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Spectral Covers

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Abstract. Spectral curves have acquired a central role in understanding the moduli spaces of vector bundles and Higgs bundles on a curve. A Higgs G-bundle on an arbitrary variety S (together with some additional data, such as a representation of G) determines a *spectral cover* $\tilde{S}$ of S and an equivariant sheaf on $\tilde{S}$. The purpose of these notes is to combine and review various results about spectral covers, focusing on the decomposition of their Picards (and the resulting Prym identities) and the interpretation of a distinguished Prym component as parameter space for Higgs bundles.

1. Introduction

Spectral curves arose historically out of the study of differential equations of Lax type. Following Hitchin's work [H1], they have acquired a central role in understanding the moduli spaces of vector bundles and Higgs bundles on a curve. Simpson's work [S] suggests a similar role for spectral covers $\widetilde{S}$ of higher-dimensional varieties S in moduli questions for bundles on S.

The purpose of these notes is to combine and review various results about spectral covers, focusing on the decomposition of their Picards (and the resulting Prym identities) and the interpretation of a distinguished Prym component as parameter space for Higgs bundles. Much of this is modeled on Hitchin's system, which we recall in Section 1, and on several other systems based on moduli of Higgs bundles, or vector bundles with twisted endomorphisms, on curves. By peeling off several layers of data that are not essential for our purpose, we arrive at the notions of an *abstract principal Higgs bundle* and a *cameral* (roughly, a principal spectral) cover. Following [D3], this leads to the statement of the main result, Theorem 12, as an equivalence between these somewhat abstract 'Higgs' and 'spectral' data, valid over an arbitrary complex variety and for a reductive Lie group G. Several more familiar forms of the equivalence can then be derived in special cases by adding choices of representation, value bundle

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and twisted endomorphism. This endomorphism is required to be *regular*, but not semesimple. Thus the theory works well even for Higgs bundles that are everywhere nilpotent. After touching briefly on the symplectic side of the story in Section 6, we discuss some of the issues involved in removing the regularity assumption, as well as some applications and open problems, in Section 7.

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We work throughout over $\mathbf{C}$. The total space of a vector bundle (= locally free sheaf) K is denoted $\mathbb{K}$. Some more notation:

Groups:	G	B	T	N	C
algebras:	$\mathfrak{g}$	$\mathfrak{b}$	$\mathfrak{t}$	$\mathfrak{n}$	$\mathfrak{c}$
Principal bundles:	$\mathcal{G}$	$\mathcal{B}$	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{C}$
bundles of algebras:	g	b	t	n	c

2. Hitchin's system

Let $\mathcal{M} := \mathcal{M}_C(n,d)$ be the moduli space of stable vector bundles of rank n and degree d on a smooth projective complex curve C. It is smooth and quasiprojective of dimension

$$\tilde{g} := n^2(g-1)+1. \tag{1}$$

Its cotangent space at a point $E \in \mathcal{M}$ is given by

$$T^*_E\mathcal{M} := H^0(\mathrm{End}(E)\otimes\omega_C), \tag{2}$$

where ω_C is the canonical bundle of C. Our starting point is:

THEOREM 1 (HITCHIN [H1]). *The cotangent bundle $T^*\mathcal{M}$ is an algebraically completely integrable Hamiltonian system.*

Complete integrability means that there is a map

$$h : T^*\mathcal{M} \longrightarrow B$$

to a $\tilde{g}$-dimensional vector space B that is Lagrangian with respect to the natural symplectic structure on $T^*\mathcal{M}$ (i.e., the tangent spaces to a general fiber $h^{-1}(a)$ over $a \in B$ are maximal isotropic subspaces with respect to the symplectic form). In this situation one gets, by contraction with the symplectic form, a trivialization of the tangent bundle:

$$T_{h^{-1}(a)} \xrightarrow{\approx} \mathcal{O}_{h^{-1}(a)} \otimes T_a^*B. \tag{3}$$

In particular, this produces a family of ('*Hamiltonian*') vector fields on $h^{-1}(a)$ that is parametrized by T_a^*B, and the flows generated by these fields on $h^{-1}(a)$ all commute. *Algebraic complete integrability* means additionally that the fibers $h^{-1}(a)$ are Zariski open subsets of abelian varieties on which the Hamiltonian flows are linear, i.e., the vector fields are constant.

We describe the idea of the proof in a slightly more general setting, following [BNR]. Let K be a line bundle on C, with total space $\mathbb{K}$. (In Hitchin's situation, K is ω_C and $\mathbb{K}$ is T^*C.) A *K-valued Higgs bundle* is a pair

$$(E,\ \phi : E \longrightarrow E \otimes K)$$

consisting of a vector bundle E on C and a K-valued endomorphism. One imposes an appropriate stability condition, and obtains a good moduli space $\mathcal{M}_K$ parametrizing equivalence classes of K-valued semistable Higgs bundles, with an open subset $\mathcal{M}_K^s$ parametrizing isomorphism classes of stable ones [S].

Let $B := B_K$ be the vector space parametrizing polynomial maps

$$p_a : \mathbb{K} \longrightarrow \mathbb{K}^n$$

of the form

$$p_a(x) = x^n + a_1 x^{n-1} + \cdots + a_n, \qquad\qquad a_i \in H^0(K^{\otimes i}).$$

In other words,

$$B := \bigoplus_{i=1}^{n} H^0(K^{\otimes i}). \tag{4}$$

The assignment

$$(E, \phi) \longmapsto \mathrm{char}(\phi) := \det(xI - \phi) \tag{5}$$

gives a morphism

$$h_K : \mathcal{M}_K \longrightarrow B_K. \tag{6}$$

In Hitchin's case, the desired map h is the restriction of h_{ω_C} to $T^*\mathcal{M}$, which is an open subset of $\mathcal{M}_{\omega_C}^s$. Note that in this case $\dim B$ is, in Hitchin's words, 'somewhat miraculously' equal to $\tilde{g} = \dim \mathcal{M}$.

The *spectral curve* $\widetilde{C} := \widetilde{C}_a$ defined by $a \in B_K$ is the inverse image in $\mathbb{K}$ of the 0-section of $\mathbb{K}^{\otimes n}$ under $p_a : \mathbb{K} \longrightarrow \mathbb{K}^n$. It is finite over C of degree n. The general fiber of h_K is given by

Proposition 2. [BNR] *For* $a \in B$ *with* integral *spectral curve* $\widetilde{C}_a$, *there is a natural equivalence between isomorphism classes of*

(i) *rank*-1*, torsion-free sheaves on* $\widetilde{C}_a$*, and*

(ii) *pairs* $(E\ ,\ \phi : E \to E \otimes K)$ *with* $\mathrm{char}(\phi) = a$.

When $\widetilde{C}_a$ is non-singular, the fiber is thus $\mathrm{Jac}(\widetilde{C}_a)$, an abelian variety. In $T^*\mathcal{M}$ the fiber is an open subset of this abelian variety. One checks that the missing part has codimension ≥ 2, so the symplectic form, which is exact, must restrict to 0 on the fibers, completing the proof.

3. Some related systems

Polynomial matrices. One of the earliest appearances of an ACIHS (algebraically completely integrable Hamiltonian system) was in Jacobi's work on the geodesic flow on an ellipsoid (or more generally, on a nonsingular quadric in R^k). Jacobi discovered that this differential equation, taking place on the tangent (= cotangent!) bundle of the ellipsoid, can be integrated explicitly in terms of hyperelliptic theta functions. In our language, the total space of the flow is an ACIHS, fibered by (Zariski open subsets of) hyperelliptic Jacobians. We are essentially in the special case of Proposition 2, where

$$C = P^1, \quad n = 2, \quad K = \mathcal{O}_{P^1}(k).$$

A variant of this system appeared in Mumford's solution [Mu1] of the Schottky problem for hyperelliptic curves.

The extension to all values of n is studied in [B] and, somewhat more analytically, in [AHP] and [AHH]. Beauville considers, for fixed n and k, the space B of polynomials

$$p = y^n + a_1(x)y^{n-1} + \cdots + a_n(x), \quad \deg(a_i) \leq ki \tag{7}$$

in variables x and y. Each p determines an n-sheeted branched cover

$$\widetilde{C}_p \to P^1.$$

The total space is the space of polynomial matrices

$$M := H^0(P^1, \mathrm{End}(\mathcal{O}^{\oplus n}) \otimes \mathcal{O}(d)), \tag{8}$$

the map $h : M \to B$ is the characteristic polynomial, and M_p is the fiber over a given $p \in B$. The result is that, for smooth spectral curves $\widetilde{C}_p$, $\mathrm{PGL}(n)$ acts freely and properly on M_p; the quotient is isomorphic to $J(\widetilde{C}_p) \smallsetminus \Theta$. (In order to obtain the entire $J(\widetilde{C}_p)$, one must allow all pairs (E, ϕ) with E of given degree, say 0. Among those, the ones with $E \approx {\mathcal{O}_{P^1}}^{\oplus n}$ correspond to the open set $J(\widetilde{C}_p) \smallsetminus \Theta$.) This system is an ACIHS, in a slightly weaker sense than before:

instead of a symplectic structure, it has a *Poisson structure*, i.e., a section β of $\wedge^2 T$, such that the **C**-linear sheaf map given by contraction with β

$$\begin{array}{ccc} \mathcal{O} & \to & \mathcal{T} \\ f & \mapsto & df \rfloor \beta \end{array}$$

sends the Poisson bracket of functions to the bracket of vector fields. Any Poisson manifold is naturally foliated, with (locally analytic) symplectic leaves. For a Poisson ACIHS, we want each leaf to inherit a (symplectic) ACIHS, so the symplectic foliation should be pulled back via h from a foliation of the base B.

The result of [BNR] suggests that analogous systems should exist when P^1 is replaced by an arbitrary base curve C. The main point is to construct the Poisson structure. This was achieved by Bottacin [Bn] and Markman [M1]; see Section 6. In the case of the polynomial matrices, though, everything—the commuting vector fields, the Poisson structure, and so on—can be written very explicitly. What makes these explicit results possible is that every vector bundle over P^1 splits. This of course fails in genus > 1, but for elliptic curves the moduli space of vector bundles is still completely understood, so here too the system can be described explicitly, as follows.

For simplicity, consider vector bundles with fixed determinant. When the degree is 0, the moduli space is a projective space P^{n-1} (or, more canonically, the fiber over 0 of the Abel–Jacobi map

$$C^{[n]} \longrightarrow J(C) = C.)$$

The ACIHS that arises is essentially the Treibich–Verdier theory [TV] of elliptic solitons. When, on the other hand, the degree is one (or, more generally, relatively prime to n), the moduli space is a single point; the corresponding system was studied explicitly in [RS].

Reductive groups. In another direction, one can replace the vector bundles by principal G-bundles $\mathcal{G}$ for any reductive group G. Again, there is a moduli space $\mathcal{M}_{G,K}$ parametrizing equivalence classes of semistable K-valued G-Higgs bundles, i.e., pairs $(\mathcal{G}, \phi)$ with $\phi \in K \otimes \mathfrak{a}d(\mathcal{G})$. The Hitchin map goes to

$$B := \bigoplus_i H^0(K^{\otimes d_i}),$$

where the d_i are the degrees of the f_i, a basis for the G-invariant polynomials on the Lie algebra $\mathfrak{g}$:

$$h : (\mathcal{G}, \phi) \longrightarrow (f_i(\phi))_i.$$

When $K = \omega_C$, Hitchin showed [H1] that one still gets a completely integrable system, and that it is algebraically completely integrable for the classical groups $\mathrm{GL}(n), \mathrm{SL}(n), \mathrm{SP}(n), \mathrm{SO}(n)$. The generic fibers are in each case isomorphic (though not quite canonically—one must choose various square roots; see Sections 5.2 and 5.3) to abelian varieties given in terms of the spectral curves $\widetilde{C}$:

$$(9)\quad \begin{cases} \mathrm{GL}(n) & \widetilde{C} \text{ has degree } n \text{ over } C\text{, the AV is } \mathrm{Jac}(\widetilde{C}). \\ \mathrm{SL}(n) & \widetilde{C} \text{ has degree } n \text{ over } C\text{, the AV is } \mathrm{Prym}(\widetilde{C}/C). \\ \mathrm{SP}(n) & \widetilde{C} \text{ has degree } 2n \text{ over } C \text{ and an involution } x \mapsto -x. \\ & \text{The map factors through the quotient } \overline{C}. \\ & \text{The AV is } \mathrm{Prym}(\widetilde{C}/\overline{C}). \\ \mathrm{SO}(n) & \widetilde{C} \text{ has degree } n \text{ and an involution, with:} \\ & \bullet \text{ a fixed component, when } n \text{ is odd.} \\ & \bullet \text{ some fixed double points, when } n \text{ is even.} \\ & \text{One must desingularize } \widetilde{C} \text{ and the quotient } \overline{C}\text{, and} \\ & \text{ends up with the Prym of the desingularized double cover.} \end{cases}$$

The algebraic complete integrability was verified in [KP1] for the exceptional group G_2. A sketch of the argument for any reductive G is in [BK], and a complete proof was given in [F1]. We will outline a proof in Section 5 below.

Higher dimensions. Finally, a sweeping extension of the notion of Higgs bundle is suggested by the work of Simpson [S]. To him, a Higgs bundle on a projective variety S is a vector bundle (or principal G-bundle ...) E with a symmetric, Ω^1_S-valued endomorphism

$$\phi : E \longrightarrow E \otimes \Omega^1_S.$$

Here *symmetric* means the vanishing of

$$\phi \wedge \phi : E \longrightarrow E \otimes \Omega^2_S,$$

a condition that is obviously vacuous on curves. He constructs a moduli space for such Higgs bundles (satisfying appropriate stability conditions), and establishes diffeomorphisms to corresponding moduli spaces of representations of $\pi_1(S)$ and of connections.

4. Decomposition of spectral Picards

4.1. The question. Let $(\mathcal{G}, \phi)$ be a K-valued principal Higgs bundle on a complex variety S. Each representation

$$\rho : G \longrightarrow \mathrm{Aut}(V)$$

determines an associated K-valued Higgs bundle

$$(\mathcal{V} := \mathcal{G} \times^G V, \quad \rho(\phi)\,),$$

which in turn determines a spectral cover $\widetilde{S}_V \longrightarrow S$.

The question, raised first in [AvM] when $S = P^1$, is to relate the Picard varieties of the $\widetilde{S}_V$ as V varies, and in particular to find pieces common to all of them. For Adler and van Moerbeke, the motivation was that many evolution differential equations (of Lax type) can be *linearized* on the Jacobians of spectral curves. This means that the 'Liouville tori', which live naturally in the complexified domain of the differential equation (and hence are independent of the representation V) are mapped isogenously to their image in $\mathrm{Pic}(\widetilde{S}_V)$ for each nontrivial V; so one should be able to locate these tori among the pieces that occur in an isogeny decomposition of each of the $\mathrm{Pic}(\widetilde{S}_V)$. There are many specific examples where a pair of abelian varieties constructed from related covers of curves are known to be isomorphic or isogenous, and some of these lead to important identities among theta functions.

Example 3. Take $G = \mathrm{SL}(4)$. The standard representation V gives a branched cover $\widetilde{S}_V \longrightarrow S$ of degree 4. On the other hand, the 6-dimensional representation $\wedge^2 V$ (= the standard representation of the isogenous group $\mathrm{SO}(6)$) gives a cover $\widetilde{\widetilde{S}} \longrightarrow S$ of degree 6, which factors through an involution

$$\widetilde{\widetilde{S}} \longrightarrow \overline{S} \longrightarrow S.$$

One has the isogeny decompositions

$$\mathrm{Pic}\,(\widetilde{S}) \sim \mathrm{Prym}(\widetilde{S}/S) \oplus \mathrm{Pic}\,(S)$$

$$\mathrm{Pic}\,(\widetilde{\widetilde{S}}) \sim \mathrm{Prym}(\widetilde{\widetilde{S}}\,/\overline{S}) \oplus \mathrm{Prym}(\overline{S}/S) \oplus \mathrm{Pic}\,(S).$$

It turns out that

$$\mathrm{Prym}(\widetilde{S}/S) \sim \mathrm{Prym}(\widetilde{\widetilde{S}}\,/\overline{S}).$$

For $S = P^1$, this is Recillas' *trigonal construction* [R]. It says that every Jacobian of a trigonal curve is the Prym of a double cover of a tetragonal curve, and vice versa.

Example 4. Take $G = \mathrm{SO}(8)$ with its standard 8-dimensional representation V. The spectral cover has degree 8 and factors through an involution, $\widetilde{\widetilde{S}} \longrightarrow \overline{S} \longrightarrow S$. The two half-spin representations V_1, V_2 yield similar covers

$$\widetilde{\widetilde{S}}_1 \longrightarrow \overline{S}_1 \longrightarrow S, \qquad \widetilde{\widetilde{S}}_2 \longrightarrow \overline{S}_2 \longrightarrow S.$$

The *tetragonal construction* [D1] says that the three Pryms of the double covers are isomorphic. (These examples, as well as Pantazis' *bigonal construction* and constructions based on some exceptional groups, are discussed in the context of spectral covers in [K] and [D2].)

It turns out in general that there is indeed a distinguished, Prym-like isogeny component common to all the spectral Picards, on which the solutions to Lax-type differential equations evolve linearly. This was noticed in some cases already in [AvM], and was greatly extended by Kanev's construction of Prym–Tyurin varieties. (He still needs S to be P^1 and the spectral cover to have generic ramification; some of his results apply only to *minuscule representations*.) Various parts of the general story have been worked out recently by a number of authors, based on either of two approaches: one, pursued in [D2, Me, MS], is to decompose everything according to the action of the Weyl group W and to look for common pieces; the other, used in [BK, D3, F1, Sc], relies on the correspondence of spectral data and Higgs bundles. The group-theoretic approach is described in the rest of this section. We take up the second method, known as *abelianization*, in Section 5.

4.2. Decomposition of spectral covers. The decomposition of spectral Picards arises from three sources. First, the spectral cover for a sum of representations is the union of the individual covers $\widetilde{S}_V$. Next, the cover $\widetilde{S}_V$ for an irreducible representation is still the union of subcovers $\widetilde{S}_\lambda$ indexed by weight orbits. And finally, the Picard of $\widetilde{S}_\lambda$ decomposes into Pryms. We start with a few observations about the dependence of the covers themselves on the representation. The decomposition of the Picards is taken up in the next subsection.

Spectral covers. There is an *infinite* collection (of irreducible representations $V := V_\mu$, hence) of spectral covers $\widetilde{S}_V$, which can be parametrized by their highest weights μ in the dominant Weyl chamber $\overline{C}$, or equivalently by the W-orbit of extremal weights, in Λ/W. Here T is a maximal torus in G, $\Lambda := \mathrm{Hom}(T, \mathbf{C}^*)$ is the *weight lattice* (also called *character lattice*) for G, and W is the Weyl group. Each of these $\widetilde{S}_V$ decomposes as the union of its subcovers $\widetilde{S}_\lambda$, parametrizing eigenvalues in a given W-orbit $W\lambda$. (Here λ runs over the weight-orbits in V_μ.)

Parabolic covers. There is a *finite* collection of covers $\widetilde{S}_P$, parametrized by the conjugacy classes in G of parabolic subgroups (or equivalently by arbitrary-dimensional faces F_P of the chamber $\overline{C}$) such that (for general S) each eigenvalue cover $\widetilde{S}_\lambda$ is birational to some parabolic cover $\widetilde{S}_P$, the one whose open face F_P contains λ.

The cameral cover. There is a W-Galois cover $\widetilde{S} \longrightarrow S$ such that each $\widetilde{S}_P$ is isomorphic to $\widetilde{S}/W_P$, where W_P is the Weyl subgroup of W that stabilizes F_P. We call $\widetilde{S}$ the *cameral cover*, since, at least generically, it parametrizes the chambers determined by ϕ (in the duals of the Cartans), or equivalently the Borel subalgebras containing ϕ. This is constructed as follows: There is a morphism $\mathfrak{g} \longrightarrow \mathfrak{t}/W$ sending $g \in \mathfrak{g}$ to the conjugacy class of its semisimple part g_{ss}. (More precisely, this is Spec of the composed ring homomorphism $\mathbf{C}[\mathfrak{t}]^W \overset{\simeq}{\leftarrow} \mathbf{C}[\mathfrak{g}]^G \hookrightarrow \mathbf{C}[\mathfrak{g}]$.) Taking fiber product with the quotient map $\mathfrak{t} \longrightarrow \mathfrak{t}/W$, we get the cameral cover $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$. The cameral cover $\widetilde{S} \longrightarrow S$ of a K-valued principal Higgs bundle on S is glued from covers of open subsets in S (on which K and $\mathcal{G}$ are trivialized) that in turn are pullbacks by ϕ of $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$.

4.3. Decomposition of spectral Picards. The decomposition of the Picard varieties of spectral covers can be described as follows:

The cameral Picard. From each isomorphism class of irreducible W-representations, choose an integral representative Λ_i. (This can always be done for Weyl groups.) The group ring $Z[W]$ that acts on $\mathrm{Pic}(\widetilde{S})$ has an isogeny decomposition

$$Z[W] \sim \bigoplus_i \Lambda_i \otimes_Z \Lambda_i^*, \tag{10}$$

which is just the decomposition of the regular representation. There is a corresponding isotypic decomposition

$$\mathrm{Pic}(\widetilde{S}) \sim \bigoplus_i \Lambda_i \otimes_Z \mathrm{Prym}\, \Lambda_i(\widetilde{S}), \tag{11}$$

where

$$\mathrm{Prym}_{\Lambda_i}(\widetilde{S}) := \mathrm{Hom}_W(\Lambda_i, \mathrm{Pic}(\widetilde{S})). \tag{12}$$

Parabolic Picards. There are at least three reasonable ways of obtaining an isogeny decomposition of $\mathrm{Pic}(\widetilde{S}_P)$, for a parabolic subgroup $P \subset G$:

- The 'Hecke' ring Corr_P of correspondences on $\widetilde{S}_P$ over S acts on $\mathrm{Pic}(\widetilde{S}_P)$, so every irreducible integral representation M of Corr_P determines a generalized Prym
$$\mathrm{Hom}_{\mathrm{Corr}_P}(M, \mathrm{Pic}(\widetilde{S}_P)),$$
and we obtain an isotypic decomposition of $\mathrm{Pic}(\widetilde{S}_P)$ as before.
- $\mathrm{Pic}(\widetilde{S}_P)$ maps, with torsion kernel, to $\mathrm{Pic}(\widetilde{S})$, so we obtain a decomposition of the former by intersecting its image with the isotypic components $\Lambda_i \otimes_Z \mathrm{Prym}_{\Lambda_i}(\widetilde{S})$ of the latter.
- Since $\widetilde{S}_P$ is the cover of S *associated* to the W-cover $\widetilde{S}$ via the permutation representation $Z[W_P \backslash W]$ of W, we get an isogeny decomposition of $\mathrm{Pic}(\widetilde{S}_P)$ indexed by the irreducible representations in $Z[W_P \backslash W]$.

It turns out [D2, Section 6] that all three decompositions agree and can be given explicitly as

$$\bigoplus_i M_i \otimes \mathrm{Prym}_{\Lambda_i}(\widetilde{S}) \subset \bigoplus_i \Lambda_i \otimes \mathrm{Prym}_{\Lambda_i}(\widetilde{S}), \qquad M_i := (\Lambda_i)^{W_P}. \tag{13}$$

Spectral Picards. To obtain the decomposition of the Picards of the original covers $\widetilde{S}_V$ or $\widetilde{S}_\lambda$, we need, in addition to the decomposition of $\mathrm{Pic}(\widetilde{S}_P)$, some information on the singularities. These can arise from two separate sources:

Accidental singularities of the $\widetilde{S}_\lambda$. For a sufficiently general Higgs bundle, and for a weight λ in the interior of the face F_P of the Weyl chamber $\overline{C}$, the natural map

$$i_\lambda : \widetilde{S}_P \longrightarrow \widetilde{S}_\lambda$$

is birational. For the *standard* representations of the classical groups of types A_n, B_n or C_n, this *is* an isomorphism. But for general λ it is *not*: In order for i_λ to be an isomorphism, λ must be a multiple of a fundamental weight [D2, Lemma 4.2]. In fact, the list of fundamental weights for which this happens is quite short; for the classical groups we have only: ω_1 for A_n, B_n and C_n, ω_n (the dual representation) for A_n, and ω_2 for B_2. Note that for D_n the list is *empty*. In particular, the covers produced by the standard representation of $\mathrm{SO}(2n)$ are singular; this fact, noticed by Hitchin in [H1], explains the need for desingularization in his result (9).

Gluing the $\widetilde{S}_V$. In addition to the singularities of each i_λ, there are the singularities created by the gluing map $\amalg_\lambda \widetilde{S}_\lambda \longrightarrow \widetilde{S}_V$. This makes explicit formulas somewhat simpler in the case, studied by Kanev [K], of *minuscule* representations, i.e., representations whose weights form a single W-orbit. These singularities account, for instance, for the desingularization required in the $\mathrm{SO}(2n+1)$ case in (9).

4.4. The distinguished Prym. Combining much of the above, the Adler–van Moerbeke problem of finding a component common to the $\mathrm{Pic}(\widetilde{S}_V)$ for all non-trivial V translates into:

> *Find the irreducible representations Λ_i of W that occur in $Z[W_P\backslash W]$ with positive multiplicity for all proper Weyl subgroups $W_P\backslash W$.*

By Frobenius reciprocity, or (13), this is equivalent to:

> *Find the irreducible representations Λ_i of W such that for every proper Weyl subgroup $W_P \subsetneqq W$, the space of invariants $M_i := (\Lambda_i)^{W_P}$ is non-zero.*

One solution is now obvious: the *reflection representation* of W acting on the weight lattice Λ has this property. In fact, Λ^{W_P} in this case is just the face F_P of $\overline{C}$. The corresponding component $\mathrm{Prym}_\Lambda(\widetilde{S})$ is called *the distinguished Prym.* We will see in Section 5 that its points correspond, modulo some corrections, to Higgs bundles.

For the classical groups, this turns out to be the only common component. For G_2 and E_6 it turns out [D2, Section 6] that a second common component exists.

The geometric significance of points in these components is not known. As far as I know, the only component other than the distinguished Prym that has arisen 'in nature' is the one associated to the one-dimensional sign representation of W; see Section 7 and [KP2].

5. Abelianization

5.1. Abstract versus K-valued objects. We want to describe the abelianization procedure in a somewhat abstract setting, as an equivalence between *principal Higgs bundles* and certain *spectral data*. Once we fix a *values* vector bundle K, we obtain an equivalence between *K-valued principal Higgs bundles* and *K-valued spectral data*. Similarly, the choice of a representation V of G will determine an equivalence of *K-valued Higgs bundles* (of a given representation type) with K-valued spectral data.

As our model of a W-cover we take the natural quotient map

$$G/T \longrightarrow G/N$$

and its partial compactification

$$\overline{G/T} \longrightarrow \overline{G/N}. \tag{14}$$

Here $T \subset G$ is a maximal torus, and N is its normalizer in G. The quotient G/N parametrizes maximal tori (= Cartan subalgebras) $\mathfrak{t}$ in $\mathfrak{g}$, while G/T parametrizes pairs $\mathfrak{t} \subset \mathfrak{b}$ with $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalgebra. An element $x \in \mathfrak{g}$ is *regular* if the dimension of its centralizer $\mathfrak{c} \subset \mathfrak{g}$ equals $\dim T$ (= the rank of $\mathfrak{g}$). The partial compactifications $\overline{G/N}$ and $\overline{G/T}$ parametrize regular centralizers $\mathfrak{c}$ and pairs $\mathfrak{c} \subset \mathfrak{b}$, respectively.

In constructing the cameral cover in Section 4.2, we used the W-cover $\mathfrak{t} \longrightarrow \mathfrak{t}/W$ and its pullback cover $\widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$. Over the open subset $\mathfrak{g}_{\text{reg}}$ of regular elements, the same cover is obtained by pulling back (14) via the map $\alpha : \mathfrak{g}_{\text{reg}} \longrightarrow \overline{G/N}$ sending an element to its centralizer:

$$\begin{array}{ccccc} \mathfrak{t} & \longleftarrow & \widetilde{\mathfrak{g}}_{\text{reg}} & \longrightarrow & \overline{G/T} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{t}/W & \longleftarrow & \mathfrak{g}_{\text{reg}} & \xrightarrow{\alpha} & \overline{G/N} \end{array} . \tag{15}$$

When working with K-valued objects, it is usually more convenient to work with the left-hand side of (15), i.e., with *eigenvalues*. When working with the abstract objects, this is unavailable, so we are forced to work with the *eigenvectors*, or the right-hand side of (15). Thus:

Definition 5. *An abstract* cameral cover *of S is a finite morphism $\widetilde{S} \longrightarrow S$ with W-action, which locally (étale) in S is a pullback of* (14).

Definition 6. *A K-valued cameral cover (for K a vector bundle on S) consists of a cameral cover $\pi : \widetilde{S} \longrightarrow S$ together with an S-morphism*

$$\widetilde{S} \times \Lambda \longrightarrow \mathbb{K} \tag{16}$$

that is W-invariant (W acts on $\widetilde{S}$ and Λ, hence diagonally on $\widetilde{S} \times \Lambda$) and linear in Λ.

We note that a cameral cover $\widetilde{S}$ determines quotients $\widetilde{S}_P$ for parabolic subgroups $P \subset G$. A K-valued cameral cover determines additionally the $\widetilde{S}_\lambda$ for $\lambda \in \Lambda$, as images in $\mathbb{K}$ of $\widetilde{S} \times \{\lambda\}$. The data of (16) is equivalent to a W-equivariant map $\widetilde{S} \longrightarrow \mathfrak{t} \otimes_{\mathbf{C}} K$.

Definition 7. *A G-principal Higgs bundle on S is a pair $(\mathcal{G}, \mathfrak{c})$ with $\mathcal{G}$ a principal G-bundle and $\mathfrak{c} \subset \mathrm{ad}(\mathcal{G})$ a subbundle of regular centralizers.*

Definition 8. *A K-valued G-principal Higgs bundle consists of $(\mathcal{G}, \mathfrak{c})$ as above, together with a section φ of $\mathfrak{c} \otimes K$.*

A principal Higgs bundle $(\mathcal{G}, \mathfrak{c})$ determines a cameral cover $\widetilde{S} \longrightarrow S$ and a homomorphism $\Lambda \longrightarrow \mathrm{Pic}(\widetilde{S})$. Let F be a parameter space for Higgs bundles with a given $\widetilde{S}$. Each non-zero $\lambda \in \Lambda$ gives a non-trivial map $F \longrightarrow \mathrm{Pic}(\widetilde{S})$. For λ in a face F_P of $\overline{C}$, this factors through $\mathrm{Pic}(\widetilde{S}_P)$. The discussion in Section 4.4 now suggests that F should be given roughly by the distinguished Prym,

$$\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S})).$$

It turns out that this guess needs two corrections. The first correction involves restricting to a coset of a subgroup; the need for this is visible even in the simplest case where $\widetilde{S}$ is étale over S, so $(\mathcal{G}, \mathfrak{c})$ is everywhere regular and semisimple (i.e., $\mathfrak{c}$ is a bundle of Cartans.) The second correction involves a twist along the ramification of $\widetilde{S}$ over S. We explain these corrections in the next two subsections.

5.2. The regular semisimple case: the shift.

Example 9. Fix a smooth projective curve C and a line bundle $K \in \mathrm{Pic}(C)$ such that $K^{\otimes 2} \approx \mathcal{O}_C$. This determines an étale double cover $\pi : \widetilde{C} \longrightarrow C$ with involution i, and homomorphisms

$$\begin{array}{rcccc} \pi^* & : & \mathrm{Pic}(C) & \longrightarrow & \mathrm{Pic}(\widetilde{C}), \\ \mathrm{Nm} & : & \mathrm{Pic}(\widetilde{C}) & \longrightarrow & \mathrm{Pic}(C), \\ i^* & : & \mathrm{Pic}(\widetilde{C}) & \longrightarrow & \mathrm{Pic}(\widetilde{C}), \end{array}$$

satisfying

$$1 + i^* = \pi^* \circ \mathrm{Nm}.$$

For $G = \mathrm{GL}(2)$ we have $\Lambda = Z \oplus Z$, and $W = \mathcal{S}_2$ permutes the summands, so

$$\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{C})) \approx \mathrm{Pic}(\widetilde{C}).$$

And, indeed, the Higgs bundles corresponding to $\widetilde{C}$ are parametrized by $\mathrm{Pic}(\widetilde{C})$: send $L \in \mathrm{Pic}(\widetilde{C})$ to $(\mathcal{G}, \mathfrak{c})$, where $\mathcal{G}$ has associated rank-two vector bundle $\mathcal{V} := \pi_* L$, and $\mathfrak{c} \subset \mathrm{End}(\mathcal{V})$ is $\pi_* \mathcal{O}_{\widetilde{C}}$.

On the other hand, for $G = \mathrm{SL}(2)$ we have $\Lambda = Z$ and $W = \mathcal{S}_2$ acts by ± 1, so

$$\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S})) \approx \{L \in \mathrm{Pic}(\widetilde{C}) \mid i^* L \approx L^{-1}\} = \ker(1 + i^*).$$

This group has four connected components. The subgroup ker(Nm) consists of two of these. The connected component of 0 is the classical Prym variety [Mu2]. Now the Higgs bundles correspond, via the above bijection $L \mapsto \pi_* L$, to

$$\{L \in \mathrm{Pic}(\widetilde{C}) \mid \det(\pi_* L) \approx \mathcal{O}_C\} = \mathrm{Nm}^{-1}(K).$$

Thus they form the *non-zero* coset of the subgroup ker(Nm). (If we return to a higher-dimensional S, it is possible for K not to be in the image of Nm, so there may be *no* SL(2)-Higgs bundles corresponding to such a cover.)

This example generalizes to all G, as follows. The equivalence classes of extensions

$$1 \longrightarrow T \longrightarrow N' \longrightarrow W \longrightarrow 1$$

(in which the action of W on T is the standard one) are parametrized by the group cohomology $H^2(W, T)$. Here the 0 element corresponds to the semidirect product. The class $[N] \in H^2(W, T)$ of the normalizer N of T in G may be 0, as it is for $G = \mathrm{GL}(n)$, $\mathrm{PGL}(n)$, and $\mathrm{SL}(2n+1)$; or it may not, as for $G = \mathrm{SL}(2n)$.

Assume first, for simplicity, that S and $\widetilde{S}$ are connected and projective. There is then a natural group homomorphism

$$c : \mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S})) \longrightarrow H^2(W, T). \tag{17}$$

Algebraically, this is an edge homomorphism for the Grothendieck spectral sequence of equivariant cohomology, which gives the exact sequence

$$0 \longrightarrow H^1(W, T) \longrightarrow H^1(S, \mathcal{C}) \longrightarrow \mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S})) \xrightarrow{c} H^2(W, T), \tag{18}$$

where $\mathcal{C} := \widetilde{S} \times_W T$. Geometrically, this expresses a *Mumford group* construction: giving $\mathcal{L} \in \mathrm{Hom}(\Lambda, \mathrm{Pic}(\widetilde{S}))$ is equivalent to giving a principal T-bundle $\mathcal{T}$ over $\widetilde{S}$; for $\mathcal{L} \in \mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S}))$, $c(\mathcal{L})$ is the class in $H^2(W, T)$ of the group N' of automorphisms of $\mathcal{T}$ that commute with the action on $\widetilde{S}$ of some $w \in W$.

To remove the restriction on S and $\widetilde{S}$, we need to replace each occurrence of T in (17, 18) by $\Gamma(\widetilde{S}, T)$, the global sections of the trivial bundle on $\widetilde{S}$ with fiber T. The natural map $H^2(W, T) \longrightarrow H^2(W, \Gamma(\widetilde{S}, T))$ allows us to think of $[N]$ as an element of $H^2(W, \Gamma(\widetilde{S}, T))$.

Proposition 10 ([D3]). *Fix an étale W-cover $\pi : \widetilde{S} \longrightarrow S$. The following data are equivalent*:

(1) *Principal G-Higgs bundles $(\mathcal{G}, \mathfrak{c})$ with cameral cover $\widetilde{S}$.*
(2) *Principal N-bundles $\mathcal{N}$ over S whose quotient by T is $\widetilde{S}$.*
(3) *W-equivariant homomorphisms $\mathcal{L} : \Lambda \longrightarrow \mathrm{Pic}(\widetilde{S})$ with $c(\mathcal{L}) = [N] \in H^2(W, \Gamma(\widetilde{S}, T))$.*

We observe that while the shifted objects correspond to Higgs bundles, the unshifted objects

$$\mathcal{L} \in \mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S})), \qquad c(\mathcal{L}) = 0$$

come from the $\mathcal{C}$-torsors in $H^1(S, \mathcal{C})$.

5.3. The regular case: the twist along the ramification.

Example 11. Modify Example 9 by letting $K \in \mathrm{Pic}(C)$ be arbitrary, and choose a section b of $K^{\otimes 2}$ that vanishes on a simple divisor $B \subset C$. We get a double cover $\pi : \widetilde{C} \longrightarrow C$ branched along B, ramified along a divisor

$$R \subset \widetilde{C}, \quad \pi(R) = B.$$

Via $L \mapsto \pi_* L$, the Higgs bundles still correspond to

$$\{L \in \mathrm{Pic}(\widetilde{C}) \mid \det(\pi_* L) \approx \mathcal{O}_C\} = \mathrm{Nm}^{-1}(K).$$

But this is no longer in $\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S}))$; rather, the line bundles in question satisfy

$$i^* L \approx L^{-1}(R). \tag{19}$$

For arbitrary G, let Φ denote the root system and Φ^+ the set of positive roots. There is a decomposition

$$\overline{G/T} \smallsetminus G/T = \bigcup_{\alpha \in \Phi^+} R_\alpha$$

of the boundary into components, with R_α the fixed locus of the reflection σ_α in α. (Via (15), these correspond to the complexified walls in $\mathfrak{t}$.) Thus each cameral cover $\widetilde{S} \longrightarrow S$ comes with a natural set of (Cartier) *ramification divisors*, which we still denote R_α, for $\alpha \in \Phi^+$.

For $w \in W$, set

$$F_w := \left\{ \alpha \in \Phi^+ \mid w^{-1}\alpha \in \Phi^- \right\} = \Phi^+ \cap w\Phi^-,$$

and choose a W-invariant form $\langle , \rangle$ on Λ. We consider the variety

$$\mathrm{Hom}_{W,R}(\Lambda, \mathrm{Pic}(\widetilde{S}))$$

of R-twisted W-equivariant homomorphisms, i.e., homomorphisms $\mathcal{L}$ satisfying

$$w^*\mathcal{L}(\lambda) \approx \mathcal{L}(w\lambda)\left(\sum_{\alpha \in F_w} \frac{\langle -2\alpha, w\lambda\rangle}{\langle \alpha, \alpha\rangle} R_\alpha\right), \qquad \lambda \in \Lambda, \quad w \in W. \tag{20}$$

This turns out to be the correct analogue of (19). (For example if $w = \sigma_\alpha$, is a reflection, F_w is $\{\alpha\}$, so this gives

$$w^*\mathcal{L}(\lambda) \approx \mathcal{L}(w\lambda)\left(\frac{\langle \alpha, 2\lambda\rangle}{\langle \alpha, \alpha\rangle} R_\alpha\right),$$

which specializes to (19).) As before, there is a class map

$$c : \mathrm{Hom}_{W,R}(\Lambda, \mathrm{Pic}(\widetilde{S})) \longrightarrow H^2(W,\ \Gamma(\widetilde{S}, T)) \tag{21}$$

that can be described via a Mumford-group construction.

To understand this twist, consider the formal object

$$\begin{aligned} \tfrac{1}{2}\,\mathrm{Ram} : \quad \Lambda \quad &\longrightarrow \quad Q \otimes \mathrm{Pic}\,\widetilde{S}, \\ \lambda \quad &\longmapsto \sum_{(\alpha \in \Phi^+)} \frac{\langle \alpha, \lambda\rangle}{\langle \alpha, \alpha\rangle} R_\alpha. \end{aligned}$$

In an obvious sense, a principal T-bundle $\mathcal{T}$ on $\widetilde{S}$ (or a homomorphism $\mathcal{L} : \Lambda \longrightarrow \mathrm{Pic}(\widetilde{S})$) is R-twisted W-equivariant if and only if $\mathcal{T}(-\frac{1}{2}\,\mathrm{Ram})$ is W-equivariant, i.e., if $\mathcal{T}$ and $\frac{1}{2}\,\mathrm{Ram}$ transform the same way under W. The problem with this is that $\frac{1}{2}\,\mathrm{Ram}$ itself does not make sense as a T-bundle, because the coefficients $\langle \alpha, \lambda\rangle / \langle \alpha, \alpha\rangle$ are not integers. (This argument shows that if $\mathrm{Hom}_{W,R}(\Lambda, \mathrm{Pic}(\widetilde{S}))$ is non-empty, it is a torsor over the untwisted $\mathrm{Hom}_W(\Lambda, \mathrm{Pic}(\widetilde{S}))$.)

Theorem 12 ([D3]). *For a cameral cover $\widetilde{S} \longrightarrow S$, the following data are equivalent*:

(1) *G-principal Higgs bundles with cameral cover $\widetilde{S}$.*
(2) *R-twisted W-equivariant homomorphisms $\mathcal{L} \in c^{-1}([N])$.*

The theorem has an essentially local nature, as there is no requirement that S be, say, projective. We also do not need the condition of generic behavior near the ramification, which appears in [F1, Me, Sc]. Thus we may consider an extreme case where $\widetilde{S}$ is 'everywhere ramified':

EXAMPLE 13. In Example 11, take the section $b = 0$. The resulting cover $\widetilde{C}$ is a 'ribbon', or length-2 non-reduced structure on C: it is the length-2 neighborhood of C in $\mathbb{K}$. The SL(2)-Higgs bundles $(\mathcal{G}, \mathfrak{c})$ for this $\widetilde{C}$ have an everywhere nilpotent $\mathfrak{c}$, so the vector bundle $\mathcal{V} := \mathcal{G} \times^{\mathrm{SL}(2)} V \approx \pi_* L$ (where V is the standard two-dimensional representation) fits in an exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{V} \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{S} \otimes K \approx \mathcal{Q}$. Such data are specified by the line bundle $\mathcal{Q}$, satisfying $\mathcal{Q}^{\otimes 2} \approx K$, and an extension class in $\mathrm{Ext}^1(\mathcal{Q}, \mathcal{S}) \approx H^1(K^{-1})$. The kernel of the restriction map $\mathrm{Pic}(\widetilde{C}) \longrightarrow \mathrm{Pic}(C)$ is also given by $H^1(K^{-1})$ (use the exact sequence $0 \longrightarrow K^{-1} \longrightarrow \pi_* \mathcal{O}^{\times}_{\widetilde{C}} \longrightarrow \mathcal{O}^{\times}_{C} \longrightarrow 0$), and the R-twist produces the required square roots of K. (For more details on the nilpotent locus, see [L] and [DEL].)

5.4. Adding values and representations. Fix a vector bundle K, and consider the moduli space $\mathcal{M}_{S,G,K}$ of K-valued G-principal Higgs bundles on S. (It can be constructed as in Simpson's [S], even though the objects we need to parametrize are slightly different from his. In this subsection we outline a direct construction.) It comes with a Hitchin map

$$h : \mathcal{M}_{S,G,K} \longrightarrow B_K \tag{22}$$

where $B := B_K$ parametrizes all possible Hitchin data. Theorem 12 gives a precise description of the fibers of this map, independent of the values bundle K. This leaves us with the relatively minor task of describing, for each K, the corresponding base, that is, the closed subvariety B_s of B parametrizing *split* Hitchin data, or K-valued cameral covers. The point is that Higgs bundles satisfy a symmetry condition, which in Simpson's setup is

$$\varphi \wedge \varphi = 0,$$

and is built into our Definition 7 through the assumption that $\mathfrak{c}$ is regular, hence abelian. Since commuting operators have common eigenvectors, this gives a splitness condition on the Hitchin data, which we describe below. (When K is a line bundle, the condition is vacuous, $B_s = B$.) The upshot is:

LEMMA 14. *The following data are equivalent:*
(a) *A K-valued cameral cover of S.*
(b) *A split, graded homomorphism $R^{\bullet} \longrightarrow Sym^{\bullet} K$.*
(c) *A split Hitchin datum $b \in B_s$.*

Here $R^{\bullet}$ is the graded ring of W-invariant polynomials on $\mathfrak{t}$:

$$R^\bullet := (\mathrm{Sym}^\bullet \mathfrak{t}^*)^W \approx \mathbf{C}[\sigma_1, \dots, \sigma_l], \qquad \deg(\sigma_i) = d_i \tag{23}$$

where $l := \mathrm{Rank}(\mathfrak{g})$ and the σ_i form a basis for the W-invariant polynomials. The Hitchin base is the vector space

$$B := B_K := \oplus_{i=1}^{l} H^0(S, Sym^{d_i} K) \approx \mathrm{Hom}(R^\bullet, \mathrm{Sym}^\bullet K).$$

For each $\lambda \in \Lambda$ (or $\lambda \in \mathfrak{t}^*$, for that matter), the expression

$$q_\lambda(x,t) := \prod_{w \in W} (x - w\lambda(t)), \qquad t \in \mathfrak{t}, \tag{24}$$

in an indeterminate x is W-invariant (as a function of t), so it defines an element $q_\lambda(x) \in R^\bullet[x]$. A Hitchin datum $b \in B \approx \mathrm{Hom}(R^\bullet, \mathrm{Sym}^\bullet K)$ sends this to

$$q_{\lambda,b}(x) \in \mathrm{Sym}^\bullet(K)[x].$$

We say that b is *split* if, at each point of S and for each λ, the polynomial $q_{\lambda,b}(x)$ factors completely, into terms linear in x.

We note that, for λ in the interior of C (the positive Weyl chamber), $q_{\lambda,b}$ gives the equation in $\mathbb{K}$ of the spectral cover $\widetilde{S}_\lambda$ of Section (4.2): $q_{\lambda,b}$ gives a morphism $\mathbb{K} \longrightarrow \mathrm{Sym}^N \mathbb{K}$, where $N := \#W$, and $\widetilde{S}_\lambda$ is the inverse image of the zero-section. (When λ is in a face F_P of $\overline{C}$, we define analogous polynomials $q^P_\lambda(x,t)$ and $q^P_{\lambda,b}(x)$ by taking the product in (24) to be over $w \in W_P \backslash W$. These give the reduced equations in this case, and q_λ is an appropriate power.)

Over B_s there is a universal K-valued cameral cover

$$\widetilde{\mathcal{S}} \longrightarrow B_s$$

with ramification divisor $R \subset \widetilde{\mathcal{S}}$. From the relative Picard, $\mathrm{Pic}(\widetilde{\mathcal{S}}/B_s)$, we concoct the relative N-shifted, R-twisted Prym

$$\mathrm{Prym}_{\Lambda,R}(\widetilde{\mathcal{S}}/B_s).$$

By Theorem 12, this can then be considered as a parameter space $\mathcal{M}_{S,G,K}$ for all K-valued G-principal Higgs bundles on S. (Recall that our objects are assumed to be everywhere *regular*!) It comes with a 'Hitchin map', namely the projection to B_s, and the fibers corresponding to smooth projective $\widetilde{S}$ are abelian varieties. When S is a smooth, projective curve, we recover this way the algebraic complete integrability of Hitchin's system and its generalizations.

6. Symplectic and Poisson structures

The total space of Hitchin's original system is a cotangent bundle, hence has a natural symplectic structure. For the polynomial matrix systems of [B] and [AHH] there is a natural Poisson structure, which one writes down explicitly.

In [Bn] and [M1], this result is extended to the systems $\mathcal{M}_{C,K}$ of K-valued GL(n) Higgs bundles on C, when $K \approx \omega_C(D)$ for an effective divisor D on C. There is a general-nonsense pairing on the cotangent spaces, so the point is to check that this pairing is 'closed', that is, satisfies the identity required for a Poisson structure. Bottacin does this by an explicit computation along the lines of [B]. Markman's idea is to consider the moduli space $\mathcal{M}_D$ of stable vector bundles on C with level-D structure. He realizes an open subset $\mathcal{M}^0_{C,K}$ of $\mathcal{M}_{C,K}$, parametrizing Higgs bundles whose covers are nice, as a quotient (by an action of the level group) of $T^*\mathcal{M}_D$, so the natural symplectic form on $T^*\mathcal{M}_D$ descends to a Poisson structure on $\mathcal{M}^0_{C,K}$. This is identified with the general-nonsense form (wherever both exist), proving its closedness.

In [Muk], Mukai constructs a symplectic structure on the moduli space of simple sheaves on a K3 surface S. Given a curve $C \subset S$, one can consider the moduli of sheaves having the numerical invariants of a line bundle on a curve in the linear system $|nC|$ on S. This has a support map to the projective space $|nC|$, which turns it into an ACIHS. This system specializes, by a 'degeneration to the normal cone' argument [DEL] to Hitchin's, allowing translation of various results about Hitchin's system (such as Laumon's description of the nilpotent cone [L]) to Mukai's.

In higher dimensions, the moduli space $\mathcal{M}$ of Ω^1-valued Higgs bundles carries a natural symplectic structure [S]. (Corlette points out in [C] that certain components of an open subet in $\mathcal{M}$ can be described as cotangent bundles.) It is not clear at the moment exactly when one should expect to have an ACIHS, with symplectic, Poisson or quasi-symplectic structure, on the moduli spaces of K-valued Higgs bundles for higher-dimensional S, arbitrary G, and arbitrary vector bundle K. A beautiful new idea [M2] is that Mukai's results extend to the moduli of those sheaves on a (symplectic, Poisson or quasi-symplectic) variety X whose support in X is *Lagrangian*. Again, there is a general-nonsense pairing. At points where the support is non-singular projective, this can be identified with another, more geometric pairing, constructed using the *cubic condition* of [DM1], which is known to satisfy the closedness requirement. This approach is quite powerful, as it includes many non-linear examples such as Mukai's, in addition to the line-bundle valued spectral systems of [Bn, M1] and also Simpson's Ω^1-valued GL(n)-Higgs bundles: just take $X := T^*S \xrightarrow{\pi} S$, with its natural symplectic form, and the support in X to be proper over S of degree n; such sheaves correspond to Higgs bundles by π_*.

The structure group $\mathrm{GL}(n)$ can of course be replaced by an arbitrary reductive group G. Using Theorem 12, this yields (in the analogous cases) a Poisson structure on the Higgs moduli space $\mathcal{M}_{S,G,K}$ described at the end of the previous section. The fibers of the generalized Hitchin map are Lagrangian with respect to this structure. Along the lines of our general approach, the necessary modifications are clear: π_* is replaced by the equivalence of Theorem 12. One thus considers only Lagrangian supports that retain a W-action, and only *equivariant* sheaves on them (with the numerical invariants of a line bundle). These two restrictions are symplectically dual, so the moduli space of Lagrangian sheaves with these invariance properties is a symplectic (respectively, Poisson) subspace of the total moduli space, and the fibers of the Hitchin map are Lagrangian as expected.

A more detailed review of the ACIHS aspects of Higgs bundles will appear in [DM2].

7. Some applications and problems

Some applications. In [H1], Hitchin used his integrable system to compute several cohomology groups of the moduli space $\mathcal{SM}$ (of rank-two, fixed odd determinant vector bundles on a curve C) with coefficients in symmetric powers of its tangent sheaf $\mathcal{T}$. The point is that the symmetric algebra $\mathrm{Sym}^{\bullet}\mathcal{T}$ is the direct image of $\mathcal{O}_{T^*\mathcal{SM}}$, and sections of the latter all pull back via the Hitchin map h from functions on the base B, since the fibers of h are open subsets in abelian varieties, and the missing locus has codimension ≥ 2. Hitchin's system is used in [BNR] to compute a couple of 'Verlinde numbers' for $\mathrm{GL}(n)$, namely the dimensions

$$h^0(\mathcal{M}, \Theta) = 1, \qquad h^0(\mathcal{SM}, \Theta) = n^g.$$

These results are now subsumed in the general Verlinde formulas; see [F2], [BL], and other references therein.

A pretty application of spectral covers was obtained by Katzarkov and Pantev [KP2]. Let S be a smooth, projective, complex variety, and $\rho : \pi_1(S) \longrightarrow G$ a Zariski dense representation into a simple G (over C). Assume that the Ω^1-valued Higgs bundle $(\mathcal{V}, \phi)$ associated to ρ by Simpson is (regular and) generically semisimple, so the cameral cover is reduced. Among other things, they show that ρ factors through a representation of an orbicurve if and only if the non-standard component $\mathrm{Prym}_\epsilon(\widetilde{S})$ is non-zero, where ϵ is the one-dimensional sign representation of W. (In a sense, this is the opposite of $\mathrm{Prym}_\Lambda(\widetilde{S})$: while $\mathrm{Prym}_\Lambda(\widetilde{S})$ is common to $\mathrm{Pic}(\widetilde{S}_P)$ for all proper Weyl subgroups, $\mathrm{Prym}_\epsilon(\widetilde{S})$ occurs in none except for the full cameral Picard.)

Another application is in [KoP]: the moduli spaces of SL(n)- or GL(n)-stable bundles on a curve have certain obvious automorphisms, coming from tensoring with line bundles on the curve, from inversion, or from automorphisms of the curve. Kouvidakis and Pantev use the dominant direct-image maps from spectral Picards and Pryms to the moduli spaces to show that there are no further, unexpected automorphisms. This then leads to a 'non-abelian Torelli theorem', stating that a curve is determined by the isomorphism class of the moduli space of bundles on it.

Compatibility? Hitchin's construction [H2] of the projectively flat connection on the vector bundle of non-abelian theta functions over the moduli space of curves does not really use much about spectral covers. Nor do other constructions of Faltings [F1] and Witten et al. [APW]. Hitchin's work suggests that the 'right' approach should be based on comparison of the non-abelian connection near a curve C with the abelian connection for standard theta functions on spectral covers $\widetilde{C}$ of C. One conjecture concerning the possible relationship between these connections appears in [A], and some related versions have been attempted by several people, so far in vain. What's missing is a compatibility statement between the actions of the two connections on pulled-back sections. If the expected compatibility turns out to hold, it would give another proof of the projective flatness. It should also imply projective finiteness and projective unitarity of monodromy for the non-abelian thetas, and may or may not bring us closer to a 'finite-dimensional' proof of Faltings' theorem (= the former Verlinde conjecture).

Irregulars? The Higgs bundles we consider in this survey are assumed to be everywhere regular. This is a reasonable assumption for line-bundle valued Higgs bundles on a curve or surface, but *not* in dim ≥ 3. This is because the complement of $\mathfrak{g}_{\text{reg}}$ has codimension 3 in $\mathfrak{g}$. The source of the difficulty is that the analogue of (15) fails over $\mathfrak{g}$. There are two candidates for the universal cameral cover: $\widetilde{\mathfrak{g}}$, defined by the left-hand side of (15), is finite over $\mathfrak{g}$ with W action, but does not have a family of line bundles parametrized by Λ. These live on $\widetilde{\widetilde{\mathfrak{g}}}$, the object defined by the right-hand side, which parametrizes pairs

$$(x, \mathfrak{b}), \qquad x \in \mathfrak{b} \subset \mathfrak{g}.$$

This suggests that the right way to analyze irregular Higgs bundles may involve spectral data consisting of a tower

$$\widetilde{\widetilde{S}} \xrightarrow{\sigma} \widetilde{S} \longrightarrow S$$

together with a homomorphism $\mathcal{L} : \Lambda \longrightarrow \operatorname{Pic}(\widetilde{\widetilde{S}})$ such that the collection of sheaves

$$\sigma_*(\mathcal{L}(\lambda)), \qquad \lambda \in \Lambda,$$

on $\widetilde{S}$ is R-twisted W-equivariant in an appropriate sense. As a first step, one may wish to understand the direct images $R^i\sigma_*(\mathcal{L}(\lambda))$ and in particular the cohomologies $H^i(F, \mathcal{L}(\lambda))$, where F, usually called a *Springer fiber*, is a fiber of σ. For regular x, this fiber is a single point. For $x = 0$, the fiber is all of G/B, so the fiber cohomology is given by the Borel–Weil–Bott theorem. The question may thus be considered as a desired extension of BWB to general Springer fibers.

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