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Complex Dynamics in Several Variables

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Contents

1. Motivation

The study of complex dynamics in several variables can be motivated in at least two natural ways. The first is by analogy with the fruitful study of complex dynamics in one variable. Since this latter subject is the subject of the parallel lectures by John Hubbard (the reader is referred to [Carleson and Gamelin 1993] for a good introduction to the subject), we focus here on the second source of motivation: the study of real dynamics.

A classical problem in the study of real dynamics is the n-body problem, which was studied by Poincaré. For instance, we can think of n planets moving in space. For each planet, there are three coordinates giving the position and three coordinates giving the velocity, so that the state of the system is determined by a total of $6n$ real variables. The evolution of the system is governed by Newton's laws, which can be expressed as a first order ordinary differential equation. In fact, the state of the system at any time determines the entire future and past evolution of the system.

To make this a bit more precise, set $k = 6n$. Then the behavior of the n planets is modeled by a differential equation

$$(\dot{x}_1, \ldots, \dot{x}_k) = F(x_1, \ldots, x_k)$$

for some $F : \mathbb{R}^k \to \mathbb{R}^k$. Here $\dot{x}$ denotes the derivative of x with respect to t.

From the elementary theory of ordinary differential equations, we know that this system has a unique solution $t \mapsto \varphi_t(x_1, \ldots, x_k)$ satisfying $\dot{\varphi} = F(\varphi)$ and $\varphi_0(x_1, \ldots, x_k) = (x_1, \ldots, x_k)$.

For purposes of studying dynamics, we would like to be able to say something about the evolution of this system over time, given some initial data. That is, given $p \in \mathbb{R}^k$, we would like to be able to say something about $\varphi_t(p)$ as t varies. For instance, a typical question might be the following.

Question 1.1. *For given initial positions and velocities, do the planets have bounded orbits for all (positive) time? That is, given* $p = (x_1, \ldots, x_k)$, *is the set* $\{\varphi_t(p) : t \geq 0\}$ *bounded?*

This question, in fact, particularly interested Poincaré. Unfortunately, the usual answer to such a question is "I don't know." Nevertheless, it is possible to say something useful about related questions, at least in some settings. For instance, one related problem is the following.

Problem 1.2. Say something interesting about the set of initial conditions for which the planets have bounded forward orbits. That is, describe the set

$$K^+ := \{p \in \mathbb{R}^k : \{\varphi_t(p) : t \geq 0\} \text{ is bounded}\}.$$

Although this question is less precise and gives less specific information than the original, an answer to it can still tell us quite a bit about the behavior of the system.

2. Iteration of Maps

In the preceding discussion, we have been taking the approach of fixing a point $p \in \mathbb{R}^k$ and following the evolution of the system over time starting from this point. An alternative approach is to think of all possible starting points evolving simultaneously, then taking a snapshot of the result at some particular instant in time.

To make this more precise, assume that the solution $\varphi_t(p)$ exists for all time t and all $p \in \mathbb{R}^k$. In this case, for fixed t, the map $\varphi_t : \mathbb{R}^k \to \mathbb{R}^k$ is a diffeomorphism of $\mathbb{R}^k$ and satisfies the group property

$$\varphi_{s+t} = \varphi_s \circ \varphi_t$$

for any s and t. The family of diffeomorphisms $(\varphi_t)_{t\in\mathbb{R}}$ is called the *flow* of the differential equation.

In order to make our study more tractable, we make two simplifications.

Simplification 1. Choose some number $\alpha > 0$, called the *period*, and define $f = \varphi_\alpha$. Then f is a diffeomorphism of $\mathbb{R}^k$ and, given $p \in \mathbb{R}^k$, the group property of φ implies that

$$\varphi_{n\alpha}(p) = \varphi_\alpha \circ \cdots \circ \varphi_\alpha(p) = f^n(p).$$

That is, studying the behavior of f under iteration is equivalent to studying the behavior of φ at regularly spaced time intervals. By concentrating on f we ignore those aspects of the behavior of the continuous flow φ that occur at time scales less than α.

Simplification 2. Set $k = 2$. Although this simplification means that we can no longer directly relate our model to the original physical problem, the ideas and techniques involved in studying such a simpler model are still rich enough to shed some light on the more realistic cases. In fact, there are interesting questions in celestial mechanics which reduce to questions about two-dimensional diffeomorphisms, but here we are focusing on the mathematical model rather than on the physical system.

We also introduce some notation. Given $p \in \mathbb{R}^2$, let $\mathcal{O}^+(p)$, $\mathcal{O}^-(p)$, and $\mathcal{O}(p)$ be respectively the *forward orbit*, *backward orbit*, and *full orbit* of p under f. In symbols,

$$\begin{aligned} \mathcal{O}^+(p) &:= \{f^n(p) : n \geq 0\}, \\ \mathcal{O}^-(p) &:= \{f^n(p) : n \leq 0\}, \\ \mathcal{O}(p) &:= \{f^n(p) : n \in \mathbb{Z}\}. \end{aligned}$$

Problem 1.2 then becomes the following.

PROBLEM 2.1. Given a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$, describe the sets

$$\begin{aligned} K^+ &:= \{p \in \mathbb{R}^2 : \mathcal{O}^+(p) \text{ is bounded}\}, \\ K^- &:= \{p \in \mathbb{R}^2 : \mathcal{O}^-(p) \text{ is bounded}\}, \\ K &:= \{p \in \mathbb{R}^2 : \mathcal{O}(p) \text{ is bounded}\}, \end{aligned}$$

For future reference, note that $K = K^+ \cap K^-$.

3. Regular Versus Chaotic Behavior

For the moment, we will make no attempt to define rigorously what we mean by regular or chaotic. Intuitively, one should think of regular behavior as being very predictable and as relatively insensitive to small changes in the system or initial conditions. On the other hand, chaotic behavior is in some sense random and can change drastically with only slight changes in the system or initial conditions. Here is a relevant quote from Poincaré on chaotic behavior:

> A very small cause, which escapes us, determines a considerable effect which we cannot ignore, and we say that this effect is due to chance.

We next give some examples to illustrate both kinds of behavior, starting with regular behavior. First we make some definitions.

A point $p \in \mathbb{R}^2$ is a *periodic point* if $f^n(p) = p$ for some $n \geq 1$. The smallest such n is the *period* of p. A periodic point p is *hyperbolic* if $(Df^n)(p)$ has no eigenvalues on the unit circle. (Here Df^n represents the derivative of f^n at p, a linear map $\mathbb{R}^2 \to \mathbb{R}^2$.) If p is a hyperbolic periodic point and both eigenvalues are inside the unit circle, p is called a *sink* or *attracting periodic point*.

Let d denote Euclidean distance in $\mathbb{R}^2$. If p is a hyperbolic periodic point, the set

$$W^s(p) = \{q \in \mathbb{R}^2 : d(f^n q, f^n p) \to 0 \text{ as } n \to \infty\} \tag{3.1}$$

is called the *stable manifold* of p: it is the set of points whose forward images become increasingly closer to the corresponding images of p. Dually, the *unstable manifold* of p is made up of those points whose *backward* images approach those of p:

$$W^u(p) = \{q \in \mathbb{R}^2 : d(f^{-n} q, f^{-n} p) \to 0 \text{ as } n \to \infty\}. \tag{3.2}$$

(The notation f^{-n} represents the n-th iterate of f^{-1}, which is well defined since we are assuming that f is a diffeomorphism.) When p is a sink, the stable manifold $W^s(p)$ is also called the *attraction basin* of p.

FACT. *When p is a sink, $W^s(p)$ is an open set containing p, and $W^u(p)$ is empty.*

A sink gives a prime example of regular behavior. Starting with any point q in the basin of attraction of a sink p, the forward orbit of q is asymptotic to the (periodic) orbit of p. Since the basin is open, this will also be true for any point q' near enough to q. Hence we see the characteristics of predictability and stability mentioned in relation to regular behavior.

For an example of chaotic behavior, we turn to a differential equation studied by Cartwright and Littlewood in 1940, and given by

$$\ddot{y} - k(1 - y^2)\dot{y} + y = b \cos t.$$

Introducing the variable $x = \dot{y}$, we can write this as a first-order system

$$\dot{y} = x, \qquad \dot{x} = g(x, y, t),$$

where g is a function satisfying $g(x, y, t + 2\pi) = g(x, y, t)$. This system has a solution φ_t as before with $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ a diffeomorphism. Although the full group property does not hold for φ since g depends on t, we still have $\varphi_{s+t} = \varphi_s \circ \varphi_t$ whenever $s = 2\pi n$ and $t = 2\pi m$ for integers n and m. Hence we can again study the behavior of this system by studying the iterates of the diffeomorphism $f = \varphi_{2\pi}$.

Rather than study this system itself, we follow the historical development of the subject and turn to a more easily understood example of chaotic behavior which was motivated by this system of Cartwright and Littlewood: the Smale horseshoe.

4. The Horseshoe Map and Symbolic Dynamics

The horseshoe map was first conceived by Steve Smale as a way of capturing many of the features of the Cartwright–Littlewood map in a system that is easily understood.

For our purposes, the horseshoe map, h is defined first on a square B in the plane with sides parallel to the axes. First we apply a linear map that stretches the square in the x-direction and contracts it in the y-direction. Then we take the right edge of the resulting rectangle and bend it around to form a horseshoe shape. The map h is then defined on B by placing this horseshoe over the original square B so that $B \cap h(B)$ consists of two horizontal strips in B. See Figure 1.

We can extend h to a diffeomorphism of $\mathbb{R}^2$ in many ways. We do it here as follows. First partition $\mathbb{R}^2 \setminus B$ into four regions by using the lines $y = x$ and $y = -x$ as boundaries. Denote the union of the two regions above and below B by B^+ and the union of the two regions to the left and right of B by B^-, as in Figure 2. Then we can extend h to a diffeomorphism of $\mathbb{R}^2$ in such a way that $h(B^-) \subseteq B^-$. In this situation, points in B^+ can be mapped to any of the three regions B^+, B, or B^-, points in B can be mapped to either B or B^-, and points

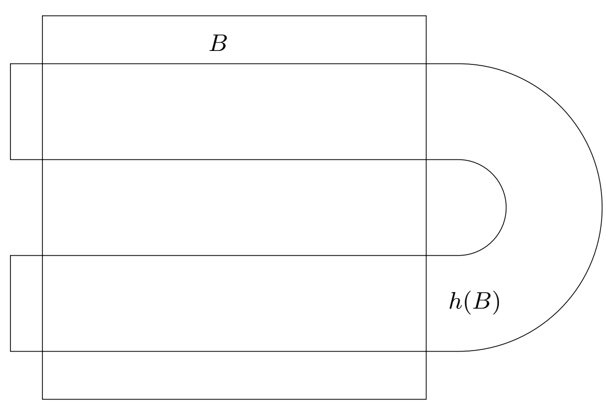

Figure 1. The image of B under the horseshoe map h.

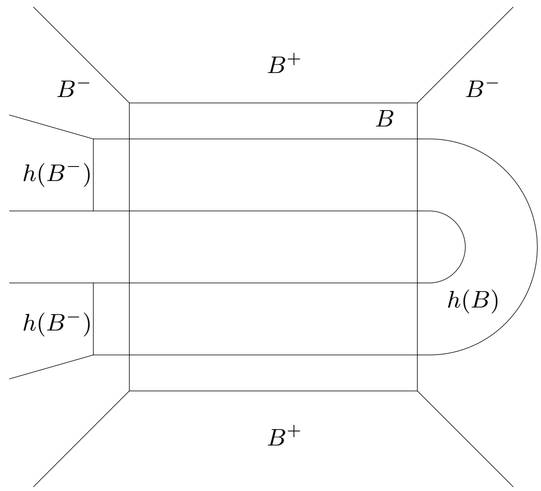

Figure 2. The sets B, B^+, and B^-.

in B^- must be mapped to B^-. Further, we require that points in B^- go to ∞ under iteration, and we require analogous conditions on f^{-1}. Note in particular that points that leave B do not return and that $K \subseteq B$.

It is not hard to see that, under these conditions,

$$K^- \cap B = B \cap hB \cap h^2 B \cap \cdots.$$

In fact, if we look at the image of the two strips $B \cap hB$ and intersect with B, the resulting set consists of four strips; each of the original two strips is subdivided into two smaller strips. Continuing this process, we see that $K^- \cap B$ is simply the set product of an interval and a Cantor set.

In fact, a simple argument shows that h has a fixed point p in the upper left corner of B, and that the unstable manifold of p is dense in the set $K^- \cap B$ and the stable manfold of p is dense in $K^+ \cap B$. The complicated structure of the stable and unstable manifolds plays an important role in the behavior of the horseshoe map.

We can describe the chaotic behavior of the horseshoe using *symbolic dynamics*. The idea of this procedure is to translate from the dynamics of h restricted to K into the dynamics of a shift map on bi-infinite sequences of symbols.

To do this, first label the two components of $B \cap hB$ with H_0 and H_1. Then, to each point $p \in K$, associate a bi-infinite sequence of 0's and 1's (that is, an element of $\{0,1\}^{\mathbb{Z}}$) using the map

$$\psi : p \mapsto s = (\ldots, s_1, s_0, s_{-1}, \ldots),$$

where

$$s_j = \begin{cases} 0 & \text{if } h^j(p) \in H_0, \\ 1 & \text{if } h^j(p) \in H_1. \end{cases}$$

We can put a metric on the space of bi-infinite sequences of 0's and 1's by

$$d(s, s') = \sum_{j=-\infty}^{\infty} |s_j - s'_j| 2^{-|j|}.$$

It is not hard to show that the metric space thus obtained is compact and that the map ψ given above produces a homeomorphism between K and this space $\{0,1\}^{\mathbb{Z}}$ of sequences. For bi-infinite sequences we have the natural concept of a *shift map*, which shifts all the entries of a sequence by one position. Formally, the *left shift map* on $\{0,1\}^{\mathbb{Z}}$ is the map that associates to a sequence $s = (s_i)_{i\in\mathbb{Z}}$ the sequence $z = (z_i)_{i\in\mathbb{Z}}$ defined by $z_i = s_{i+1}$. The definition of $\psi(p)$ implies that if σ is the left-shift map defined on bi-infinite sequences, then $\psi(h(p)) = \sigma(\psi(p))$.

Here are a couple of simple exercises that illustrate the power of using symbolic dynamics.

Exercise 4.1. Show that periodic points are dense in K. Hint: Periodic points correspond to periodic sequences.

Exercise 4.2. Show that there are periodic points of all periods.

5. Hénon Maps

The horseshoe was one motivating example for what are known as Axiom A diffeomorphisms [Bowen 1978]. The features that make the horseshoe easy to analyze dynamically are the uniform expansion in the horizontal direction and the uniform contraction in the vertical direction. This behavior is captured in the notion of hyperbolicity. We say that a diffeomorphism is *hyperbolic* over a set $X \subset \mathbb{R}^2$ if for each $x \in X$ there is a direction in which length is uniformly expanded and a direction in which length is uniformly contracted. These directions can depend on the point x, but the angle between them must be bounded away from zero. Hyperbolicity is the key ingredient in the definition of Axiom A. Like the horseshoe, Axiom A diffeomorphisms admit a symbolic description. Another important property of Axiom A diffeomorphisms is structural stability. This implies that small changes in the parameters do not change the symbolic description of the diffeomorphism.

Axiom A diffeomorphisms received a great deal of attention in the 1960s and 70s. Much current work focuses either on how Axiom A fails, as in the work of Newhouse, or on how some Axiom A ideas can be applied in new settings, as in the work of Benedicks and Carleson [1991] or Benedicks and Young [1993]. For more information and further references, [Ruelle 1989] provides a fairly gentle introduction, while [Palis and de Melo 1982; Shub 1978; Palis and Takens 1993] are more advanced. See also [Yoccoz 1995].

A model system for the study of non-Axiom A behavior that has received a great deal of attention is the so-called Hénon map. This is actually a family of

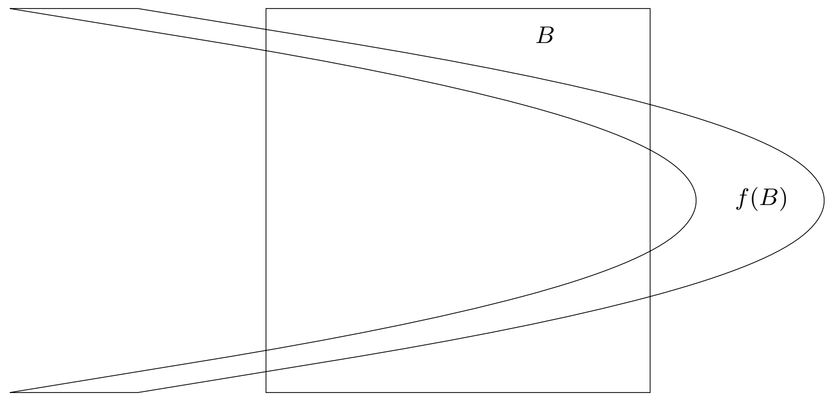

Figure 3. A square B and its image $f(B)$ for some parameter values a and b.

diffeomorphisms $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f_{a,b}(x, y) = (-x^2 + a - by,\ x)$$

for $b \neq 0$. These maps arise from a simplification of a simplification of a map describing turbulent fluid flow.

We can get some idea of the behavior of the map $f_{a,b}$ and the ways in which it relates to the horseshoe map by considering the image of a large box B under $f_{a,b}$. For simplicity, we write f for $f_{a,b}$. From Figure 3, we see that for some values of a and b, the Hénon map f is quite reminiscent of the horseshoe map h.

Since the map f is polynomial in x and y, we can also think of x and y as being complex-valued. In this case, $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic diffeomorphism of $\mathbb{C}^2$. This is also in some sense a change in the map f, but all of the dynamics of f restricted to $\mathbb{R}^2$ are contained in the dynamics of the maps on $\mathbb{C}^2$, so we can still learn about the original map by studying it on this larger domain.

We next make a few observations about f. First, note that f is the composition $f = f_3 \circ f_2 \circ f_1$ of the three maps

$$\begin{aligned} f_1(x, y) &= (x, by), \\ f_2(x, y) &= (-y, x), \\ f_3(x, y) &= (x + (-y^2 + a),\ y) \end{aligned} \tag{5.1}$$

For $0 < b < 1$, the images of B under the maps f_1 and $f_2 f_1$ are depicted in Figure 4, while f is depicted in Figure 3 with some $a > 0$.

From the composition of these functions, we can easily see that f has constant Jacobian determinant $\det(DF) = b$. Moreover, when $b = 0$, f reduces to a quadratic polynomial on $\mathbb{C}$.

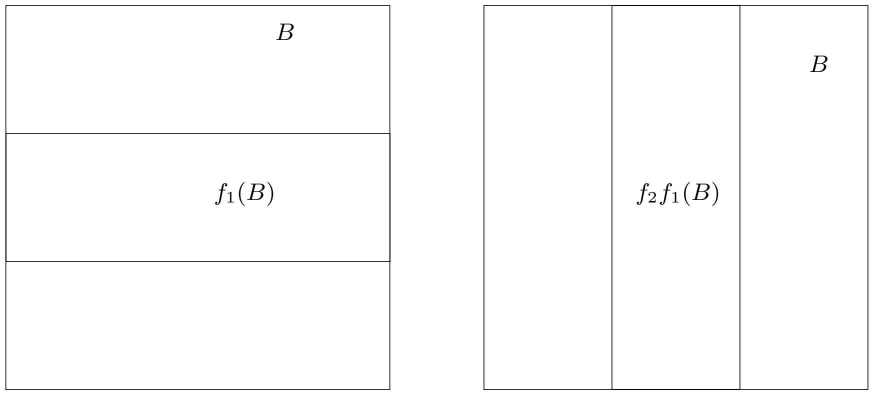

Figure 4. The Hénon map can be decomposed as $f = f_3 \circ f_2 \circ f_1$, where the component functions are defined in (5.1). Left: $f_1(B)$ sitting inside B. Right: $f_2 f_1(B)$ sitting inside B.

A simple argument shows that there is an $R = R(a, b)$ such that if we define the three sets

$$\begin{aligned} B &= \{|x| < R, |y| < R\}, \\ B^+ &= \{|y| > R, |y| > |x|\}, \\ B^- &= \{|x| > R, |x| > |y|\}, \end{aligned}$$

then we have the same dynamical relations as for the corresponding sets for the horseshoe map. That is, points in B^+ can be mapped to B^+, B, or B^-, points in B can be mapped to B or B^-, and points in B^- must be mapped to B^-.

Recall from Problem 2.1 that K^+ denotes the set of points with bounded forward orbit, K^- the set of points with bounded backward orbit, and K to be the intersection of these two sets.

When a is large, the tip of $f_{a,b}(B)$ is outside B. For certain larger values of a, Devaney and Nitecki [1979] proved that $f_{a,b}$ "is" a horseshoe. By this we mean that f is hyperbolic on the set X of points that remain in B for all time, and the dynamics of f restricted to X are topologically conjugate to those on the standard horseshoe of Section 4. Using complex techniques, Oberste-Vorth [1987] improved this result by showing that it works for any a such that the tip of $f_{a,b}(B)$ is outside of B.

In Theorem 14.3 we will describe an optimal result in this direction.

Example 5.1. To compare the dynamics of f in the real and complex cases, consider $f_{a,b}$ with a and b real. As an ad hoc definition, let $K_{\mathbb{R}}$ be the set of $p \in \mathbb{R}^2$ with bounded forward and backward orbits, and let $K_{\mathbb{C}}$ be the set of $p \in \mathbb{C}^2$ with bounded forward and backward orbits. Then another result of Oberste-Vorth [1987] is that $K_{\mathbb{C}} = K_{\mathbb{R}}$.

Thus we already have a mental picture of K for these parameter values. We can also get a picture of K^+ and K^- in the complex case, since we can extend the analogy between f and the horseshoe map by replacing the square B by a bidisk $B = D(R) \times D(R)$ contained in $\mathbb{C}^2$, where $D(R)$ is the disk of radius R centered at 0 in $\mathbb{C}$. In the definitions of B^+ and B^-, we can interpret x and y as complex-valued, in which case the definitions of these sets still make sense. Moreover, the same mapping relations hold among B^+, B^-, and B as before. In this case, $B \cap K^+$ is topologically equivalent to the set product of a Cantor set and a disk, $B \cap K^-$ is equivalent to the product of a disk and a Cantor set, and $B \cap K$ is equivalent to the product of two Cantor sets.

THESIS. *A surprising number of properties of the horseshoe* (*when properly interpreted*) *hold for general complex Hénon diffeomorphisms.*

The "surprising" part of the above thesis is that the horseshoe map was designed to be simple and easily understood, yet it sheds much light on the less immediately accessible Hénon maps.

6. Properties of Horseshoe and Hénon Maps

We again consider some properties of the horseshoe map in terms of its periodic points. The investigation of periodic points plays an important role in the study of many dynamical systems. In Poincaré's words,

> What renders these periodic points so precious to us is that they are, so to speak, the only breach through which we might try to penetrate into a stronghold hitherto reputed unassailable.

As an initial observation, recall that from symbolic dynamics, we know that the periodic points are dense in K. In fact, it is not hard to show that these periodic points are all *saddle points;* that is, if p has period n, then $(Dh^n)(p)$ has one eigenvalue larger than 1 in modulus, and one smaller. After recalling if necessary the definition of stable and unstable manifolds from (3.1) and (3.2), you should attempt to prove the following fact:

EXERCISE 6.1. For any periodic saddle point of the horseshoe map h, $W^s(p)$ is dense in K^+ and $W^u(p)$ is dense in K^-.

Now suppose $p \in K^+$, and let $n \in \mathbb{N}$ and $\varepsilon > 0$. By exercise 4.2, there is a periodic point q with period n, and by this last exercise, the stable manifold for q comes arbitrarily close to p. In particular, we can find $p' \in W^s(q)$ with $d(p, p') < \varepsilon$. Hence in any neighborhood of p, there are points that are asymptotic to a periodic point of any given period. We can contrast this with a point p in the basin of attraction for a sink. In this case, for a small enough neighborhood of p, every point will be asymptotic to the same periodic point.

This example illustrates the striking difference between regular and chaotic behavior. In the case of a sink, the dynamics of the map are relatively insensitive to the precise initial conditions, at least within the basin of attraction.

But in the horseshoe case, the dynamics can change dramatically with an arbitrarily small change in the initial condition. In a sense, chaotic behavior occurs throughout K^+.

A second basic example of Axiom A behavior is the solenoid [Bowen 1978, p. 4]. Take a solid torus in $\mathbb{R}^3$ and map it inside itself so that it wraps around twice. The image of this new set then wraps around 4 times. The solenoid is the set that is the intersection of all the forward images of this map. Moreover, the map extends to a diffeomorphism of $\mathbb{R}^3$ and displays chaotic behavior on the solenoid, which is the attractor for the diffeomorphism.

EXAMPLE 6.2. Consider $f_{a,b} : \mathbb{C}^2 \to \mathbb{C}^2$ when a and b are small. It is shown in [Hubbard and Oberste-Vorth 1995] that $f_{a,b}$ has both a fixed sink and an invariant set having the topological and dynamical structure of a solenoid, so that it displays both regular and chaotic behavior in different regions.

Note that if q is a sink, then $W^s(q) \subseteq K^+$ is open, and hence $W^s(q) \subseteq \mathring{K}^+$. On the interior of K^+, there is no chaos. To see this, suppose $p \in \mathring{K}^+$, and choose $\varepsilon > 0$ such that $\overline{\mathbb{B}_\varepsilon(p)} \subseteq K^+$. A simple argument using the form of f and the definitions of B, B^+, and B^- shows that any point in K^+ must eventually be mapped into B. Hence by compactness, there is an n sufficiently large that $f^n(\overline{\mathbb{B}_\varepsilon(p)}) \subseteq B$. Since B is bounded, we see by Cauchy's integral formula that the norm of the derivatives of f^n are uniformly bounded on $\mathbb{B}_\varepsilon(p)$ independently of $n \geq 0$. This is incompatible with chaotic behavior. For more information and further references, see [Bedford and Smillie 1991b].

To start our study of sets where chaotic behavior can occur, we define $J^+ := \partial K^+$ and $J^- := \partial K^-$, where K^+ and K^- are as in problem 2.1. The following theorem gives an analog of exercise 6.1 in the case of a general complex Hénon mapping, and is contained in [Bedford and Smillie 1991a].

THEOREM 6.3. *If p is a periodic saddle point of the Hénon map f, then $W^s(p)$ is dense in J^+, and $W^u(p)$ is dense in J^-.*

We will see in Corollary 13.4 that a Hénon map f has saddle periodic points of all but finitely many periods, so just as in the argument after exercise 6.1, we see that chaotic behavior occurs throughout J^+, and a similar argument applies to J^- under backward iteration.

7. Dynamically Defined Measures

In the study of dynamics in one variable, there are many tools available coming from classical complex analysis, potential theory, and the theory of quasiconformal mappings. In higher dimensions, not all of these tools are available, but one tool that remains useful is potential theory. The next section will provide some background for the ways in which this theory can be used to study dynamics; but before talking about potential theory proper, we first discuss some measures associated with the horseshoe map h. With notation as in section 4, we define the level-n set of h to be the set $h^{-n}B \cap h^n B$. Since the forward images of B are horizontal strips and the backward images of B are vertical strips, we see that the level-n set consists of 2^{2n} disjoint boxes.

Assertion. *For j sufficiently large, the number of fixed points of h^j in a component of the level-n set of h is independent of the component chosen.*

In fact, there is a unique probability measure m on K that assigns equal weight to each level-n square, and the above assertion can be rephrased in terms of this measure. Let P_k denote the set of $p \in \mathbb{C}^2$ such that $h^k(p) = p$. Then it follows from the above assertion that

$$\frac{1}{2^k} \sum_{p \in P_k} \delta_p \to m \tag{7.1}$$

in the topology of weak convergence.

We can use a similar technique to study the distribution of unstable manifolds. Again we consider the horseshoe map h, and we suppose that p_0 is a fixed saddle point of h and that S is the component of $W^u(p_0) \cap B$ containing p_0. In this case, S is simply a horizontal line segment through p_0. Next, let T be a line segment from the top to the bottom of B so that T is transverse to every horizontal line. See Figure 5. From the discussion of h, we know that $h^n(S) \cap B$ consists of 2^n horizontal line segments, so $h^n(S)$ intersects T in 2^n points.

We can define a measure on T using an averaging process as before. This time we average over points in $h^n(S) \cap T$ to obtain a measure m_T^-. Thus we have

$$\frac{1}{2^n} \sum_{p \in h^n(S) \cap T} \delta_p \to m_T^-, \tag{7.2}$$

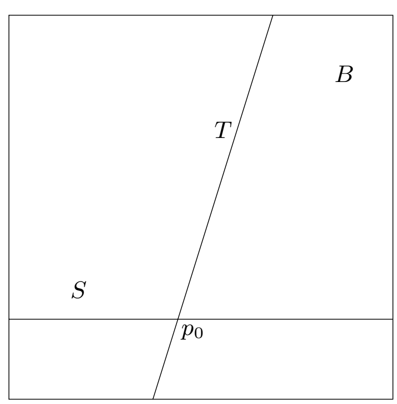

Figure 5. S is a component of $W^u(p_0) \cap B$, and T is a line transversal to all such components.

where again the convergence is in the weak sense. This gives a measure on T which assigns equal weight to each level-n segment; i.e., to each component of $h^n(B) \cap T$.

Note that if T' is another segment like T, the unstable manifolds of h give a way to transfer the above definition to a measure on T'. That is, given a point $p \in h^n(S) \cap T$, we can project along the component of $h^n(S) \cap B$ containing p to obtain a point $p' \in T'$. Using this map we obtain a measure $\varphi(m_T^-)$ on T'. It is straightforward to show that this is the same measure as $m_{T'}^-$ obtained by using T' in place of T in (7.2).

This family of measures on transverse segments is called a *transversal measure* and we denote it by m^-. Using an analogous construction with stable manifolds, we can likewise define a measure m^+ defined on "horizontal" segments.

Finally, we can take the product of these two measures to get a measure $m = m^- \times m^+$ defined on B. Then one can show that this product measure is the same as the measure m defined in (7.1). Hence there are at least two dynamically natural ways to obtain this measure.

The two transversal measures m^+ and m^- and the product measure m can be defined for general Axiom A diffeomorphisms [Ruelle and Sullivan 1975].

8. Potential Theory

To provide some physical motivation for the study of potential theory, consider two electrons moving in $\mathbb{R}^d$, each with a charge of -1. Then the repelling force between them is proportional to $1/r^{d-1}$. If we fix one electron at the origin, the total work in moving the other electron from the point z_0 to the point z_1 is independent of the path taken and is given by $P(z_1) - P(z_0)$, where P is a potential function that depends on the dimension:

$$\begin{cases} P(z) = |z| & \text{if } d = 1, \\ P(z) = \log|z| & \text{if } d = 2, \\ P(z) = -\|z\|^{-(d-2)} & \text{if } d \geq 3. \end{cases} \tag{8.1}$$

From the behavior of P at 0 and ∞ we see that, if $d \leq 2$, the amount of work needed to bring a unit charge in from the point at infinity is infinite, while this work is finite for $d \geq 3$. On the other hand, if $d \geq 2$, the amount of work needed to bring two electrons together is infinite, but for $d = 1$ this work is finite.

We can think of a collection of electrons as a charge, and we can represent charges by measures μ on $\mathbb{R}^d$. Then, for $S \subseteq \mathbb{R}^d$, the amount of charge on S is $\mu(S)$.

EXAMPLE 8.1. A unit charge at the point z_0 corresponds to the Dirac delta mass δ_{z_0}.

By using measures to represent charges, we can use convolution to define potential functions for general charge distributions. That is, given a measure μ on $\mathbb{R}^d$, we define

$$P_\mu(z) = \int_{\mathbb{R}^d} P(z-w)\,d\mu(w), \tag{8.2}$$

where P is the appropriate potential function from (8.1). Note that this definition agrees with the previous definition of potential functions in the case of point charges. Note also that the assignment $\mu \mapsto P_\mu$ is linear in μ.

In order to be able to use potential functions to study dynamics, we first need to understand a little more about their properties. In particular, we would like to know which functions can be the potential function of a finite measure.

In the case $d = 1$, the definition of P_μ and the triangle inequality imply that potential functions are convex, hence also continuous. We also have $P_\mu(x) = c|x| + O(1)$, where $c = \mu(\mathbb{R})$. In fact, any function f satisfying these two conditions is a potential function of some measure. Hence a natural question is: How do we recover the measure from f?

In particular, given a convex function f of one real variable, we can consider the assignment

$$f \mapsto \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} f\right) dx, \tag{8.3}$$

where the right-hand side is interpreted in the sense of distributions. By convexity, this distribution is positive. That is, it assigns a positive number to positive test functions, and a positive distribution is actually a positive measure. Hence we have an explicit correspondence between convex functions and positive measures, and with the additional restriction on the growth of potential functions given in the previous paragraph, we have an explicit correspondence between potential functions and finite positive measures. (The $\frac{1}{2}$ in the above formula occurs because we have normalized by dividing by the volume of the unit sphere in $\mathbb{R}$, which consists of the two points 1 and -1.)

In the case $d = 2$, the integral definition of P_μ implies that potential functions satisfy the *subaverage property.* That is, given a potential function f, any z_0 in the plane, and a disk D centered at z_0, the value $f(z_0)$ is bounded above by the average of f on ∂D. That is, if σ represents one-dimensional Legesgue measure normalized so that the unit circle has measure 1, and if r is the radius of D, then

$$f(z_0) \le \frac{1}{r}\int_{\partial D} f(\zeta)\,d\sigma(\zeta).$$

Moreover, (8.2) implies that potential functions are *upper-semicontinuous;* a real-valued function f is said to be upper-semicontinuous if its sub-level sets $f^{-1}(-\infty, a)$, for all $a \in \mathbb{R}$, are open. A function that is upper-semicontinuous and satisfies the subaverage property is called *subharmonic.*

Finally, if f is subharmonic and satisfies $f(z) = c\log|z| + O(1)$ for some $c > 0$, then f is said to be a *potential function.* Just as before, a potential function has the form P_μ for some measure μ.

In fact, if f is subharmonic and of class C^2, the Laplacian of f is always positive. This is an analog of the fact that the second derivative of a convex function is positive. If f is subharmonic but not C^2, then Δf is a positive distribution, hence a positive measure. Thus the Laplacian gives us a correspondence between potential functions and finite measures much like that in (8.3):

$$f \mapsto \frac{1}{2\pi}(\Delta f)\, dx\, dy,$$

where this is to be interpreted in the sense of distributions and again we have normalized by dividing by the volume of the unit sphere.

EXAMPLE 8.2. Applying the above assignment to the function $\log|z|$ produces the delta mass δ_0 in the sense of distributions.

Now suppose that $K \subset \mathbb{R}^2 = \mathbb{C}$ is compact. Put a unit charge on K and allow it to distribute itself through so that the mutual repulsion of the electrons is minimized. The distribution of charge on K is described by a measure μ. We will find μ by finding its potential function P_μ, which is usually written G. The function G satisfies the following properties:

1. G is subharmonic.
2. G is harmonic outside K.
3. $G = \log|z| + O(1)$.
4. G is constant on K.

If G satisfies properties 1 through 3, and also property

4′. $G \equiv 0$ on K,

we say that G is a *Green function* for K. If G exists, it is unique, and in this case we can take the Laplacian of G in the sense of distributions. Thus, we say that

$$\mu_K := \frac{1}{2\pi}\Delta G\, dx\, dy$$

is the *equilibrium measure* for K.

EXAMPLE 8.3. Let D be the unit disk. Then the Green function for D is

$$G(z) = \log^+|z|,$$

where $\log^+|z| := \max\{\log|z|, 0\}$, and the equilibrium measure is

$$\mu_D = \frac{1}{2\pi}(\Delta \log^+|z|)\, dx\, dy,$$

which is simply arc length measure on ∂D, normalized to have mass 1.

9. Potential Theory in One-Variable Dynamics

In this section we discuss some of the ways in which potential theory can be used to understand the dynamics of polynomial maps of the complex plane. These ideas were introduced in [Brolin 1965]. In the next section we will see how they help us understand higher-dimensional complex dynamics.

For this section, let f be a monic polynomial in one variable of degree $d \geq 2$, and let $K \subseteq \mathbb{C}$ be the set of z such that the forward orbit of z is bounded. Then K has a Green function G_K, given by the formula

$$G_K(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ \bigl|f^n(z)\bigr|.$$

It is difficult to understand Brolin's paper without knowing this formula. However, it was in fact first written down by Sibony in his UCLA lecture notes after Brolin's paper had already been written.

It is not hard to show that the limit in the definition of G_K converges uniformly on compact sets, and since each of the functions on the right-hand side is subharmonic, the limit is also subharmonic. Moreover, on a given compact set outside of K, each of these functions is harmonic for sufficiently large n, so that the limit is harmonic on the complement of K. The property $G = \log|z| + O(1)$ follows by noting that for $|z|$ large we have $|z|^d/c \leq |f(z)| \leq c|z|^d$ for some $c > 1$, then taking logarithms and dividing by d, then using an inductive argument to bound $\bigl|\log^+|f^n(z)|/d^n - \log|z|\bigr|$ independently of d. Finally, the property $G \equiv 0$ on K is immediate since $\log^+|z|$ is bounded for $z \in K$. In fact, G_K has the additional property of being continuous.

Hence we see that G_K really is the Green function for K, and we can define the equilibrium measure

$$\mu := \mu_K = \frac{1}{2\pi}(\Delta G_K)\,dx\,dy.$$

The following theorem provides a beautiful relationship between the measure μ and the dynamical properties of f. It says that we can recover μ by taking the average of the point masses at the periodic points of period n and passing to the limit or by taking the average of the point masses at the inverse images of any nonexceptional point and passing to the limit. (A point p is said to be *nonexceptional* for a polynomial f if the set $\{f^{-n}(p) : n \geq 0\}$ contains at least three points. It is a theorem that there is at most one exceptional point for any polynomial.)

THEOREM 9.1 [Brolin 1965; Tortrat 1987]. *Let f be a monic polynomial of degree d, and let $c \in \mathbb{C}$ be a nonexceptional point. Then*

$$\mu = \lim_{n\to\infty} \frac{1}{d^n} \sum_{z\in A_n} \delta_z,$$

in the sense of convergence of measures, where A_n is either the set of z satisfying $f^n(z) = c$ (counted with multiplicity), or the set of z satisfying $f^n(z) = z$ (counted with multiplicity).

PROOF. We prove only the case $f^n(z) = c$ here. Let

$$\mu_n = \frac{1}{d^n} \sum_{f^n(z)=c} \delta_z.$$

Then we want to show that $\mu_n \to \mu_K$. Since the space of measures with the topology of weak convergence is compact, it suffices to show that if some subsequence of μ_n converges to a measure μ^*, then $\mu^* = \mu_K$. By renaming, we may assume that μ_n converges to μ^*.

We can show $\mu^* = \mu_K$ by showing the convergence of the corresponding potential functions. The potential function for μ_n is

$$G_n(z) = \frac{1}{d^n} \sum_{f^n(w)=c} \log|z-w| = \frac{1}{d^n} \log \left| \prod_{f^n(w)-c=0} (z-w) \right| = \frac{1}{d^n} \log|f^n(z) - c|.$$

Here the sum and products are taken over the indicated sets with multiplicities, and the last equality follows from the fact that we are simply multiplying all the monomials corresponding to roots of the monic polynomial $f^n(z) - c$.

Let $G^*(z) := \lim_{n\to\infty} G_n(z)$. Then G^* is the potential function for μ^*, and

$$G^*(z) = \lim_{n\to\infty} \frac{1}{d^n} \log\bigl|f^n(z) - c\bigr|,$$

while

$$G_K(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+\bigl|f^n(z)\bigr|,$$

and we need to show that $G^*(z) = G_K(z)$. If $z \notin K$, then $f^n(z)$ tends to ∞ as n increases, so that $G^*(z) = G_K(z)$ in this case. Since G^* is the potential function for μ^*, it is upper-semicontinuous, so it follows that $G^*(z) \geq 0$ for $z \in \partial K$. On the other hand, since $G^* = G$ on the set where $G = \varepsilon$, the maximum principle for subharmonic functions implies that $G \leq \varepsilon$ on the region enclosed by this set. Letting ε tend to 0 shows that $G^* \leq 0$ on K.

Finally, using some knowledge of the possible types of components for the interior of K, one can show that, if c is nonexceptional, the measure μ^* assigns no mass to the interior of K. This implies that G^* is harmonic on K, since μ^* is the Laplacian of G^*. Hence both the maximum and minimum principles apply to G^* on K, which implies that $G^* \equiv 0$ on K.

Thus $G^* \equiv G_K$ and hence $\mu^* \equiv \mu_K$ as desired. □

REMARK. This theorem provides an algorithm for drawing a picture of the Julia set J for a polynomial f. Start with a nonexceptional point c, and compute points on the backward orbits of c. These points will accumulate on the Julia set for f, and by discarding points in the first several backwards iterates of c, we can obtain a reasonably good picture of the Julia set. This algorithm has the disadvantage that these backwards orbits tend to accumulate most heavily on points in J that are easily accessible from infinity. That is, it favors points at which a random walk starting at infinity is most likely to land and avoids points such as inward pointing cusps.

Exercise 9.2. Since G is harmonic both in the complement of K and in the interior of K, we see that $\operatorname{supp}\mu \subseteq J$, where $J = \partial K$ is the Julia set. Show that $\operatorname{supp}\mu = J$. Hint: Use the maximum principle.

Note that an immediate corollary of this exercise and Theorem 9.1 is that periodic points are dense in J.

10. Potential Theory and Dynamics in Two Variables

In Theorem 9.1, we took the average of point masses distributed over either the set $\{z : f^n(z) = c\}$, or the set $\{z : f^n(z) = z\}$. In the setting of polynomial diffeomorphisms of $\mathbb{C}^2$, there are two natural questions motivated by these results.

(i) What happens when we iterate one-dimensional submanifolds (forwards or backwards)?
(ii) Is the distribution of periodic points described by some measure μ?

In $\mathbb{C}$, we can loosely describe the construction of the measure μ as first counting the number of points in the set $\{z : f^n(z) = c\}$ or $\{z : f^n(z) = z\}$, then using potential theory to describe the location of these points.

Before we consider such a procedure in the case of question (i) for $\mathbb{C}^2$, we first return to the horseshoe map and recall the measure m^- defined in Section 7. Suppose that B is defined as in that section, that p is a fixed point for the horseshoe map h, that S is the component of $W^u(p) \cap B$ containing p, and that T is a line segment from the top to the bottom of B as before. Then orient T and S so that these orientations induce the standard orientation on $\mathbb{R}^2$ at the point of intersection of T and S.

Now apply h to S. Then $h(S)$ and T will intersect in two points, one of which is the original point of intersection, and one of which is new. See Figure 6. Because of the form of the horseshoe map, the intersection of $h(S)$ and T at the new point will not induce the standard orientation on $\mathbb{R}^2$, but rather the opposite orientation. In general, we can apply h^n to S, then assign $+1$ to each point of intersection that induces the standard orientation, and -1 to each point that induces the opposite orientation. Unfortunately, the sum of all such points of intersection for a given n will always be 0, so this doesn't give us a way to count these points of intersection.

A second problem with real manifolds is that the number of intersections may change with small perturbations of the map. For instance, if the map is changed so that $h(S)$ is tangent to T and has no other other intersections with T, then for small perturbations g near h, $g(S)$ may intersect T in zero, one, or two points.

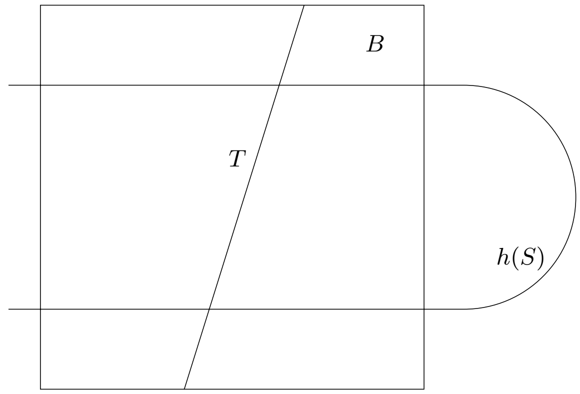

Figure 6. T and $h(S)$.

Now suppose that B is a bidisk in $\mathbb{C}^2$ (that is, the product of the disk with itself), that h is a complex horseshoe map, and that T and S are complex submanifolds. In this case, there is a natural orientation on T at any point given by taking a vector v in the tangent space of T over this point and using the set $\{v, iv\}$ to define the orientation at that point. We can use the same procedure on S, then apply h^n as before. In this case, the orientation induced on $\mathbb{C}^2$ by $h^n(S)$ and T is always the same as the standard orientation. Hence assigning $+1$ to such an intersection and taking the sum gives the number of points in $T \cap h^n(S)$.

Additionally, if both S and T are complex manifolds, the number of intersections between $h^n(S)$ and T, counted with multiplicity, is constant under small perturbations.

Thus, in studying question (i), we will use complex one-dimensional submanifolds.

Recall that, in the case of one variable, the Laplacian played a key role by allowing us to relate the potential function G to the measure μ. Here we consider an extension of the Laplacian to $\mathbb{C}^2$ in order to achieve a similar goal.

For a function f of two real variables x and y, the exterior derivative of f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

which is invariant under smooth maps. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, multiplication by i induces the map $(i)^*$ on the cotangent bundle, and this map takes dx to dy and dy to $-dx$. Hence, defining $d^c = (i)^* d$, we have

$$d^c f = \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx,$$

which is invariant under smooth maps preserving the complex structure, that is, holomorphic maps. Hence dd^c is also invariant under holomorphic maps.

Expanding dd^c gives

$$dd^c f = d\left(\frac{\partial f}{\partial x}\right) dy - d\left(\frac{\partial f}{\partial y}\right) dx = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx\, dy,$$

which is nothing but the Laplacian. This shows that the Laplacian, when viewed as a map from functions to two-forms, is invariant under holomorphic maps, and also shows that this procedure can be carried out in any complex manifold of any dimension. Moreover, it also shows that the property of being subharmonic is invariant under holomorphic maps.

EXERCISE 10.1. Let D_r be the disk of radius r centered at 0 in the plane, and compute

$$\int_{D_r} dd^c \log |z|.$$

(Hint: This is equal to $\int_{\partial D_r} d^c \log |z|$.)

We next need to extend the idea of subharmonic functions to $\mathbb{C}^2$.

DEFINITION 10.2. A function $f : \mathbb{C}^2 \to \mathbb{R}$ is *plurisubharmonic* if h is upper-semicontinuous and if the restriction of h to any one-dimensional complex line satisfies the subaverage property.

Intrinsically, an upper-semicontinuous function h is plurisubharmonic if and only if $dd^c h$ is nonnegative, where again we interpret this in the sense of distributions.

In fact, in the above definition we could replace the phrase "complex line" by "complex submanifold" without changing the class of functions, since subharmonic functions are invariant under holomorphic maps. As an example of the usefulness of this and the invariance property of dd^c, suppose that φ is a holomorphic embedding of $\mathbb{C}$ into $\mathbb{C}^2$ and that h is smooth and plurisubharmonic on $\mathbb{C}^2$. Then we can either evaluate $dd^c h$ and pull back using φ, or we can first pull back and then apply dd^c. In both cases we get the same measure on $\mathbb{C}$.

11. Currents and Applications to Dynamics

In this section we provide a brief introduction to the theory of currents. A current is simply a linear functional on the space of smooth differential forms; that is, a current μ acts on a differential form of a given degree, say $\varphi = f_1\, dx + f_2\, dy$ in the case of a one-form, to give a complex number $\mu(\varphi)$, and this assignment is linear in φ. This is a generalization of a measure in the sense that a measure acts on zero-forms (functions) by integrating the function against the measure.

As an example, suppose that $M \subseteq \mathbb{C}^2$ is a submanifold of real dimension n. Then integration over M is a current $[M]$ acting on n-forms φ; it is given simply by

$$[M](\varphi) = \int_M \varphi.$$

In this example the linearity is immediate, as is the relationship to measures. In particular, if $p \in \mathbb{C}^2$, then $[p] = \delta_p$, the delta mass at p, acts on zero-forms.

EXAMPLE 11.1. Suppose $P : \mathbb{C} \to \mathbb{C}$ is a polynomial having only simple roots, and let R be the set of roots of P. Then $[R]$ is a current acting on 0-forms, and

$$[R] = \frac{1}{2\pi} dd^c \log |P|.$$

This formula is still true for arbitrary polynomials if we account for multiplicities in constructing $[R]$.

We can extend this last example to the case of polynomials from $\mathbb{C}^2$ to $\mathbb{C}$. This is the content of the next proposition.

PROPOSITION 11.2 (POINCARÉ–LELONG FORMULA). *If* $P : \mathbb{C}^2 \to \mathbb{C}$ *is a polynomial and* $V = \{P = 0\}$, *then*

$$[V] = \frac{1}{2\pi} dd^c \log |P|,$$

where again $[V]$ *is interpreted with weights according to multiplicity.*

Suppose now that $f : \mathbb{C}^2 \to \mathbb{C}^2$ is a Hénon diffeomorphism (see Section 5), and define

$$G^+(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \bigl|\pi_1(f^n(p))\bigr|,$$
$$G^-(p) = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \bigl|\pi_2(f^{-n}(p))\bigr|,$$

where π_j is projection to the j-th coordinate. As in the case of the function G defined for a one-variable polynomial, it is not hard to check that G^+ is plurisubharmonic, is identically 0 on K^+, is pluriharmonic on $\mathbb{C}^2 \setminus K^+$ (i.e., is harmonic on any complex line), and is positive on $\mathbb{C}^2 \setminus K^+$. In analogy with the function G, we say that G^+ is the Green function of K^+. Likewise, G^- is the Green function of K^-.

Note that for n large and $p \notin K$, the value of $\bigl|\pi_1 f^n(p)\bigr|$ is comparable to the square of $\bigl|\pi_2 f^n(p)\bigr|$, and hence we may replace $\bigl|\pi_1 f^n(p)\bigr|$ by $\bigl\|f^n(p)\bigr\|$ in the formula for G^+, and likewise for G^-.

Again in analogy with the one-variable case, and using the equivalence between the Laplacian and dd^c outlined earlier, we define

$$\mu^+ = \frac{1}{2\pi} dd^c G^+, \quad \mu^- = \frac{1}{2\pi} dd^c G^-.$$

Then μ^+ and μ^- are currents supported on $J^+ = \partial K^+$ and $J^- = \partial K^-$, respectively. Moreover, $\mu^\pm$ restrict to measures on complex one-dimensional submanifolds in the sense that we can pull back $G^\pm$ from the submanifold to an open set in the plane, then take dd^c on this open set.

As an analog of the case $f^n(z) = c$ of Theorem 9.1, we have the following theorem.

THEOREM 11.3. *Let V be the (complex) x-axis in $\mathbb{C}^2$, i.e., the set where π_2 vanishes, and let f be a complex Hénon map. Then*

$$\lim_{n\to\infty} \frac{1}{2^n}[f^{-n}V] = \mu^+.$$

PROOF. Note that the set $f^{-n}V$ is the set where the polynomial $\pi_1 f^n$ vanishes. Hence the previous proposition implies that

$$[f^{-n}V] = \frac{1}{2\pi} dd^c \log|\pi_1 f^n|.$$

Passing to the limit and using an argument like that in Theorem 9.1 to replace log by $\log^+$, we obtain the theorem. See [Bedford and Smillie 1991a] or [Fornæss and Sibony 1992a] for more details. □

Here is a more comprehensive form of this theorem:

THEOREM 11.4. *If S is a complex disk and f is a complex Hénon map, then*

$$\lim_{n\to\infty} \frac{1}{2^n}[f^{-n}S] = c\mu^+,$$

where $c = \mu^-[S]$. An analogous statement is true with μ^+ and μ^- interchanged and f^n in place of f^{-n}.

As a corollary, we obtain the following theorem from [Bedford and Smillie 1991b].

COROLLARY 11.5. *If p is a periodic saddle point, $W^u(p)$ is dense in J^-.*

PROOF. Replacing f by f^n, we may assume that p is fixed. Let S be a small disk in $W^u(p)$ containing p. Then the forward iterates of S fill out the entire unstable manifold. Moreover, by the previous theorem, the currents associated with these iterates converge to $c\mu^-$ where $c = \mu^+[S]$. If $c \neq 0$, the proof is complete since then $W^u(p)$ must be dense in $\operatorname{supp}\mu^- = J^-$.

But c cannot be 0: if it were, $G^+|_S$ would be harmonic, hence identically 0 by the minimum principle since G is nonnegative on S and 0 at p. Hence S would be contained in K, which is impossible since the iterates of S fill out all of $W^u(p)$, which is not bounded. Thus $c \neq 0$, so $W^u(p)$ is dense in J^-. □

This corollary gives some indication of why pictures of invariant sets on complex slices in $\mathbb{C}^2$ show essentially the full complexity of the map. If we start with any complex slice transverse to the stable manifold of a periodic point p, then the forward iterates of this slice accumulate on the unstable manifold of p, hence on all of J^- by the corollary. All of this structure is then reflected in the original slice, giving rise to sets that are often self-similar and bear a striking resemblance to Julia sets in the plane.

12. Currents and Hénon Maps

In this section we continue the study of the currents μ^+ and μ^- in order to obtain more dynamical information.

We first consider this in the context of the horseshoe map. Recall that B is a square in the plane and that we have defined measures m^+ and m^-, and their product measure m, in Section 7.

In fact, m^+ and m^- generalize to μ^+ and μ^- in the case that the Hénon map is a horseshoe. More explicitly, let D_λ be a family of complex disks in $\mathbb{C}^2$ indexed by the parameter λ, such that each D_λ intersects $\mathbb{R}^2$ in a horizontal segment in B and such that these segments fill out all of B. Then we can recover μ^+, at least restricted to B, by

$$\mu^+|_B = \int [D_\lambda]\, dm^+(\lambda).$$

In analogy with the construction of m as a product measure using m^+ and m^-, we would like to combine μ^+ and μ^- to obtain a measure μ. Since μ^+ and μ^- are currents, the natural procedure to try is to take $\mu = \mu^+ \wedge \mu^-$. While forming the wedge product is not well-defined for arbitrary currents, it is well-defined in this case using the fact that these currents are obtained by taking dd^c of a continuous plurisubharmonic function and applying a theorem of pluripotential theory. In this way we get a measure μ on $\mathbb{C}^2$.

DEFINITION 12.1. $\mu = \mu^+ \wedge \mu^-$.

DEFINITION 12.2. $J = J^+ \cap J^-$.

We next collect some useful facts about μ.

(1) μ is a probability measure. For a proof of this, see [Bedford and Smillie 1991a].

(2) μ is invariant under f. To see this, note that, since

$$G^\pm = \lim_{n\to\infty} \frac{1}{2^n} \log^+ \|f^{\pm n}\|,$$

we have $G^\pm\bigl(f(p)\bigr) = 2^\pm G^\pm(p)$. Since $\mu^\pm = (1/2\pi)dd^c G^\pm$, this implies that $f^*(\mu^\pm) = 2^\pm \mu^\pm$, and hence

$$f^*(\mu) = f^*(\mu^+) \wedge f^*(\mu^-) = 2\mu^+ \wedge \tfrac{1}{2}\mu^- = \mu^+ \wedge \mu^- = \mu.$$

(3) $\operatorname{supp}\mu \subseteq J$. This is a simple consequence of the fact that the support of μ is contained in the intersection of $\operatorname{supp}\mu^+ = J^+$ and $\operatorname{supp}\mu^- = J^-$ and the definition of J.

In order to examine the support of μ more precisely, we turn our attention for a moment to Shilov boundaries. Let X be a subset either of $\mathbb{C}$ or $\mathbb{C}^2$. We say that a set B is a *boundary* for X if B is closed and if for any holomorphic polynomial P we have

$$\max_X |P| = \max_B |P|.$$

With the right conditions, the intersections of any set of boundaries is a again a boundary by a theorem of Shilov, so we can intersect them all to obtain the smallest such boundary. This is called the *Shilov boundary* for X.

EXAMPLE 12.3. Let $X = D_1 \times D_1$, where D_1 is the unit disk. Then the Shilov boundary for X is $(\partial D_1) \times (\partial D_1)$, while the topological boundary for X is

$$\partial X = (D_1 \times \partial D_1) \cup (\partial D_1 \times D_1).$$

The following theorem is contained in [Bedford and Taylor 1987].

THEOREM 12.4. $\operatorname{supp} \mu = \partial_{\text{Shilov}} K$.

We have already defined J as the intersection of J^+ and J^-, and the choice of notation is designed to suggest an analogy with the Julia set in one variable. However, in two variables, the support of μ is also a natural candidate for a kind of Julia set. Hence we make the following definition.

DEFINITION 12.5. $J^* = \operatorname{supp} \mu$.

For subsets of $\mathbb{C}$ there is no distinction between the topological boundary and the Shilov boundary; hence we could have defined the Julia set J as either the topological or the Shilov boundary of K.

13. Heteroclinic Points and Pesin Theory

In the previous section, we discussed some of the formal properties of μ arising from considerations of the definition and of potential theory. In this section we concentrate on the less formal properties of μ and on the relation of μ to periodic points. The philosophy here is that since μ^+ and μ^- describe the distribution of one-dimensional objects, μ should describe the distribution of zero-dimensional objects.

An example of a question using this philosophy is the following. For a periodic point p, we know that μ^+ describes the distribution of $W^s(p)$ and μ^- describes the distribution of $W^u(p)$. Does μ describe (in some sense) the distribution of intersections $W^s(p) \cap W^u(q)$?

DEFINITION 13.1. Let p and q be saddle periodic points of a diffeomorphism f. A point in the set $(W^u(p) \cap W^s(q)) \setminus \{p, q\}$ is called a *heteroclinic point*. If $p = q$, then such a point is called a *homoclinic point*.

Unfortunately, the techniques discussed so far do not allow us to prove the existence of even one heteroclinic point. We will imagine how it might be possible for the unstable manifold of p to avoid the stable manifold of q. The stable and unstable manifolds are conformally equivalent to copies of $\mathbb{C}$, so we have parametrizations $\varphi_u : \mathbb{C} \to W^u(p)$ and $\varphi_s : \mathbb{C} \to W^s(q)$.

Now, if π_j represents projection onto the j-th coordinate, then $\pi_1\varphi_u : \mathbb{C} \to \mathbb{C}$ is an entire function, and as such can have an omitted value. As an example, $\pi_1\varphi_u(z)$ could be equal to e^z and hence would omit the value 0. It could happen that there is a second saddle point q such that $W^s(q)$ is the x-axis, in which case $W^u(p) \cap W^s(q) = \varnothing$.

At first glance, it may seem that this contradicts some of our earlier results. It might seem that Theorem 11.4 should imply that $W^u(p)$ intersects transversals which cross J^-, but in fact, that statement is a statement about convergence of distributions. Each of the distributions must be evaluated against a test function, and the test function must be positive on an open set. Thus there is still room for $W^u(p)$ and $W^s(p)$ to be disjoint.

Hence, in order to understand more about heteroclinic points, we need a better understanding of the stable and unstable manifolds. One possible approach is to use what is known as Ahlfors' three-island theorem. This theorem concerns entire maps $\psi : \mathbb{C} \to \mathbb{C}$. Roughly, it says that if we have n open regions in the plane and consider their inverse images under ψ, then some fixed proportion of them will have an inverse image that is compact and that maps injectively under ψ onto the corresponding original region.

If we apply this theorem to the map $\pi_1\varphi_u$ giving $W^u(p)$, we can divide the plane into increasingly more and smaller islands, and we can do this in such a way that at each stage we obtain more of $W^u(p)$ as the injective image of regions in the plane. The result is that we get a picture of $W^u(p)$ which is locally laminar.

Since $W^u(p)$ is dense in J^-, this gives us one possible approach to studying μ^-, and we can use a similar procedure to study μ^+. However, recall that our goal here is to describe heteroclinic points. Thus in order for this approach to apply, we need to be able to get the disks for μ^+ to intersect the disks for μ^-. Unfortunately, we don't get any kind of uniformity in the disks using this approach, so getting this intersection is difficult.

An alternate approach is to use the theory of nonuniform hyperbolicity (see [Young 1995] for more information). This is an extension of parts of the hyperbolic theory to a very general situation. This theory applies to the measure μ, which is to say that we have expanding and contracting directions at μ-almost every point, though the expansion and contraction need not be uniform and these directions need not depend continuously on the point. This is enough to produce stable and unstable manifolds through μ-almost every point.

We can then identify the stable and unstable manifolds obtained using this theory with the disks obtained in the previous nonuniform laminar picture to guarantee that we get intersections between stable and unstable manifolds and hence heteroclinic points. Putting all of this together, we obtain the following theorem, contained in [Bedford, Lyubich, and Smillie 1993a].

Theorem 13.2. *J^* is the closure of the set of all periodic saddle points, and also the closure of the union of all $W^u(p) \cap W^s(q)$ over all periodic saddles p and q.*

This theorem can be viewed as an analog of the theorem in one variable dynamics saying that the Julia set is the closure of the repelling periodic points. For this reason, the set J^* is perhaps a better analogue of the Julia set in the two dimensional case than is J.

Recall that $J^* = \partial_{\text{shilov}} K \subseteq \partial K = J$. When f is an Axiom A diffeomorphism, it is a theorem that $J^* = J$. However, it is an interesting open question whether this equality holds in general. If it were the case that $J \neq J^*$, then there would be a saddle periodic point q and another point p such that $p \in \overline{W^s(q)} \cap \overline{W^u(q)}$, but $p \notin \overline{W^s(q) \cap W^u(q)}$.

In fact, using the ideas of Pesin theory, one can get precise information about the number of periodic points of a given period and how their distribution relates to the measure μ. This is contained in the following theorem and corollary from [Bedford, Lyubich, and Smillie 1993b].

Theorem 13.3. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a complex Hénon map, and let P_n be either the set of fixed points of f^n or the set of saddle points of minimal period n. Then*

$$\lim_{n\to\infty} \frac{1}{2^n} \sum_{p \in P_n} \delta_p = \mu.$$

For the following corollary, let P_n be the set of saddle points of f of minimal period n, and let $|P_n|$ denote the number of points contained in this set.

Corollary 13.4. *There are periodic saddle points of all but finitely many periods. More precisely, we have* $\lim_{n\to\infty} |P_n|/2^n = 1$.

Recall that the horseshoe map had periodic points of all periods, so while we haven't achieved that result for general Hénon maps, we have still obtained a good deal of information about periodic points and heteroclinic points.

14. Topological Entropy

Recall that the horseshoe map is topologically equivalent to the shift map on two symbols. One could also ask if it is topologically equivalent to the shift on four symbols. That is, if h is the horseshoe map defined on the square B, then $h(B) \cap h^{-1}(B)$ consists of four components, and we can label these components with four symbols. However, with this labeling scheme, one can check by counting that not all sequences of symbols correspond to an orbit of a point in the way that sequences of two symbols did. In fact, the number of symbol sequences of length 2 corresponding to part of an orbit is 8, and the number of such sequences of length 3 is 16. Allowing longer sequences and letting

$S(n)$ denote the number of sequences of length n which correspond to part of an orbit, we obtain the formula

$$\lim_{n\to\infty} \frac{1}{n} \log S(n) = \log 2.$$

The number $\log 2$ is the *topological entropy* of the horseshoe map, and can be defined as the maximum growth rate over all finite partitions. In general, the shift map on N symbols has entropy $\log N$ and, since entropy is a topological invariant, we see that all of these different shift maps are topologically distinct.

In the case of a general Hénon map, we have the following theorem, contained in [Smillie 1990].

THEOREM 14.1. *The topological entropy of a complex Hénon map is* $\log 2$.

Topological entropy is a useful idea because it is connected to many different aspects of polynomial diffeomorphisms. It is a measure of area growth and of the growth rate of the number of periodic points, both of which are closely related to the degree of the map as a polynomial. Moreover, it is related to measure-theoretic entropy in the sense that, for any probability measure ν, the measure-theoretic entropy $h_\nu(f)$ is bounded from above by the topological entropy $h_{\text{top}}(f)$. Moreover, μ is the *unique* measure for which $h_\mu(f) = h_{\text{top}}(f)$ [Bedford, Lyubich, and Smillie 1993a].

We can also consider topological entropy for real Hénon maps, that is, $f_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ as in Section 5 with $a, b \in \mathbb{R}$. In contrast to the theorem above, in this case we have $0 \le h_{\text{top}}(f_{\mathbb{R}}) \le \log 2$, and all values are possible. However, one can show that not all values are possible for Axiom A diffeomorphisms, but only logarithms of algebraic numbers [Milnor 1988]. Moreover, we also have the following theorem [Bedford, Lyubich, and Smillie 1993a].

THEOREM 14.2. *For a Hénon map f with real coefficients, the following are equivalent.*

1. $h_{\text{top}}(f_{\mathbb{R}}) = \log 2$.
2. $J^* \subseteq \mathbb{R}^2$.
3. $K \subseteq \mathbb{R}^2$.
4. *All periodic points are real.*

Moreover, these conditions imply that $J = J^*$.

PROOF. Condition 1 implies that $f_{\mathbb{R}}$ has a measure μ' of maximal entropy with $\operatorname{supp} \mu' \subseteq \mathbb{R}^2$. By uniqueness we have $\mu' = \mu^*$, so $\operatorname{supp} \mu^* \subseteq \mathbb{R}^2$, thus giving condition 2.

Condition 2 implies that $J^* = \partial_{\text{Shilov}} K$ is contained in $\mathbb{R}^2$, which implies that K is contained in $\mathbb{R}^2$. This gives condition 3, and in fact, since polynomials in $\mathbb{R}^2$ are dense in the set of continuous functions of $\mathbb{R}^2$, this also implies that $\partial_{\text{Shilov}} K = K$, and hence $J^* = K$ and thus $J^* = J$ since $J^* \subseteq J \subseteq K$.

Condition 3 immediately implies condition 4.

Condition 4 together with theorem 13.2 implies that $J^* \subseteq \mathbb{R}^2$, which implies that $\operatorname{supp} \mu^* \subseteq \mathbb{R}^2$, which implies condition 1. □

These conditions are true for the set of real Hénon maps that are horseshoes. We can identify such maps with their parameter values in $\mathbb{R}^2$, in which case the set of horseshoe maps is an open set in $\mathbb{R}^2$. Since topological entropy is continuous for C^∞ diffeomorphisms, we see that maps on the boundary of this set also satisfy the above conditions.

Let's call a real Hénon map that satisfies the conditions of Theorem 14.2 a *maximal entropy map*. Real horseshoes are maximal entropy maps. As we will see shortly, there are maximal entropy maps that are not horseshoes. The following result shows that any maximal entropy map is either Axiom A or fails to be Axiom A in a very specific way. Recall that a homoclinic tangency is an intersection of $W^u(p)$ and $W^s(p)$ at some point $q \neq p$ for some saddle point p. This intersection is a homoclinic tangency if the stable and unstable manifolds are tangent at q, and this is a quadratic tangency if the manifolds have quadratic contact at q. A diffeomorphism with a homoclinic tangency violates one of the defining properties of hyperbolicity and hence is not Axiom A.

THEOREM 14.3. *If the conditions in Theorem* 14.2 *hold*, *then*

(i) *periodic points are dense in* K,
(ii) *every periodic point is a saddle with expansion constants bounded below*,
(iii) *either* f *is Axiom A or* f *has a quadratic homoclinic tangency*.

Theorem 14.3 gives a picture of how the property of being a horseshoe is lost as the parameters change. (Recall that horseshoes are by definition Axiom A.) Suppose we have a one-parameter family f_t of real Hénon maps that starts out as a horseshoe then loses the Axiom A property. The set of parameters for which the map is a horseshoe is open because of structural stability, so there is some first parameter value t_0 at which the map is not Axiom A. What happens at this parameter value? The function $h_{\text{top}}(f_{\mathbb{R}})$ is continuous, so f_{t_0} is a maximal entropy map but it is not Axiom A. According to the previous theorem there are pieces of stable and unstable manifolds that have a quadratic tangency. Let us assume that for t past t_0 the pieces of stable and unstable manifolds pull through each other. This means that the intersection point of the stable and unstable manifolds is moving out of $\mathbb{R}^2$ and into $\mathbb{C}^2$. This causes a decrease in the topological entropy $h_{\text{top}}(f_{\mathbb{R}})$. Since the topological entropy is continuous as a function in parameter space and is an invariant of topological conjugacy, we pass through uncountably many topological conjugacy classes as we vary the parameter. This presents a striking contrast to the horseshoe example, in which small variations in the parameters produce topologically conjugate diffeomorphisms.

15. Suggestions for Further Reading

We have presented here one point of view on complex dynamics in several variables. For other viewpoints on polynomial diffeomorphisms the reader can consult [Hubbard and Oberste-Vorth 1994; 1995; Fornæss and Sibony 1992a; 1994]. Other related directions include the study of rational maps on complex projective spaces [Fornæss and Sibony 1992b; Hubbard and Papadopol 1994; Ueda 1986; 1991; 1994; 1992; $\geq$ 1998]. There is also interesting work on non-polynomial diffeomorphisms of $\mathbb{C}^2$ by Buzzard [1995; 1997; $\geq$ 1998].

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