Simons Laufer Mathematical Sciences Institute

Hot Topics: Life after the Telescope Conjecture

December 9 - 13, 2024

Monday, December 9			
9:15 AM - 9:30 AM	Eisenbud Auditorium		Welcome
9:30 AM - 10:30 AM	Eisenbud Auditorium	Doug Ravenel	History of the Telescope Conjecture
10:30 AM - 11:00 AM	Atrium		Break
11:00 AM - 12:00 PM	Eisenbud Auditorium	Tomer Schlank	Overview
12:00 PM - 2:00 PM	Atrium		Lunch
2:00 PM - 3:00 PM	Eisenbud Auditorium	Maxine Calle	Cyclotomic spectra
3:00 PM - 3:30 PM	Atrium		Tea
3:30 PM - 4:30 PM	Eisenbud Auditorium	Markus Land	Chromatically localized algebraic K-theory

Tuesday, December 10			
9:30 AM - 10:30 AM	Eisenbud Auditorium	Hana Jia Kong	Cyclotomic boundedness I
10:30 AM - 11:00 AM	Atrium		Break
11:00 AM - 12:00 PM	Eisenbud Auditorium	Christian Ausoni	Boundedness of the Adams summand
12:00 PM - 2:00 PM	Atrium		Lunch
2:00 PM - 3:00 PM	Eisenbud Auditorium	Kate Ponto	Trace methods beyond connective rings
3:00 PM - 3:30 PM	Atrium		Tea
3:30 PM - 4:30 PM	Eisenbud Auditorium	Cary Malkiewich	Cochains on a circle
4:30 PM - 6:20 PM	Atrium		Reception

Wednesday, December 11			
9:30 AM - 10:30 AM	Eisenbud Auditorium	Allen Yuan	Ambidexterity and cyclotomic extensions
10:30 AM - 11:00 AM	Atrium		Break
11:00 AM - 12:00 PM	Eisenbud Auditorium	Shay Ben Moshe	Cyclotomic redshift

Thursday, December 12			
9:30 AM - 10:30 AM	Eisenbud Auditorium	Irina Bobkova	Calculating TC for homotopy fixed points of the Adams summand
10:30 AM - 10:40 AM	Front Courtyard		Group Photo
10:40 AM - 11:00 AM	Atrium		Break
11:00 AM - 12:00 PM	Eisenbud Auditorium	Eva Belmont	Asymptotic constancy I
12:00 PM - 2:00 PM	Atrium		Lunch
2:00 PM - 3:00 PM	Eisenbud Auditorium	Ishan Levy	Asymptotic constancy II
3:00 PM - 3:30 PM	Atrium		Tea
3:30 PM - 4:30 PM	Eisenbud Auditorium	Burklund, Hahn, Levy, Schlank	Q&A

Friday, December 13			
9:30 AM - 10:30 AM	Eisenbud Auditorium	Andrew Senger	Examples at low primes and large heights
10:30AM - 11:00AM	Atrium		Tea
11:00 AM - 12:00 PM	Eisenbud Auditorium	Jeremy Hahn	Assembling the disproof
12:00 PM - 2:00 PM	Atrium		Lunch
2:00 PM - 3:00 PM	Eisenbud Auditorium	Lior Yanovski	Future directions I
3:00 PM - 3:30 PM	Atrium		Tea
3:30 PM - 4:30 PM	Eisenbud Auditorium	Robert Burklund	Future directions II

Life after the Telescope Conjecture
Hot Topics Workshop
SLMath/MSRI, Berkeley, CA, December 9-13, 2024

List of talks, abstracts and references

1. **Talk 1 Monday: History of the Telescope Conjecture.**
   - Speaker: Douglas Ravenel
   - Abstract: In this talk, the speaker will explain the origins of, and motivation for, the Telescope Conjecture.
   - Suggested references: [Rav84, Bar20, Rav87, Mah82, Mah81, MRS01, Mil81, BBB$^+$21]

2. **Talk 2 Monday: Overview.**
   - Speaker: Tomer Schlank
   - Abstract: An overview of the week and of the disproof, outlining all the pieces we will need to develop and how they fit together. Alternatively, the speaker may discuss one of several height 1 proofs.
   - Suggested references: [BHLS23] or [Mil81, Mah82]

3. **Talk 3 Monday: Cyclotomic spectra.**
   - Speaker: Maxine Calle
   - Abstract: The fundamental work of Dundas–Goodwillie–McCarthy relates the algebraic K-theory of a connective ring spectrum R to its topological cyclic homology $\mathrm{TC}(R)$, which is in turn a functor of the cyclotomic spectrum $\mathrm{THH}(R)$.

     In this talk, the speaker will introduce a modern definition of the category of cyclotomic spectra, due to Nikolaus and Scholze. A basic example is given by $\mathrm{THH}(R)$ when R is a ring spectrum. The speaker will introduce several invariants of cyclotomic spectra, namely TP, TC^-, TC, and TR. They will also briefly mention THH with coefficients in a bimodule, and the related formalism of p-polygonic spectra. Finally, they will give the statement of the Dundas–Goodwillie–McCarthy theorem.
   - Suggested references: [NS18], [DGM13], [KMN23]

4. **Talk 4 Monday: Chromatically localized algebraic K-theory.**
   - Speaker: Markus Land
   - Abstract: In several ways, the algebraic K-theory of a height n ring simplifies after localization at a telescope $T(n+1)$. For us, the most fundamental will be Land–Mathew–Meier–Tamme purity, which is intimately tied to Clausen–Mathew–Naumann–Noel descent. The speaker will explain the purity theorem from [LMMT24], which states that

     $$L_{T(n+1)}\mathrm{K}(R) \simeq L_{T(n+1)}\mathrm{K}(L_{T(n)\oplus T(n+1)}R),$$

     and may sketch a few ingredients of the proof. The most basic example of purity is Mitchell's theorem, which states that $L_{T(n+1)}\mathrm{K}(R) = 0$ whenever R is a discrete ring and $n \geq 1$; the speaker will note how this simplifies the use of the Dundas–Goodwillie–McCarthy theorem.
   - Suggested references: [LMMT24], [CMNN20]

5. **Talk 1 Tuesday : Cyclotomic boundedness.**
   - Speaker: Hana Jia Kong
   - Abstract: Hesselholt–Madsen proved the Lichtenbaum–Quillen conjecture for p-adic local fields K (with $p > 2$) by proving the stronger statement that $V(1)_*\mathrm{TR}(\mathcal{O}_K)$ is bounded.

     Bounded TR is now best interpreted as boundedness in the Antieau–Nikolaus t-structure. The speaker will introduce this t-structure, and note that $\mathrm{THH}(\mathbb{F}_p)$ is bounded. The main aim of the talk should be to characterize cyclotomic boundedness in more concrete terms, as the combination

of the Segal conjecture and canonical vanishing. Bounded cyclotomic rings admit a Bökstedt class, and the speaker will discuss its basic properties.
- Suggested references: Sections 2.2, 2.3 and 2.4 of [AN21]

6. **Talk 2 Tuesday: Boundedness of the Adams summand.**
   - Speaker: Christian Ausoni
   - Abstract: This talk will explain the theorem, by Ausoni–Rognes, that $V(2)_*\mathrm{TR}(\ell)$ is bounded for primes $p > 3$.[1] Here, $\ell = \mathrm{BP}\langle 1\rangle$ is the Adams summand of p-local complex K-theory. This was the first example of a higher height Lichtenbaum–Quillen theorem, and the speaker will also explain how to deduce chromatic redshift.

     The computations going into the Ausoni–Rognes result will be explained explicitly enough that they may be adapted to compute $V(2)_*\,\mathrm{TC}(\ell^{hp^k\mathbb{Z}})$ on Thursday (speakers may want to coordinate these two talks).
   - Suggested references: [AR02], [HRW22] and [BHLS23, §6 & 7]

7. **Talk 3 Tuesday: Trace methods beyond connective rings.**
   - Speaker: Kate Ponto
   - Abstract: This talk will explain how many of the results comparing K-theory and TC can be extended beyond the setting of connective rings, and in particular to -1-connective rings, because the universal localizing invariant of -1-connective rings are built out of those of connective rings. In the case of rings that are fixed points of $\mathbb{Z}$-actions on connective rings, this talk will explain how this can be accomplished using the work of Land–Tamme on the K-theory of pullbacks.
   - Notes for the speaker:
     - The most important thing to cover is [Lev22, Theorem B], and the corresponding parts of Land–Tamme that go into the proof of it. This will be applied in the example where E is the fiber of the map from K to TC, and $R \to S$ is the map $\ell \to \pi_0\ell$ (where $\pi_0\ell$ denotes the Eilenberg–MacLane spectrum $\mathbb{Z}_{(p)}$). The $\mathbb{Z}$-action is by Adams operations on ℓ, and is trivial on $\pi_0\ell$.
     - More generally, there is a version that works for arbitrary -1-connective rings [Lev22, Theorem C], which you might want to discuss depending on available time. It might be worth noting that the relevant property used to prove these results happens at the level of localizing motive in the sense of Blumberg–Gepner–Tabuada [BGT13].
   - Suggested references: [LT19], [Lev22]

8. **Talk 4 Tuesday: Cochains on a circle.**
   - Speaker: Cary Malkiewich
   - Abstract: This talk will study the cochains on the circle as an $\mathbb{E}_\infty$-ring in cyclotomic spectra. In particular, its study is largely controlled by studying the free loop space of the p-adic circle $B\mathbb{Z}_p$. As a consequence, the coassembly map for the TC of the fixed points by a trivial $\mathbb{Z}$-action is usually not an isomorphism.
   - Suggested references: [BHLS23, §3]

9. **Talk 1 Wednesday: Ambidexterity and cyclotomic extensions.**
   - Speaker: Allen Yuan
   - Abstract: This talk will present the property of higher semi-additivity in general and in particular for $Sp_{T(n)}$. In addition higher semi-additivity will be used to defined the telescopic cyclotomic extensions. The notion of cyclotomic completion will be discussed.
   - Notes for the speaker:
     - state ambidexterity
     - construct the cyclotomic extensions using ambidexterity
     - define and discuss cyclotomically complete spectra.
   - Suggested references: [CSY22], [CSY21] and [BCSY24]

[1] The speaker may assume that $p > 5$, so that $V(2)$ exists as a homotopy commutative and associative ring, or alternatively may make use of the motivic spectral sequence.

10. **Talk 2 Wednesday: Cyclotomic redshift.**
    - Speaker: Shay Ben-Moshe
    - Abstract: In this talk we prove that $T(n+1)$-localized algebraic K-theory satisfies descent for π-finite p-group actions on stable ∞-categories of chromatic height up to n. As a consequence, we use this to show that cyclotomic extensions "redshift" and to present cyclotomic completion of a $T(n+1)$-localized algebraic K-theory as an co-assembly map.
    - Suggested references: [BCSY23]

11. **Talk 1 Thursday: Calculating** TC **of** $\ell^{hp^k\mathbb{Z}}$**.**
    - Speaker: Irina Bobkova
    - Abstract: This talk will disprove the telescope conjecture at height 2 and primes $p > 5$, by a direct computational approach. Specifically, the speaker will prove that, for k sufficiently large and $\mathbb{Z}$ acting by the Adams operation Ψ^{p+1},
    $$V(2)_* \operatorname{TC}(\ell^{hp^k\mathbb{Z}})$$
    has non-finite homotopy groups.
    - Notes for the speaker: Some key points to emphasize:
      - Using the module structure over $\operatorname{TC}^-(\mathbb{S}^{\mathrm{B}\mathbb{Z}})$, the homotopy fixed point spectral sequence computing $V(2)_* \operatorname{TC}^-(\ell^{hp^k\mathbb{Z}})$ may be viewed as a family of spectral sequences indexed over $p^k\mathbb{Z}_p$. At $0 \in p^k\mathbb{Z}_p$, the spectral sequence is the S^1 homotopy fixed point spectral sequence for $\operatorname{TC}^-(\ell)^{hp^k\mathbb{Z}}$. Accounting for algebra structure, only finitely many differentials control this homotopy fixed point spectral sequeence, and therefore the 0 fiber controls a neighborhood of $0 \in p\mathbb{Z}_p$.
      - TP can be handled in a similar manner, and we can control φ and can using these same techniques.
      - The key non-formal step is controlling the class ζ in π_{-1}.
    - Suggested references: Oberwolfach report 34/2023, [BHLS23, §7], final video/notes from Mark Behrens' graduate course

12. **Talk 2 Thursday: Asymptotic constancy I.**
    - Speaker: Eva Belmont
    - Abstract: TBA
    - Notes for the speaker: This and the following talk form a pair and the speakers should coordinate to divide the following material over two talks:
      - Statements of asymptotic constancy
      - locally unipotent actions
      - A.3 and 4.1
      - The Dehn twist trivialization.
      - Bootstrapping from constancy for THH as a spectrum to constancy for THH as a cyclotomic spectrum.
    - Suggested references: [BHLS23, §4]

13. **Talk 3 Thursday: Asymptotic constancy II.**
    - Speaker: Ishan Levy
    - Abstract: TBA
    - Notes for the speaker: The contents of this talk should be coordinated in conjunction with the previous talk.

14. **Talk 4 Thursday: Q&A Session.** This will be an hour reserved for the audience to ask questions.

15. **Talk 1 Friday: Examples at low primes and large heights.**
    - Speaker: Andrew Senger
    - Abstract: To apply asymptotic constancy, we require fp-type n ring spectra that both satisfy Lichtenbaum–Quillen and admit locally unipotent $\mathbb{Z}$-actions. For our study of the telescope conjecture, these $\mathbb{Z}$-actions should be related to cyclotomic extensions.

At primes $p \geq 5$ and height $n+1=2$, we may use the Adams summand $\mathrm{BP}\langle 1\rangle$, the computations of Ausoni–Rognes, and geometrically defined Adams operations. At primes $p < 5$ and height $n+1=2$, we may still use geometrically defined Adams operations on the Adams summand, but need a replacement for the Ausoni–Rognes proof of Lichtenbaum–Quillen. At a general prime and height, this talk will summarize how $\mathrm{BP}\langle n\rangle$ can be constructed as an $\mathbb{E}_3$-ring with $(\mathbb{E}_1 \otimes \mathbb{A}_2)$-Adams operations, and how one may prove the Lichtenbaum–Quillen property for it.

- Suggested references: [BHLS23, §5,§7]

16. **Talk 2 Friday: Assembling the disproof.**
    - Speaker: Jeremy Hahn
    - Abstract: TBA
    - Suggested references:
17. **Talk 3 Friday: Future directions.**
    - Speaker: Lior Yanovski
    - Abstract: TBA
        - using this to get growth in stable stems
    - Suggested references:
18. **Talk 4 Friday: Future directions.**
    - Speaker: Robert Burklund
    - Abstract: TBA
        - Galois lifting and the universality of hyperdescent as an obstruction
    - Suggested references:

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Bibliography by Agnès France Marie Beaudry.
October 8, 2024

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