. However, for certain values of t, a parallel hypersurface f_t contains focal points of the original hypersurface f, corresponding to singularities of the map f_t. In this case, a Lie invariant notion of tautness has yet to be established.

As noted earlier, the concept of tautness can be defined for maps that are not immersions. However, when the parallel map f_t is not an immersion, it is not necessarily true that f_t is a taut map of M into S^n, even though the original immersion $f : M \to S^n$ is taut. An example of this phenomenon is most easily described in Euclidean space rather than in the sphere.

Let M be a two-dimensional torus T^2 and let f be an embedding of T^2 as the torus of revolution obtained by revolving the circle with center $(2, 0)$ and radius 1 in the xy-plane about the y-axis in $\mathbb{R}^3$. Then f is a taut embedding (see [Banchoff 1970] or Section 2). Let ξ be the field of unit outer normals on the torus of revolution. For $0 < t < 1$, the parallel hypersurface f_t is also a torus of revolution that is tautly embedded in $\mathbb{R}^3$. However, for $t \geq 1$, the parallel hypersurface f_t has singularities. If $t > 1$, two latitude circles of the original torus of revolution are mapped by f_t to single points where the profile circle intersects the axis of revolution. In this case, it is easy to see that the map f_t no longer has the STPP. Specifically, if p is a point on the positive x-axis that is outside the image of f_t, then L_p has two local maxima and two local minima at the points where the x-axis intersects the surface $f_t(T^2)$. Thus, the map f_t does not have the STPP, so it is not a taut map of T^2 into $\mathbb{R}^3$. Classically, these surfaces $f_t(T^2)$ with $t > 1$ were known as *spindle tori* (see [Cecil and Ryan 1985, pp. 151–165] for more detail).

There is a natural way to extend the notion of tautness to the Lie sphere geometric setting in the case where f is not an immersion by using the concept of minimal total absolute curvature, as we will describe below. This formulation of tautness is invariant under parallel transformation, but it has not been proved to be invariant under Möbius transformations, unlike the definition of a taut map given above. When the map f is an immersion, the two definitions give the same results.

Suppose that $f : M \to S^n \subset \mathbb{R}^{n+1}$ is an embedding of a compact $(n-1)$-dimensional manifold M and that ξ is a field of unit normals to $f(M)$ in S^n. We can consider ξ as a map from M into S^n by parallel translating each vector $\xi(x)$ to the origin in $\mathbb{R}^{n+1}$. We have the normal exponential map $F : M \times \mathbb{R} \to S^n$ defined by

$$F(x, t) = f_t(x) = \cos t\, f(x) + \sin t\, \xi(x). \tag{1.4}$$

If $p \in S^n$ is not a focal point of $f(M)$, then the linear height function l_p has a finite number $\mu(p)$ of critical points on M. In fact, $\mu(p)$ is the number of points $x \in M$ such that $p = F(x,t)$ for some $t \in [0, 2\pi)$. That is, $\mu(p)$ is the number of geodesics normal to $f(M)$ that go through the point p. The focal set C of the embedding f has measure zero in S^n. The *total absolute curvature* τ of f (see [Cecil and Ryan 1985, pp. 12–13], for example) is defined by the expression

$$\tau = \frac{1}{c_n} \int_{S^n - C} \mu(p)\, da, \tag{1.5}$$

where c_n is the volume of the unit sphere S^n. Of course, in the usual development of the theory, the embedding $f : M \to S^n \subset \mathbb{R}^{n+1}$ is tight (and hence taut, since f is spherical) if and only if $\tau = \beta(M)$, the sum of the $\mathbb{Z}_2$-Betti numbers of M.

Now consider briefly how the notion of a hypersurface is generalized in the setting of Lie sphere geometry (see Section 5, [Pinkall 1985a] or [Cecil 1992, pp. 65–78] for more detail). We consider T_1S^n, the bundle of unit tangent vectors to S^n, as the $(2n-1)$-dimensional submanifold of $S^n \times S^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ given by

$$T_1S^n = \{(x, \xi) : |x| = 1,\ |\xi| = 1,\ \langle x, \xi \rangle = 0\}.$$

The manifold T_1S^n has a *contact structure*, that is, a globally defined one-form ω such that $\omega \wedge d\omega^{n-1}$ never vanishes on T_1S^n. An immersion $\lambda : M \to T_1S^n$ of an $(n-1)$-dimensional manifold M into T_1S^n is called a *Legendre submanifold* if $\lambda^*\omega = 0$ on M. A Legendre submanifold is determined by two maps f and ξ from M into S^n that satisfy the following three conditions:

(L1) $\langle f(x), f(x) \rangle = 1$, $\langle \xi(x), \xi(x) \rangle = 1$, $\langle f(x), \xi(x) \rangle = 0$, for all $x \in M$.
(L2) There is no nonzero $X \in T_xM$, for any $x \in M$, such that $f_*(X)$ and $\xi_*(X)$ are both zero.
(L3) $\langle f_*(X), \xi(x) \rangle = 0$ for all $X \in T_xM$, for all $x \in M$.

Condition (L1) is precisely what is necessary for $\lambda = (f, \xi)$ to be a map into T_1S^n. Condition (L2) is what is needed for λ to be an immersion, and (L3) is equivalent to $\lambda^*\omega = 0$ on M.

If $f : M \to S^n$ is an immersed hypersurface with field of unit normals ξ, it is easy to check that $\lambda = (f, \xi)$ is a Legendre submanifold. However, for a general Legendre submanifold λ, the map f into the first factor S^n may not be an immersion.

Suppose now that $\lambda = (f, \xi)$ is a Legendre submanifold with maps f and ξ satisfying the conditions (L1)–(L3), and that M is compact. For each $x \in M$, there is a well-defined "normal geodesic" $\gamma(t) = f_t(x)$, where f_t is given by 1.3. We can define the normal exponential map $F : M \times \mathbb{R} \to S^n$ by the same formula 1.4 used in the case where the map f is assumed to be an immersion. As before, we define the focal points of λ to be the critical values of F, and, again by Sard's Theorem, the set C of focal points has measure zero in S^n. If $p \in S^n$ is not a focal point, we define $\mu(p)$ to be the number of points $x \in M$ such that

$p = F(x,t)$ for some $t \in [0, 2\pi)$, that is, the number of normal geodesics that pass through p. Then we can define the total absolute curvature τ of λ by the same formula 1.5 used in the case where f is an immersion. We then define the Legendre submanifold λ to be taut if $\tau(\lambda) = \beta(M)$.

If $\lambda = (f, \xi)$ is a Legendre submanifold, the parallel hypersurface λ_t at signed distance t is defined by $\lambda_t = (f_t, \xi_t)$, where f_t is given by 1.3 and

$$\xi_t(x) = -\sin t\, f(x) + \cos t\, \xi(x).$$

One can show [Pinkall 1985a] that λ and λ_t have precisely the same focal set in S^n, and that, if p is not a focal point, then $\mu(p)$ is the same for λ_t as for λ, since λ_t and λ determine exactly the same family of normal geodesics. Thus, the total absolute curvature τ is invariant under parallel transformation. Although the usual proof of invariance under Möbius transformations does not work for this formulation of tautness when f is not an immersion, Möbius invariance may in fact hold. If so, that would be a satisfactory resolution of the question of the Lie invariance of tautness for Legendre submanifolds.

We close this section by noting that the torus of revolution in the example above was obtained from the taut circle by embedding the xy-plane into $\mathbb{R}^3$ and then forming a surface of revolution. This construction can be generalized to higher dimensions. Let M be a taut compact hypersurface in $\mathbb{R}^{k+1}$ that is disjoint from a hyperplane $\mathbb{R}^k$ in $\mathbb{R}^{k+1}$. Now embed $\mathbb{R}^{k+1}$ into $\mathbb{R}^{n+1}$ as a totally geodesic subspace. Then the hypersurface W obtained by revolving M about the axis $\mathbb{R}^k$ is a taut embedding of $M \times S^{n-k}$ into $\mathbb{R}^{n+1}$. See [Cecil and Ryan 1985, p. 187] for more detail.

2. Isoparametric Submanifolds

An important class of examples of taut submanifolds are the isoparametric submanifolds in $\mathbb{R}^n$ or in S^n. We begin with a discussion of isoparametric hypersurfaces and later treat the case of codimension greater than one.

A hypersurface $f : M \to \mathbb{R}^n$ (or S^n) is said to be *isoparametric* if it has constant principal curvatures. An isoparametric hypersurface in $\mathbb{R}^n$ must be an open subset of a hyperplane, hypersphere, or spherical cylinder $S^k \times \mathbb{R}^{n-k-1}$. This was shown by Levi-Civita [1937] for $n = 3$ and by B. Segre [1938] for arbitrary n. As E. Cartan demonstrated in a remarkable series of papers [Cartan 1938; 1939a; 1939b; 1940], the situation is much more interesting for isoparametric hypersurfaces in S^n. Despite the depth and beauty of Cartan's work, however, this topic was largely ignored until it was revived in the 1970's by Nomizu [1973; 1975] and Münzner [1980; 1981].

Among other things, Cartan showed that isoparametric hypersurfaces come as a parallel family of hypersurfaces; that is, if $f : M \to S^n$ is an isoparametric hypersurface, then so is any parallel hypersurface f_t. Of course, f_t is not an immersion if $\mu = \cot t$ is a principal curvature of M. Then, however, f_t factors through an immersion of the space of leaves M/T_μ of the principal foliation T_μ. Thus, f_t is a submanifold of codimension $m+1$ in S^n, where m is the multiplicity of μ. Much more can be said about this parallel family, however. Münzner [1980; 1981] showed that a parallel family of isoparametric hypersurfaces in S^n always consists of the level sets in S^n of a homogeneous polynomial defined on $\mathbb{R}^{n+1}$. This implies that any local piece of an isoparametric hypersurface can be extended to a unique compact isoparametric hypersurface. Further, he showed that regardless of the number of distinct principal curvatures of M, there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, and these are minimal submanifolds of the sphere. This minimality of the focal submanifolds was established independently by Nomizu [1973; 1975]. Münzner uses this information and a difficult topological argument to prove the following important theorem. The key fact here is that any isoparametric hypersurface divides the sphere into two ball bundles over the two focal submanifolds.

THEOREM 2.1. *The number g of distinct principal curvatures of an isoparametric hypersurface in S^n must be* 1, 2, 3, 4, *or* 6.

Cartan classified isoparametric hypersurfaces with $g \leq 3$ principal curvatures. Of course, if $g = 1$, then M must is umbilic and it must be a great or small sphere. If $g = 2$, then M must be a standard product of two spheres

$$S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad \text{with } r^2 + s^2 = 1.$$

In the case $g = 3$, Cartan [1939a] showed that all the principal curvatures must have the same multiplicity $m \in \{1, 2, 4, 8\}$, and the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of a projective plane FP^2 into S^{3m+1}, where F is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions), $\mathbb{O}$ (Cayley numbers) for $m = 1, 2, 4, 8$, respectively. Thus, up to congruence, there is only one such family for each value of m. This was a remarkable result with a difficult proof. These isoparametric hypersurfaces with three principal curvatures are often referred to as *Cartan hypersurfaces*.

Isoparametric hypersurfaces with $g = 4$ or 6 principal curvatures have yet to be classified. In the case $g = 4$, Ferus, Karcher and Münzner [Ferus et al. 1981] use representations of Clifford algebras to construct for any positive integer m_1 an infinite series of isoparametric hypersurfaces with four principal curvatures having respective multiplicities (m_1, m_2, m_1, m_2), where m_2 is nondecreasing and unbounded in each series. In each case, one of the focal submanifolds is a Clifford–Stiefel manifold (see also [Pinkall and Thorbergsson 1989a; Wang 1988]). This class contains all known examples with $g = 4$ with the exception of two homogeneous examples.

Other notable facts in the case $g = 4$ are that the four principal curvatures can only have two distinct multiplicities m_1 and m_2, and several restrictions on these multiplicities have been obtained [Münzner 1980; 1981; Abresch 1983; Tang 1991; Fang 1995a; 1996]. Also, many of the isoparametric families with $g = 4$

are inhomogeneous [Ozeki and Takeuchi 1975; 1976; Ferus et al. 1981]. Wang [1988] found many results concerning the topology of the Clifford examples, and Wu [1994] showed that there are only finitely many diffeomorphism classes of compact isoparametric hypersurfaces with four distinct principal curvatures.

In the case $g = 6$, Münzner showed that all of the principal curvatures must have the same multiplicity m, and Abresch [1983] showed that m must be 1 or 2. There is only one homogeneous family in each case, and in the case $m = 1$, Dorfmeister and Neher [1985] showed that an isoparametric hypersurface must be homogeneous. However, for $m = 2$, it is still unknown whether or not M must be homogeneous. In the case $m = 1$, Miyaoka [1993a] has shown that a homogeneous isoparametric hypersurface M^6 in S^7 can be obtained as the inverse image under the Hopf fibration $h : S^7 \to S^4$ of an isoparametric hypersurface with three principal curvatures of multiplicity 1 in S^4. She also shows that the two focal submanifolds of M^6 are not congruent, even though they are lifts under h^{-1} of congruent Veronese surfaces in S^4. Thus, these focal submanifolds are two noncongruent minimal taut homogeneous embeddings of $\mathbb{RP}^2 \times S^3$ in S^7. Peng and Hou [1989] gave explicit forms for the isoparametric polynomials of degree six for the homogeneous isoparametric hypersurfaces with $g = 6$. Recently, Fang [1995b] has obtained results concerning the topology of isoparametric and compact proper Dupin hypersurfaces with six principal curvatures.

In a series of papers, Dorfmeister and Neher [1983a; 1983b; 1983c; 1983d; 1985; 1990] gave an algebraic approach to the study of isoparametric hypersurfaces and isoparametric triple systems.

All isoparametric hypersurfaces in S^n are taut. This was established in [Cecil and Ryan 1981] using the results of Münzner. In particular, any normal geodesic to an isoparametric hypersurface is also normal to each parallel hypersurface and to the focal submanifolds. Using Münzner's results, one can show that every nonfocal point $p \in S^n$ lies on exactly $2g$ normal geodesics from points in M; that is, the height function l_p has exactly $2g$ critical points on M. Since Münzner showed that the sum of the $\mathbb{Z}_2$-Betti of an isoparametric hypersurface with g principal curvatures is $2g$, the hypersurface M is taut. A similar argument shows that the focal submanifolds are taut.

In the early 1980's, a theory of isoparametric submanifolds of codimension greater than 1 was introduced independently by several mathematicians [Carter and West 1985b; West 1989; Harle 1982; Strübing 1986; Terng 1985]. An immersed submanifold $\phi : V \to \mathbb{R}^n$ (or S^n) is called *isoparametric* if its normal bundle $N(V)$ is flat and if, for any locally defined normal field ξ that is parallel with respect to the normal connection $\nabla^\perp$, the eigenvalues of the shape operator A_ξ are constant. Recall that $N(V)$ is flat if and only if for every $x \in V$ all the shape operators A_η, where $\eta \in N_x(V)$, are simultaneously diagonalized. In the decade after this definition of isoparametric submanifolds was formulated, intense research by several mathematicians produced a remarkable theory; see [Palais and Terng 1988; Terng 1993] for more detail. Among other things, Terng

[1985] showed that a compact isoparametric submanifold in Euclidean space must lie in a standard hypersphere. Palais and Terng [1987] showed that the only homogeneous isoparametric submanifolds in Euclidean spaces are the principal orbits of the isotropy representations of symmetric spaces, which had been studied extensively by Bott and Samelson [1958], who showed that the orbits are taut. The orbits of the isotropy representations of symmetric spaces, also called *R-spaces*, were studied independently by Takeuchi and Kobayashi [1968], who also showed that they were taut. The class of R-spaces contains many special subclasses of homogeneous submanifolds that were shown to be taut by various special arguments. (See, for example, [Kobayashi 1967; Tai 1968; Wilson 1969; Kuiper 1970; 1980; Ferus 1982; Kühnel 1994].) Later, Hsiang, Palais, and Terng [1988] obtained many facts about the geometry and topology of isoparametric submanifolds, including the fact that they and their focal submanifolds are taut. Thorbergsson [1991] then used the extensive results that had been obtained for isoparametric submanifolds along with the theory of Tits buildings to prove that all isoparametric submanifolds of codimension greater than one in the sphere are homogeneous. (See [Olmos 1993] for an alternate proof of this result.)

In a series of papers, Carter and West [1978; 1981; 1982; 1990] studied the relationship between isoparametric and totally focal submanifolds. Recall that a submanifold $\phi : V \to \mathbb{R}^n$ is said to be *totally focal* if the critical points of every Euclidean distance function L_p are either all nondegenerate or all degenerate. An isoparametric submanifold is totally focal, and the main result of [Carter and West 1990] is that a totally focal submanifold must be isoparametric. However, Terng and Thorbergsson [Terng and Thorbergsson 1997] have noted that there is a gap in the proof of this assertion, specifically in the proof of Theorem 5.1 of [Carter and West 1990].

A slight variation of the notion of isoparametric submanifolds is the following. A submanifold $\phi : V \to \mathbb{R}^n$ (or S^n) is said to have *constant principal curvatures* if, for any smooth curve γ on V and any parallel normal vector field $\xi(t)$ along γ, the shape operator $A_{\xi(t)}$ has constant eigenvalues along γ. If the normal bundle $N(M)$ is flat, then having constant principal curvatures is equivalent to being isoparametric. Heintze, Olmos, and Thorbergsson [Heintze et al. 1991] showed that a submanifold with constant principal curvatures is either isoparametric or a focal submanifold of an isoparametric submanifold. In a related work, Olmos [Olmos 1994] defines the *rank* of a submanifold in Euclidean space to be the maximal number of linearly independent (locally defined) parallel normal vector fields. He then shows that a compact homogeneous irreducible submanifold M substantially embedded in Euclidean space with rank greater than one must be an orbit of the isotropy representation of a simple symmetric space.

In closing this section, we note that many results on real hypersurfaces with constant principal curvatures in complex space forms are surveyed in the article by Niebergall and Ryan [1997] in this volume. There is also a theory of isoparametric hypersurfaces in pseudo-Riemannian space forms [Nomizu 1981; Hahn

1984; 1988; Magid 1985; Kashani 1993b; 1993a; 1992]. Wu [1992] extended the theory of isoparametric submanifolds of arbitrary codimension to submanifolds of hyperbolic space (see also [Zhao 1993]), and Verhóczki [1992] developed a theory of isoparametric submanifolds for Riemannian manifolds that do not have constant curvature. West [1993] and Mullen [1994] have formulated a theory of isoparametric systems on symmetric spaces, while Terng and Thorbergsson [1995] have generalized the notion of isoparametric to submanifolds of symmetric spaces using the concept of equifocal submanifolds. In a different direction, Carter and Şentürk [1994] have considered the space of immersions parallel to a given immersion whose normal bundle has trivial holonomy group. In yet another direction, Niebergall and Ryan [1993; 1994a; 1994b; 1996] have generalized the notions of isoparametric and Dupin hypersurfaces to the context of affine differential geometry.

More generally, Q.-M. Wang [1987; 1988; 1986] has extended Cartan's theory of isoparametric functions to arbitrary Riemannian manifolds. A smooth function $\phi : \tilde{M} \to \mathbb{R}$ on a Riemannian manifold $\tilde{M}$ is said to be *transnormal* if there is a smooth function b such that $|d\phi|^2 = b(\phi)$. The function ϕ is said to be *isoparametric* if it is transnormal and if there exists a smooth function a such that the Laplacian $\Delta\phi = a(\phi)$. This agrees with the definition of an isoparametric function used by Cartan in his work in the case where $\tilde{M}$ is a real space form. Wang shows that if $\tilde{M}$ is complete and ϕ is a transnormal function on $\tilde{M}$, the focal varieties of ϕ (of which there are at most two) are smooth submanifolds of $\tilde{M}$, and each regular level set of ϕ is a tube over either of the focal varieties. Moreover, if $\tilde{M}$ is S^n or $\mathbb{R}^n$, a transnormal function must, in fact, be isoparametric. However, this is not true if $\tilde{M}$ is a hyperbolic space H^n.

Finally, Solomon [1990a; 1990b; 1992] has studied the spectrum of the Laplacian of isoparametric hypersurfaces in S^n with three or four principal curvatures, while Eschenburg and Schroeder [Eschenburg and Schroeder 1991] have studied the behavior of the Tits metric on isoparametric hypersurfaces.

3. Dupin Submanifolds

In this section, we introduce the notion of Dupin submanifolds, beginning with hypersurfaces. Let $f : M \to \mathbb{R}^n$ be an immersed hypersurface. Let ξ be a locally defined field of unit normals to $f(M)$. A *curvature surface* of M is a smooth submanifold S such that, for each point $x \in S$, the tangent space T_xS is equal to a principal space of the shape operator A of M at x. This generalizes the classical notion of a line of curvature on a surface in $\mathbb{R}^3$. The hypersurface M is said to be *Dupin* if

(a) along each curvature surface, the corresponding principal curvature is constant.

The hypersurface M is called *proper Dupin* if, in addition,

(b) the number g of distinct principal curvatures is constant on M.

Several remarks about these definitions are in order. The proofs can be found in [Cecil and Ryan 1985, Section 2.4]. First, if the dimension of a curvature surface S is greater than one, the corresponding principal curvature is automatically constant on S. This is proved using the Codazzi equation. Second, Condition (b) is equivalent to requiring that each continuous principal curvature have constant multiplicity on M.

There is an open dense subset of M on which the multiplicities of the continuous principal curvatures of M are locally constant. (See [Singley 1975], for example.) Suppose now that a continuous principal curvature μ has constant multiplicity m on an open subset $U \subset M$. Then μ and its principal distribution T_μ are smooth on U. Furthermore, again using the Codazzi equation, one can show that T_μ is integrable, and thus it is called the *principal foliation* corresponding to μ. The leaves of this principal foliation are the curvature surfaces corresponding to μ on U. The principal curvature μ is constant along each of its curvature surfaces in U if and only if these curvature surfaces are open subsets of m-dimensional Euclidean spheres or planes. The *focal map* f_μ corresponding to μ is the map that maps $x \in M$ to the focal point $f_\mu(x)$ corresponding to μ, i.e.,

$$f_\mu(x) = f(x) + \frac{1}{\mu(x)}\xi(x).$$

A direct calculation shows that μ is constant along each of its curvature surfaces in U if and only if the focal map f_μ factors through an immersion of the $(n-1-m)$-dimensional space of leaves M/T_μ into $\mathbb{R}^n$.

In summary, on an open subset U on which the number of distinct principal curvatures is constant, Condition (a) is equivalent to requiring that each curvature surface in each principal foliation be an open subset of a Euclidean sphere or plane with dimension equal to the multiplicity of the corresponding principal curvature. On U, Condition (a) is also equivalent to the condition that each focal map be a submanifold of codimension greater than one in $\mathbb{R}^n$.

Like tautness, both the Dupin and proper Dupin conditions are invariant under Möbius transformations and under stereographic projection from S^n to $\mathbb{R}^n$ (see [Cecil and Ryan 1985, pp. 147–148]). These conditions are also invariant under parallel transformations and thus under the group of Lie sphere transformations. A proof of these claims must be formulated in the setting of Lie sphere geometry in order to handle the fact that a parallel hypersurface f_t to $f(M)$ may not be an immersion on all of M [Pinkall 1985a]. See also [Cecil 1992, p. 87].

An important class of compact proper Dupin hypersurfaces in $\mathbb{R}^n$ is obtained by taking the images under stereographic projection of isoparametric hypersurfaces in S^n. Of course, for these examples, the number g of distinct principal curvatures must be 1, 2, 3, 4, or 6. In fact, Thorbergsson [1983a] has shown that this restriction on g holds for any compact proper Dupin hypersurface embedded in $\mathbb{R}^n$ (see Section 4). Thorbergsson first showed that a compact proper Dupin hypersurface M embedded in S^n must be taut and then used this fact to show that M divides the sphere into two ball bundles over the first focal submanifolds on either side of M. One can then invoke Münzner's theorem to get the restriction on g. All of the restrictions on the multiplicities of the principal curvatures that follow from this topological situation also apply in this case as well. For some time, it was conjectured [Cecil and Ryan 1985, p. 184] that every compact proper Dupin hypersurface embedded in S^n was Lie equivalent to an isoparametric hypersurface. This is not true, however, as was shown independently by Pinkall and Thorbergsson [1989a] and by Miyaoka and Ozawa [1989]. Their constructions of counterexamples to this conjecture will be discussed in Section 6.

In contrast with Thorbergsson's result, Pinkall [1985a] showed how to construct a proper Dupin hypersurface with an arbitrary number of distinct principal curvatures having any prescribed multiplicities. (See also [Cecil and Ryan 1985, p. 179].) This is done using the following basic constructions. Start with a Dupin hypersurface W^{n-1} in $\mathbb{R}^n$ and then consider $\mathbb{R}^n$ as the linear subspace $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+1}$. The following constructions yield a Dupin hypersurface M^n in $\mathbb{R}^{n+1}$.

(1) Let M^n be the cylinder $W^{n-1} \times \mathbb{R}$ in $\mathbb{R}^{n+1}$.
(2) Let M^n be the hypersurface in $\mathbb{R}^{n+1}$ obtained by rotating W^{n-1} around an axis $\mathbb{R}^{n-1} \subset \mathbb{R}^n$.
(3) Project W^{n-1} stereographically onto a hypersurface $V^{n-1} \subset S^n \subset \mathbb{R}^{n+1}$. Let M^n be the cone over V^{n-1} in $\mathbb{R}^{n+1}$.
(4) Let M^n be a tube in $\mathbb{R}^{n+1}$ around W^{n-1}.

These constructions introduce a new principal curvature of multiplicity one that is easily seen to be constant along its lines of curvature. The other principal curvatures are determined by the principal curvatures of W^{n-1}, and the Dupin property is preserved for these principal curvatures. These constructions can easily be generalized to produce a new principal curvature of multiplicity m by considering $\mathbb{R}^n$ as a subset of $\mathbb{R}^n \times \mathbb{R}^m$ rather than $\mathbb{R}^n \times \mathbb{R}$. By repeated use of these constructions, Pinkall [1985a] proved the following basic existence theorem, and we repeat his proof here.

THEOREM 3.1. *Given positive integers* $m_1, \dots, m_g$ *with* $m_1 + \cdots + m_g = n-1$, *there exists a proper Dupin hypersurface* M^{n-1} *in* $\mathbb{R}^n$ *with* g *distinct principal curvatures having respective multiplicities* $m_1, \dots, m_g$.

PROOF. The proof is by an inductive construction that will be clear once the first few cases are handled. First we construct a proper Dupin hypersurface M^3 in $\mathbb{R}^4$ with three principal curvatures of multiplicity one. Begin with an open subset U of a torus of revolution in $\mathbb{R}^3$ on which neither principal curvature vanishes. Take M^3 to be the cylinder $U \times \mathbb{R}$ in $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Then M^3 has three distinct principal curvatures at each point, one of which is identically zero. These curvatures are clearly constant along their corresponding lines of curvature. Next, to construct a proper Dupin hypersurface in $\mathbb{R}^5$ with three distinct principal curvatures having respective multiplicities 1, 1, 2, take a cylinder $U \times \mathbb{R}^2$ in $\mathbb{R}^3 \times \mathbb{R}^2$. Finally, to get a proper Dupin hypersurface V^4 in $\mathbb{R}^5$ with four principal curvatures, first invert the hypersurface M^3 above in a three-sphere in $\mathbb{R}^4$ chosen so that the image of M^3 contains an open set W^3 on which no principal curvature vanishes. Now take V^4 to be the cylinder $W^3 \times \mathbb{R}$. □

These constructions only yield a compact proper Dupin hypersurface if the original manifold W^{n-1} is itself a sphere [Cecil 1989]. Otherwise, the number of distinct principal curvatures is not constant on a compact manifold M^n obtained in this way, because there are points where the new principal curvature is equal to one of the original principal curvatures. For example, in the cylinder construction, the new principal curvature is identically zero, while the other principal curvatures of M^n are equal to those of W^{n-1}. Thus, if one of the principal curvatures of W^{n-1} is zero at some points but not identically zero, the number of distinct principal curvatures is not constant on M^n. For a tube of radius ε over W^{n-1}, there are always only two distinct principal curvatures at the points on the set $W^{n-1} \times \{\pm\varepsilon\}$ in M^n, regardless of the number of distinct principal curvatures on W^{n-1}. In the surface of revolution construction, the new principal curvature is equal to one of the original principal curvatures if the focal point corresponding to the original principal curvature lies on the axis of revolution.

A second problem is that the constructions may not yield an immersed hypersurface in $\mathbb{R}^{n+1}$. For example, in the tube construction, a singularity occurs if the radius of the tube is the reciprocal of one of the principal curvatures of W^{n-1} at some point, that is, if the tube contains a focal point of W^{n-1}. In the surface of revolution construction, a singularity occurs if W^{n-1} intersects the axis of revolution. These problems are resolved by working in the context of Lie sphere geometry. See [Cecil 1989; Cecil 1992, Section 4.2] for more detail.

A proper Dupin hypersurface that is locally Lie equivalent to a hypersurface M^n obtained by one of these constructions is said to be *reducible*, and a proper Dupin hypersurface that does not contain any reducible open subset is said to be *locally irreducible*. These are useful concepts in attempting to obtain local classifications of proper Dupin hypersurfaces (see Section 7).

In order to extend the notion of Dupin to submanifolds of codimension greater than one, we first need the definition, due to Reckziegel [1979], of a curvature surface in that case. Suppose that $\phi : V \to \mathbb{R}^n$ is a submanifold of codimension greater than one, and let B^{n-1} denote the unit normal bundle of $\phi(V)$. Then a *curvature surface* is a connected submanifold $S \subset V$ for which there is a parallel section $\eta : S \to B^{n-1}$ such that, for each $x \in S$, the tangent space T_xS is equal to some smooth eigenspace of the shape operator A_η. We then define $\phi(V)$ to be *Dupin* if, along each curvature surface, the corresponding principal curvature of A_η is constant. The Dupin submanifold $\phi(V)$ is *proper Dupin* if the number of distinct principal curvatures of A_ξ is constant on the unit normal bundle B^{n-1}. From the definitions, it is clear that an isoparametric submanifold is always Dupin, but it may not be proper Dupin. (See [Terng 1993, pp. 464–469] for more discussion.) Dupin submanifolds of codimension greater than one are handled quite naturally in the setting of Lie sphere geometry (see Section 5).

Pinkall [1985a, p. 439] showed that every extrinsically symmetric submanifold of a real space form is Dupin. Takeuchi [Takeuchi 1991] then determined which of these are proper Dupin.

4. Relationship Between the Taut and Dupin Conditions

All of the results in this section apply to submanifolds of S^n as well as to those in $\mathbb{R}^n$.

Thorbergsson [1983a] showed that if M is a complete proper Dupin hypersurface embedded in $\mathbb{R}^n$, then M is tautly embedded. Pinkall [1986] then showed the following.

Theorem 4.1. *Every taut submanifold in* $\mathbb{R}^n$ *is Dupin.*

This result was also obtained independently by Miyaoka [1984b]. (See also [Cecil and Ryan 1985, p. 195] for a proof.) Note that a taut submanifold need not be proper Dupin. For example, a tube M^3 of sufficiently small radius ε over a torus of revolution $T^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ is taut but not proper Dupin, since there are only two distinct principal curvatures on the set $T^2 \times \pm\{\varepsilon\}$ but three distinct principal curvatures elsewhere on M. Many isoparametric submanifolds of codimension greater than one in S^n are also not proper Dupin; that is, the number of distinct principal curvatures is not constant on the unit normal bundle.

Pinkall also showed that, if M is an embedded submanifold of $\mathbb{R}^n$ and M_ε is a tube of sufficiently small radius ε as to be embedded in $\mathbb{R}^n$, then M is taut with respect to $\mathbb{Z}_2$-coefficients if and only if M_ε is taut with respect to $\mathbb{Z}_2$-coefficients. This can be combined with Thorbergsson's result to yield the following.

Theorem 4.2. *Let M be a complete proper Dupin submanifold of* $\mathbb{R}^n$. *Then M is taut with respect to* $\mathbb{Z}_2$*-coefficients.*

Using tautness, Thorbergsson then showed that a compact proper Dupin hypersurface embedded in S^n divides the sphere into two ball bundles over the first focal submanifolds on either side of M in S^n. He could then invoke Münzner's [1980; 1981] results to obtain the following.

Theorem 4.3. *Let M be a compact proper Dupin hypersurface in* $\mathbb{R}^n$. *Then*:

(1) *The number g of distinct principal curvatures of M must be* 1, 2, 3, 4, *or* 6.
(2) *The sum of the* $\mathbb{Z}_2$*-Betti numbers of M is* $2g$.

Later, Grove and Halperin [1987] derived many other results about the topology of compact proper Dupin hypersurfaces.

One consequence of Theorem 4.3 is that any compact proper Dupin hypersurface M with $g \geq 3$ principal curvatures must be irreducible [Cecil 1989, p. 297; Cecil 1992, p. 148]. To see this, suppose that M is reducible to a compact proper Dupin hypersurface V in a lower-dimensional Euclidean space. If $g \geq 3$, then M must be obtained from V by the surface of revolution construction, since a compact proper Dupin hypersurface obtained via the other constructions always has some points where the number of distinct principal curvatures is two. In that case, M is diffeomorphic to $V \times S^m$ for some positive integer m, and the sum $\beta(M)$ of the $\mathbb{Z}_2$-Betti numbers of M is

$$\beta(M) = \beta(V \times S^m) = 2\beta(V). \tag{4.1}$$

An analysis of the surface of revolution construction shows that the number k of distinct principal curvatures on V must be $g-1$ or g. Since $k = 2\beta(V)$ by Theorem 4.3, and $\beta(V) \neq \beta(M)$, it is impossible for V and M to have the same number of distinct principal curvatures. So $k = g-1$, and 4.1 implies that $2g = 4k = 4g-4$. This implies that $g = 2$, a contradiction, which shows that M is not reducible.

In the case where M is compact, we can use a theorem of Ozawa [1986] to prove something slightly stronger than Theorem 4.1. As we noted in Section 3, if a curvature surface has dimension greater than one, the corresponding principal curvature is always constant along it, even without the assumption of tautness. Thus, Pinkall's proof of Theorem 4.1 consisted in showing that tautness implies that each principal curvature is constant along each of its curvature surfaces of dimension one (lines of curvature). Note that we are using Pinkall's definition of Dupin, which does not insist that given any principal space T_μ at any point $x \in M$ there exists a curvature surface S through x whose tangent space at x is T_μ. However, using the next theorem, due to Ozawa [1986], we will show as a corollary that tautness implies the existence of such a curvature surface at each $x \in M$. (See [Terng and Thorbergsson 1997] in this volume for a generalization of Ozawa's theorem for taut immersions into arbitrary complete Riemannian manifolds.)

THEOREM 4.4. *Let M be a taut compact submanifold of $\mathbb{R}^n$, and let L_p be a Euclidean distance function on M. Let $x \in M$ be a critical point of L_p and let S be the connected component of the critical set of L_p that contains x. Then S is*

(1) *a smooth compact manifold of dimension equal to the nullity of the Hessian of L_p at the critical point x,*
(2) *nondegenerate as a critical manifold, and*
(3) *taut in $\mathbb{R}^n$.*

Using this result, we can prove the following theorem.

Theorem 4.5. *Let M be a taut compact submanifold of $\mathbb{R}^n$. Then*:

(1) *M is a Dupin submanifold.*

(2) *Given any principal space T_μ of any shape operator A_ξ at any point $x \in M$, there exists a curvature surface S through x whose tangent space at x is equal to T_μ.*

Proof. Let $f : M \to \mathbb{R}^n$ be a taut embedding. Let ξ be any unit normal vector at any given point $x \in M$, and let μ be a principal curvature of A_ξ. Let $p = f(x) + (1/\mu)\xi$ be the focal point of (M, x) determined by the principal curvature μ of A_ξ. Then the distance function L_p has a degenerate critical point at x, and the nullity of the Hessian of L_p at x is equal to the multiplicity m of μ as a principal curvature of A_ξ [Milnor 1963, p. 36]. By Theorem 4.4, the connected component S of the critical set of L_p containing x is a smooth submanifold of dimension m. We will now show that S is the desired curvature surface and that the corresponding principal curvature is constant along S, i.e., that M is Dupin.

The function L_p has a constant value, which must be $1/\mu^2$, on the critical submanifold S. Thus, for every point $y \in S$, the vector $p - f(y)$ is normal to $f(M)$ at $f(y)$, and it has length $1/\mu$. So we can extend the normal vector ξ to a unit normal vector field to $f(M)$ along S, which we also denote by ξ, by setting $\xi(y) = \mu\big(p - f(y)\big)$. Note that p is a focal point of (M, y) for every point $y \in S$, and Theorem 4.4 implies that the number μ is a principal curvature of $A_{\xi(y)}$ of multiplicity $m = \dim S$ for every point $y \in S$. Thus, the principal curvature μ is constant along S. We now complete the proof of Theorem 4.5 by showing that T_yS equals the principal space $T_\mu(y)$ at each point $y \in S$ and that the normal field ξ is parallel along S with respect to the normal connection. Consider the focal map

$$f_\mu(y) = f(y) + \frac{1}{\mu}\xi(y),$$

for $y \in S$. Then $f_\mu(y) = p$ for all $y \in S$. Let X be any tangent vector to S at any point $y \in S$. Then $(f_\mu)_*X = 0$, since f_μ is constant on S. On the other hand,

$$(f_\mu)_*X = f_*X + \frac{1}{\mu}\xi_*X,$$

and $\xi_*X = D_X\xi = f_*(-A_\xi X) + \nabla^\perp_X\xi$. Therefore,

$$(f_\mu)_*X = f_*\Big(X - \frac{1}{\mu}A_\xi X\Big) + \frac{1}{\mu}\nabla^\perp_X\xi.$$

Since $(f_\mu)_*X = 0$, we see that $A_\xi X = \mu X$ and $\nabla^\perp_X\xi = 0$. Thus, ξ is parallel along S and $T_yS \subset T_\mu(y)$. But since T_yS and $T_\mu(y)$ have the same dimension, they must be equal. So S is the curvature surface through y corresponding to μ. □

Certainly, a major open problem in this area is whether the converse to Theorem 4.5 is true. That is, suppose M is a compact Dupin submanifold embedded in $\mathbb{R}^n$ with the property that, given any principal space T_μ of any shape operator A_ξ at any point $x \in M$, there exists a curvature surface S through x whose tangent space at x is equal to T_μ. Must M be taut? Thorbergsson's proof in the case where M is proper Dupin relies on the fact that all of the curvature surfaces are spheres, and this is not true if M is not proper Dupin.

Tautness has been established for *Dupin submanifolds with constant multiplicities* by Terng [1987; 1993, p. 467]. These are Dupin submanifolds M such that the multiplicities of the principal curvatures of any parallel normal field $\xi(t)$ along any piecewise smooth curve on M are constant.

5. Submanifolds in Lie Sphere Geometry

In this section, we give a brief description of the method for studying submanifolds of Euclidean space and the sphere using Lie sphere geometry (see [Cecil 1992; Cecil and Chern 1987; Chern 1991; Pinkall 1985a] for more detail). As we noted earlier, the Dupin property is invariant under stereographic projection from $\mathbb{R}^n$ to S^n (see [Cecil and Ryan 1985, pp. 147–148]). At times, it is simpler to work in S^n, and we will give our description in those terms here. The formulation given here in terms of projective geometry has some advantages over the formulation given in Section 1 in terms of the unit tangent bundle to S^n. In practice, it is helpful to keep both models in mind.

Let $\mathbb{R}^{n+3}_2$ be a real vector space of dimension $n+3$ endowed with a metric of signature $(n{+}1,\, 2)$,

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + \cdots + x_{n+2}y_{n+2} - x_{n+3}y_{n+3}. \tag{5.1}$$

Let $e_1, \ldots, e_{n+3}$ denote the standard orthonormal basis with respect to this metric, with e_1 and e_{n+3} timelike. Let P^{n+2} be the real projective space of lines through the origin in $\mathbb{R}^{n+3}_2$, and let Q^{n+1} be the quadric hypersurface determined by the equation $\langle x, x\rangle = 0$. This hypersurface is called the *Lie quadric*. We consider S^n to be the unit sphere in the Euclidean space $\mathbb{R}^{n+1}$ spanned by the vectors $e_2, \ldots, e_{n+2}$.

The points in Q^{n+1} are in bijective correspondence with the set of all oriented hyperspheres and point spheres in S^n. Specifically, the oriented hypersphere with center $p \in S^n$ and signed radius ρ corresponds to the point $[(\cos\rho, p, \sin\rho)]$ in Q^{n+1}, where the square brackets denote the point in P^{n+2} given by the homogeneous coordinates within the parentheses. The point spheres in S^n correspond to those points with $\rho = 0$.

The Lie quadric contains projective lines but no linear subspaces of P^{n+2} of higher dimension. The line $[x, y]$ determined by two points $[x]$ and $[y]$ of Q^{n+1} lies on the quadric if and only if $\langle x, y\rangle = 0$. In terms of the geometry of S^n, this means that the two hyperspheres corresponding to $[x]$ and $[y]$ are in oriented contact. The points on a line on the quadric correspond to the parabolic pencil of oriented hyperspheres in S^n in oriented contact at a point (p, ξ) in the unit tangent bundle T_1S^n to S^n, where ξ is a unit tangent vector to S^n at the point p. This leads to a natural diffeomorphism from T_1S^n to the manifold Λ^{2n-1} of projective lines on Q^{n+1} given by $(p, \xi) \to [k_1, k_2]$, where $k_1 = (1, p, 0)$ and $k_2 = (0, \xi, 1)$. In terms of the geometry of spheres in S^n, k_1 corresponds to the point sphere in the parabolic pencil and k_2 corresponds to the great sphere in the pencil. We will refer to the elements of T_1S^n as *contact elements*.

A *Lie sphere transformation* is a projective transformation of P^{n+2} that maps Q^{n+1} to itself. In terms of the geometry of S^n, a Lie sphere transformation maps oriented hyperspheres to oriented hyperspheres. Furthermore, a Lie sphere transformation preserves oriented contact of spheres, since it takes lines on Q^{n+1} to lines on Q^{n+1}. The group of Lie sphere transformations is isomorphic to $O(n+1, 2)/\{\pm I\}$, where $O(n+1, 2)$ is the orthogonal group for the metric in 5.1. A *Möbius transformation* is a Lie sphere transformation that takes point spheres to point spheres. As a transformation on S^n itself, a Möbius transformation is conformal. The Lie sphere group is generated by Möbius transformations and parallel transformations P_t, which fix the center of each sphere but add t to its signed radius.

The manifold Λ^{2n-1} has a *contact structure*, that is, a globally defined one-form ω such that $\omega \wedge d\omega^{n-1}$ never vanishes on Λ^{2n-1}. The condition $\omega = 0$ defines a codimension-one distribution D on Λ^{2n-1} that has integral submanifolds of dimension $n-1$ but none of higher dimension. A *Legendre submanifold* is one of these integral submanifolds of maximal dimension, that is, an immersion $\lambda : M^{n-1} \to \Lambda^{2n-1}$ such that $\lambda^*\omega = 0$.

A Legendre submanifold is determined by two functions k_1, k_2 from an $(n-1)$-dimensional manifold M to $\mathbb{R}^{n+3}_2$ satisfying these conditions:

(L1) For all $x \in M$, the vectors $k_1(x)$ and $k_2(x)$ are linearly independent and $\langle k_i(x), k_j(x)\rangle = 0$, for $i, j = 1, 2$.

(L2) There is no nonzero $X \in T_xM$, for any $x \in M$, such that $dk_1(X)$ and $dk_2(X)$ are both in $\mathrm{Span}\{k_1(x), k_2(x)\}$.

(L3) $\langle dk_1(X), k_2(x)\rangle = 0$, for all $X \in T_xM$, for all $x \in M$.

The Legendre submanifold is then defined by $\lambda(x) = [k_1(x), k_2(x)]$. Conditions (L1)–(L3) are preserved if one reparametrizes by taking $\tilde{k}_1 = \alpha k_1 + \beta k_2$ and $\tilde{k}_2 = \gamma k_1 + \delta k_2$, where $\alpha, \beta, \gamma, \delta$ are smooth real-valued functions on M such that $\alpha\delta - \beta\gamma$ never vanishes.

Condition (L1) means that k_1 and k_2 determine a line on the quadric for each $x \in M$. Condition (L2) means that λ is an immersion, and (L3) means that $\lambda^*\omega = 0$. These conditions correspond precisely to the conditions (L1)–(L3) given in Section 1, where we used T_1S^n rather than Λ^{2n-1} as our contact manifold.

An immersion $f : M^{n-1} \to S^n$ with field of unit normals $\xi : M^{n-1} \to S^n$ naturally induces a Legendre submanifold $\lambda = [k_1, k_2]$, where

$$k_1 = (1, f, 0), \quad k_2 = (0, \xi, 1). \tag{5.2}$$

For each $x \in M^{n-1}, [k_1(x)]$ is the point sphere in the pencil of spheres in S^n corresponding to $\lambda(x)$, and $[k_2(x)]$ is the great sphere in the pencil.

An immersed submanifold $\phi : V \to S^n$ of codimension greater than one also induces a Legendre submanifold whose domain is the bundle B^{n-1} of unit normal vectors to $\phi(V)$. For a unit normal ξ to $\phi(V)$ at a point $\phi(v)$, we define $\lambda(v, \xi)$ to be the line on Q^{n+1} corresponding to the contact element $(\phi(v), \xi)$. In this case, the point sphere map $k_1(v, \xi) = (1, \phi(v), 0)$ has constant rank equal to the dimension of V. For a general Legendre submanifold λ, the point sphere map does not have constant rank.

A Lie sphere transformation β maps lines on Q^{n+1} to lines on Q^{n+1}, so it naturally induces a map $\tilde{\beta}$ from Λ^{2n-1} to itself. If λ is a Legendre submanifold, then $\tilde{\beta}\lambda$ is also a Legendre submanifold, which is denoted $\beta\lambda$ for short. These two Legendre submanifolds are said to be *Lie equivalent*. If β is a Möbius transformation, then the two Legendre submanifolds are said to be *Möbius equivalent*. Finally, if β is the parallel transformation P_t and λ is the Legendre submanifold induced by an immersed hypersurface $f : M \to S^n$, then $P_t\lambda$ is the Legendre submanifold induced by the parallel hypersurface f_{-t} (see [Cecil 1992, p. 88]).

Suppose that $\lambda = [k_1, k_2]$ is a Legendre submanifold. Let $x \in M$ and let r and s be real numbers at least one of which is nonzero. The sphere corresponding to the point

$$[K] = [rk_1(x) + sk_2(x)]$$

is called a *curvature sphere* of λ at x if there exists a nonzero $X \in T_xM$ such that

$$rdk_1(X) + sdk_2(X) \in \operatorname{Span}\{k_1(x), k_2(x)\}.$$

This definition is invariant under a reparametrization of λ by a pair $\{\tilde{k}_1, \tilde{k}_2\}$.

To see the relationship between curvature spheres and principal curvatures, suppose now that $\lambda = [k_1, k_2]$ as in 5.2. At a given $x \in M$, we can write the distinct curvature spheres in the form

$$[K_i] = [\mu_i k_1 + k_2], \quad \text{for } 1 \leq i \leq g. \tag{5.3}$$

When the map f in 5.2 is an immersion, these μ_i are the principal curvatures of f at x. In terms of the geometry of S^n, the curvature sphere at x corresponding to a principal curvature μ_i is the oriented hypersphere in oriented contact with $f(M)$ at $f(x)$ and centered at the focal point determined by the principal curvature μ_i.

We refer to the μ_i in 5.3 as the *principal curvatures* of λ. These principal curvatures are not Lie invariant, and they depend on the special parametrization 5.2 for λ. However, Miyaoka [1989a] pointed out that the cross-ratio of any four of these principal curvatures is Lie invariant. These cross-ratios are known as the *Lie curvatures* of λ. Such a cross-ratio is Lie invariant because it is equal to the cross-ratio of the corresponding four curvature spheres on the line $\lambda(x)$. Since a Lie sphere transformation β is a projective transformation and it maps the curvature spheres of λ to the curvature spheres of $\beta\lambda$, it preserves these cross-ratios.

A Legendre submanifold is said to be *Dupin* if, along each curvature surface, the corresponding curvature sphere map is constant, and it is *proper Dupin* if the number of distinct curvature spheres is constant. Pinkall [1985a] showed that both of these concepts are invariant under Lie sphere transformation. In the case where a Legendre submanifold is induced from a submanifold of S^n, these definitions agree with those given in Section 3.

Recall from Section 3 that a proper Dupin submanifold is reducible if it is Lie equivalent to a proper Dupin submanifold obtained as a result of one of Pinkall's four standard constructions. Pinkall [1985a] found a simple formulation for reducibility in terms of Lie sphere geometry as follows.

THEOREM 5.1. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a proper Dupin submanifold with distinct curvature spheres $K_1, \dots K_g$. Then λ is reducible if and only if, for some i with $1 \le i \le g$, the image of the curvature sphere map K_i is contained in an n-dimensional linear subspace of P^{n+2}.*

Another important question in classifying proper Dupin submanifolds is when is the submanifold Lie equivalent to an isoparametric hypersurface in S^n. This condition also has a natural formulation in Lie sphere geometry. Recall that a line in P^{n+2} is said to be *timelike* if it contains only timelike points. This means that that an orthonormal basis for the two-plane in $\mathbb{R}^{n+3}_2$, determined by the timelike line, consists of two timelike vectors. An example is the line $[e_1, e_{n+3}]$. The following theorem was obtained in [Cecil 1990].

THEOREM 5.2. *Let $\lambda : M^{n-1} \to \Lambda^{2n-1}$ be a Legendre submanifold with g distinct curvature spheres $K_1, \dots, K_g$ at each point. Then λ is Lie equivalent to the Legendre submanifold induced by an isoparametric hypersurface in S^n if and only if there exist g points $P_1, \dots, P_g$ on a timelike line in P^{n+2} such that $\langle K_i, P_i \rangle = 0$ for $1 \le i \le g$.*

For more general considerations of Lie contact structures on manifolds, see [Miyaoka 1991b; 1991a; 1993b].

6. Compact Proper Dupin Submanifolds

In this section, we consider compact proper Dupin submanifolds embedded in the sphere S^n. Since a tube of sufficiently small radius ε over a compact proper Dupin submanifold of codimension greater than one is a compact proper Dupin hypersurface, we will for the most part restrict our attention to the codimension-one case. Let M be a compact proper Dupin hypersurface embedded in S^n with

g distinct principal curvatures. As noted in Theorem 4.3, the number g must be 1, 2, 3, 4, or 6. Of course, in the case $g = 1$, the hypersurface M is totally umbilic and must be a great or small hypersphere in S^n. The case $g = 2$ was handled in [Cecil and Ryan 1978]:

THEOREM 6.1. *A compact, connected proper Dupin hypersurface embedded in S^n with two distinct principal curvatures is Möbius equivalent to an isoparametric hypersurface, that is, a standard product of two spheres.*

Next Miyaoka [1984a] handled the case $g = 3$, where the full Lie sphere group was needed to get equivalence with an isoparametric hypersurface:

THEOREM 6.2. *A compact, connected proper Dupin hypersurface embedded in S^n with three distinct principal curvatures is Lie equivalent to an isoparametric hypersurface.*

For some time after that, it was widely thought that every compact, connected proper Dupin hypersurface embedded in S^n is Lie equivalent to an isoparametric hypersurface [Cecil and Ryan 1985, p. 184]. Further evidence for this conjecture was provided by Grove and Halperin [1987], who found many restrictions on the topology of a compact proper Dupin hypersurface and on the multiplicities of its principal curvatures. These restrictions included almost all of the known restrictions for isoparametric hypersurfaces due to Münzner [1980; 1981] and Abresch [1983]. However, in 1989, the conjecture was shown to be false by Pinkall and Thorbergsson [1989a] and Miyaoka and Ozawa [1989], who independently gave different methods for constructing counterexamples to the conjecture in the case $g = 4$. The method of Miyaoka and Ozawa also works in the case $g = 6$.

These constructions are also described in detail in [Cecil 1992]. The construction of Pinkall and Thorbergsson begins with the isoparametric hypersurfaces constructed using representations of Clifford algebras by Ferus, Karcher, and Münzner [Ferus et al. 1981]. For each of these isoparametric hypersurfaces, one of the focal submanifolds is a so-called Clifford–Stiefel manifold. Here we will only describe the simplest case where the Clifford algebra is $\mathbb{R}$.

Let $\mathbb{R}^{2n+2} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and let S^{2n+1} denote the unit sphere in $\mathbb{R}^{2n+2}$. The Stiefel manifold V of orthogonal two-frames in $\mathbb{R}^{n+1}$ of length $1/\sqrt{2}$ is given by

$$V = \{(u,v) \in \mathbb{R}^{2n+2} : u \cdot v = 0, |u| = |v| = 1/\sqrt{2}\}.$$

The submanifold V lies in S^{2n+1} with codimension two, so V has dimension $2n - 1$. Note that

$$V \subset F^{-1}(0) \cap G^{-1}(0),$$

where F and G are the real-valued functions defined on S^{2n+1} by

$$F(u,v) = (v \cdot v - u \cdot u)/2, \quad G(u,v) = -u \cdot v.$$

The gradients $\xi = (-u, v)$ and $\eta = (-v, -u)$ of F and G are two orthogonal fields of unit normals to V in S^{2n+1}. One can show by a direct calculation

[Cecil 1992, pp. 115–117; Pinkall and Thorbergsson 1989a] that, at every point of V, the shape operators A_ξ and A_η have three distinct principal curvatures $-1, 0, 1$, with respective multiplicities $n-1$, 1, $n-1$, although A_ξ and A_η are not simultaneously diagonalizable. From this one can calculate that if ζ is any unit normal at any point of V, then A_ζ has these same eigenvalues and multiplicities. Thus, the principal curvatures

$$-1 = \cot(3\pi/4), \quad 0 = \cot(\pi/2), \quad 1 = \cot(\pi/4)$$

are constant on the unit normal bundle $B(V)$. Therefore, a tube M_t^{2n} of radius t around V in S^{2n+1} has four constant principal curvatures (see [Cecil and Ryan 1985, pp. 131–132])

$$\cot(3\pi/4 - t),\ \cot(\pi/2 - t),\ \cot(\pi/4 - t),\ \cot(-t), \tag{6.1}$$

with respective multiplicities $n-1$, 1, $n-1$, 1. This is the family of isoparametric hypersurfaces with focal submanifold V. For a general Clifford algebra, one considers the Clifford–Stiefel manifold of "Clifford orthogonal" pairs of vectors in defining the focal submanifold [Pinkall and Thorbergsson 1989a].

For a proper Dupin submanifold with $g = 4$, we can order the principal curvatures,

$$\mu_1 < \mu_2 < \mu_3 < \mu_4, \tag{6.2}$$

and thereby determine a unique Lie curvature Ψ by the cross-ratio

$$\Psi = \frac{(\mu_4 - \mu_3)(\mu_1 - \mu_2)}{(\mu_4 - \mu_2)(\mu_1 - \mu_3)}. \tag{6.3}$$

Note that the ordering 6.2 implies that $0 < \Psi < 1$. Münzner [1980] showed that the principal curvatures of an isoparametric hypersurface with $g = 4$ must always take the values in 6.1 for an appropriate value of t, and one can directly compute that Ψ always takes the constant value $\frac{1}{2}$ on an isoparametric hypersurface with four principal curvatures. In terms of projective geometry, this means that the four curvature spheres along each line $\lambda(x)$ form a harmonic set.

The construction of Pinkall and Thorbergsson then proceeds as follows. Let α and β be positive real numbers satisfying $\alpha^2 + \beta^2 = 1$, and let $T_{\alpha,\beta}$ be the linear transformation of $\mathbb{R}^{2n+2}$ defined by

$$T_{\alpha,\beta}(u, v) = \sqrt{2}(\alpha u, \beta v).$$

The image $W^{\alpha,\beta} = T_{\alpha,\beta} V$ is contained in S^{2n+1}, and it is proper Dupin. To see this, one first notes that $W^{\alpha,\beta}$ has codimension two in S^{2n+1}, just as V does. This means that the point sphere map k_1 in 5.2 is a curvature sphere in both cases, and so by 5.3 they both have a constant principal curvature $\kappa_4 = \infty$ of multiplicity one (see [Cecil 1992, p. 97] for more detail on this point). For the other principal curvatures, note that a hypersphere Σ that is tangent to V along a curvature surface S lies in a hyperplane π in $\mathbb{R}^{2n+2}$ that is tangent to V along S. The image $T_{\alpha,\beta}(\pi)$ cuts S^{2n+1} in a hypersphere $\tilde{\Sigma}$ that is tangent to $W^{\alpha,\beta}$ along

$T(S)$. Thus, $\tilde{\Sigma}$ is a curvature sphere of $W^{\alpha,\beta}$ with corresponding curvature surface $T(S)$, and $\tilde{\Sigma}$ is constant along $T(S)$. Therefore, we have established a bijective correspondence between the curvature surfaces of V and those of $W^{\alpha,\beta}$ and have shown that the Dupin condition is satisfied on $W^{\alpha,\beta}$. Hence, $W^{\alpha,\beta}$ is a proper Dupin submanifold with four distinct principal curvatures, including $\kappa_4 = \infty$.

The other principal curvatures of $W^{\alpha,\beta}$ can be computed in the same way as for V. Then one can find a certain unit normal ζ to $W^{\alpha,\beta}$ such that the shape operator A_ζ has principal curvatures

$$\kappa_1 = -\alpha/\beta, \quad \kappa_2 = 0, \quad \kappa_3 = \beta/\alpha,$$

with respective multiplicities $n-1$, 1, $n-1$. When these principal curvatures are taken along with $\kappa_4 = \infty$, the cross-ratio Ψ equals α^2. Thus, if $\alpha \neq 1/\sqrt{2}$, that is, if $T_{\alpha,\beta} \neq I$, then the Lie curvature Ψ is not $\frac{1}{2}$ at this point ζ, and $W^{\alpha,\beta}$ is not Lie equivalent to an isoparametric hypersurface. In fact, one can show that the Lie curvature is not constant on $W^{\alpha,\beta}$ if $\alpha \neq 1/\sqrt{2}$.

The construction of Miyaoka and Ozawa [Miyaoka and Ozawa 1989] (see also [Cecil 1992, pp. 120–128]) uses the *Hopf fibration* of S^7 over S^4. Let $\mathbb{R}^8 = \mathbb{H}\times\mathbb{H}$, where $\mathbb{H}$ is the division ring of quaternions. The Hopf fibering of the unit sphere S^7 in $\mathbb{R}^8$ over the unit sphere S^4 in $\mathbb{R}^5 = \mathbb{H}\times\mathbb{R}$ is given by

$$h(u,v) = (2u\bar{v},\, |u|^2 - |v|^2), \quad \text{for } u, v \in \mathbb{H}.$$

Miyaoka and Ozawa begin by showing that, if M is a taut compact submanifold of S^4, then $h^{-1}M$ is taut in S^7. They use this to show that if M is proper Dupin in S^4 with g distinct principal curvatures, then $h^{-1}M$ is proper Dupin in S^7 with $2g$ principal curvatures. Finally, they show that if M is proper Dupin but not isoparametric, then the Lie curvatures of $h^{-1}M$ are not constant, and so $h^{-1}M$ is not Lie equivalent to an isoparametric hypersurface with $2g$ principal curvatures in S^7. Taking $g = 2$ or 3, respectively, yields a compact proper Dupin hypersurface with 4 or 6 principal curvatures in S^7 that is not Lie equivalent to an isoparametric hypersurface.

As noted in Section 2, Miyaoka [1993a] has recently shown that if M is an isoparametric hypersurface with three principal curvatures in S^4, then $h^{-1}M$ is an isoparametric hypersurface with six principal curvatures in S^7. According to Dorfmeister and Neher [Dorfmeister and Neher 1985], this family of isoparametric hypersurfaces with six principal curvatures in S^7 is unique up to congruence.

A problem for further study is to determine the strength of the assumption that the Lie curvatures are constant on a proper Dupin submanifold. Miyaoka [Miyaoka 1989a; Miyaoka 1989b] proved that this assumption on a compact proper Dupin hypersurface M in S^n with $g = 4$ or 6 principal curvatures, together with an additional assumption regarding the intersections of curvature surfaces from different principal foliations, implies that M is Lie equivalent to an isoparametric hypersurface. It is not known whether this additional assumption on the intersections of the curvature surfaces can be dropped. However, the compactness assumption of Miyaoka is definitely needed because there exist noncompact proper Dupin hypersurfaces with $g = 4$ on which $\Psi = c$, for every value $0 < c < 1$ [Cecil 1990; 1992, pp. 106–108]. These examples are all obtained as open subsets of a tube in S^n over an isoparametric hypersurface with three principal curvatures $V^{k-1} \subset S^k \subset S^n$, and they cannot be completed to be compact proper Dupin hypersurfaces. They are also reducible as Dupin hypersurfaces.

7. Local Results on Dupin Submanifolds

Most local classifications of proper Dupin submanifolds have been obtained in the context of Lie sphere geometry. We will state these results for hypersurfaces of S^n, since we can always arrange that the point sphere map of a Legendre submanifold is locally an immersion by taking a parallel submanifold if necessary.

The known results depend on the number g of distinct principal curvatures, and they are progressively harder to prove as g increases. In fact, results have only been obtained up to the case $g = 4$, and much remains to be done in that case. Of course, a connected proper Dupin hypersurface in S^n with one distinct principal curvature must be an open subset of a hypersphere. In the case $g = 2$, Pinkall [1985a] has obtained a complete classification, which we now describe. A proper Dupin hypersurface in S^n (or $\mathbb{R}^n$) with two distinct principal curvatures of respective multiplicities p and q is called a *cyclide of Dupin of characteristic* (p, q). The compact cyclides embedded in $\mathbb{R}^3$ can all be obtained through stereographic projection from a standard product of two circles in the unit sphere $S^3 \subset \mathbb{R}^4$. This construction obviously can be generalized to higher dimensions. Cecil and Ryan [1978] showed that a connected, compact cyclide M^{n-1} of characteristic (p, q) embedded in S^n must be Möbius equivalent to a standard product of spheres

$$S^p \times S^q \subset S^n(1) \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} = \mathbb{R}^{n+1}, \quad \text{for } r^2 + s^2 = 1,\ n = p + q + 1.$$

Varying the value of r in this product produces a family of parallel hypersurfaces. These are Lie equivalent by parallel transformation, but they are not Möbius equivalent for different values of r.

The proof of Cecil and Ryan uses the compactness assumption in an essential way, whereas the classification of Dupin surfaces in $\mathbb{R}^3$ obtained in the nineteenth century does not need such an assumption (see [Cecil and Ryan 1985, pp. 151–166], for example). Using Lie sphere geometry, Pinkall obtained the following classification in arbitrary dimensions, which does not need the assumption of compactness.

THEOREM 7.1. (a) *Every connected cyclide of Dupin is contained in a unique compact, connected cyclide.*

(b) *Any two cyclides of the same characteristic are locally Lie equivalent.*

A consequence of part (a) of the theorem is that any connected piece of a Dupin cyclide of characteristic (p, q) immersed in S^n determines a unique compact Legendre submanifold with domain $S^p \times S^q$.

Pinkall's result can be used to obtain the following Möbius geometric characterization of the cyclides in $\mathbb{R}^n$ [Cecil 1991; 1992, p. 154].

THEOREM 7.2. (a) *Every connected cyclide of Dupin M^{n-1} of characteristic (p, q) embedded in $\mathbb{R}^n$ is Möbius equivalent to an open subset of a surface of revolution obtained by revolving a q-sphere $S^q \subset \mathbb{R}^{q+1} \subset \mathbb{R}^n$ about an axis of revolution $\mathbb{R}^q \subset \mathbb{R}^{q+1}$ or a p-sphere $S^p \subset \mathbb{R}^{p+1} \subset \mathbb{R}^n$ about an axis $\mathbb{R}^p \subset \mathbb{R}^{p+1}$.*

(b) *Two such surfaces are Möbius equivalent if and only if they have the same value of $\rho = |r|/a$, where r is the signed radius of the profile sphere S^q and $a > 0$ is the distance from the center of S^q to the axis of revolution.*

Note that the profile sphere is allowed to intersect the axis of revolution, thereby resulting in singularities. However, in the context of Lie sphere geometry, the corresponding Legendre map is an immersion.

The classical cyclides of Dupin in $\mathbb{R}^3$ are the only surfaces for which all lines of curvature are circles (or straight lines). Using exterior differential systems, Ivey [1995] showed that any surface in $\mathbb{R}^3$ containing two orthogonal families of circles is a cyclide of Dupin.

Finally, we note that recently the cyclides of Dupin have been used in computer-aided geometric design of surfaces. See, for example, [Degen 1994; Pratt 1990; 1995; [Srinivas and Dutta 1994a]; 1994b; 1995a; 1995b].

The case $g = 3$ has proved to be much more difficult than the case of two principal curvatures. In his dissertation, Pinkall [1981] did obtain a complete local classification up to Lie sphere transformation for Dupin hypersurfaces with three principal curvatures in $\mathbb{R}^4$, but it is quite involved. (See also [Pinkall 1985b; Cecil and Chern 1989; Cecil 1992, pp. 171–190].) Recall from Section 3 that a proper Dupin hypersurface is *reducible* if it locally Lie equivalent to a proper Dupin hypersurface obtained from one of Pinkall's standard constructions. It is *locally irreducible* if it does not contain any reducible open subset. Pinkall found that any two irreducible proper Dupin hypersurfaces in $\mathbb{R}^4$ are locally Lie equivalent, each being Lie equivalent to an open subset of an isoparametric hypersurface in S^4. However, he found a one-parameter family of Lie equivalence classes among the reducible proper Dupin hypersurfaces with $g = 3$ in $\mathbb{R}^4$. In higher dimensions, the focus has been on classifying the locally irreducible Dupin hypersurfaces and little has been done in attempting to classify the reducible ones up to Lie equivalence, although this appears to be a problem where some progress could be made.

The first result in higher dimensions is due to Niebergall [1991], who showed that every proper Dupin hypersurface in $\mathbb{R}^5$ with three principal curvatures is reducible. Recently, Cecil and Jensen [1997] have generalized the results of Pinkall and Niebergall as follows.

Theorem 7.3. *Let $f : M \to S^n$ be a proper Dupin hypersurface with three distinct principal curvatures of multiplicities m_1, m_2, and m_3. If the hypersurface is locally irreducible, then $m_1 = m_2 = m_3$, and M is Lie equivalent to an isoparametric hypersurface in a sphere.*

We now give a brief outline of the proof of this theorem. We work in the context of Lie sphere geometry and consider a proper Dupin submanifold $\lambda : M^{n-1} \to \Lambda^{2n-1}$ with three curvature spheres. As in Section 5, we can parametrize the Dupin submanifold as $\lambda = [k_1, k_2]$, where $[k_1]$ and $[k_2]$ are two curvature sphere maps. We can also arrange that the third curvature sphere map have the form $[k_3] = [k_1 + k_2]$. We first compute the derivatives of the $[k_i]$ using the method of moving frames. In the case where M has dimension three, Pinkall found one function c with the property that all of the terms arising in the exterior differentiation of the frame fields could eventually be expressed in terms of c and its derivatives. If c is identically zero, then λ is reducible. If c is never zero on M, then one can arrange that $c = 1$ with an appropriate choice of frame, and all such hypersurfaces are locally Lie equivalent to Cartan's isoparametric hypersurface in S^4.

In the general case where the three curvature spheres have respective multiplicities m_1, m_2, and m_3, there are $m_1 \cdot m_2 \cdot m_3$ functions F^{α}_{ap}, where

$$1 \le a \le m_1, \quad m_1 + 1 \le p \le m_1 + m_2, \quad m_1 + m_2 + 1 \le \alpha \le n - 1,$$

which are defined in a similar way to the one function c of Pinkall. They can be arranged in vector form, $v_{p\alpha} = (F^{\alpha}_{ap})$, for $1 \le a \le m_1$, with $v_{a\alpha}$ and v_{ap} defined in a similar way. One first shows that if a column or row of any of the arrays $[v_{p\alpha}]$, $[v_{a\alpha}]$, $[v_{ap}]$ is identically zero on an open subset $V \subset M$, then the restriction of λ to V is reducible. This result is applied to show that unless all of the multiplicities are equal, λ must contain a reducible open subset, i.e., it is not locally irreducible. In the case where all the multiplicities equal m, one next shows that all of the vectors in all of the arrays have the same length ρ. This one function ρ actually plays the same role that c did in the case $m = 1$. If λ is locally irreducible, then ρ is nonzero on M, and it can be made locally constant with an appropriate choice of frame. The proof is completed by showing that there exist three points P_1, P_2, P_3 on a certain timelike line in P^{n+2} such that $\langle k_i, P_i \rangle = 0$, for $1 \le i \le 3$, where the $[k_i]$ are the curvature sphere maps of λ. By Theorem 5.2, this implies that λ is Lie equivalent to an isoparametric hypersurface.

The next case $g = 4$ is still more complicated, but many aspects of the approach outlined above apply. As in Section 6, one can order the principal curvatures as in 6.2 and determine a unique Lie curvature Ψ by 6.3 satisfying $0 < \Psi < 1$. As in the $g = 3$ case, one can reparametrize the Dupin submanifold as $\lambda = [k_1, k_2]$, where $[k_1]$ and $[k_2]$ are two of the curvature sphere maps, and then arrange that a third curvature sphere map satisfies $[k_3] = [k_1 + k_2]$. Then the fourth curvature sphere $[k_4]$ is determined by the Lie curvature Ψ. Niebergall [1992] used this framework to find some sufficient conditions for a proper Dupin hypersurface with $g = 4$ in S^5 to be Lie equivalent to an isoparametric hypersurface.

For $g = 4$ or 6, it is not true that every locally irreducible Dupin hypersurface is Lie equivalent to an isoparametric hypersurface. The examples of Pinkall and Thorbergsson [1989a] and Miyaoka and Ozawa [1989] discussed in Section 6 are locally irreducible and are not Lie equivalent to an isoparametric hypersurface. On the other hand, Miyaoka [1989a; 1989b] has shown that a compact proper Dupin hypersurface embedded in $\mathbb{R}^n$ is Lie equivalent to an isoparametric hypersurface if it has constant Lie curvatures and it satisfies certain global conditions regarding the intersections of leaves of its various principal foliations. As in the $g = 3$ case, in trying to obtain local results, one can replace the global conditions of Miyaoka with the assumption of local irreducibility. This yields the following question. In the cases $g = 4$ or 6, is every locally irreducible proper Dupin hypersurface with constant Lie curvatures Lie equivalent to an isoparametric hypersurface?

Note that the hypothesis of local irreducibility is definitely necessary here because of the noncompact reducible proper Dupin hypersurfaces with constant Lie curvature of [Cecil 1990] mentioned in Section 6.

A more general problem is to attempt to identify key local Lie invariants of Dupin submanifolds within the context of moving Lie frames. Niebergall [1992] made some progress in this direction in the case of a proper Dupin hypersurface M^4 in S^5 with four principal curvatures. He assumed that the Lie curvature is constant and then found four other invariants, analogous to the F^α_{ap} in the $g = 3$ case, that when suitably prescribed yield the conclusion that M^4 is Lie equivalent to an isoparametric hypersurface. Moreover, the vanishing of any three of these invariants implies that M^4 is reducible. At this point, however, the geometric meaning of these invariants is not yet clear, nor is it obvious that these are the only invariants to be considered. This is clearly a complicated problem, but a systematic study may yield new results.

Another problem is to attempt to obtain a complete local classification of reducible Dupin hypersurfaces with three principal curvatures up to Lie equivalence. As mentioned above, Pinkall obtained such a classification in the case of $M^3 \subset \mathbb{R}^4$. In that case, while there is only one class of irreducible Dupin hypersurfaces, the reducible ones determine a one-parameter family of Lie equivalence classes. It may be possible to obtain a similar classification in the higher dimensional reducible case by using the framework established to prove Theorem 7.3.

The approach of Lie sphere geometry can also be used to obtain results in Möbius (conformal) geometry. As noted in Theorem 7.2, one can derive a local Möbius classification in the case $g = 2$ from Pinkall's Lie-geometric classification. Pinkall and Thorbergsson [1989a] introduced a Möbius invariant, called the *Möbius curvature*, that can distinguish among the Lie equivalent parallel hypersurfaces in a family of isoparametric hypersurfaces. Recently, C.-P. Wang used the method of moving frames to determine a complete set of Möbius invariants for surfaces in $\mathbb{R}^3$ without umbilic points [Wang 1992] and for hypersurfaces in $\mathbb{R}^4$ with three distinct principal curvatures at each point [Wang 1995]. He then applied this result to derive a local classification of Dupin hypersurfaces in $\mathbb{R}^4$ with three principal curvatures up to Möbius transformation. A natural problem is to try to extend Wang's result to proper Dupin hypersurfaces in $\mathbb{R}^n$, for $n > 4$, and to thoroughly investigate the geometric significance of his invariants, including the Möbius curvature, in $\mathbb{R}^4$.

Ferapontov [1995a; 1995b] has explored the relationship between Dupin and isoparametric hypersurfaces and Hamiltonian systems of hydrodynamic type. Ferapontov poses several research problems in that context.

8. Classifications of Taut Submanifolds

In this section, we survey the known classification results on taut submanifolds. To a reasonable extent, we have attempted to make this section self-contained, although some references to the previous sections are inevitable.

In the paper [Banchoff 1970] that introduced the STPP, it was shown that a taut embedding of S^1 into $\mathbb{R}^n$ must be a metric circle in a plane. In the same paper, Banchoff also obtained a complete classification of compact taut (two-dimensional) surfaces in Euclidean spaces. We now give a brief outline of his proof. As noted in Section 1, Banchoff observed that, because tautness is invariant under stereographic projection, there exists a substantial taut nonspherical embedding of a compact manifold M into $\mathbb{R}^n$ if and only if there exists a substantial taut spherical embedding of M into $S^n \subset \mathbb{R}^{n+1}$. As a consequence, one can invoke Kuiper's result on the bound on the codimension to obtain Theorem 1.1. In particular, if $f : M^2 \to \mathbb{R}^n$ is a substantial taut embedding, then $n \leq 5$, and if $n = 5$, then f is an embedding of P^2 as a Veronese surface in $S^4 \subset \mathbb{R}^5$. Next, if $f : M^2 \to \mathbb{R}^4$ is a taut nonspherical embedding, then $f(M^2)$ must be the image under stereographic projection of a spherical Veronese surface in $\mathbb{R}^5$. Thus, the problem is reduced to finding all compact taut surfaces in S^3. Again by stereographic projection, this is equivalent to finding all compact taut surfaces in $\mathbb{R}^3$.

A key step in Banchoff's classification of compact taut surfaces in $\mathbb{R}^3$ is showing that if $f : M^2 \to \mathbb{R}^3$ is a taut embedding of a compact surface M^2, then $f(M^2)$ lies in between the two spheres S_1 and S_2 tangent to $f(M^2)$ at $f(x)$ and centered at the focal points $p_i = f(x) + (1/\mu_i)\xi(x)$, for $i = 1, 2$, respectively, where ξ is a field of unit normals to $f(M^2)$, and the μ_i are the principal curvatures of f. Thus, if $f(M^2)$ has one umbilic point, it must be a metric sphere, because it lies between two identical spheres S_1 and S_2 at the umbilic point. This implies that $f(M^2)$ must be either a metric sphere or smooth torus, because any embedding of a surface of higher genus would necessarily have an umbilic point. Suppose now that $f(M^2)$ is a taut torus with no umbilic points. Then Banchoff shows that the principal curvatures must be constant along their corresponding lines of curvature, i.e., $f(M^2)$ is Dupin, and so it is a cyclide of Dupin in $\mathbb{R}^3$.

Cecil [1976] then generalized Banchoff's argument to the noncompact case to again show that a taut $f(M^2) \subset \mathbb{R}^3$ must be Dupin. This implies that $f(M^2)$ must a plane, circular cylinder, or parabolic ring cyclide. The latter surface is obtained by by inverting a torus of revolution in a sphere centered at a point on the torus; its name comes from the fact that its focal set consists of a pair of parabolas [Cecil and Ryan 1985, pp. 151–166]. These results are combined in the following theorem, which is a complete classification of taut surfaces in $\mathbb{R}^n$.

THEOREM 8.1. *Let $f : M^2 \to \mathbb{R}^n$ be a substantial taut embedding of a surface M^2.*

(a) *If M^2 is compact, then $f(M^2)$ is a metric sphere or a cyclide of Dupin in $\mathbb{R}^3$, a spherical Veronese surface in $S^4 \subset \mathbb{R}^5$, or a surface in $\mathbb{R}^4$ related to one of these by stereographic projection.*
(b) *If M^2 is noncompact, then it is a plane, circular cylinder or parabolic ring cyclide in $\mathbb{R}^3$, or it is the image in $\mathbb{R}^4$ of a punctured spherical Veronese surface under stereographic projection from S^4.*

The first results after Banchoff's paper dealt with taut embeddings of submanifolds with relatively simple topology. Nomizu and Rodríguez [1972] proved the following.

THEOREM 8.2. *Let M be a complete Riemannian manifold of dimension $k \geq 2$, isometrically immersed in $\mathbb{R}^n$. If every nondegenerate Euclidean distance function L_p has index 0 or k at any of its critical points, then M is embedded as a k-plane or a metric k-sphere $S^k \subset \mathbb{R}^{k+1} \subset \mathbb{R}^n$.*

The proof is accomplished by showing that M is a totally umbilic submanifold. This is a consequence of the following elementary but important argument. Let $f : M \to \mathbb{R}^n$ be the isometric immersion. Let ξ be any unit normal to $f(M)$ at any point $f(x)$. We want to show that the shape operator A_ξ is a scalar multiple of the identity. If $A_\xi = 0$, we are done. If not, then by replacing ξ by $-\xi$, if necessary, we can assume that A_ξ has a positive eigenvalue. Let λ be the largest eigenvalue of A_ξ, and let t satisfy $1/\lambda < t < 1/\mu$, where μ is the next largest positive eigenvalue of A_ξ (just consider $t > 1/\lambda$ if there are no other positive eigenvalues). If $q = f(x) + t\xi$, then L_q has a nondegenerate critical point at x with index equal to the multiplicity m of λ. It may be that L_q is not a Morse function. If so, there exists a Morse function L_p, with p near q, such that L_p has a critical point near x of the same index m. Now since $m > 0$, the hypotheses imply that $m = k$, and thus $A_\xi = \lambda I$, as desired.

An immediate consequence is the following result, obtained independently by Carter and West [1972]. The result is true in the case $k = 1$ by the work of Banchoff mentioned earlier. See [Cecil 1974] for a similar characterization of metric spheres in hyperbolic space.

THEOREM 8.3. *Let $f : S^k \to \mathbb{R}^n$ be a taut embedding. Then $f(S^k)$ is a metric sphere in a $(k+1)$-dimensional affine subspace $\mathbb{R}^{k+1} \subset \mathbb{R}^n$.*

Approaching the problem from a different point of view, Hebda [1981] asked which ambient spaces admit taut embeddings of hyperspheres. He found that a complete simply connected n-dimensional manifold that admits a taut embedding of S^{n-1} is either homeomorphic to S^n, diffeomorphic to $\mathbb{R}^n$ or diffeomorphic to $S^{n-1} \times \mathbb{R}$.

Next Carter and West [1972] obtained the following characterization of spherical cylinders.

THEOREM 8.4. *Let $f : M^{n-1} \to \mathbb{R}^n$ be a taut embedding of a noncompact manifold such that $H_k(M^{n-1}; \mathbb{Z}_2) = \mathbb{Z}_2$ for some k with $0 < k < n-1$, and such that $H_i(M^{n-1}; \mathbb{Z}_2) = 0$ for $i \neq 0, k$. Then M^{n-1} is diffeomorphic to $S^k \times \mathbb{R}^{n-k-1}$, and f is a standard product embedding.*

Thorbergsson [1983b; 1986] then obtained the following characterization of highly connected taut submanifolds of arbitrary codimension. (See [Jorge and Mercuri 1984] for a related result involving minimal submanifolds.)

THEOREM 8.5. *Let M be a compact $2k$-dimensional taut submanifold of $\mathbb{R}^n$ that is $(k-1)$-connected but not k-connected, and that does not lie in any totally umbilic hypersurface of $\mathbb{R}^n$. Then either*

(a) *$n = 2k+1$ and M is a cyclide of Dupin diffeomorphic to $S^k \times S^k$, or*
(b) *$n = 3k+1$ and M is a spherical Veronese embedding of the projective plane FP^2, where F is the division algebra $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ for $k = 1, 2, 4, 8$, respectively.*

In a related paper, Hebda [1984] constructs tight smooth embeddings of arbitrarily many copies of $S^k \times S^k$ into $\mathbb{R}^{2k+1}$. No taut embeddings of these manifolds exist by Theorem 8.5.

The next case in terms of the homology is when M has the same homology as $S^k \times S^m$, where $k \neq m$. In that case, Cecil and Ryan [1978; 1985, p. 202] obtained the following.

THEOREM 8.6. *A taut hypersurface $M \subset \mathbb{R}^n$ with the same $\mathbb{Z}_2$-homology as $S^k \times S^{n-k-1}$ is a cyclide of Dupin.*

To prove this, one first uses the Index Theorem for L_p functions to prove that, at each point of M, the number of distinct principal curvatures must be either 2 or 3. The most difficult part of the proof is to then show that the number of distinct principal curvatures must be constant on M. Then, since taut implies

Dupin, M is a proper Dupin hypersurface with $g = 2$ or 3 principal curvatures. Then it is fairly easy given the homology of M to show that $g = 2$, and so M must be a cyclide of Dupin.

Ozawa [1986] generalized this result by showing that, if an embedding of $S^k \times S^m$ into S^n, where $k < m$, is taut and substantial, then the codimension of the embedding is either 1 or $m - k + 1$. He also showed that the r-times connected sum of $S^k \times S^m$, for $k < m$, cannot be tautly embedded into any Euclidean space if $r > 1$.

There is a related result that takes into account the intrinsic geometry of M. Recall that a Riemannian manifold (M, g) is *conformally flat* if every point has a neighborhood conformal to an open subset in Euclidean space. Schouten [1921] showed that a hypersurface M of dimension $n \geq 4$ and immersed in $\mathbb{R}^{n+1}$ is conformally flat in the induced metric if and only if at least $n - 1$ of the principal curvatures coincide at each point. (This characterization fails when $n = 3$ [Lancaster 1973].) Using Schouten's result, Theorem 8.6, and some basic results on tautness, Cecil and Ryan [1980] proved the following.

THEOREM 8.7. *Let M be a taut hypersurface of dimension $n \geq 4$ in $\mathbb{R}^{n+1}$. Then M is conformally flat in the induced metric if and only if it is one of the following*:

(a) *a hyperplane or metric sphere*;
(b) *a cylinder over a circle or over an $(n-1)$-sphere*;
(c) *a cyclide of Dupin (diffeomorphic to $S^1 \times S^{n-1}$)*;
(d) *a parabolic ring cyclide (diffeomorphic to $S^1 \times S^{n-1} - \{p\}$)*.

Concerning taut embeddings of three-manifolds, Pinkall and Thorbergsson have proved the following [1989b].

THEOREM 8.8. *A compact taut three-dimensional submanifold in Euclidean space is diffeomorphic to one of the following seven manifolds*: S^3, $\mathbb{RP}^3$, *the quaternion space* $S^3/\{\pm 1, \pm i, \pm j, \pm k\}$, *the three-torus* T^3, $S^1 \times S^2$, $S^1 \times \mathbb{RP}^2$, *or* $S^1 \times_h S^2$, *where h denotes an orientation-reversing diffeomorphism of S^2. All of these manifolds admit taut embeddings.*

Pinkall and Thorbergsson actually determine much more about the geometric structure of these taut embeddings. Because of the invariance of tautness under stereographic projection, it makes sense to attempt to classify *spherically substantial* embeddings, that is, those that do not lie in any hypersphere. In the following description of the results of Pinkall and Thorbergsson, the codimension means the spherically substantial codimension.

A taut embedding of S^3 must be a metric hypersphere. The projective space $\mathbb{RP}^3$ can be tautly embedded with codimension 2 as the Stiefel manifold $V_{2,3} \subset S^5 \subset \mathbb{R}^6$ and with codimension 5 as SO(3) in the unit sphere in the space of 3×3 matrices. It is not known whether the codimensions 3 and 4 are possible. The quaternion space is realized as Cartan's isoparametric hypersurface in S^4, where it is unique up to Lie equivalence, and no other codimensions are possible. The three-torus can be tautly embedded with codimension one as a tube in $\mathbb{R}^4$ around a torus of revolution $T^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$, and with codimension 2 as $T^2 \times S^1 \subset \mathbb{R}^5$. One can tautly embed $S^1 \times S^2$ with codimension 1 as a cyclide of Dupin (see Theorem 8.6), and no other codimension is possible. The manifold $S^1 \times \mathbb{RP}^2$ can be tautly embedded with codimension 3 as the product of a metric circle and a Veronese surface. It can be tautly embedded with codimension 2 as a rotational submanifold with profile submanifold $\mathbb{RP}^2$, and the only codimensions possible are 2 and 3. Finally, $S^1 \times_h S^2$ can be tautly embedded with codimension 2 as the "complexified unit sphere"

$$\{e^{i\theta}x : \theta \in \mathbb{R}, x \in S^2 \subset \mathbb{R}^3\} \subset S^5 \subset \mathbb{C}^3.$$

This is one of the focal submanifolds of a homogeneous family of isoparametric hypersurfaces with four principal curvatures in S^5, the other being a Stiefel manifold $V_{2,3}$ (see [Cecil and Ryan 1985, pp. 299–304], for example). No other codimensions are possible for a taut embedding of $S^1 \times_h S^2$.

As we have seen, many examples of taut embeddings, such as the R-spaces, are homogeneous spaces. However, the question of which homogeneous spaces admit taut embeddings remains open. Thorbergsson [1988] found some necessary topological conditions for the existence of a taut embedding, which allowed him to prove that certain homogeneous spaces do not admit taut embeddings. In the same vein, Hebda [1988] found certain necessary cohomological conditions for the existence of a taut embedding, and he used these results to give examples of manifolds that cannot be tautly embedded. In the case where M is a compact homogeneous submanifold substantially embedded in Euclidean space with flat normal bundle, Olmos [1994] has shown that the following statements are equivalent: (i) M is taut; (ii) M is Dupin; (iii) M has constant principal curvatures; (iv) M is an orbit of the isotropy representation of a symmetric space; (v) the first normal space of M coincides with the normal space.

There have been various generalizations of tautness in terms of the distance functions used and the ambient space considered. Carter, Mansour, and West [Carter et al. 1982] introduced a notion of *k-cylindrical taut* immersion $f : M \to \mathbb{R}^n$ by using distance functions from k-planes in $\mathbb{R}^n$ (see also [Carter and Şentürk 1994; Carter and West 1985a]). For $k = 0$, this is equivalent to tautness, and for $k = n-1$ it is equivalent to tightness. This theory turns out to closely related to the theory of convex sets and many of the results concern embeddings of spheres. (See also [Wegner 1984] for more on cylindrical distance functions.)

There is also a theory of tight and taut immersions into hyperbolic space H^n [Cecil and Ryan 1979a; 1979b]. This involves consideration of three types of distance functions on $\mathbb{H}^n$ whose level sets are respectively spheres, horospheres, and hypersurfaces equidistant from a hyperplane. Buyske [1989] introduced a notion of tautness in semi-Riemannian space forms and treated the issue of tightness of focal sets of certain taut submanifolds obtained by Lie sphere transformations.

Beltagy [1986] extended the TPP and STPP to subsets of a general Riemannian manifold without focal points. Finally, in a paper published in this volume, Terng and Thorbergsson [1997] have extended the notion of tautness to submanifolds of complete Riemannian manifolds.

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