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Scalar Curvature and Geometrization Conjectures for 3-Manifolds

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Abstract. We first summarize very briefly the topology of 3-manifolds and the approach of Thurston towards their geometrization. After discussing some general properties of curvature functionals on the space of metrics, we formulate and discuss three conjectures that imply Thurston's Geometrization Conjecture for closed oriented 3-manifolds. The final two sections present evidence for the validity of these conjectures and outline an approach toward their proof.

Introduction

In the late seventies and early eighties Thurston proved a number of very remarkable results on the existence of geometric structures on 3-manifolds. These results provide strong support for the profound conjecture, formulated by Thurston, that every compact 3-manifold admits a canonical decomposition into domains, each of which has a canonical geometric structure.

For simplicity, we state the conjecture only for closed, oriented 3-manifolds.

Geometrization Conjecture [Thurston 1982]. *Let M be a closed, oriented, prime 3-manifold. Then there is a finite collection of disjoint, embedded tori T_i^2 in M, such that each component of the complement $M \setminus \bigcup T_i^2$ admits a geometric structure, i.e., a complete, locally homogeneous Riemannian metric.*

A more detailed description of the conjecture and the terminology will be given in Section 1. A complete Riemannian manifold N is *locally homogeneous* if the universal cover $\tilde{N}$ is a complete homogenous manifold, that is, if the isometry group $\operatorname{Isom}\tilde{N}$ acts transitively on $\tilde{N}$. It follows that N is isometric to $\tilde{N}/\Gamma$, where Γ is a discrete subgroup of $\operatorname{Isom}\tilde{N}$, which acts freely and properly discontinuously on $\tilde{N}$.

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Thurston showed that, in dimension three, there are eight possible geometries, all of which are realized. Namely, the universal covers are either the constant curvature spaces $\mathbb{H}^3$, $\mathbb{E}^3$, $\mathbb{S}^3$, or products $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, or twisted products $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, Nil, Sol (see Section 1B).

It is perhaps easiest to understand the context and depth of this conjecture by recalling the classical uniformization (or geometrization) theorem for surfaces, due to Poincaré and Koebe. If M is a closed, oriented surface, the uniformization theorem asserts that M carries a smooth Riemannian metric of constant curvature K equal to -1, 0 or $+1$. This means that it carries a geometric structure modeled on $\mathbb{H}^2$, $\mathbb{E}^2$ or $\mathbb{S}^2$, respectively. Further, knowledge of the sign of the curvature and the area of the surface gives a complete topological description of the surface, via the Gauss–Bonnet formula

$$2\pi\chi(M) = \int_M K\,dV.$$

The validity of the Geometrization Conjecture in dimension three would similarly provide a deep topological understanding of 3-manifolds, as well as a vast array of topological invariants, arising from the geometry of the canonical metrics.

There is a noteworthy difference between these pictures in dimensions two and three, however. In two dimensions, there is typically a nontrivial space of geometric structures, that is, of constant curvature metrics—the moduli space, or the related Teichmüller space. Only the case $K = +1$ is rigid, that is, there is a unique metric (up to isometry) of constant curvature $+1$ on $\mathbb{S}^2$. The moduli space of flat metrics on a torus is a two-dimensional variety, and that of hyperbolic metrics on surfaces of higher genus g is a variety of dimension $3g-3$. As we will indicate briefly below, these moduli spaces also play a crucial role in Thurston's approach to and results on the Geometrization Conjecture.

In dimension three, the geometric structures are usually rigid. The moduli spaces of geometric structures, if not a point, tend to arise from the moduli of geometric structures on surfaces. In any case, the question of uniqueness or moduli of smooth geometric structures on a smooth 3-manifold is by and large understood; what remains is the question of existence.

The Geometrization Conjecture may be viewed as a question about the existence of canonical or distinguished Riemannian metrics on 3-manifolds that satisfy certain topological conditions. This type of question has long been of fundamental interest to workers in Riemannian geometry and analysis on manifolds. For instance, it is common folkore that Yamabe viewed his work on what is now known as the Yamabe problem [1960] as a step towards the resolution of the Poincaré conjecture. Further, it has been a longstanding open problem to understand the existence and moduli space of Einstein metrics (that is, metrics of constant Ricci curvature) on closed n-manifolds. Most optimistically, one would like to find necessary and sufficient topological conditions that guarantee the existence of such a metric. The Thurston conjecture, if true, provides the answer to this in dimension three. (Einstein metrics in dimension three are metrics of constant curvature).

One of the most natural means of producing canonical metrics on smooth manifolds is to look for metrics that are critical points of a natural functional on the space of all metrics on the manifold. In fact, the definition of Einstein metrics is best understood from this point of view.

Briefly, let $\mathcal{M}_1$ denote the space of all smooth Riemannian structures of total volume 1 on a closed n-manifold M. Two Riemannian metrics g_0 and g_1 are *equivalent* or *isometric* if there is a diffeomorphism f of M such that $f^*g_0 = g_1$; we also say that they *define the same structure* on M. Given a metric $g \in \mathcal{M}_1$, let $s_g : M \to \mathbb{R}$ be its scalar curvature (the average of all the curvatures in the two-dimensional subspaces of TM), and let dV_g be the volume form determined by the metric and orientation. The total scalar curvature functional $\mathcal{S}$ is defined by

$$\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}, \qquad \mathcal{S}(g) = \int_M s_g \, dV_g.$$

Hilbert showed that the critical points of this functional are exactly the *Einstein metrics*, that is, metrics that satisfy the Euler–Lagrange equation

$$Z_g := \mathrm{Ric}_g - \frac{s_g}{n} g = 0,$$

where Ric_g is the Ricci curvature of g (Section 2) and n is the dimension of M. It is an elementary exercise to show that in dimension three (and only in dimension three) the solutions of this equation are exactly the metrics of constant curvature, that is, metrics having geometric structure $\mathbb{H}^3$, $\mathbb{E}^3$ or $\mathbb{S}^3$.

In fact, $\mathcal{S}$ is the only functional on $\mathcal{M}_1$ known to the author whose critical points are exactly the metrics of constant curvature in dimension three. The fact that $\mathcal{S}$ is also the simplest functional that one can form from the curvature invariants of the metric makes it especially appealing.

The three geometries $\mathbb{H}^3$, $\mathbb{E}^3$, $\mathbb{S}^3$ of constant curvature are by far the most important of the eight geometries in understanding the geometry and topology of 3-manifolds. $\mathbb{H}^3$ and $\mathbb{S}^3$, in particular, play the central roles.

In this article, we will outline an approach toward the Geometrization Conjecture, based on the study of the total scalar curvature on the space of metrics on M^3. We formulate and discuss (Section 4) three conjectures on the geometry and topology of the limiting behavior of metrics on a 3-manifold that attempt to realize a critical point of $\mathcal{S}$. This conjecture, if true, implies that the geometrization of a 3-manifold can be implemented or performed by studying the convergence and degeneration of such a sequence of metrics.

R. Hamilton has developed another program toward resolution of the Geometrization Conjecture, by studying the singularity formation and long-time existence and convergence behavior of the Ricci flow on $\mathcal{M}$. This has of course already been spectacularly successful in certain cases [Hamilton 1982].

This article is intended partly as a brief survey of ideas related to the Thurston conjecture and of the approach to this conjecture indicated above. A number of new results are included in Sections 4–6, in order to substantiate this approach. However, by and large, only statements of results are provided, with references to proofs elsewhere, mainly in [Anderson a; b; c]. The paper is an expanded, but basically unaltered, version of talks given at the September 1993 MSRI Workshop.

1. Review of 3-manifolds: Topology, Geometry and Thurston's Results

1A. Topology. Throughout the paper, M will denote a closed, oriented 3-manifold and N will denote a compact, oriented 3-manifold with (possibly empty) compact boundary. There are two basic topological decompositions of M, obtained by examining the structure of the simplest types of surfaces embedded in M, namely spheres and tori.

THEOREM 1.1 (SPHERE DECOMPOSITION [Kneser 1929; Milnor 1962]). *Let M be a closed, oriented 3-manifold. Then M has a finite decomposition as a connected sum*

$$M = M_1 \# M_2 \# \cdots \# M_k,$$

where each M_i is prime. The collection $\{M_i\}$ is unique, up to permutation of the factors. (A closed 3-manifold is *prime* if it is not the three-sphere and cannot be written as a nontrivial connected sum of closed 3-manifolds.)

This sphere decomposition (or *prime decomposition*) is obtained by taking a suitable maximal family of disjoint embedded two-spheres in M, none of which bounds a three-ball, and cutting M along those spheres. The summands M_i are formed by gluing in three-balls to the boundary spheres. (This implicitly uses the Alexander–Schoenflies theorem, which says that any two-sphere embedded in $\mathbb{S}^3$ bounds a 3-ball.)

The sphere decomposition is canonical in the sense that the summands are unique up to homeomorphism. However, the collection of spheres is not necessarily unique up to isotopy; it is unique up to diffeomorphism of M.

A 3-manifold M is *irreducible* if every smooth two-sphere embedded in M bounds a three-ball in M. Clearly an irreducible 3-manifold is prime. The converse is almost true: a prime orientable 3-manifold is either irreducible or is $\mathbb{S}^2 \times \mathbb{S}^1$ [Hempel 1983].

The topology of an irreducible 3-manifold M is coarsely determined by the cardinality of the fundamental group. For then the sphere theorem [Hempel 1983] implies that $\pi_2(M) = 0$. Let $\tilde{M}$ be the universal cover of M, so that $\pi_1(\tilde{M}) = \pi_2(\tilde{M}) = 0$. If $\pi_1(M)$ is finite, $\tilde{M}$ is closed, and thus a homotopy three-sphere (that is, a simply connected closed 3-manifold), by elementary algebraic topology. If $\pi_1(M)$ is infinite, $\tilde{M}$ is open, and thus contractible (by the Hurewicz theorem); therefore M is a $K(\pi, 1)$, that is, M is aspherical.

Thus, the prime decomposition of Theorem 1.1 can be rewritten as

$$M = (K_1 \# K_2 \# \cdots \# K_p) \# (L_1 \# L_2 \# \cdots \# L_q) \# \Bigl(\mathop{\#}_1^r (\mathbb{S}^2 \times \mathbb{S}^1)\Bigr),$$

where the factors K_i are closed, irreducible and aspherical, while the factors L_j are closed, irreducible and finitely covered by homotopy three-spheres. Thus, one needs to understand the topology of the factors K_i and L_j.

It is worth emphasizing that the sphere decomposition is perhaps the simplest topological procedure that is performed in understanding the topology of 3-manifolds. In contrast, in dealing with the geometry and analysis of metrics on 3-manifolds, we will see that this procedure is the most difficult to perform or understand.

From now on, we make the further assumption that M and N are irreducible.

Before stating the torus decomposition theorem, we introduce several definitions. Let S be a compact, oriented surface embedded in N (and thus having trivial normal bundle), with $\partial S \subset \partial N$. The surface S is *incompressible* if, for every closed disc D embedded in N with $D \cap S = \partial D$, the curve ∂D is contractible in S. This happens if and only if the inclusion map induces an injection $\pi_1(S) \to \pi_1(N)$ of fundamental groups (see [Jaco 1980, Lemma III.8]; his definition of incompressibility disagrees with ours for $S = \mathbb{S}^2$). If S is not incompressible, it is *compressible*. A 3-manifold N is *Haken* if it contains an incompressible surface of genus $g \geq 1$.

Incompressible tori play the central role in the torus decomposition of a 3-manifold, just as spheres do in the prime decomposition. Note, however, that when one cuts a 3-manifold along an incompressible torus, there is no canonical way to cap off the boundary components thus created, as is the case for spheres. For any toral boundary component, there are many ways to glue in a solid torus, corresponding to the automorphisms of T^2; typically, the topological type of the resulting manifold depends on the choice. Thus, when a 3-manifold is split along incompressible tori, one leaves the compact manifolds with toral boundary fixed. This leads to another definition: a compact 3-manifold N is *torus-irreducible* if every incompressible torus in N is isotopic to a boundary component of N.

THEOREM 1.2 (TORUS DECOMPOSITION [Jaco and Shalen 1979; Johannson 1979]). *Let M be a closed, oriented, irreducible 3-manifold. Then there is a finite collection of disjoint incompressible tori $T_i^2 \subset M$ that separate M into a finite collection of compact 3-manifolds with toral boundary, each of which is either torus-irreducible or Seifert fibered. A minimal such collection (with respect to cardinality) is unique up to isotopy.*

A 3-manifold N is *Seifert fibered* if it admits a foliation by circles with the property that a foliated tubular neighborhood $D^2 \times \mathbb{S}^1$ of each leaf is either the trivial foliation of a solid torus $D^2 \times \mathbb{S}^1$ or its quotient by a standard action of a cyclic group. The quotient or leaf space of the foliation is a two-dimensional orbifold with a finite number of isolated cone singularities. The orbifold or cone points correspond to the exceptional fibers, that is, fibers whose foliated neighborhoods are nontrivial quotients of $D^2 \times \mathbb{S}^1$.

The tori appearing in the Geometrization Conjecture give a torus decomposition of M. Thus, the Geometrization Conjecture asserts that the torus-irreducible and Seifert fibered components of a closed, oriented, irreducible 3-manifold admit canonical geometric structures.

Of course, it is possible that the collection of incompressible tori is empty. In this case, M is itself a closed irreducible 3-manifold that is either Seifert fibered or torus-irreducible.

The Geometrization Conjecture thus includes the following important special cases (recall M is closed, oriented and irreducible):

Hyperbolization Conjecture. *If $\pi_1(M)$ is infinite and M is atoroidal, then M is hyperbolic, that is, admits a hyperbolic metric.* (M is *atoroidal* if $\pi_1(M)$ has no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(T^2)$.)

Elliptization Conjecture. *If $\pi_1(M)$ is finite, then M is spherical, that is, admits a metric of constant positive curvature.*

In fact, these are the only remaining open cases of the Geometrization Conjecture. If M has a nontrivial torus decomposition (equivalently, if M contains an incompressible torus), then in particular M is Haken. Thurston [1982; 1986; 1988] has proved the conjecture for Haken manifolds; see Theorem 1.4 below, and also [Morgan 1984]. If M has no incompressible tori, recent work on the Seifert fibered space conjecture [Gabai 1992; Casson and Jungreis 1994] implies that M is either Seifert fibered or atoroidal. It is known that Seifert fibered spaces have geometric structures (Section 1B). In the remaining case, M is atoroidal, and so satisfies the hypotheses of either the elliptization or the hyperbolization conjectures. Note that the elliptization conjecture implies the Poincaré conjecture.

For later sections, we will require a generalization of Seifert fibered spaces. Let N be a compact manifold, possibly with boundary. Then N is a *graph manifold* if there is a finite collection of disjoint embedded tori $T_i \subset N$ such that each component N_j of $N \setminus \bigcup T_i$ is an $\mathbb{S}^1$ bundle over a surface. To such a decomposition one assigns a graph G as follows: the vertices of G are the components of $N \setminus \bigcup T_i$, and two vertices are joined by an edge if the associated components are separated by a torus $T \in \{T_i\}$. This description of the graph is somewhat of a simplification; consult [Waldhausen 1967] for full details.

Of course, Seifert fibered spaces are graph manifolds, as one sees by letting $\{T_i\}$ be the boundaries of tubular neighborhoods of the exceptional fibers. A graph manifold need not admit a globally defined free, or locally free, $\mathbb{S}^1$ action. However, by definition, there are always free $\mathbb{S}^1$ actions defined on the components N_j. These $\mathbb{S}^1$ actions commute on the intersections of their domains of definition (neighborhoods of $\{T_i\}$), and thus extend to give free T^2 actions in this region. These locally defined $\mathbb{S}^1$ and T^2 actions give a well-defined partition of N into orbits, called the *orbit structure* of the graph manifold. In most cases, although not always, this orbit structure is unique up to isotopy [Waldhausen 1967].

We note that, as a consequence of their structure, irreducible graph manifolds of infinite fundamental group necessarily have a $\mathbb{Z} \oplus \mathbb{Z}$ contained in the fundamental group; in fact, with few exceptions, they have incompressible tori. For further details, see [Waldhausen 1967; Cheeger and Gromov 1986; Rong 1990; 1993].

1B. Geometries of 3-Manifolds. We summarize here the basic features of the eight 3-manifold geometries. For details, see [Scott 1983; Thurston 1996, Section 3.8]. A *geometric structure* on a simply connected space X is a homogenous space structure on X, that is, a transitive action of a Lie group G on X. Thus, X is given by G/H, for H a closed subgroup of G. In order to avoid redundancy, it is assumed that the identity component H_0 of the stabilizer H is a compact subgroup of G, and that G is maximal. Further, G is assumed to be unimodular; this is equivalent to the existence of compact quotients of X.

The possible geometric structures may be divided into three categories.

Constant curvature geometries. Here X is the simply connected space form $\mathbb{H}^3$ of constant curvature -1, or $\mathbb{E}^3$ of curvature 0, or $\mathbb{S}^3$ of curvature $+1$. The corresponding geometries are called hyperbolic, Euclidean, and spherical. The groups G are $\mathrm{PSL}(2,\mathbb{C})$, $\mathbb{R}^3 \times \mathrm{SO}(3)$, and $\mathrm{SO}(4)$. In all cases, $H_0 = \mathrm{SO}(3)$.

Product geometries. Here $X = \mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$. The groups G are given by the orientation-preserving subgroups of $\mathrm{Isom}\,\mathbb{H}^2 \times \mathrm{Isom}\,\mathbb{E}^1$ and $\mathrm{SO}(3) \times \mathrm{Isom}\,\mathbb{E}^1$, with stabilizer $H_0 = \mathrm{SO}(2)$.

Twisted product geometries. Here there are three possibilities, called $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, Nil and Sol. For $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the space X is the universal cover of the unit sphere bundle of $\mathbb{H}^2$, and $G = \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}$, with $H_0 = \mathrm{SO}(2)$. For the Nil geometry, X is the three-dimensional nilpotent Heisenberg group (consisting of upper triangular 3×3 matrices with diagonal entries 1), and G is the semidirect product of X with $\mathbb{S}^1$, acting by rotations on the quotient of X by its center. Again $H_0 = \mathrm{SO}(2)$. For the Sol geometry, X is the three-dimensional solvable Lie group, $H_0 = \{e\}$, and G is an extension of X by an automorphism group of order eight.

A 3-manifold N is *geometric* if it admits one of these eight geometric structures. Geometric 3-manifolds modeled on six of these geometries, namely all but the hyperbolic and Sol geometries, are Seifert fibered. Thus, topologically, such manifolds are circle “bundles” over two-dimensional orbifolds, with isolated cone singularities. In particular, all such manifolds have finite covers that are $\mathbb{S}^1$ bundles over closed surfaces of genus $g \geq 0$.

These six geometries divide naturally into a pair of threes, corresponding to whether the $\mathbb{S}^1$ bundle is trivial or not. 3-manifolds with product geometries $\mathbb{H}^2\times\mathbb{R}$, $\mathbb{E}^3$, and $\mathbb{S}^2\times\mathbb{R}$ are, up to finite covers, trivial circle bundles over oriented surfaces of genus g, where $g \geq 2$, $g = 1$, and $g = 0$, respectively. 3-manifolds with the twisted product geometries $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, Nil, and $\mathbb{S}^3$ are, up to finite covers, nontrivial circle bundles over surfaces, again of genus $g \geq 2$, $g = 1$, and $g = 0$, respectively.

Conversely, it is not difficult to prove [Scott 1983] that any Seifert fibered space admits a geometric structure modeled on one of these six geometries.

Geometric 3-manifolds modeled on the Sol geometry all have finite covers that are torus bundles over $\mathbb{S}^1$, with holonomy given by a hyperbolic automorphism of T^2, that is, an element of $\mathrm{SL}(2,\mathbb{Z})$ with distinct real eigenvalues. Again, conversely, all such T^2 bundles admit a Sol geometry. Note that Sol-manifolds are graph manifolds, that is, they may be split by incompresssible tori into a union of Seifert fibered spaces.

Thus, seven of the eight geometric 3-manifolds are topologically $\mathbb{S}^1$-fibered over surfaces or T^2-fibered over $\mathbb{S}^1$. Since, in any reasonable sense, most 3-manifolds do not admit such fibrations, the hyperbolic geometry is by far the most prevalent of the eight geometries (see Section 1C).

It is well known [Scott 1983] that the same 3-manifold cannot have geometric structures modeled on two distinct geometries.

Of course, it is not true that the geometric structure itself, that is, the homogeneous metric, is unique in general. In this respect, we recall:

THEOREM 1.3 (MOSTOW RIGIDITY [Mostow 1968; Prasad 1973]). *Let N be a 3-manifold carrying a complete hyperbolic metric of finite volume. Then the hyperbolic metric is unique, up to isometry. Further, if N and N' are 3-manifolds with isomorphic fundamental groups, and if N and N' carry complete hyperbolic metrics of finite volume, then N and N' are diffeomorphic.*

In particular, invariants of the hyperbolic metric such as the volume and the spectrum are topological invariants of the 3-manifold.

There is a similar rigidity for spherical 3-manifolds, in the sense that any metric of curvature $+1$ on the manifold is unique, up to isometry [Wolf 1977]. The fundamental group in this case does not determine the topological type of the manifold. There are further topological invariants, such as the Reidemeister torsion. The other six geometries are typically not rigid, but have moduli closely related to the moduli of constant curvature metrics on surfaces.

1C. Thurston's Results on the Geometrization Conjecture. As already mentioned, many cases of the Geometrization Conjecture have been proved by Thurston [1982; 1986; 1988] (see also [Morgan 1984] for a detailed survey). In particular:

THEOREM 1.4 (GEOMETRIZATION OF HAKEN MANIFOLDS). *A closed, oriented, irreducible Haken manifold that is atoroidal admits a hyperbolic structure. A compact, oriented, irreducible, and torus-irreducible* 3*-manifold whose boundary consists of a finite number of tori admits a complete hyperbolic metric of finite volume.*

We indicate in a few lines the approach to the proof of this result. It was shown by Haken [1961] and Waldhausen [1968] that, if the manifold M is Haken, one may successively split it along incompressible surfaces into a hierarchy, that is, a collection of (possibly disconnected) compact submanifolds with boundary:

$$M = M_k \supset M_{k-1} \supset \cdots \supset M_1 \supset M_0 = \text{union of balls},$$

where each M_i has an incompressible surface S_i with $\partial S_i \subset \partial M_i$, and M_{i-1} is obtained from M_i by splitting along S_i. If M is atoroidal, so is each M_i. Thurston proves that, for an appropriate hierarchy, the manifolds M_i admit complete, geometrically finite hyperbolic metrics, typically of infinite volume. This is proved by induction on the length of the hierarchy. Thus, suppose that M_{i-1} admits a complete, geometrically finite hyperbolic metric. The manifold M_i is obtained by gluing together certain of the ends of M_{i-1}. The most difficult part of the proof is showing that the hyperbolic metric on M_{i-1} may be deformed appropriately so that the ends to be glued are isometric, so that M_i acquires a complete, geometrically finite hyperbolic metric. Thurston has developed a wealth of new geometric ideas and methods to carry this out.

McMullen [1989; 1990] has given a different proof of this gluing process, in the case where M does not fiber over $\mathbb{S}^1$.

Theorem 1.4 implies that the torus-irreducible pieces of a nonempty torus decomposition carry hyperbolic structures. As we saw in Section 1B, all the Seifert fibered pieces also carry geometric structures. Thus, the Geometrization Conjecture is proved in the case of manifolds that have a nonempty torus decomposition.

Nevertheless, many, perhaps most, 3-manifolds are not Haken. Thurston has established the Geometrization Conjecture for many further classes of non-Haken 3-manifolds. For instance, suppose M is a closed oriented 3-manifold, and N is the complement of a knot K in M. One may obtain new closed 3-manifolds by *Dehn surgery* on N, that is, by gluing in a solid torus to the boundary of N. The possible Dehn surgeries are classified by classes in $\mathrm{SL}(2,\mathbb{Z})$. Thurston [1979] showed that if N admits a complete hyperbolic metric g_∞ of finite volume, all but finitely many Dehn surgeries yield closed manifolds that admit hyperbolic structures. All of these hyperbolic manifolds obtained by closing the cusp of

(N, g_∞) have volume strictly less than that of (N, g_∞). If M itself is not Haken, then Dehn surgeries on N will often yield closed non-Haken 3-manifolds.

2. Preliminaries on the Space of Metrics

Let $\mathbf{M}$ denote the space of all smooth Riemannian metrics on the closed oriented 3-manifold M. Thus, $\mathbf{M}$ is an open convex cone in the space $S^2(M)$ of symmetric bilinear forms on M. The diffeomorphism group $\operatorname{Diff} M$ of M acts naturally on $\mathbf{M}$ by pullback, $(\psi, g) \mapsto \psi^* g$. The two metrics g and $\psi^* g$ are isometric, and ψ is an isometry between them. Since all intrinsic notions associated with the metric are invariant under isometries, it is natural to divide $\mathbf{M}$ by the action of $\operatorname{Diff}(M)$. We let $\mathcal{M} = \mathbf{M}/\operatorname{Diff} M$ be the space of Riemannian structures on M, that is, isometry classes of Riemannian metrics. The space $\mathcal{M}$ is no longer an infinite-dimensional manifold, since the action of $\operatorname{Diff} M$ is not free; fixed points of the action correspond to metrics with nontrivial isometry group, that is, maps $\phi \in \operatorname{Diff} M$ such that $\phi^* g = g$. This rarely presents a problem, however. We denote by $\mathcal{M}_1$ the subset of $\mathcal{M}$ consisting of metrics of volume 1 on M.

The tangent space $T_g \operatorname{Diff} M$ to the orbit of $\operatorname{Diff} M$ in $\mathbf{M}$ is the image of the map δ^* that associates to a vector field X on M the element $\delta^*(X) = \mathcal{L}_X g \in S^2(M)$, where $\mathcal{L}$ denotes the Lie derivative. Since δ^* is (underdetermined) elliptic, there is a splitting

$$T_g \mathbf{M} = T_g \operatorname{Diff} M \oplus N_g \operatorname{Diff} M = \operatorname{Im} \delta^* \oplus \operatorname{Ker} \delta,$$

where δ is the divergence operator, the formal adjoint of δ^* on $S^2(M)$, given by $\delta(\alpha) = -(D_{e_i}\alpha)(e_i, \cdot)$, where D is the covariant derivative of the metric g and $\{e_i\}$ is an orthonormal basis. We note that the action of $\operatorname{Diff} M$ on $\mathbf{M}$ has a slice, that is, a locally defined submanifold of $\mathbf{M}$, transverse to the orbits of $\operatorname{Diff} M$ in a neighborhood of any $g \in \mathbf{M}$ [Ebin 1970].

The space $\mathbf{M}$ will be endowed with a normed $L^{k,p}$ topology, given by

$$\|h\|_{k,p}^p = \int \big(|h|^p + |Dh|^p + \cdots + |D^k h|^p\big)\, dV, \tag{2.1}$$

for $h \in T_g \mathbf{M} = S^2(M)$. Here all norms, derivatives, and the volume form, are taken with respect to the metric g. This corresponds, locally, to the Sobolev topology on functions defined on domains of $\mathbb{R}^n$, namely the first k derivatives are in L^p. The exact values of k and p may depend on the problems at hand, but the minimal requirement is that

$$\alpha = k - \frac{n}{p} > 0, \qquad 1 < p < \infty, \tag{2.2}$$

corresponding to the Sobolev embedding $L^{k,p} \subset C^\alpha$. The completion of $\mathbf{M}$ with respect to this topology will also be denoted by $\mathbf{M}$, and gives $\mathbf{M}$ the structure of a Banach manifold, or Hilbert manifold when $p = 2$. Further, these norms are invariant under the action of Diff M, and thus descend to define a topology on $\mathcal{M}$.

Suppose $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ is a smooth function in the $L^{k,p}$ topology on $\mathcal{M}$ and suppose $\{g_i\} \in \mathcal{M}$ is a sequence that approaches a critical value of $\mathcal{F}$. By slightly perturbing g_i if necessary, we can assume that $\|d\mathcal{F}_{g_i}\|_{(L^{k,p})^*} \to 0$, that is,

$$\sup_{\|h\|_{k,p}=1} |d\mathcal{F}_{g_i}(h)| = \sup_{\|h\|_{k,p}=1} \left| \frac{d}{dt} \left(\mathcal{F}(g_i(t)) \right) \right| \to 0, \tag{2.3}$$

where $g_i(t) = g_i + th$ and the norm is taken with respect to the metric g_i. The dual space $(L^{k,p})^*$ is naturally identified, locally, with $L^{-k,q}$, where $p^{-1}+q^{-1} = 1$. One sees that (2.3) contains less information, that is, is a weaker condition, the larger k and p are. Thus, in general, one would like to choose values for k and p as small as possible, in order that (2.3) give as much information as possible. Of course, the pair (k, p) must satisfy (2.2), and also be chosen so that $\mathcal{F}$ is smooth in the $L^{k,p}$ topology.

Next, we fix some notation for later sections. Given a metric g, let R denote the Riemann curvature tensor, given by $R = \sum R_{ijkl}\, dx^i \otimes dx^j \otimes dx^k \otimes dx^l$ in local coordinates. The sectional curvature $K_P = K_{ij} = R_{ijji}$ in the direction of a two-plane P in T_xM spanned by orthonormal vectors e_i and e_j may be defined as the Gauss curvature at x of the geodesic surface in M tangent to P at x. Knowledge of the sectional curvature K_P for all two-planes P determines the curvature tensor.

The Ricci curvature Ric is a symmetric bilinear form on TM, obtained by contracting R; more precisely, $\mathrm{Ric}(v, w) = \sum R(v, e_i, e_i, w)$, for an orthonormal basis e_i. The scalar curvature s is the contraction of the Ricci curvature, $s = \sum \mathrm{Ric}(e_i, e_i)$. In dimension two, these curvatures are all the same, up to multiplicative constants. In dimension three, but not in higher dimensions, the Ricci curvature determines the full curvature R. For instance, if λ_i are the eigenvalues of Ric, with eigenvectors e_i, then for distinct indices (i, j, k) we have

$$K_{ij} = \tfrac{1}{2}(\lambda_i + \lambda_j - \lambda_k).$$

The covariant derivative associated to g will be denoted by D. The Laplacian or Laplace–Beltrami operator $\triangle$ associated with g will be taken to have negative spectrum (so $\triangle f = f''$ on $\mathbb{R}$).

3. Functionals on the Space of Metrics

We consider functionals $\mathcal{F} : \mathcal{M} \to \mathbb{R}$ on the space of metrics on M. There are functionals that measure the global size or extent of the Riemannian manifold (M, g); for example, volume, diameter, radius, etc. We will only consider functionals that are Lagrangian, in the sense that

$$\mathcal{F}(g) = \int_M \mathcal{L}_{\mathcal{F}}(g)\, dV_g,$$

where $\mathcal{L}_{\mathcal{F}}(g)$ is the (scalar) Lagrangian density for $\mathcal{F}$ at g, and dV_g is the volume form of the metric g. The invariance of $\mathcal{F}$ under the action of $\operatorname{Diff} M$ requires that the Lagrangian satisfy the invariance property

$$\mathcal{L}_{\mathcal{F}}(\psi^* g) = \psi^* \mathcal{L}_{\mathcal{F}}(g) = \mathcal{L}_{\mathcal{F}}(g) \circ \psi, \tag{3.1}$$

for all $\psi \in \operatorname{Diff} M$. We say that $\mathcal{L}_{\mathcal{F}}(g)$ is a k-th order Lagrangian if it is a smooth function of the k-jet of g, that is, depends only on g and its first k derivatives. Thus, one method to produce Lagrangians is to consider functions of g and its derivatives in some coordinate system that are in fact invariant, in the sense of (3.1), under changes of coordinates.

Recall that any Riemannian manifold admits (geodesic) normal coordinates x_i at any prescribed point p. In these coordinates, the metric satisfies $g_{ij} = \delta_{ij}$ and $\partial g_{ij}/\partial x_k = 0$ at p. In other words, at p the metric osculates, to first order, a flat Euclidean metric. An important and well-known consequence of this is that there are no nonconstant invariant Lagrangians of order ≤ 1 [Lovelock and Rund 1972; Palais 1968]. Thus, one is required to seek Lagrangians of order at least two. We note that most other problems in the calculus of variations can be expressed in terms of first-order Lagrangians: for example, harmonic functions or maps, geodesics, minimal surfaces, Yang–Mills fields, etc.

Consider the Taylor expansion of a metric $g = g_{ij}$ in a normal coordinate system about p. A fundamental fact, due to Cartan, is that the order-k Taylor coefficients can be (universally) expressed in terms of polynomials in the components of the curvature tensor R and its covariant derivatives $\nabla^m R$, for $m \leq k-2$. For example, in normal coordinates, one has, by Riemann,

$$g_{ij} = \delta_{ij} + \tfrac{2}{3} \sum_{k,l} R_{iklj} x_k \, x_l + O(|x|^3).$$

Thus, we seek Lagrangians whose expressions in normal coordinates are of the form $\mathcal{L}_{\mathcal{F}}(g) = \phi(R, \nabla R, \ldots, \nabla^{k-2} R)$, that is, ϕ is a function of the components of the curvature tensor and its covariant derivatives. The orthogonal group $O(n)$ acts freely and transitively on the possible normal coordinates (which are determined uniquely by an orthonormal frame at p); the action of $O(n)$ extends naturally to an action on the curvature tensor R and its derivatives. Thus, we seek functions ϕ that are $O(n)$-invariant. If one considers functions ϕ that are polynomials P in the components of the arguments, then one seeks to classify $O(n)$-invariant polynomials $P(T_1, T_2, \ldots, T_{k-2})$ on a sum of tensor spaces over R^n (the tensor spaces being the spaces of curvature tensors R, covariant derivatives ∇R, and so on). Now the fundamental theorem of invariance theory for $O(n)$ states that any such polynomial is a linear combination of terms, each obtained by fully contracting an even tensor product of the $\{T_i\}$ to a scalar [Atiyah et al. 1973; Palais 1968].

Thus, for $k = 2$, the simplest Lagrangian one can take is the full contraction of the curvature tensor R, that is, the scalar curvature. Thus, in a precise sense, the simplest metric functional on $\mathcal{M}$ is the total scalar curvature functional

$$\mathcal{S} : \mathcal{M} \to \mathbb{R}, \qquad \mathcal{S}(g) = \int_M s_g \, dV_g.$$

Next, again for $k = 2$, one could take contractions on $R \otimes R$. It turns out there are three possibilities, namely $|R|^2$, $|\mathrm{Ric}|^2$ and s^2. This gives rise to the functionals $\mathcal{R}^2$, $\mathcal{R}ic^2$, and $\mathcal{S}^2$, corresponding to the L^2 norms of the tensors R, Ric, and s.

One could also consider higher-order functionals of the curvature and its covariant derivatives. Since they become rapidly more complicated, especially regarding the expressions of their Euler–Lagrange equations, we will not pursue their discussion here.

Consider the equation for a critical point of $\mathcal{F}$, that is, the Euler–Lagrange equation associated to the Lagrangian $\mathcal{L}_{\mathcal{F}}$. For a k-th order Lagrangian, this will have the form

$$\nabla \mathcal{F}(g) = A^{ij}(g, \partial g, \ldots, \partial^m g) = 0,$$

where $m \leq 2k$ and, generically, $m = 2k$. The two-tensor $A = A^{ij}$ is symmetric and, as a consequence of the invariance of $\mathcal{L}_{\mathcal{F}}$, is divergence-free: $\delta A = 0$. Thus, for the second-order Lagrangians mentioned above, the Euler–Lagrange equations will typically be a fourth-order system of partial differential equations in the metric g.

The scalar curvature functional has the remarkable property that its Euler–Lagrange equation is of second order in g; in fact, when restricted to $\mathcal{M}_1$, the gradient $\nabla|_{\mathcal{M}_1}\mathcal{S}$ (with respect to the L^2 metric on $\mathcal{M}_1$, that is, the metric (2.1) with $k = 0$ and $p = 2$) is given by

$$\nabla|_{\mathcal{M}_1}\mathcal{S}(g) = \frac{s}{n} g - \mathrm{Ric} \equiv -Z.$$

The two-tensor Z is just the trace-free part of the Ricci curvature. Further, in dimensions three and four, it is known [Lovelock and Rund 1972] that $\mathcal{S}$ is the unique functional (expressed in terms of a second-order invariant Lagrangian) whose Euler–Lagrange operator is of second order in g. The Euler–Lagrange equations for $\mathcal{R}^2$, $\mathcal{R}ic^2$ and $\mathcal{S}^2$ are all of order four in g; see [Berger 1970; Besse 1987, p.133; Anderson 1993; a] for a discussion.

Finally, we briefly discuss the appropriate topologies on the space $\mathcal{M}_1$ for these functionals. All of the functionals discussed above are smooth in the $L^{2,2}$ topology on $\mathcal{M}_1$; compare (2.1). For the functionals $\mathcal{R}^2$, $\mathcal{R}ic^2$ and $\mathcal{S}^2$, this is the smallest topology in which they are smooth. The functional $\mathcal{S}$ is also smooth in the weaker topology $L^{1,q}$, for $q > 3$.

4. Conjectures on the Realization of the Sigma Constant

As indicated in the Introduction, researchers in Riemannian geometry and analysis on manifolds (and of course in mathematical physics and general relativity) have long been interested in the existence and moduli of Einstein metrics. In light of the discussion in Section 3, it is natural to seek such metrics variationally, as critical points of the total scalar curvature functional.

A number of immediate problems are encountered in the variational approach to existence. The functional $\mathcal{S}$ is bounded neither below nor above. Further, it is well-known that any critical point has infinite index and co-index, that is, there are infinite-dimensional subspaces of $T_g\mathcal{M}$ on which $\mathcal{S}$ can be infinitesimally, and thus locally, decreased or increased. Thus $\mathcal{S}$ is far from satisfying any of the usual compactness properties used in obtaining existence of critical points, such as the Palais–Smale condition, mountain-pass lemmas, etc.

There is, however, a well-known minimax procedure to obtain critical values of $\mathcal{S}$. It goes as follows. Given a metric $g \in \mathcal{M}_1$, let $[g]$ denote the conformal class of g, that is, $[g] = \{g' \in \mathcal{M}_1 : g' = \psi^2\, g\}$, for some smooth positive function ψ. The functional $\mathcal{S}$ is bounded below on $[g]$; define

$$\mu[g] = \inf_{g\in[g]} \mathcal{S}(g).$$

The number $\mu[g]$ is called the *Yamabe constant* (or *Sobolev quotient*) of $[g]$. An elementary comparison argument [Aubin 1976] shows that

$$\mu[g] \le \mu(S^n, g_{\mathrm{can}})$$

for any conformal class $[g]$, where g_{can} is the canonical metric of constant positive sectional curvature and volume 1 on the n-sphere S^n. Thus, define the *Sigma constant* by

$$\sigma(M) = \sup_{[g]\in\mathcal{C}} \mu[g],$$

where $\mathcal{C}$ is the space of conformal structures on M. Thus, $\sigma(M)$ is a smooth invariant of the manifold M. (I don't know who first considered this minimax approach. One guesses that certainly Yamabe was aware of it, and it may well have been considered earlier. I have found no definite references, besides the relatively recent [Kobayashi 1987; Schoen 1989]).

It is reasonable to expect, and certainly conjectured (in [Besse 1987, p. 128], for example), that $\sigma(M)$ is a critical value of $\mathcal{S}$, that is, any metric $g_0 \in \mathcal{M}_1$ such that $s_{g_0} \equiv \mu[g_0] = \sigma(M)$ is an Einstein metric. In full generality, this remains unknown, due partly to the possible lack of uniqueness of metrics realizing $\mu[g]$.

REMARKS 4.1. (i) Clearly, $\sigma(M) \le \sigma(S^n) = n(n-1)(\mathrm{vol}\, S^n)^{2/n}$, where the volume is that of the unit sphere. Further, $\sigma(S^n)$ is realized by the canonical metric on S^n of volume 1.

(ii) If $\sigma(M) \leq 0$, it is easy to prove that a metric $g \in \mathcal{M}_1$ realizing $\sigma(M)$ is Einstein; see [Besse 1987, p. 128], and also Section 5.

(iii) If $\dim M = 2$, the Gauss–Bonnet theorem gives

$$2\pi\chi(M) = \int_M s_g \, dV_g = \mathcal{S}(g),$$

for all g. Thus, $\mathcal{S}$ is a constant functional on $\mathcal{M}_1$, whose value is a topological invariant of the surface M. In some sense, one can think of $\sigma(M)$ as a generalization of the Euler characteristic to higher-dimensional manifolds, especially in dimension three.

Comparatively little is known regarding the Sigma constant in dimension three, and even less in higher dimensions. Two important and well-known open questions are:

Question 4.2. *If M is a homotopy three-sphere, is $\sigma(M) > 0$?*

Question 4.3. *If M is a hyperbolic 3-manifold, does the hyperbolic metric realize $\sigma(M)$, modulo renormalization to volume 1?*

In fact, not a single example is known of a 3-manifold with $\sigma(M) < 0$. There are, however, two important positive results on $\sigma(M)$, due to Gromov–Lawson [1983] and Schoen–Yau [Schoen 1984]. Namely, if a 3-manifold M is a $K(\pi, 1)$, or contains a $K(\pi, 1)$ factor in its prime decomposition (see Section 1), then $\sigma(M) \leq 0$. This is equivalent to saying that M admits no metric of positive scalar curvature.

On the other hand, if M has a "small" fundamental group, then $\sigma(M) > 0$, assuming the Poincaré conjecture is true. More precisely, if M is a connected sum of a finite number of manifolds, each of which is either $\mathbb{S}^2 \times \mathbb{S}^1$ or a quotient of $\mathbb{S}^3$ by a group of isometries, then $\sigma(M) > 0$ [Gromov and Lawson 1980; Schoen and Yau 1979a]. Here one explicitly constructs metrics of positive scalar curvature on such manifolds, starting from the canonical metrics on the component manifolds, which clearly have positive scalar curvature.

The minimax procedure to obtain $\sigma(M)$ has two parts: first minimize in a conformal class, then maximize over all conformal classes. Fortunately, the first step has been solved [Yamabe 1960; Trudinger 1968; Aubin 1976; Schoen 1984]:

Theorem 4.4 (Solution to the Yamabe problem). *For any conformal class $[g] \in \mathcal{C}$, the Yamabe constant $\mu[g]$ is realized by a smooth metric $g_\mu \in [g]$ whose scalar curvature s_μ is identically equal to $\mu[g]$.*

The metrics g_μ realizing $\mu[g]$ are called *Yamabe metrics*. The solution to the Yamabe Problem amounts to showing that the equation

$$4\,\frac{n-1}{n-2}\,\triangle u - s_g\, u = -\mu[g]\, u^{(n+2)/(n-2)}, \tag{4.1}$$

where $g \in [g]$ is a fixed background metric, has a smooth, positive solution on M. Equation (4.1) is the Euler–Lagrange equation of the variational problem $\mathcal{S}|_{[g]}$. In fact, the metric $g_\mu := u^{4/(n-2)}\, g$ gives then the desired solution to the Yamabe problem. Equation (4.1) is a nonlinear elliptic (scalar) equation. In particular, this is a determined problem in the sense that there is one equation imposed on an unknown function u. The subtlety of the problem arises from the fact that the exponent $2n/(n-2)$ is borderline for the Sobolev embedding $L^{1,2} \to L^{2n/(n-2)}$.

The second part of the minimax procedure, maximizing over the conformal classes, is considerably more difficult. In fact, at least in dimensions three and four, it is known that there are topological obstructions to the existence of Einstein metrics. In dimension three, since Einstein metrics have constant curvature, no reducible 3-manifold admits an Einstein metric, that is, neither $\mathbb{S}^2 \times S^1$ nor any 3-manifold that is a nontrivial connected sum admits an Einstein metric. More generally, among the 3-manifolds admitting Seifert geometries discussed in Section 1B, only those admitting $\mathbb{S}^3$ or $\mathbb{E}^3$ geometries admit Einstein metrics. Similarly, torus bundles over $\mathbb{S}^1$, corresponding to Sol geometry, do not admit Einstein metrics.

Thus, if one tries to realize the value $\sigma(M)$ on $\mathcal{M}_1$ by taking an appropriate maximizing sequence $\{g_i\} \in \mathcal{M}_1$ for $\mathcal{S}$, the sequence $\{g_i\}$ has in general no subsequence converging to a limit metric in $\mathcal{M}_1$. Whether such a sequence $\{g_i\}$ should have convergent subsequences or not depends on the topology of the underlying 3-manifold M. It is thus an interesting (and difficult) challenge to relate the possible degenerations of a sequence $\{g_i\}$ to the topology of the underlying manifold M.

The following three conjectures describe the geometry and topology of metrics that attempt to realize the Sigma constant on a 3-manifold M. For all three conjectures, the following assumption is made.

Assumption. M is a closed, *irreducible*, oriented 3-manifold.

Conjecture I (The Negative Case). *Suppose $\sigma(M) < 0$. Then there is a finite collection of disjoint, embedded incompressible tori T_i^2 in M such that the complement $M \setminus \bigcup T_i^2$ is a finite union of complete hyperbolic manifolds M_j of finite volume, together with a (possibly empty) finite union of graph manifolds S_k with toral boundaries.*

Further,

$$|\sigma(M)| = 6\Big(\sum \operatorname{vol}_{-1} M_j\Big)^{2/3}, \tag{4.2}$$

where vol_{-1} *denotes volume in the hyperbolic metric. In particular, if M is atoroidal, it admits a hyperbolic metric that realizes the Sigma constant (modulo renormalization): $|\sigma(M)| = 6(\operatorname{vol}_{-1} M)^{2/3}$.*

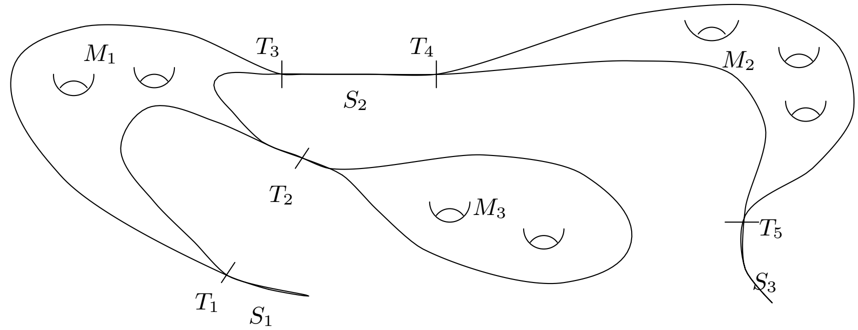

Figure 1. Conjecture I.

Conjecture II (The Zero Case). *Suppose $\sigma(M) = 0$. Then M is a graph manifold, with infinite π_1. The Sigma constant $\sigma(M)$ is realized if and only if M is a flat 3-manifold; in particular, M must be finitely covered by a three-torus.*

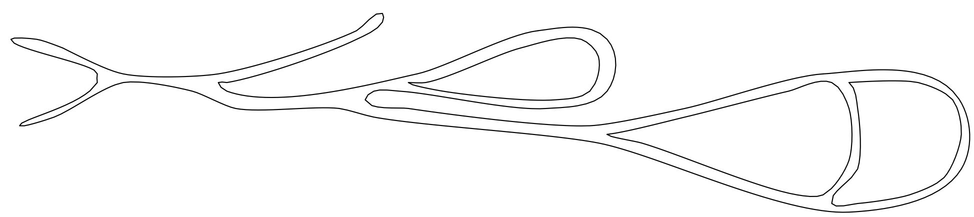

Figure 2. Conjecture II.

Conjecture III (The Positive Case). *Suppose $\sigma(M) > 0$. Then M is diffeomorphic to $\mathbb{S}^3/\Gamma$, for $\Gamma \subset \mathrm{SO}(4)$. The Sigma constant is realized by the standard metric of constant curvature on $\mathbb{S}^3/\Gamma$, that is, $\sigma(M) = 6(\mathrm{vol}_{+1} M)^{2/3}$, where vol_{+1} denotes volume in the metric of sectional curvature $+1$.*

It is worth discussing these conjectures in some further detail. Let $\{g_i\}$ be a maximizing sequence of Yamabe metrics for M, as described above, so that $\mu[g_i] \to \sigma(M)$. Figures 1 and 2 are schematic representations of the (near) limiting behavior of $\{g_i\}$ according to Conjectures I and II.

Conjecture I corresponds to the conjecture that, after possibly passing to a subsequence and altering g_i if necessary by a diffeomorphism ψ_i of M, the sequence $\{g_i\}$, when restricted to the domains $M_j \subset M$, can be chosen to converge (smoothly) to a complete hyperbolic metric (more precisely, a metric of constant negative curvature) of finite volume on $\bigcup M_j$. On the complementary part of M, namely the graph manifolds S_k, the sequence $\{g_i\}$ degenerates, so that there is no well-defined limiting Riemannian metric on $\bigcup S_k$. Briefly, the sequence $\{g_i\}$ collapses to a point each orbit $\mathcal{O}_x$ of the graph manifold structure (see Section 1), that is, $\mathrm{diam}_{g_i} \mathcal{O}_x \to 0$ as $i \to \infty$. In particular, the total volume of the graph manifold pieces $\bigcup S_k$ converges to 0, while the total volume of the complement

$\bigcup M_j$ converges to 1. This phenomenon of collapse will be discussed in further detail in Section 5.

The incompressible tori T_i in Conjecture I lie in the region of transition between these two types of behavior, that is, between the regions of convergence and the regions of collapse. This transition region represents also the transition between the "thick" parts and the (arbitrarily) "thin" parts of the manifolds (M, g_i), as $i \to \infty$.

These tori give a partial torus decomposition of M; one obtains a full torus decomposition by adding further (disjoint) tori in the graph manifold pieces, corresponding (generally) to the edges of the graph manifold structure, that is, to the decomposition of the graph manifolds into Seifert fibered pieces.

Conjecture I implies that the Sigma constant is realized by the union of the hyperbolic pieces; the graph manifold pieces play no role in the value of $\sigma(M)$. In particular, the set of hyperbolic pieces M_j is nonempty.

In the atoroidal case, there is no degeneration of $\{g_i\}$, that is, the sequence can be chosen to converge (smoothly) to a smooth metric of constant negative curvature on the closed manifold M.

The power $\frac{2}{3}$ in (4.2) is necessary, since the invariants $\sigma(M)$ and $\operatorname{vol} M$ of (4.2) behave differently under rescaling of the metric. Thus, (4.2) is a scale-invariant equality.

Conjecture II can now be understood by means of the discussion above—namely, there are no hyperbolic pieces. With the possible exception of the special case where M is a flat 3-manifold, the sequence $\{g_i\}$ fully collapses the graph manifold M along the orbits of the graph structure. The manifold M becomes arbitrarily long and thin in the metrics g_i, as $i \to \infty$, that is, $\operatorname{diam}_{g_i} M \to \infty$ and $\operatorname{inj}_x(g_i) \to 0$ for all $x \in M$, where inj_x denotes the injectivity radius at x (see Section 5).

It is necessary to explain how the condition that π_1 is infinite arises, since there are graph manifolds, such as $\mathbb{S}^3/\Gamma$, for which $\sigma(M) > 0$.

We claim that irreducible graph manifolds M of infinite π_1 satisfy $\sigma(M) = 0$. Indeed, it can be deduced from the results of Cheeger and Gromov (see Section 5) that an arbitrary graph manifold necessarily has $\sigma(M) \geq 0$. Further, as seen in Section 1, an irreducible graph manifold with infinite π_1 necessarily has a $\mathbb{Z} \oplus \mathbb{Z}$ contained in π_1. By a well-known result of Schoen and Yau [1979b], such manifolds do not admit metrics of positive scalar curvature; thus $\pi_1(M)$ infinite implies $\sigma(M) = 0$, as claimed.

On the other hand, irreducible graph manifolds of finite π_1 are known to be the spaces $\mathbb{S}^3/\Gamma$ and thus have $\sigma(M) > 0$.

Conjecture II thus amounts to the converse, that the only irreducible manifolds with $\sigma(M) = 0$ are graph manifolds; the discussion above then implies these manifolds have infinite π_1.

Conjecture III now needs no further explanation. Conjecturally, there should exist a sequence of Yamabe metrics g_i that converges smoothly to a metric of constant positive curvature on $\mathbb{S}^3/\Gamma$ and that realizes the Sigma constant.

We note that the decomposition of M via the collection $(\{M_j\}, \{T_i\}, \{S_k\})$ is unique, up to isotopy. This follows from the uniqueness of the torus decomposition (Theorem 1.2) and Mostow rigidity (Theorem 1.3). See also [Jaco and Shalen 1979].

These conjectures imply results about the geometry of 3-manifolds, namely about the structure of metrics realizing the Sigma constant, as described above, as well as the topology of 3-manifolds. We turn to a discussion of the topological consequences.

In fact Conjectures I–III imply the Geometrization Conjecture (page 49). For example, let us indicate how they imply the Poincaré conjecture, or more generally the Elliptization Conjecture (page 54). Thus, suppose M is a 3-manifold with finite fundamental group. Using the prime decomposition (Section 1), we may assume that M is irreducible. Then Conjecture I implies that $\sigma(M) \geq 0$, since Conjecture I implies that π_1 must be infinite if $\sigma(M) < 0$. For the same reason, Conjecture II implies $\sigma(M) > 0$. Thus, Conjecture III implies that M is S^3/Γ. Note that all three conjectures are needed to reach this conclusion; Conjecture III alone does not suffice, as Question 4.2 above indicates.

To see how Conjectures I–III imply the hyperbolization conjecture, suppose that M is irreducible, atoroidal, and has infinite fundamental group. Conjecture III implies that $\sigma(M) \leq 0$. By the discussion in Section 1B on graph manifolds, a graph manifold with infinite π_1 cannot be atoroidal; thus Conjecture II implies that $\sigma(M) < 0$. Finally, Conjecture I implies that M is hyperbolic.

In fact, it is not necessary to use Conjecture III in a proof of the hyperbolization conjecture. Namely, the assumptions of the hyperbolization conjecture imply that M is a $K(\pi, 1)$ (see Section 1A), so that by the above-mentioned results from [Gromov and Lawson 1983; Schoen 1984] one can conclude that $\sigma(M) \leq 0$. Thus, Conjectures I and II, together with known results, alternately imply the hyperbolization conjecture.

As pointed out in Section 1A, the remaining cases of the Geometrization Conjecture for closed, oriented, irreducible 3-manifolds have been proved by Thurston, Gabai, Casson, and Jungreis. We indicate briefly how these remaining cases also follow from Conjectures I–III. Conjecture III fully describes the topology of irreducible 3-manifolds with $\sigma(M) > 0$. If M is a 3-manifold as above with $\sigma(M) = 0$, Conjecture II implies that M is a graph manifold with infinite π_1. Such manifolds admit torus decompositions, all of whose components are Seifert fibered (including Sol manifolds). In fact the theory of collapse of 3-manifolds exhibits this splitting into Seifert fibered components, with the exception of Sol manifolds, which might not split by tori in the process of collapse. In other words, the torus decomposition of the graph manifolds M can be detected from the geometry of a collapsing sequence of metrics on M. We refer to [Cheeger and Gromov 1986; Rong 1990] for further details. As indicated in Section 1, Seifert fibered spaces are geometric. If $\sigma(M) < 0$, the tori of Conjecture I of the conjecture decompose M into hyperbolic and graph manifold pieces, and by the remarks above all graph manifold pieces are unions of geometric manifolds along tori.

Conjectures I–III thus amount to the conjecture that an appropriate sequence $g_i \in \mathcal{M}_1$ such that $\mu[g_i] \to \sigma(M)$ *implements* the Geometrization Conjecture, provided M is irreducible. This will be discussed in some further detail in the next sections.

Taken together, these conjectures imply that the Sigma constant $\sigma(M)$ of an irreducible, oriented 3-manifold behaves in a remarkably similar way to the Euler characteristic $\chi(M)$ of an oriented two-manifold (which is also just the total scalar curvature). One sees immediately that Conjectures I, II, III, corresponding to $\sigma(M)$ negative, zero, or positive, bear a strong resemblence to the classification of surfaces with negative, zero, or positive Euler characteristic. Of course, the Sigma constant alone cannot determine the topology of M, since for instance all graph manifolds, in particular Seifert fibered spaces, with infinite π_1, have $\sigma = 0$. With this "degeneracy" removed, one has quite a sharp correspondence between the value of $\sigma(M)$ and the topology of M. For instance, the conjectures imply there are only finitely many irreducible atoroidal 3-manifolds with a given (necessarily nonzero) value of $\sigma(M)$.

For atoroidal manifolds with $\sigma(M) < 0$, the Sigma constant is related to the hyperbolic volume by (4.2). Thurston [1979] has developed a beautiful theory describing the basic structure of the values of the hyperbolic volume; see also [Gromov 1981b].

5. Convergence and Degeneration of Riemannian Metrics

Let $\{g_i\}$ be a sequence of Yamabe metrics in $\mathcal{M}_1$ such that

$$\mu[g_i] \to \sigma(M), \tag{5.1}$$

so that $\{g_i\}$ attempts to realize the Sigma constant on M. If $\{g_i\}$ converges to a metric $g \in \mathcal{M}_1$, then g is an Einstein metric on M, at least when $\sigma(M) \leq 0$, and conjecturally in general.

Since an arbitrary 3-manifold does not admit an Einstein metric, $\{g_i\}$ cannot converge in general to a metric $g \in \mathcal{M}_1$. Of course, when M admits an Einstein metric g_0, since it is generally unique, $\{g_i\}$ should converge to g_0. In general, however, there must exist subsets on M on which $\{g_i\}$ degenerates. How can one relate the degeneration to the topology of M?

An examination of Conjectures I–III shows that they imply that the essential two-spheres and tori in M are obstructions, and the only obstructions, to the existence of Einstein metrics. For instance, the conjectures imply that if M is irreducible and atoroidal, then $\{g_i\}$ should converge to an Einstein metric on

M; this would necessarily be of constant positive or negative curvature, since the atoroidal condition rules out flat metrics. Further, the conjectures implicitly describe the behavior of the degeneration of $\{g_i\}$ in a neighborhood of the essential tori in M: the metrics become very long and thin in this region. This will be described in more detail below. In the next section, we will describe the conjectural degeneration of $\{g_i\}$ in neighborhoods of essential two-spheres.

To summarize, the sequence $\{g_i\}$ should conjecturally degenerate along the two-spheres and tori corresponding to the sphere and torus decompositions of the 3-manifold, and should converge on the complement; this complement is also called the *characteristic variety* [Jaco and Shalen 1979].

To understand if a sequence of metrics converges, or understand how it degenerates, one needs to understand how to control the behavior of a sequence of metrics. First, we note that there is little or no reason to expect that one can control the convergence or degeneration of an arbitrary sequence $\{g_i\}$ satisfying (5.1). From (5.1), one controls the scalar curvature of the metric, as well as the gradient $\nabla \mathcal{S}$ of $\{g_i\}$, restricted to the space of Yamabe metrics, in a weak topology (say $L^{-2,2}$, the dual of $L^{2,2}$). Controlling the scalar curvature of a metric gives good control of the metric in its conformal class, as indicated in Theorem 4.4; one seeks to control only the behavior of a function (the conformal factor), given that it satisfies an elliptic differential equation of the type (4.1). On the other hand, since the equivalence class of a metric (modulo the action of diffeomorphisms) depends locally on three unknown functions, one cannot expect to control the metric, or understand general degenerations, with the scalar curvature function alone. The condition $|\nabla \mathcal{S}| \to 0$ in a topology such as the $L^{-2,2}$ topology is also too weak to lead to definite conclusions about the behavior of $\{g_i\}$.

An analogous visual picture can be obtained by considering minimizing sequences for the Plateau problem, that is, the problem of finding the disc of least area spanning a given smooth curve in $\mathbb{R}^3$. It is well-known that the Plateau problem has a smooth solution. However, the behavior of minimizing sequences, that is, the geometry or configuration of discs in $\mathbb{R}^3$ with given boundary whose area converges to the least area, can be quite bizarre; one may have very long, thin filaments whose contribution to the area is arbitrarily small. Thus, although the limit is well behaved (that is, smooth), the minimizing sequence may have very dense regions of bad behavior that are irrelevant to the geometry of the limit.

Under what circumstances can one control the convergence or degeneration of a sequence in $\mathcal{M}_1$? Arguing heuristically for the moment, the full curvature R of g involves all second derivatives of g (in local coordinates), so that one may expect that a bound on $|R|$ gives a bound on $|g|_{L^{2,\infty}}$. Assuming that one has a Sobolev inequality for g, it follows that g is bounded in $C^{1,\alpha}$ norm, for any $\alpha < 1$, again in local coordinates. If $g_i \in \mathcal{M}_1$ and there is a fixed coordinate system (atlas) in which $(g_i)_{kl}$ is bounded in $C^{1,\alpha}$, it follows from the Arzela–

Ascoli theorem that a subsequence converges, in the $C^{1,\alpha'}$ topology, for $\alpha' < \alpha$, to a limit metric g of class $C^{1,\alpha}$.

This heuristic reasoning can be made rigorous, and leads to the fundamental theory of Cheeger–Gromov on the behavior of metrics with uniform curvature bound. Although the result is valid in all dimensions, we will summarize it in dimension three only.

THEOREM 5.1 [Cheeger 1970; Gromov 1981a; Cheeger and Gromov 1986; 1990]. *Let* $\{g_i\}$ *be a sequence of metrics in* $\mathcal{M}_1$. *Suppose there is a uniform bound*

$$|R_{g_i}| \leq \Lambda. \tag{5.2}$$

Then there is a subsequence, also called $\{g_i\}$, *and diffeomorphisms* ψ_i *of* M *such that exactly one of the following cases occurs*:

I. (Convergence) *The metrics* $\psi_i^* g_i$ *converge in the* $C^{1,\alpha'}$ *topology*, $\alpha' < \alpha$, *to a* $C^{1,\alpha}$ *metric* g_0 *on* M, *for any* $\alpha < 1$.

II. (Collapse) *The metrics* $\psi_i^* g_i$ *collapse* M *along a graph manifold structure. Thus*, M *is necessarily a graph manifold. The metrics* $\psi_i^* g_i$ *collapse the orbits* $\mathcal{O}_x$ (*namely circles or tori*) *of a* (*sequence of*) *orbit structures to a point*, *as* $i \to \infty$, *that is*, $diam_{\psi_i^*(g_i)} \mathcal{O}_x \to 0$ *for all* $x \in M$.

III. (Cusps) *There is a maximal domain* $\Omega \subset M$ *such that* $\psi_i^* g_i|_\Omega$ *converges*, *in the* $C^{1,\alpha'}$ *topology*, $\alpha' < \alpha$, *to a complete* $C^{1,\alpha}$ *metric* g_0 *of volume at most* 1 *on* Ω. *The complement* $M \setminus \Omega$ *is collapsed along a sequence of orbit structures*, *as in case* II. *In particular*, *a neighborhood of* $M \setminus \Omega$ *has the structure of a graph manifold*.

A sequence of metrics h_i defined on a domain V converges in the $C^{1,\alpha}$ topology to a limit metric h if there is a smooth coordinate atlas on V for which the component functions of h_i converge to the component functions of h; here the convergence is with respect to the usual $C^{1,\alpha}$ topology for functions defined on domains in $\mathbb{R}^3$. The convergence in cases I and III above is also in the weak $L^{2,p}$ topology, for any $p < \infty$. In the regions of collapse, the metrics g_i become long and thin: the injectivity radius converges to 0 in these regions, while the diameter of these regions diverges to infinity. The region Ω may not be connected; in fact, in general, it might have infinitely many components. We refer to [Cheeger and Gromov 1986; 1990] or to [Anderson 1993; a] for a more detailed discussion.

These results can be understood as a generalization of the coarse features of Teichmüller theory to higher dimensions and variable curvature. The three possibilities in Theorem 5.1 correspond to the three basic behaviors of sequences in the moduli of metrics of constant curvature on surfaces of genus 0, 1, and $g \geq 2$, respectively.

Teichmüller spaces play an important part in Thurston's results and work on the Geometrization Conjecture. Speaking very loosely, one is given hyperbolic structures on pieces of a 3-manifold, and studies the deformations and degenerations of hyperbolic structures on these pieces and their boundaries, in order to obtain hyperbolic structures on larger manifolds by a smooth gluing. Thus, the convergence and degeneration of hyperbolic metrics, on 3-manifolds and on surfaces, plays a central role.

In attempting to approach the Geometrization Conjecture by studying the space of all metrics on a 3-manifold, Theorem 5.1 plays an analogous central role.

Suppose, for instance, that there is a sequence $\{g_i\}$ of Yamabe metrics satisfying both (5.1) and (5.2), that is,

$$\mu[g_i] \to \sigma(M) \quad \text{and} \quad |R_{g_i}| \le \Lambda, \tag{5.3}$$

for some Λ. As usual we are assuming that M is closed, oriented, and irreducible. We will outline how, in this case, one may approach and in fact come quite close to a proof of Conjectures I–III.

Let $\mathcal{C}$ be the space of conformal classes, represented by a choice of Yamabe metric (it is not known whether Yamabe metrics are unique in their conformal classes when $\mu > 0$). Although $\mathcal{C}$ is not an infinite-dimensional submanifold of $\mathcal{M}_1$, it does have a formal tangent space at every $g \in \mathcal{C}$, and we may write

$$T_g\mathcal{M}_1 = T_g\mathcal{C} \oplus N_g\mathcal{C}, \tag{5.4}$$

where $N_g\mathcal{C}$ is the normal space to $T_g\mathcal{C}$ in $T_g\mathcal{M}_1$, with respect to the L^2 metric on $\mathcal{M}_1$. (Note that $N_g\mathcal{C}$ is *not* tangent to the conformal class of metrics $[g] \subset \mathcal{M}_1$). Now $T_g\mathcal{C} = \{h \in S^2(M) : s(h) = \text{const}\} = \{h \in S^2(M) : \triangle(s'(h)) = 0\}$. One has the classical formula [Besse 1987, p. 63]

$$s'(h) = -\triangle \operatorname{tr}(h) + \delta\delta h - \langle \operatorname{Ric}, h\rangle. \tag{5.5}$$

Thus, $N_g\mathcal{C} = \operatorname{Im}(\triangle \circ s')^*$, which implies

$$N_g\mathcal{C} = \big\{\alpha \in S^2(M) : \alpha = D^2u - (\triangle u)\, g - u \operatorname{Ric} \ \text{for } u \text{ with} \textstyle\int u\, dV_g = 0\big\}. \tag{5.6}$$

Applying this to the metrics g_i, we may then write

$$\nabla|_{\mathcal{M}_1}\mathcal{S}_{g_i} = Z_{g_i} = D_i^2 u_i - (\triangle_i u_i)\, g_i - u_i \operatorname{Ric}_i + Z_i^T, \tag{5.7}$$

where $Z_i^T \in T_{g_i}\mathcal{C}$ is the tangential projection of Z. Since $\{\mu[g_i]\}$ approaches a critical (maximal) value of $\mathcal{S}|_{\mathcal{C}}$, we may assume, as in (2.3),

$$Z_i^T \to 0 \quad \text{in } L^{-2,2}(T\mathcal{C}). \tag{5.8}$$

We apply Theorem 5.1 to $\{g_i\}$, and consider the three cases individually. It is implicitly assumed that appropriate subsequences and diffeomorphisms of M are taken where necessary.

Case I. Suppose the metrics g_i converge. Then the limit g is a $C^{1,\alpha} \cap L^{2,p}$ metric on M, which satisfies the equation

$$Z = D^2 u - (\triangle u)\, g - u \,\mathrm{Ric}, \tag{5.9}$$

weakly. Taking the trace of (5.9) gives

$$\triangle u = -\tfrac{1}{2} s u = -\tfrac{1}{2}\sigma(M)\, u, \tag{5.10}$$

so that u is an eigenfunction of the Laplacian, with eigenvalue $-\frac{1}{2}\sigma(M)$. Elliptic regularity allows one to conclude that any weak $L^{2,p}$ solution g of (5.9) and (5.10) is smooth. Since the Laplacian has negative spectrum, we conclude that, if $\sigma(M) \leq 0$, then u is a constant. Further, since the integral of u is 0, by (5.6), it follows in this case that $u \equiv 0$. Thus, by (5.9), $Z = 0$, that is, g is an Einstein metric realizing $\sigma(M)$. It is conjectured that also in the positive case, $\sigma(M) > 0$ the only solution of (5.9) is again given by $Z = 0$.

Case II. If the metrics $\{g_i\}$ collapse, M is necessarily an (irreducible) graph manifold. As indicated in Section 4, M necessarily satisfies $\sigma(M) \geq 0$, and $\sigma(M) = 0$ if and only if $\pi_1(M)$ is infinite.

Case III. Suppose the metrics $\{g_i\}$ converge to a collection of complete, non-compact Riemannian manifolds (M_j, g_j) of finite volume, and collapse the complement. Then, arguing as in Case I, on each M_j the metric g_j satisfies

$$Z = D^2 u - (\triangle u)\, g - u \,\mathrm{Ric}\,. \tag{5.11}$$

Taking the trace as in Case I implies that u is an eigenfunction of $\triangle$. Now we point out that in this case, we must have $\sigma(M) \leq 0$. Namely, if $\sigma(M) > 0$, so $\mu[g_i] > 0$ for i sufficiently large, it follows from (4.1) that g_i has a uniform bound on its Sobolev constant. This implies that the volume of unit balls in (M, g_i) is bounded below, that is, (M, g_i) does not become thin at any point. This is of course not the case due to the presence of cusps, that is, neighborhoods of infinity of M_j do not satisfy a (uniform) Sobolev inequality. Thus, $\sigma(M) \leq 0$ and the arguments above again imply that $u = 0$. It follows that g_j is a metric of constant negative curvature. With some further arguments, which are not difficult, it is possible to prove that the collection $\{M_i\}$ is finite and the metric $g = \{g_i\}$ on $\bigcup M_j$ realizes $\sigma(M)$. Since the complement of $\bigcup M_j$ is collapsed, it has the structure of a finite union of connected graph manifolds.

Thus, as indicated in outline form above, a significant portion of Conjectures I–III is resolved if there is a sequence of Yamabe metrics $\{g_i\}$ with $\mu[g_i] \to \sigma(M)$ having uniformly bounded curvature. More precisely, Conjecture III can be resolved modulo the conjecture that solutions of (5.9) are Einstein. Further, Conjecture II can be resolved.

Conjecture I requires however further consideration, for it remains to be proved that the tori in the hyperbolic cusps are incompressible in M. This is, of course, an important issue. For instance, Thurston has shown that every 3-manifold M has many hyperbolic knot or link complements, that is, there are knots or links L in M whose complements $\Omega = M \setminus L$ admit complete hyperbolic metrics g_L of finite volume. In this case, the tori in the hyperbolic cusps are compressible in M; they just form the boundary components of a tubular neighborhood of L in M.

All of these metrics g_L can also be considered as critical points of $\{g_i\}$, in the sense discussed above. Namely, it is not hard to see that there are sequences $g_i \in \mathcal{M}_1$ such that $g_i|_\Omega \to g_L$ smoothly (and uniformly on compact sets), while $\|\nabla \mathcal{S}_{g_i}\|_{L^{-2,2}} \to 0$ as $i \to \infty$. These metrics g_L are not tied tightly to, and so do not reflect easily, the global topology of M; consider for instance that the three-sphere has many hyperbolic knot complements. The Thurston theory indicates that the hyperbolic knot or link complements tend to have large volume (when the curvature is -1); see Section 1C. This corresponds to large values of $|\mathcal{S}|$, that is of $|\mu|$, when the volume is normalized to 1. Since we are dealing with the case $\sigma(M) < 0$, the absolute value $|\sigma(M)|$ represents the *smallest* possible value of $|\mu[g]|$. Thus, it is not unreasonable to expect the critical metric corresponding to the value $\sigma(M)$ to have a special behavior.

In fact, we have the following result:

THEOREM 5.2. *Let (M_j, g_j) be a finite collection of hyperbolic manifolds realizing $\sigma(M)$, as discussed in Case* III (page 72). *Then each torus T_k in the cusp region of any hyperbolic component M_j is incompressible in M.*

The idea of the proof is that if one of the tori T in the hyperbolic cusps is compressible, T must bound a solid torus U in M. By explicitly constructing metrics on U that smoothly match the hyperbolic metric at T, one can prove that there is a smooth metric $g' \in \mathcal{M}_1$ with $\mu[g'] > \sigma(M)$. This contradicts the definition of $\sigma(M)$. Full details appear in [Anderson b]. This argument is similar in spirit, although quite different in proof, to Thurston's cusp closing theorem [Thurston 1979; Gromov 1981b].

In spite of the possible optimism implicit in the discussion above, Conjectures I–III remain very difficult to prove. The assumption that one can find sequences of Yamabe metrics $\{g_i\}$ satisfying (5.1) for which the curvature remains uniformly bounded is very strong, and it is not at all clear how to realize it. In fact, in general it is not true that a 3-manifold admits a sequence $\{g_i\}$ satisfying (5.1) with uniformly bounded curvature. Indeed, in all the arguments of this section, the (crucial) hypothesis of irreducibility has not been used. Thus, on manifolds of the form $M = N \# N$, where $N \neq \mathbb{S}^3$, or $M = \mathbb{S}^2 \times \mathbb{S}^1$, it is impossible to find metrics satisfying (5.1) with uniformly bounded curvature. Of course, when studying sequences in the space $\mathcal{M}_1$, it is not easy to distinguish the topology of the underlying manifold M.

Thus, there remains the fundamental problem of being able to understand if the sets in M where the curvature of $\{g_i\}$ diverges to infinity can be related with the essential two-spheres in M, that is, with the prime decomposition of M. This will be discussed further in the next section.

To summarize: assuming the conjecture that the only solutions of (5.9) are Einstein, we have outlined a proof that Conjectures I–III follow from the following conjecture:

CONJECTURE IV. *Let M be a closed, oriented, irreducible 3-manifold. Then there is a maximizing sequence of unit volume Yamabe metrics $\{g_i\}$ having uniformly bounded curvature.*

REMARK 5.3. Since the full curvature is needed to control the convergence or degeneration of metrics, the scalar curvature alone being too weak, it is perhaps natural to consider other functionals on $\mathcal{M}_1$, besides $\mathcal{S}$, that incorporate the full curvature. Thus, in dimension three, one may consider

$$\mathcal{R}^2 = \int_M |R|^2 \, dV,$$

the L^2 norm of the curvature tensor (see Section 3). This functional is clearly bounded below, so one can study the existence, regularity, and geometry of metrics that realize the infimum of $\mathcal{R}^2$. Such a program has been carried out in [Anderson 1993; a], where we obtain essentially the same results as in Theorem 5.1 (the L^∞ case). In addition, we show there that minimizing metrics are smooth, in contrast to the L^∞ case. However, the geometry of the minimizing metrics appears to be complicated: the Euler–Lagrange equations are of fourth order in g. Einstein metrics are solutions of the equations, but it would appear likely that there are other solutions as well on compact manifolds. In trying to implement the Geometrization Conjecture, as discussed above, by studying minimizing sequences of $\mathcal{R}^2$, one runs into the same difficulties as above, namely the behavior of the sequence, or the limit, near what one would conjecture to be essential two-spheres.

6. Essential Two-Spheres and "Black Holes": A Relation with General Relativity

We now discuss briefly some arguments supporting the validity of Conjecture IV. The scope of this paper will require this section to contain a number of oversimplificatons and to be even more terse than Section 5. Many important points will be neglected in order to keep the overall spirit of the argument simple.

Suppose, as above, that $g_i \in \mathcal{M}_1$ is a sequence of Yamabe metrics such that

$$\mu[g_i] \to \sigma(M). \tag{6.1}$$

Suppose further that $\sup |R_{g_i}| = R_i \to \infty$ as $i \to \infty$. Choose points $x_i \in M$ such that $|R_{g_i}|(x_i) = R_i$. In order to understand the degeneration of $\{g_i\}$ in a neighborhood of x_i, we rescale or "blow up" the metrics g_i by the factor R_i, that is, we consider the metrics

$$\tilde{g}_i = R_i g_i. \tag{6.2}$$

Then consider the behavior of the pointed Riemannian manifolds $(M, \tilde{g}_i, x_i)$ centered at x_i. Distances in $\tilde{g}_i$ are $\sqrt{R_i}$ times larger than distances in g_i, so that in effect, we are studying the degeneration behavior of $\{g_i\}$ in smaller and smaller neighborhoods of x_i, magnified to unit size. In particular, $\operatorname{diam}_{\tilde{g}_i} M = \sqrt{R_i}\operatorname{diam}_{g_i} M \to \infty$. The scaling properties of curvature imply that $|R_{\tilde{g}_i}| \leq 1$ and $|R_{\tilde{g}_i}|(x_i) = 1$.

Using an appropriate version of Theorem 5.1, one can conclude that, modulo diffeomorphisms, a subsequence of $\{\tilde{g}_i\}$ either converges uniformly on domains of bounded diameter to a limit metric $\tilde{g}$, or degenerates, that is, collapses (again uniformly on domains of bounded diameter), as described in Case II of Theorem 5.1. For simplicity, we will not deal with the latter case here. Thus, we assume that $\{\tilde{g}_i\}$ converges to a complete $C^{1,\alpha}$ metric $\tilde{g}$, defined on a 3-manifold X, with base point x. The triple $(X, \tilde{g}, x)$ is sometimes also called a *geometric limit* of $(M, \tilde{g}_i, x_i)$.

The gradient $\nabla \mathcal{S}_{g_i}$ is given by (5.7), so that for the rescaled metrics $\tilde{g}_i$, one has

$$\tilde{Z}_i = \tilde{D}_i^2 u_i - (\tilde{\triangle}_i u_i)\,\tilde{g}_i - u_i \widetilde{\mathrm{Ric}}_i + \tilde{Z}_i^T. \tag{6.3}$$

Note that u_i is scale-invariant. Now recall from (5.8) that $\|Z_i^T\|_{L^{-2,2}}(T\mathcal{C}) \to 0$ as $i \to \infty$. However, neither the $L^{-2,2}$ norm nor the functional $\mathcal{S}$ are scale-invariant. Taking the scaling behavior into account, one easily computes that

$$\|\tilde{Z}_i^T\|_{L^{-2,2}}(T\mathcal{C}) = R_i^{3/4}\,\|Z_i^T\|_{L^{-2,2}(T\mathcal{C})}. \tag{6.4}$$

Thus, it is no longer true, as in Section 5, that (6.1) implies automatically that $\|\tilde{Z}_i^T\|_{L^{-2,2}(T\mathcal{C})} \to 0$. The question of whether there exist sequences $\{g_i\}$ satisfying (6.1) and for which the tangential gradient $\tilde{Z}_i^T$ converges to 0 in $L^{-2,2}(\tilde{g}_i)$ norm is important. Because it involves deeper technicalities, it will not be discussed further here. We therefore suppose there exists a maximizing sequence $\{g_i\}$ of Yamabe metrics for which

$$\tilde{Z}_i^T \to 0 \quad \text{in } L^{-2,2}(\tilde{g}_i). \tag{6.5}$$

It follows that the limit metric $\tilde{g}$ satisfies the equation

$$Z = D^2 u - (\triangle u)\, g - u\,\mathrm{Ric} \tag{6.6}$$

weakly, that is, in $L^{2,p}$. Here we have dropped the tildes. Note that, since the scalar curvature of $\{g_i\}$ is uniformly bounded, by scaling, the scalar curvature $\tilde{s}$ of the limit metric $\tilde{g}$ is identically 0 (in L^p) on X. In particular, for $\tilde{g}$, we have $Z = \mathrm{Ric}$. Thus, setting $h = 1 + u$ in (6.6) gives

$$h\,\mathrm{Ric} = D^2 h, \qquad \triangle h = 0. \tag{6.7}$$

It can be shown, using elliptic regularity, that weak $L^{2,p}$ solutions of the system (6.7) are in fact smooth. (Again, we have to assume that the function h thus obtained is not identically 0).

The equations (6.7) are classical equations arising in general relativity, called the *static vacuum Einstein equations*. Let $X_{\pm} = \{x \in X : \pm h(x) \geq 0\}$, and consider the product four-manifold $M^4_{\pm} = X_{\pm} \times \mathbb{S}^1$, with warped product metric

$$g' = g_X + h^2 d\theta^2. \tag{6.8}$$

Then $(M^4_{\pm}, g')$ is a Ricci-flat four-manifold (its Ricci curvature vanishes identically), and is thus a vacuum solution of the Einstein equations. The length of the circle S^1_x at $x \in M$ is given by $|h(x)|$; we note that the space M_+ or M_- may be singular on the locus where $h = 0$.

The equations (6.7) are defined on a space-like hypersurface of a Lorentzian four-manifold. In (6.8), we have changed the Lorentz signature $(-h^2\, d\theta^2)$ to a Riemannian signature; this has no effect on computations of curvature and the like.

Summarizing, blow-ups of a sequence of metrics $\{g_i\}$ satisfying (6.5) and degenerating in a neighborhood of a point sequence $x_i \in M$ (of maximal curvature) have geometric limits that are solutions of the static vacuum Einstein equations.

The canonical solution of the static vacuum Einstein equations is the Schwarzschild metric g_s, given by

$$g_s = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, ds^2_{\mathbb{S}^2} + \left(1 - \frac{2m}{r}\right) d\theta^2, \tag{6.9}$$

with $h = (1 - 2m/r)^{1/2}$; here $m > 0$ is a free parameter, called the mass of the metric g_s. The metric on the space-like hypersurface

$$g_s = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, ds^2_{\mathbb{S}^2} \tag{6.10}$$

is defined on $[2m, \infty) \times \mathbb{S}^2$, and is clearly spherically symmetric. Although the metric (6.9) or (6.10) may appear to be singular at $r = 2m$, this is only an apparent singularity, and can be removed by a change of coordinates. From (6.10), one sees that the set $B = h^{-1}(0)$ where $r = 2m$ is a totally geodesic two-sphere of constant curvature, while the metric (6.10) is asymptotically flat, that is, asymptotic to the flat metric on $\mathbb{R}^3$ for large r. The "horizon" $h^{-1}(0)$ is interpreted as the surface of an (isolated) black hole in general relativity. Asymptotically flat solutions to the equations (6.10) are often considered in the physics literature, since they serve as models of isolated black holes. The four-manifold Riemannian metric (6.9) is a smooth Ricci flat metric on $\mathbb{S}^2 \times \mathbb{R}^2$, asymptotic to the flat metric on $\mathbb{R}^3 \times \mathbb{S}^1$ at ∞.

This picture serves as a model for the general behavior of (appropriate) Yamabe sequences satisfying (6.1), in the neighborhoods of sets where the curvature goes to infinity. Namely, the two-sphere $B = h^{-1}(0)$ in the blow-up metric

$\tilde{g}$, when rescaled to the original sequence $\{g_i\}$, is being collapsed to a point. The fact that $\tilde{g}$ is asymptotically flat indicates, when blown down to $\{g_i\}$, that the curvature remains uniformly bounded in regions away from the central $\mathbb{S}^2$. Thus, $\{g_i\}$ is collapsing an $\mathbb{S}^2 \subset M$ to a point, giving rise to a limit metric g defined on the union of two manifolds (3-balls) M_1 and M_2 identified at one point. In other words, $\{g_i\}$ *performs* a surgery on the two-sphere $B \subset M$.

We illustrate how this behavior actually arises in a concrete example. It is known [Kobayashi 1987; Schoen 1989] that

$$\sigma(\mathbb{S}^2 \times \mathbb{S}^1) = \sigma(\mathbb{S}^3). \tag{6.11}$$

In fact, $\mathbb{S}^2 \times \mathbb{S}^1$ admits a sequence of conformally flat Yamabe metrics $g_i \in \mathcal{M}_1$ with $\mu[g_i] \to \sigma(\mathbb{S}^3)$. These metrics behave in the following way. View $\mathbb{S}^1$ as $I = [-1, 1]$ with endpoints identified. The metrics g_i have spherical ($\mathbb{S}^2$) symmetry, and on domains of the form $\mathbb{S}^2 \times I_{\varepsilon_i}$, for $I_{\varepsilon_i} = [-1+\varepsilon_i,\, 1-\varepsilon_i]$, they converge smoothly to the canonical metric of volume 1 on $\mathbb{S}^3 \smallsetminus \{p \cup p'\}$, where p and p' are antipodal points on $\mathbb{S}^3$; here $\varepsilon_i \to 0$ as $i \to \infty$. In particular, $\{\mathbb{S}^2 \times \mathbb{S}^1, g_i\}$ converges to $\mathbb{S}^3/\{p \cup p'\}$, that is, $\mathbb{S}^3$ with two points identified, in the Gromov–Hausdorff topology [Gromov 1981a]. Note that the curvature is remaining uniformly bounded in the region $\mathbb{S}^2 \times I_\varepsilon$, for any fixed $\varepsilon > 0$. Let $J_{\varepsilon_i} = S^1 \smallsetminus I_{\varepsilon_i}$. On the complement $\mathbb{S}^2 \times J_{\varepsilon_i}$, whose diameter converges to 0, the curvature is blowing up, that is, diverging to $+\infty$. If one rescales, or blows up, the metrics $\{g_i\}$, as in the procedure described above, the blown-up metrics $\tilde{g}_i$ converge to the Schwarzschild metric. (Note that the Schwarzschild metric (6.10) is conformally flat.)

We see here, in this concrete example, and conjecturally in general, how $\{g_i\}$ is implementing the prime decomposition of the 3-manifold M. Of course, there remains the basic issue of proving that two-spheres that arise in this fashion are essential in M. We will not discuss this further here, beyond saying that it is natural to attempt to prove an analogue of Theorem 5.2 with spheres in place of tori.

There are however many other solutions to the equations (6.7) besides the Schwarzschild metric. Of course, there are the flat solutions on $\mathbb{R}^3$, where h is constant or linear. It is a classical result of Lichnerowicz [1955] that there are no complete, nonflat, asymptotically flat solutions of (6.7) with $h > 0$ everywhere. In fact, the assumption of asymptotic flatness can be dropped: There are no complete, nonflat solutions to (6.7) with $h > 0$ everywhere [Anderson c]. Thus, the cases of interest are where h vanishes somewhere.

In this regard there is a beautiful uniqueness theorem in the physics literature [Israel 1967; Robinson 1977; Bunting and Masood-ul-Alam 1987]:

THEOREM 6.1 (BLACK-HOLE UNIQUENESS THEOREM). *Let (X, g) be a smooth solution to the vacuum Einstein equations*

$$h\,\mathrm{Ric} = D^2 h, \qquad \triangle h = 0, \tag{6.12}$$

with $\partial X = h^{-1}(0)$ compact. If g is asymptotically flat, then (X, g) is the Schwarzschild solution.

This implies in particular that ∂X is connected: there are no smooth static solutions with multiple black holes. It is interesting to note that there are asymptotically flat solutions to (6.12) with multiple black holes, having only cone singularities along a line segment (geodesic) joining the black holes. In particular, these singularities are not curvature singularities: the curvature is uniformly bounded everywhere. The line singularity is interpreted as a "strut" keeping the black holes in equilibrium from their mutual gravitational attraction [Kramers et al. 1980].

Theorem 6.1 can be proved more generally under the single assumption that $h^{-1}(0)$ is compact, that is, it is not necessary to assume the metric asymptotically flat; the metric must be still be assumed to be smooth up to ∂X, see [Anderson c].

We note that there are examples of solutions to (6.12) with smooth non-connected, and in fact noncompact, boundary. The so-called B1 solution [Ehlers and Kundt 1962; Kramers et al. 1980] has 3-manifold metric given by

$$g_b = \left(1 - \frac{b}{r}\right)^{-1} dr^2 + \left(1 - \frac{b}{r}\right) d\phi^2 + r^2\, d\theta^2. \tag{6.13}$$

Here the function h of (6.12) is given by $h = r \sin\theta$, for $r \in [b, \infty)$ and for $\phi \in [0, \pi]$, $\theta \in [0, 2\pi]$ the standard spherical coordinates on $\mathbb{S}^2$. This metric has the property that $h^{-1}(0)$ is two copies of $\mathbb{R}^2$, each asymptotic to a flat cylinder. This metric is not asymptotically flat in the usual sense, that is, not asymptotic to the flat metric on $\mathbb{R}^3$. However, it is asymptotic to the flat metric on $\mathbb{R}^2 \times \mathbb{S}^1$. Note that h is unbounded. Note further that when forming the four-manifold metric as in (6.8) with h as above, one obtains exactly the Schwarzschild metric on $\mathbb{S}^2 \times \mathbb{R}^2$. In fact, the metrics (6.10) and (6.13) are just different (orthogonal) three-dimensional slices to the four-dimensional Schwarzschild metric (6.9).

It seems that one should be able to classify completely the smooth solutions to the static vacuum Einstein equations with smooth boundary $B = h^{-1}(0)$, and that are complete away from B. We venture the following.

CONJECTURE 6.2. *Let (X, g) be a smooth complete solution to the static Einstein vacuum equations* (6.12), *with $B = h^{-1}(0)$ smooth, that is, g smooth up to B. Then (X, g) is either flat, or the Schwarzschild solution, or the* B1 *solution.*

There is a wealth of examples in the physics literature on singular solutions to the static vacuum equations (6.12); see [Ehlers and Kundt 1962; Kramers et al. 1980], for instance. These solutions are typically not complete, or have curvature singularities on B, that is, the curvature blows up on approach to some region in B. Since, in the context of our discussion, the spaces $(X, \tilde{g})$ arise as limits of spaces of bounded curvature, one might hope that these singular solutions could be ruled out.

In sum, the partly heuristic arguments presented above provide some evidence to support Conjecture IV, establishing a relation between the sets where the curvature of a maximizing sequence of Yamabe metrics diverges to infinity, and the sets of essential two-spheres in M.

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